Homework

V19263: Basic Course Dynamical Systems

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Problem 9: Consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\dot{x} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} x,$$

with $a, b \in \mathbb{R}$. Transform this linear differential equation into polar coordinates:

$$x = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

with $r > 0$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Choose $b \neq 0$ arbitrarily and sketch phase portraits in (r, ϕ)-coordinates and in x-coordinates for $a < 0$, $a = 0$, $a > 0$.

Problem 10: Let $\Phi_{s,t} : \mathbb{R}^N \to \mathbb{R}^N$ be a *periodic* evolution with period $p > 0$, i.e.

$$\text{for all } t, s \in \mathbb{R} : \quad \Phi_{t+p,s+p} = \Phi_{t,s}.$$

Consider the *stroboscope* map $\Pi : \mathbb{R}^N \to \mathbb{R}^N$,

$$\Pi(x) = \Phi_{p,0}(x).$$

Prove:

(i) for all $k \in \mathbb{N} : \quad \Phi_{kp,0} = \Pi^k$;

(ii) for each $t \in \mathbb{R}$ there exists a change of coordinates $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ such that for all $k \in \mathbb{N} : \quad \Phi_{t+kp,t} = \Psi^{-1}\Pi^k\Psi$. Determine Ψ.

Problem 11: Consider the initial-value problem

$$\dot{x}(t) = x(t)^2, \qquad x(0) = x_0 = 1.$$

We know from class that the solution blows up in finite time. Discuss the discretizations

(i) explicit Euler: $x_{n+1} = x_n + \varepsilon x_n^2$,

(ii) implicit Euler: $x_{n+1} = x_n + \varepsilon x_{n+1}^2$.

In particular, calculate numerical solutions for several (small) $\varepsilon > 0$ and compare with the exact solution. Do the discretizations yield global solutions? Why? Do the discretizations break down after finite time? How? Explain!

Free extra: On which time interval do the discretized solutions converge to the exact solution for $\varepsilon \searrow 0$?

Problem 12: The map

$$\Phi_t(x_1, x_2) = (x_1 + t, x_2 + \sigma t), \qquad \sigma \in \mathbb{R},$$

with

$$\Phi_t(x_1 + k, x_2 + n) = \Phi_t(x_1, x_2) + (k, n), \qquad \forall k, n \in \mathbb{Z},$$

defines a flow on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let σ be rational. Describe typical trajectories as well as their α- and ω-limits.

Free extra: What happens for irrational σ ?