Homework

V19263: Basic Course Dynamical Systems

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Problem 25: Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + D(y - x), \\ \dot{y} &= Ay + D(x - y), \end{aligned}$$

with $x, y \in \mathbb{R}^n$, which models two symmetrically coupled oscillators. D denotes a diagonal matrix, $D = \operatorname{diag}(d_1, \ldots, d_n)$, with strictly positive entries, $d_i > 0$. Furthermore, let $\Re\mathrm{e}\,\operatorname{spec}(A) < 0$.

(i) Prove: If $x(0) = y(0)$ then $x(t), y(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

(ii) Find matrices A, D (with the above constraints) such that $x(t) \longrightarrow \infty$ as $t \longrightarrow +\infty$ for some initial condition $x(0)$, $y(0)$.

Hint: In the second part, choose $n = 2$ and consider the invariant subspace $\{x = -y\}$.

Problem 26: [Lissajous figures] Let A be a symmetric real (2×2)-matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}.$$

Consider the Hamilton system with Hamilton function $H(x, \dot{x}) = \frac{1}{2}(\dot{x}^{\mathrm{T}}\dot{x} + x^{\mathrm{T}}Ax)$:

$$(*) \qquad \ddot{x} = -Ax.$$

(i) Transform $(*)$ into a system of decoupled pendulum equations (ω_1, ω_2 real):

$$(**) \qquad \begin{cases} \ddot{y}_1 + \omega_1^2 y_1 = 0, \\ \ddot{y}_2 + \omega_2^2 y_2 = 0, \end{cases}$$

(ii) Sketch the solution $(x_1(t),\ x_2(t))$ of $(*)$ for

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

as well as for a matrix A of your choice. Use initial conditions $x_1 = x_2 = \dot{x}_1 = -\dot{x}_2 = 1$.

(iii) What are the ω-limit sets of your trajectories?

Problem 27: [Variation-of-constants formula for maps] Consider a matrix $A \in \mathbb{R}^{N \times N}$, and a sequence of vectors $\big(f(n) \in \mathbb{R}^N,\ n \in \mathbb{N}\big)$.

Complete and prove the formula

$$x(n) \;=\; A^{\bigcirc} x_0 + \sum_{k=\bigcirc}^{\bigcirc} A^{\bigcirc} f(\bigcirc),$$

for the unique solution $x : \mathbb{N} \to \mathbb{R}^N$ to the initial-value problem

$$x(0) \;=\; x_0, \qquad x(n+1) \;=\; A\,x(n) + f(n), \qquad \forall n \in \mathbb{N}.$$

Problem 28: Let f be a differentiable vector field such that each trajectory is bounded.

Prove or disprove: The ω-limit depends continuously on the initial condition, i.e. if

$$\lim_{n\to\infty} \operatorname{dist}(x_n, x) \;=\; 0,$$

then

$$\lim_{n\to\infty} \operatorname{dist}(\omega(x_n), \omega(x)) \;=\; 0.$$

Here, the distance is defined as

$$\operatorname{dist}(A, B) \;:=\; \inf_{a\in A} \inf_{b\in B} \operatorname{dist}(a, b).$$