Homework

V19263: Basic Course Dynamical Systems

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Problem 5: Consider a flow $\Phi(t, x) = \Phi_t(x)$ on the real axis, $x \in \mathbb{R}$.

(i) Prove or disprove: every perodic orbit is an equilibrium, i.e.

$$\forall\, x_0 \in \mathbb{R} \ \Big(\ \exists\, p > 0 \ : \ \Phi(p, x_0) = x_0 \quad \Longrightarrow \quad \forall\, t \in \mathbb{R} \ : \ \Phi(t, x_0) = x_0 \ \Big)$$

(ii) What are possible α-limits and ω-limits of an orbit of Φ? (Consider bounded and unbounded orbits.)

Problem 6: Consider a flow Φ on $\mathbb{R}^n$, $n \geq 2$. The orbit $\Phi_t(x_0)$ of x_0 is assumed to have arbitrarily small periods, i.e.

$$\forall\, \varepsilon > 0 \ \exists\, 0 < p < \varepsilon \ : \ \Phi_p(x_0) = x_0.$$

Prove: x_0 is an equilibrium.

Problem 7: Let Φ be a flow on $X = \mathbb{R}^n$. Sets of the form

$$\Gamma(x_0) \ = \ \{ \ (t, \Phi(t, x_0)) \ : \ t \in \mathbb{R} \ \} \ \subset \ \mathbb{R} \times X$$

are called *extended solution curves*. Define the time shift S_ϑ on the extended phase space $\mathbb{R} \times X$ by

$$S_\vartheta \ : \quad \mathbb{R} \times X \to \mathbb{R} \times X, \qquad (t, x) \mapsto (t + \vartheta, x).$$

(i) Prove: the shift S_ϑ maps extended solution curves onto extended solution curves, for any fixed ϑ.

(ii) Which extended solution curves remain fixed under *a particular* S_ϑ? Which extended solution curves remain fixed under *all* S_ϑ?

Problem 8: The flow $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ of the mathematical pendulum corresponds to the vector field

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -\sin x \end{pmatrix}.$$

Prove: the flow Φ is *equivariant* under translations by 2π, i.e.

$$\Phi\left(t, \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}\right) = \Phi\left(t, \begin{pmatrix} x \\ v \end{pmatrix}\right) + \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \qquad \text{for all } t, x, v.$$

Hint: You may use the uniqueness of the solution of the initial-value problem without proof.