Homework

V19028: Dynamical Systems II

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Problem 25: [(infinite) adding machine] Consider the space

$$\Sigma_2^+ \;=\; \left\{ s = (s_j)_{j\in\mathbb{N}\cup\{0\}} \;\middle|\; s_j \in \{0,1\} \right\}$$

of (one sided) sequences on the two symbols $\{0,1\}$. The topology on Σ_2^+ is the product topology, as for the space of two-sided sequences defined in class. It is generated by the cylinder sets

$$N_k(s) \;:=\; \left\{ \; \tilde{s} \in \Sigma_2^+ \;\middle|\; \tilde{s}_j = s_j \;\; \forall j \le k \; \right\}, \qquad s \in \Sigma_2^+, \quad k \in \mathbb{N},$$

and an equivalent metric would be

$$\operatorname{dist}(s,\tilde{s}) \;:=\; \sum_{j\ge 0} 2^{-j}\; |s_j - \tilde{s}_j| \,.$$

For an arbitrary but fixed element $g \in \Sigma_2^+$ define the (addition) map

$$T_g \;:\; \Sigma_2^+ \longrightarrow \Sigma_2^+, \qquad a = (a_n)_{n\in\mathbb{N}\cup\{0\}} \longmapsto T_g(a) = (T_g(a)_n)_{n\in\mathbb{N}\cup\{0\}}$$

recursively by

$$\begin{aligned} r_0 &:= 0, \\ T_g(a)_n + 2r_{n+1} &:= a_n + g_n + r_n, \qquad n = 0,1,\ldots \end{aligned}$$

Here the temporary variable $r \in \Sigma_2^+$ denotes the overflow of the addition. If one takes *finite* sequences as binary representations of integer numbers then $T_g(a)$ is just the sum of a and g.

(i) Let $g_0 = 0$. Prove that Σ_2^+ does not contain a dense orbit of T_g. (Find, for example, nontrivial invariant subsets of Σ_2^+.)

(ii) Let $g = (1,0,0,0,\ldots)$. Find a dense orbit.

Free extra: Prove that T_g has a dense orbit for arbitrary g with $g_0 = 1$. In this case, *each* orbit of T_g is dense in Σ_2^+. However there are no periodic orbits and no sensitive dependence on initial conditions, thus no "chaos".

Problem 26: Let $\mu > 0$ be fixed. Consider a map $u : [0,1] \to [0,1]$ and the horizontal cone $S^+ = \{ (\xi, \eta) \, : \, |\eta| \leq \mu |\xi| \, \}$.

(i) Prove that u is a horizontal curve (i.e. u is Lipschitz continuous with Lipschitz constant μ) if, and only if, its graph lies inside every horizontal cone attached to it (i.e. graph $(u) \subset S^+ + (x, u(x))$ for all $x \in [0,1]$).

(ii) Let u be differentiable. Prove that u is a horizontal curve if, and only if, every tangent vector lies in S^+.

(iii) Formulate similar cone conditions for maps $v : \mathbb{R}^n \to \mathbb{R}^n$.

Problem 27: Consider the C^1-iteration Φ on the square Q. Let the assumptions of the theorem about the C^1-horseshoe be satisfied. Thus, we have shift dynamics on the invariant set $I^\Phi \subset Q$.

Prove that all nearby Iterations Ψ also contain a shift, i.e. there exists an $\varepsilon > 0$ such that every C^1-iteration Ψ on Q with

$$\|\Phi - \Psi\|_{C^1} < \varepsilon$$

gives rise to shift dynamics on a invariant set $I^\Psi \subset Q$.

Find a homeomorphism of $\Phi|_{I^\Phi}$ to $\Psi|_{I^\Psi}$.

Remark: Assume that the vertical stripes do not touch the vertical boundaries of Q, i.e. $0 < x < 1$ for all $(x, y) \in V_a$, and likewise for the horizontal stripes $\Phi(V_a)$.

Problem 28: [Proof of hyperbolic horseshoe] Consider a C^1-iteration Φ on the square Q satisfying the assumptions of the theorem on the C^1-horseshoe. In particular, this includes a positive parameter $\mu < 1/2$ and forward/backward invariant cones $S^+ = \{(\xi, \eta) \, : \, |\eta| \leq \mu |\xi|\}$, $S^- = \{(\xi, \eta) \, : \, |\xi| \leq \mu |\eta|\}$,

$$D\Phi(p) \, S^+ \subset \operatorname{int} S^+ \cup \{0\}, \qquad D\Phi^{-1}(p) \, S^- \subset \operatorname{int} S^- \cup \{0\},$$

for each $p \in \bigcup_{a \in A} V_a$, with the expansion/contraction properties

$$\begin{aligned} |\xi_1| &\geq \mu^{-1} |\xi| \qquad \text{for all} \quad (\xi, \eta) \in S^+, \\ |\eta_1| &\leq \mu |\eta| \qquad \text{for all} \quad (\xi_1, \eta_1) \in S^-, \end{aligned}$$

with $\binom{\xi_1}{\eta_1} = D\Phi(p) \binom{\xi}{\eta}$. Thus there exists a horseshoe on some invariant set $I \subset Q$. Assume additionally

$$\mu^2 < \inf_{p \in I} |\det D\Phi(p)| \quad \text{and} \quad \mu^2 < \inf_{p \in I} |\det D\Phi^{-1}(p)|.$$

Prove: there exists a unique hyperbolic structure on I.

Hint: Consider line bundles $L^\pm(p)$ in $S^\pm$ on I, say given by $\eta = \alpha^+(p)\xi$ and $\xi = \alpha^-(p)\eta$ with $|\alpha^\pm| \leq \mu$. Then prove that the action of $D\Phi^{\pm 1}$ on $L^\pm$ is a contraction.