Homework

V19263: Basic Course Dynamical Systems

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Problem 17: Consider the differential equation

$$\dot{z} = z^2$$

as a vectorfield on $\mathbb{C} \cong \mathbb{R}^2$. Write the differential equation as a vector field on $\mathbb{R}^2$ in coordinates (x, y), $z = x + \mathrm{i}y$. Solve the differential equation explicitly for all initial values $z(0) = z_0 \in \mathbb{C}$ and determine the maximal time of existence $t_+(z_0)$.

Does $t_+(z_0)$ continuously depend on z_0?

Problem 18: Consider a continuously differentiable vector field $f : X \times \mathbb{R} \to X = \mathbb{R}^n$. Let $x(t, t_0)$ denote the solution at time t of the associated initial-value problem

$$\dot{x}(t) = f(x(t), t), \qquad x(t_0) = x_0.$$

Prove: For any fixed τ such that $x(\tau + t_0, t_0)$ exists, there exists a neighborhood U of t_0 such that the map

$$(t_0 - \varepsilon, t_0 + \varepsilon) \to X, \qquad s \mapsto x(\tau + s, s) = \Phi_{\tau+s,s} x_0,$$

is differentiable with respect to s, for $s \in U$. Which differential equation is solved by $v(t) := \mathrm{D}_{t_0} x(t + t_0, t_0)$?

Problem 19: Consider the Banach space BC^1 of continuously differentiable vector fields $f : X \to X = \mathbb{R}^n$ with

$$\|f\|_{BC^1} := \sup_{x \in X} (|f(x)| + |f'(x)|) < \infty.$$

Let f, g be vector fields in BC^1 and $x(f, t)$ denote the solution at time t of the differential equation

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0.$$

Is the map

$$x(t, \cdot) : BC^1 \to X, \qquad f \mapsto x(t, f),$$

differentiable with respect to $f \in BC^1$, for fixed t ? If so then which differential equation is solved by the variation $v(t) := \mathrm{D}_f x(t, f) g$?

Problem 20: Find a counterexample to the following claim:

$$e^A e^B = e^B e^A,$$

for all real (2×2)-matrices A, B.