Homework

V19028: Dynamical Systems II

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Free extra problems

Problem 53: Consider a diffeomorphism Ψ with a transverse homoclinic point. For every point x_0 and given $\epsilon > 0$ define the return times to the ε-neighborhood of x_0:

$$R_\varepsilon(x_0) := \{t \in \mathbb{Z} | \Psi^t(x_0) \in U_\varepsilon(x_0)\}.$$

(i) Prove the existence of Poisson trajectories, that is of points x_0 such that the set of return times is unbounded and has arbitrarily large gaps for small enough ε:

$$\exists \varepsilon_0 > 0 \; \forall \varepsilon < \varepsilon_0 \; \forall N \in \mathbb{N} \; \exists t_1, t_2 \in R_\varepsilon(x_0) \\ \text{such that} \quad t_2 - t_1 \geq N \text{ and } \forall t_1 < t < t_2 \; : \; t \notin R_\varepsilon(x_0).$$

(ii) Prove the existence of non-periodic recurrent trajectories, that is of non-periodic points x_0 such that the set of return times is unbounded but with bounded gaps:

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall t \in R_\varepsilon(x_0) \; \exists \tilde{t} \in R_\varepsilon(x_0) \quad \text{such that} \quad 0 < \tilde{t} - t < N.$$

The Cherry flow is an example of a flow on the torus with a saddle equilibrium. The minimal set is nowhere dense and in fact the closure of a Poisson-stable non-recurrent semi-trajectory. The following problems outline its construction.

Problem 54: Consider the constant vector field f on the torus. Chose a global transverse section S. Find a local modification $\tilde{f}$ of f with the following properties:

(i) The modified vector field $\tilde{f}$ contains a saddle equilibrium and a sink.

(ii) The section S is still transverse to $\tilde{f}$ and intersects every trajectory except the two equilibria and the heteroclinic orbit between them.

(iii) The return map given by flow induced by $\tilde{f}$ can be extended to a map $\psi : S \to S$ of degree one with monotone lift $\Psi : \mathbb{R} \to \mathbb{R}$.

Problem 55: Consider a monotone map $\Psi : \mathbb{R} \to \mathbb{R}$ with $\Psi(x + 2\pi) = \Psi(x) + 2\pi$ for all $x \in \mathbb{R}$, not necessarily injective. Define a rotation number $\rho(\Psi)$ in analogy to homeomorphisms. Show that $\rho(\Psi)$ depends continuously on Ψ.

Problem 56: Consider the vector field on the torus with two equilibria as above. Find a modification such that

(i) The induced return map Ψ is expansive outside an interval which is contracted to a single point.

(ii) The rotation number $\rho(\Psi)$ is irrational.

The flow to this vector field is called *Cherry flow*.

Problem 57: Consider the Cherry flow constructed above. Discuss the complement of the set of trajectories converging to an equilibrium. Show that its closure intersects the section S in a Cantor set.