Homework

V19028: Dynamical Systems II

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Problem 21: Consider the space

$$\Sigma_2 \;=\; \left\{ s = (s_j)_{j\in\mathbb{Z}} \;\middle|\; s_j \in \{0,1\} \right\}$$

of sequences on the two symbols $\{0,1\}$, as described in class. The topology on Σ_2 is the product topology; it is generated by the cylinder sets

$$N_k(s) \;:=\; \{\; \tilde{s} \in \Sigma_2 \;\mid\; \tilde{s}_j = s_j \;\text{ for all }\; |j| \le k \;\}, \qquad s \in \Sigma_2, \quad k \in \mathbb{N},$$

i.e. the open sets are all possible unions of arbitrary cylinder sets. For every $\lambda \in (0,1)$ define the metric dist_λ on Σ_2 by

$$\mathrm{dist}_\lambda\,(s,\tilde{s}) \;:=\; \sum_{j\in\mathbb{Z}} \lambda^{|j|}\;|s_j - \tilde{s}_j|\,.$$

(i) Prove that for arbitrary $\lambda \in (0,1)$ the metric dist_λ induces the above defined topology on Σ_2.

(ii) Prove that for arbitrary $\lambda, \tilde{\lambda} \in (0,1)$, $\lambda \ne \tilde{\lambda}$, the metric dist_λ is *not* equivalent to $\mathrm{dist}_{\tilde{\lambda}}$.

Reminder: The topology induced by a metric μ on a space X is generated by all its ε-Balls, i.e. the open sets are all possible unions of arbitrary sets of the form

$$B_\varepsilon(x) \;:=\; \{\; \tilde{x} \in X \;\mid\; \mu(x,\tilde{x}) < \varepsilon \;\}, \qquad x \in X, \quad \varepsilon > 0.$$

Two metrics $\mu, \tilde{\mu}$ on X are called equivalent, if there exists a positive constant $C > 0$ such that

$$\frac{1}{C}\mu(x,y) \;<\; \tilde{\mu}(x,y) \;<\; C\mu(x,y) \qquad \text{for all} \quad x,y \in X.$$

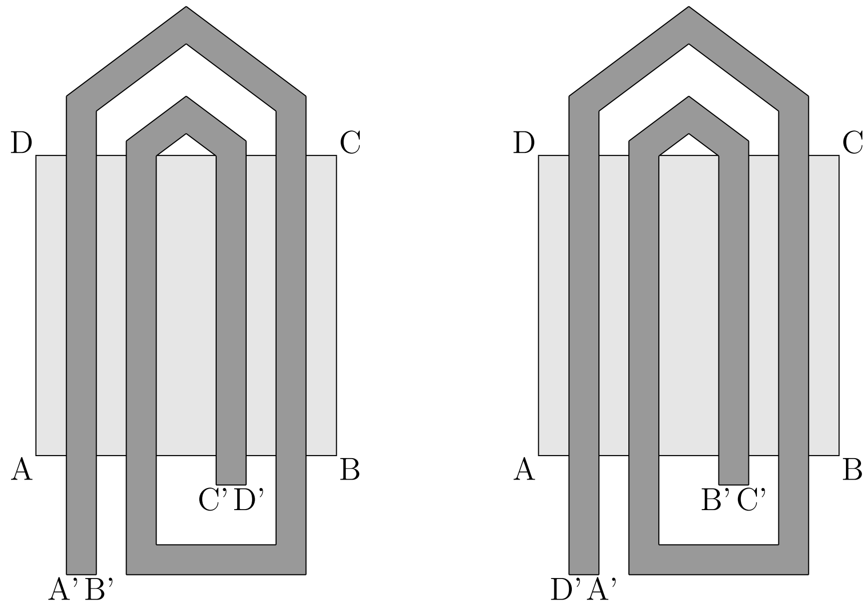
Equivalent metrics necessarily induce the same topology.

Problem 22: Consider again the space Σ_2 with its topology from the previous problem.

(i) Prove that the cylinder sets $N_k(s)$ are also closed. Thus there are *open and closed.*

(ii) Use this fact to prove that Σ_2 is totally disconnected, i.e. for arbitrary $s, \tilde{s} \in \Sigma_2$, $s \ne \tilde{s}$, there are open sets $U, \tilde{U} \subset \Sigma_2$, such that $s \in U$, $\tilde{s} \in \tilde{U}$, $U \cap \tilde{U} = \emptyset$, and $U \cup \tilde{U} = \Sigma_2$.

Problem 23: Which of the following "paper-clip" maps gives rise to shift dynamics? (You can assume the maps to be affine linear, in the regions of intersection.)



Problem 24: Let the assumptions of the theorem about the C^0-horseshoe be satisfied for the iteration Φ on the square Q. Thus, there exists a homeomorphism τ conjugating the shift $\sigma : S \to S$ to $\Phi : I \to I$, on an invariant subset $I := \tau(S) \subset Q$. Let the horizontal and vertical Lipschitz-curves $U(s)$ and $V(s)$ be defined as in class, that is

$$
\begin{aligned}
U(s) &:= \left\{ q \in Q \;\middle|\; \Phi^{-k}(q) \in V_{s_k} \;\; \forall k \geq 1 \right\}, \\
V(s) &:= \left\{ q \in Q \;\middle|\; \Phi^{-k}(q) \in V_{s_k} \;\; \forall k \leq 0 \right\},
\end{aligned}
$$

for any sequence $s = (s_k)_{k\in\mathbb{Z}} \in S$ and the primary vertical stripes $\{V_a \,|\, a \in A\}$ of the horseshoe construction.

Consider the unstable and stable sets of points $p \in I$,

$$
\begin{aligned}
W^{\mathrm{u}}(p) &:= \left\{ q \in Q \;\middle|\; \Phi^k(q) \in \bigcup_{a\in A} V_a \;\; \forall k \leq -1, \quad \lim_{k\to-\infty} \operatorname{dist}\left(\Phi^k(p), \Phi^k(q)\right) = 0 \right\} \\
W^{\mathrm{s}}(p) &:= \left\{ q \in Q \;\middle|\; \Phi^k(q) \in \bigcup_{a\in A} V_a \;\; \forall k \geq 0, \quad \lim_{k\to\infty} \operatorname{dist}\left(\Phi^k(p), \Phi^k(q)\right) = 0 \right\}.
\end{aligned}
$$

Let $p = \tau(s)$ be a point of the invariant set with corresponding sequence s. Prove:

(i) $U(s) \subset W^{\mathrm{u}}(p)$ and $V(s) \subset W^{\mathrm{s}}(p)$;

(ii) $U(s)$ and $V(s)$ are the connected components of $W^{\mathrm{u}}(p)$ and $W^{\mathrm{s}}(p)$ containing p. Thus, they are the local unstable and stable (Lipschitz) manifolds of p.