Homework

V19263: Basic Course Dynamical Systems

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Problem 13: Consider the initial-value problem

$$\dot{x} = f(x) := x, \qquad x(0) = x_0 := 1$$

on the time interval $0 \le t \le T$ for fixed $T > 0$. Calculate an approximate solution $x(T)$ analytically

(i) by Picard iteration, i.e. determine the n-th Picard iterate $x^{[n]}(T)$;

$$x^{[k+1]}(t) = x_0 + \int_0^t f(x^{[k]}(\tau))\,\mathrm{d}\tau, \qquad x^{[0]}(t) \equiv x_0$$

(ii) by explicit Euler scheme, i.e. determine the value $x_n^{[n]}$ after n Euler steps of stepsize $h^{[n]} = T/n$,

$$x_{k+1}^{[n]} = x_k^{[n]} + h^{[n]} f(x_k^{[n]}), \qquad x_0^{[n]} = x_0$$

Compare!

Problem 14: The initial-value problem

$$\dot{x} = f(x) := x^2, \qquad x(0) = x_0 := 1$$

has a solution for $-\infty < t < 1$ with “blow-up”, $\lim_{t\to 1} x(t) = +\infty$.

Let $(x_k)_{k\in\mathbb{N}}$ be the series of Picard iterates:

$$\begin{aligned} x_0(t) &\equiv x_0, \\ x_{n+1}(t) &= x_0 + \int_0^t f(x_n(s))\,\mathrm{d}s. \end{aligned}$$

(i) Prove: $x_k(t)$ is defined for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

(ii) Calculate $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ explicitly.

(iii) Determine all $t \ge 0$ such that $x_k(t)$ converges to the solution $x(t)$ of the initial-value problem, as $k \to \infty$.

Problem 15: Let $f : X \to X = \mathbb{R}^n$ be locally Lipschitz continuous and $J(x_0) = (t_-(x_0), t_+(x_0))$ the maximal interval of existence of the solution to the initial-value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

Is the map $x_0 \mapsto t_+(x_0) \in (0, \infty]$ continuous?

Problem 16: Show that

$$w(t) = \int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(\varsigma)\,\mathrm{d}\varsigma\right)\,\mathrm{d}s$$

is the solution of the initial-value problem

$$\dot{w}(t) = (\alpha(t) + w(t))\beta(t), \qquad w(0) = 0.$$