Homework

V19263: Basic Course Dynamical Systems

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Problem 1: Differentiate the following maps

(i) $\Phi \,:\, \mathcal{BC}^2([0,1],[0,1]) \times \mathcal{BC}^0([0,1],[0,1]) \longrightarrow \mathcal{BC}^0([0,1],[0,1]), \qquad (g,f) \longmapsto g\circ f,$
i.e. $\Phi(f,g)(x) = f(g(x))$

(ii) $\Psi \,:\, \mathcal{BC}^2([0,1],[0,1]) \longrightarrow \mathcal{BC}^0([0,1],[0,1]), \qquad f \longmapsto f\circ f.$

Here, $\mathcal{BC}^0$ denotes the Banach space of bounded continuous functions and $\mathcal{BC}^k$ denotes the Banach space of bounded functions with k bounded countinuous derivatives.

Free extra: Is Ψ still differentiable if it is considerd as a map $\mathcal{BC}^2([0,1],[0,1]) \to \mathcal{BC}^2([0,1],[0,1])$?

Problem 2: Differentiate the map

$$\Phi : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathcal{BC}^1([0,1]\times\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R}^n)\times\mathcal{BC}^0([0,1],\mathbb{R}^n) \longrightarrow \mathcal{BC}^0([0,1],\mathbb{R}^n),$$

$$\Phi(t_0,x_0,\lambda,f,x)(t) = x_0 + \int_{t_0}^t f(s,x(s),\lambda)\mathrm{d}s$$

Problem 3: We define the set of Lipschitz continuous function on $[0,1]$,

$$\mathcal{C}^{0,1}([0,1],\mathbb{R}) \;:=\; \{\, f\,:\, [0,1]\to\mathbb{R} \;\text{ such that }\; \exists L\forall x,y \,:\, |f(x)-f(y)| \le L|x-y| \,\}\,.$$

Prove that $\mathcal{C}^{0,1}([0,1],\mathbb{R})$ with the norm

$$\|f\|_{\mathcal{C}^{0,1}} \;:=\; \sup_{0\le x\le 1}|f(x)| + \sup_{0\le x<y\le 1}\frac{|f(y)-f(x)|}{y-x}$$

is a Banach space.

Problem 4: Prove that for every $\varepsilon>0$ the function

$$f \,:\, \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

$$f(x) = \begin{cases} x & \text{for } |x|<\varepsilon \\ \varepsilon x/|x| & \text{for } |x|\ge\varepsilon \end{cases}$$

is Lipschitz continuous with Lipschitz constant 1.