Homework

V19028: Dynamical Systems II

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Problem 41: [Arnol'd, (Russian) sample examination problems] It is known from experience that light is refracted at the interface between media such that the sines of the angles of the incident and refracted rays with the normal to the interface are inversely proportional to the indices of refraction of the media:

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{n_2}{n_1}.$$

Pass to the continuous limit and find the form of light rays (geodesics) in the plane $(x, y) \in \mathbb{R}^2$ if the index of refraction is $n = n(y)$.

Draw the rays emanating in different directions from the origin in a plane with index of refraction $n = n(y) = y^4 - y^2 + 1$.

Remark: This explains the formation of a mirage: the air over a desert has its maximum of refraction in a certain finite height, due to more rarefied air in higher and lower layers.

Acoustic channels in the ocean are a similar phenomenon: the maximum of refraction is found at a depth of 500m-1000m.

The half plane $\{y > 0\}$ with the index of refraction $n(y) = 1/y$ gives a model of Lobachevskian geometry.

Problem 42: Show that the system

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y, \end{aligned}$$

has infinitely many local center manifolds of $(x, y) = (0, 0) \in \mathbb{R}^2$. Show that the part of the center manifold in the half plane $\{x > 0\}$ is unique.

Free extra: Modify the vector field outside a neighborhood of the origin to obtain any of these local center manifolds as a global one.

Problem 43: Let $\mathcal{M}^c = \operatorname{graph} \psi$ be a center manifold of the flow Φ_t to the vector field

$$\dot{x} = Ax + g(x).$$

Here $\psi : E^c \to E^h$ where $\mathbb{R}^n = E^c \oplus E^h$ is the eigenspace decomposition w.r.t. A.

Prove that $f|_{\mathcal{M}^c}$ defines a vector field $\mathcal{M}^c \to T\mathcal{M}^c$ on the center manifold.

Let $\pi^{c/h} : \mathbb{R}^n \to E^{c/h}$ be the eigenprojections w.r.t. A. Decompose $f|_{\mathcal{M}^c}$ w.r.t. $(x^c, x^h) = (\pi^c x, \pi^h x)$ and determine the reduced vector field on E^c. Calculate reduced vector field in problem 42.

Problem 44: Let $\mathcal{M}^c = \operatorname{graph} \psi$ be a $\mathcal{C}^1$ center manifold of the flow Φ_t to the vector field

$$\dot{x} = Ax + g(x).$$

Here $\psi : E^c \to E^h$ where $\mathbb{R}^n = E^c \oplus E^h$ is the eigenspace decomposition w.r.t. A.

Prove that $\mathcal{M}^c$ is tangential to E^c, i.e. $\psi'(0) = 0$.

Hint: Use the invariance of $\mathcal{M}^c$ under the flow Φ_t.