

Homework

V19028: Dynamical Systems II

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Problem 45: Consider the system of differential equations

$$\begin{aligned}\dot{x} &= xy, \\ \dot{y} &= -y - x^2.\end{aligned}$$

Use a (local) center manifold to decide whether the equilibrium $x = y = 0$ is asymptotically stable.

Note: Use the invariance of the center manifold to calculate the necessary terms of its Taylor expansion.

Problem 46: Consider the parameter-dependent vector field

$$\begin{aligned}\dot{x} &= 2(x + y)^2 + x - y - \lambda, \\ \dot{y} &= -x + y - \lambda.\end{aligned}$$

Calculate the local center manifold of the origin and the reduced vector field (that is their Taylor expansions of sufficient order) to determine the bifurcation diagram.

Problem 47: Consider a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the flow φ_t with equilibrium $x_0 = 0$. Let $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear involution ($R \circ R = \text{id}$). Assume that the vector field f is equivariant under the symmetry R , that is

$$f \circ R = R \circ f.$$

- (i) Let the assumptions of the theorem on the existence of a *global* center manifold in $x_0 = 0$ be satisfied. Prove that the center manifold $W^c(x_0)$ inherits the symmetry of the vector field, that is

$$R(W^c) = W^c.$$

- (ii) Let the assumptions of the theorem on the existence of a *local* center manifold in $x_0 = 0$ be satisfied. Prove that there exists a symmetric local center manifold $W_{\text{loc}}^c(x_0)$, i.e.

$$R(W_{\text{loc}}^c) = W_{\text{loc}}^c.$$

Free extra: The reduced vector field on the (symmetric) center manifold inherits the symmetry, i.e. it is equivariant under R restricted to the center eigenspace.

Problem 48: Consider a parameter-dependent vector field

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x, y; \lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & B(\lambda) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + g(x, y; \lambda),$$

with $x \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$, some hyperbolic matrix $B(\lambda)$, and terms $g(x, y; \lambda) = \mathcal{O}(|x|^2 + \|y\|^2)$ of higher order. Assume that f is symmetric under the reflection at the y -plane,

$$f \circ R = R \circ f \quad \text{with} \quad R = \text{diag}(-1, 1, \dots, 1).$$

Discuss the reduced vector field on a local center manifold.