Homework

V19028: Dynamical Systems II

Bernold Fiedler, Stefan Liebscher

due date: Thursday, February 7, 2008

Problem 49: Consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\lambda_1 x \\ -\lambda_2 y \end{pmatrix}$$

with $0 \leq \lambda_1 \leq \lambda_2$. Determine all flow-invariant manifolds tangential to the eigenspace E_{λ_1} and their smoothness class $\mathcal{C}^k$ depending on λ_1, λ_2.

Relate your observations to the question of regularity of invariant manifolds corresponding to general eigenvalue splittings

$$\Re\mathrm{e}\ \mathrm{spec}\, A|_{E^s} \ \leq\ -\beta_- \ <\ -\eta_- \ \leq\ \Re\mathrm{e}\ \mathrm{spec}\, A|_{E^c} \ \leq\ \eta_+ \ <\ \beta_+ \ \leq\ \Re\mathrm{e}\ \mathrm{spec}\, A|_{E^u}$$

of the linearization A at an equilibrium.

Problem 50: Consider the system

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= Ax_{\mathrm{c}} + f(x_{\mathrm{c}} + x_{\mathrm{h}}), \\ \dot{x}_{\mathrm{h}} &= Bx_{\mathrm{h}} + g(x_{\mathrm{c}} + x_{\mathrm{h}}), \end{aligned}$$

with $f, g \in C^\kappa$, $f(x) = \mathcal{O}(|x|^2)$, $g(x) = \mathcal{O}(|x|^2)$, and $\mathrm{spec}(A) \subset \mathbf{i}\mathbb{R}$, $\mathrm{spec}(B) \cap \mathbf{i}\mathbb{R} = \emptyset$. Assume the existence of a local center manifold $x_{\mathrm{h}} = h(x_{\mathrm{c}})$, $h \in C^\kappa$.

Prove that the κ-th derivative of h is uniquely determined. Describe a method to calculate the Taylor expansion of h.

Hint: Compare the Taylor expansions of $Bh(x_{\mathrm{c}}) + g(x_{\mathrm{c}} + h(x_{\mathrm{c}}))$ and $Dh(x_{\mathrm{c}})\,[Ax_{\mathrm{c}} + f(x_{\mathrm{c}} + h(x_{\mathrm{c}}))]$ and use the fact that $h(x_{\mathrm{c}}) = \mathcal{O}(|x_{\mathrm{c}}|^2)$.

Problem 51: Discuss the "cusp bifurcation"

$$\dot{x} \ =\ x^3 + \lambda x + \mu, \qquad x, \lambda, \mu \in \mathbb{R}.$$

In particular, determine number and stability of equilibria for all parameters (λ, μ) and the parameter curves along which degenerate equilibria (i.e. saddle-node bifurcations) occur.

Problem 52: Discuss the "local map" $\Pi_{\text{loc}} : \Sigma_{\text{in}} \to \Sigma_{\text{out}}$ given by the flow to the vector field

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \lambda & -\omega & 0 \\ \omega & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with $\lambda < 0 < \mu$ and

$$\begin{aligned} \Sigma_{\text{in}} &= \{ \, x^2 + y^2 = 1, \; x > 1/\sqrt{2}, \; |z| \leq 1 \, \}, \\ \Sigma_{\text{out}} &= \{ \, x^2 + y^2 \leq 1, \; z = 1 \, \}. \end{aligned}$$