Homework

V19028: Dynamical Systems II

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Problem 33: Consider the iteration on the 2-torus $T = (\mathbb{R}/\mathbb{Z})^2$ defined by the matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find a horseshoe for a suitable iterate B^k, $k > 0$.

Hint: Identify the torus with the unit square centered at $(0, 0)$ and investigate the images of a parallelogram parallel to the eigenvectors of B.

Problem 34: Let Φ be a diffeomorphism of the plane $\mathbb{R}^2$ with a transverse homoclinic orbit. In class, we found a shift on two symbols for an iterate Φ^n. Prove that for every $m \in \mathbb{N}$ the shift of m symbols is conjugate to some iterate Φ^n on a suitable subset of $\mathbb{R}^2$.

Problem 35: Sketch the stable and unstable manifolds to an equlibrium with a transverse homoclinic point of a diffeomorphism of your choice. Extend the manifolds as far as possible such that the picture remains consitent with all your knowledge about dynamical systems, in particular the λ-lemma.

Problem 36: [Horocycles] Consider the POINCARÉ-model of hyperbolic geometry, that is the upper half plane $\mathcal{H} = \{z = (x, y) \in \mathbb{R}^2 \,:\, y > 0\}$ with the arclength element

$$\mathrm{d}s^2 \;=\; \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

The geodesics of $\mathcal{H}$ are the vertical straight lines, $\{z = (x, y) \,:\, x = c,\ y > 0\}$, $c \in \mathbb{R}$, and the (Euclidean) circles whith centers on the x-axis, $\{z = (x, y) \,:\, (x-c)^2 + y^2 = r,\ y > 0\}$, $c \in \mathbb{R}$, $r > 0$.

(i) Consider the geodesic flow Φ. Choose initial condition $z_0 = (0, 1)$, $\dot{z}_0 = (0, 1)$. What is the orbit to this initial condition? What is the stable set of $(z_0, \dot{z}_0)$, i.e the set of points $(w_0, \dot{w}_0)$, such that $z(t) - w(t) \to 0$ as $t \to \infty$?

(ii) Prove that the following transformations are isometries of $\mathcal{H}$:

$$\begin{array}{lrcll}
\text{translation} & \tau_a(x, y) & = & (x + a, y), & \\
\text{scaling} & \theta_b(x, y) & = & (bx, by), & b > 0, \\
\text{inversion in the unit circle} & \sigma(x, y) & = & \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), &
\end{array}$$

(iii) What are the stable/unstable sets of an arbitrary point in the tangent bundle of $\mathcal{H}$ with respect to the geodesic flow Φ?