Homework

V19263: Basic Course Dynamical Systems

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Problem 37: Let $A \subseteq B \subseteq X = \mathbb{R}^N$ be sets and φ_t a flow on X. The set A is called *chain-recurrent* with respect to B if for every $y_0 \in A$ and every $\varepsilon > 0$, $T > 0$ there exists a positive number $n \in \mathbb{N}$, a sequence of times $t_0, \dots, t_{n-1} \geq T$, and points $y_1, \dots, y_{n-1} \in B$ such that

$$\operatorname{dist}(\varphi_{t_i}(y_i), y_{i+1}) < \varepsilon, \qquad i = 0, \dots, n-1 \pmod n, \text{ i.e. } y_n := y_0).$$

The set A is called *recurrent*, if we can chose chains of length $n = 1$ for all points, i.e. if $y_0 \in \omega(y_0)$ for all $y_0 \in A$.

Prove: For any $x_0 \in X$, the ω-limit $\omega(x_0)$ is chain-recurrent with respect to X, but it is not necessarily recurrent.

Free extra: Let the trajectory $\varphi_t(x_0)$ be bounded. Prove or disprove: The ω-limit $\omega(x_0)$ is chain-recurrent with respect to *itself*.

Problem 38: Let $x = 0$ be an *isolated* equilibrium of a flow f in $\mathbb{R}^2$. Prove or disprove:

(i) $x = 0$ is stable but not asymptotically stable if, and only if, every neighborhood of $x = 0$ contains (at least) one periodic orbit (with positive minimal period).

(ii) If there exists a C^2 Lyapunov function V with $\nabla V(0) = 0$ and strictly indefinite Hessian matrix $\nabla^2 V(0)$ then $x = 0$ is unstable.

Problem 39: Let f be a differentiable vector field on $\mathbb{R}^N$.

Prove or disprove: A trajectory coincides with its ω-limit set if, and only if, the trajectory is an equilibrium or a periodic orbit.

Problem 40: Consider the Van-der-Pol oscillator

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

with positive parameter $\alpha > 0$. Prove the existence of a periodic orbit.

Hint: Apply the theorem of Poincaré & Bendixson. Use polar coordinates (r, φ) to discuss the dynamics near the origin and rescaled coordiantes (ϱ, φ), $\varrho = r^{-1}$, to discuss the dynamics far away from the origin.