Homework

V19263: Basic Course Dynamical Systems

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Voluntary problems

Problem 45: Prove or disprove: Every three-dimensional divergence-free differentiable vector field

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \operatorname{div} f \equiv 0,$$

possesses a regular First Integral $I : \mathbb{R}^3 \to \mathbb{R}$.

Note: a First Integral is called regular if, and only if, its gradient vanishes only at zeros of the vector field, i.e

$$\forall x \in \mathbb{R}^3 \;\; \nabla I(x) = 0 \implies f(x) = 0.$$

Problem 46: Find the general solution to the differential equation

$$\dot{x}(t) = h\left(\frac{x}{t}\right),$$

by separation of variables. In particular, calculate the solution to the initial-value problem

$$\dot{x}(t) = \frac{x^2(t)}{t^2} + \frac{x(t)}{t} + 1, \qquad x(1) = x_0.$$

Hint: Derive an equation for $y(t) := x(t)/t$, as an intermediate step.

Problem 47: Consider the autonomous differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^N$$

with Lipschitz-continuous f. Suppose that all $x \in \mathbb{R}^N$ satisfy the inequality

$$(*) \qquad f(x)^T x \geq \|x\|^3_{\mathbb{R}^N}.$$

Prove: The maximal existence time $t_+(x_0)$ of the solution $x(t)$ is bounded, for every non-zero initial condition $x_0 \neq 0$.

Free extra: Is $x = 0$ automatically an equilibrium if $(*)$ is satisfied?

Problem 48: Consider the differential equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (1 - x^2 - y^2)y - x. \end{aligned}$$

Prove: there exists a *unique* periodic orbit.

Problem 49: Consider two linear dynamical systems in the plane,

$$(1) \qquad \dot{x} = Ax, \qquad\qquad (2) \qquad \dot{x} = Bx,$$

$x = (x_1, x_2) \in \mathbb{R}^2$, $A = (a_{ij})_{1\le i,j\le 2}$, $B = (b_{ij})_{1\le i,j\le 2}$. Combine both system to the piecewise linear system

$$(3) \qquad \dot{x} = \begin{cases} Ax & \text{for } x_1x_2 > 0 \\ Bx & \text{for } x_1x_2 < 0 \end{cases}$$

We call a function $x : (0,T) \to \mathbb{R}$ a solution of (3) to the initial condition $x(0)$ if

- x is Lipschitz-continuous and piecewise differentiable.
- x solves (3) at each point $x(t)$ of the set

$$\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1x_2 \neq 0 \} \cup \{ (0,0) \}$$

- x intersects the complement of Ω only at discrete times, i.e. the set $\{t \in I \,|\, x(t) \notin \Omega\}$ is discrete in $\mathbb{R}$.

(i) Prove: If $a_{12}b_{12} > 0$ and $a_{21}b_{21} > 0$ then system (3) has a global and unique solution for any initial condition.

(ii) Prove or disprove: If, under the assumptions of (i), the origin is asymptotically stable in (1) as well as (2), i.e. if $\Re\mathrm{e}\,\mathrm{spec}\, A < 0$ and $\Re\mathrm{e}\,\mathrm{spec}\, B < 0$, then the origin is asymptotically stable in (3).

Problem 50: Consider a closed, sealed-off lake with prey and predator fish of total masses x and y, respectively. Suppose their dynamics obeys the Volterra-Lotka equations

$$\begin{aligned} \dot{x} &= x(\mu - \nu y), \\ \dot{y} &= y(-\varrho + \sigma x), \end{aligned}$$

with positive fixed parameters $\mu, \nu, \varrho, \sigma$.

Show that the time-averaged prey population

$$\overline{x} := \lim_{t\to\infty} \frac{1}{t} \int_0^t x(\tau)\, \mathrm{d}\tau$$

coincides with the equilibrium value of x. Does the same hold for the predator population?

Very (ε-)cautious fishing changes μ into $\tilde{\mu} = \mu - \varepsilon$ and ϱ into $\tilde{\varrho} = \varrho + \varepsilon$, with $\varepsilon > 0$. Why? Does the time-averaged prey population increase or decrease, due to fishing? What happens to the total population $\overline{x + y}$?

Hint: $x = \sigma^{-1}(\dot{y}/y + \tilde{\varrho})$.

Problem 51: Consider the prey-predator system

$$\begin{aligned} \dot{x} &= x(1 - ax - y), \\ \dot{y} &= y(-c + x - by), \end{aligned}$$

with $(x, y) \in \mathbb{R}^2_+$, and parameters $a > 0$, $b > 0$, $c > 0$, $ac < 1$.

Prove:

(i) there exists a unique equilibrium (x_*, y_*);

(ii) $\omega((x_0, y_0)) = \{(x_*, y_*)\}$, for all initial conditions $x_0 > 0$, $y_0 > 0$.

Hint: Remember the case $a = b = 0$, and compare with it.

Problem 52: Use your favorite numerical integrator (e.g. `dstool`) to plot the solution of the Van-der-Pol oscillator

$$\begin{aligned} \varepsilon\dot{x} &= -y + x(1 - x^2) \\ \dot{y} &= x \end{aligned}$$

with initial conditions $x(0) = 1$, $y(0) = 0$ up to time $t = 10$. Choose parameters $\varepsilon = 0.1, 0.01, 0.001, 0.0007$, and 0.0002. Use stepsize $h = 10^{-3}$. Compare explicit Euler, Runge Kutta, and another solver of your choice. Describe and discuss observations and problems.

Free extra: What happens for $\varepsilon = 0.0007$ to the numerical solution calculated by the Runge-Kutta solver with fixed stepsize $h = 10^{-3}$?

Problem 53: The theorem Poincaré & Bendixson given in class has a rather complicated formulation that (in class) was not justified by examples.

Construct a planar C^1 vector field that contains a bounded trajectory $\gamma(x_0)$ such that the ω-limit set $\omega(x_0)$ contains a point $y_0 \in \omega(x_0)$ whose ω-limit set $\omega(y_0)$ contains at least 2 equilibria.

Hint: Due to the theorem of Poincaré & Bendixson and the compactness of ω-limit sets the set $\omega(y_0)$ must be a *continuum* of equilibria.

Problem 54: Can an unstable equilibrium position become stable upon linearization? Can it become asymptotically stable? Can an asymptotically stable equilibrium become unstable?

Problem 55: Let $y(t)$ be a continuous function solving the integral equation

$$y(t) = \int_0^t y(s)^\alpha \sin(2007\, y(s))\, \mathrm{d}s,$$

for some fixed $\alpha \geq 0$ and all $t \in [0, 1]$.

Prove: $y(t) \equiv 0$ is constant for all $t \in [0, 1]$. For which $\alpha \in [-1, 0)$ does the claim hold?

Problem 56: Can matrices $A, B \in \mathbb{R}^{n\times n}$ fail to commute if $e^A = e^B = e^{A+B} = \mathrm{id}$?

Problem 57: Use your favorite numerical integrator (e.g. `dstool`) to plot the "time-2π map" (POINCARÉ map)

$$\begin{array}{rcl} P: S^1\times\mathbb{R} & \to & S^1\times\mathbb{R} \\ \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} & \mapsto & \begin{pmatrix} x(2\pi) \\ \dot{x}(2\pi) \end{pmatrix} \end{array}$$

of the swing ($x \approx 0$), respectively the inverted pendulum ($x \approx \pi$)

$$\ddot{x} + (\omega^2 + \alpha^2 \sin t)\sin x \;=\; 0.$$

a) near $x = 0$, $\dot{x} = 0$ width $\alpha = 0.2$ and $\omega = 0.05$ resp. $\omega = 0.5$.

b) near $x = \pi$, $\dot{x} = 0$ width $\alpha = 0.2$ and $\omega = 0.05$ resp. $\omega = 0.005$. (Here you should reduce the displayed region to $|\dot{x}| \le 0.03$!).

Examine the stability properties of the swing / pendulum.

Problem 58: [Arnol'd] We want to prove analytically that the upper (unstable) equilibrium of the pendulum can be stabilized by vertical vibrations.

Let l be the length of the pendulum. The vertical forcing has amplitude $a \ll l$ and period 2τ. As a simplification, we assume that the acceleration is piecewise constant, $\pm c = \pm 8a/\tau^2$.

The resulting equation of motion yields

$$\ddot{x}(t) = (\omega^2 \pm \alpha^2)x(t),$$

with $\omega^2 = g/l$ and $\alpha^2 = c/l$. The sign switches with period 2τ. Thus $\alpha^2 = 8a/(l\tau^2) > \omega^2$, for a sufficiently fast forcing ($\tau \ll 1$).

What is the minimal forcing frequency to stabilize the inverted pendulum?

Problem 59: [Arnol'd, (Russian) sample examination problems] It is known from experience that when light is refracted a the interface between media, the sines of the angles formed by the incident and refracted rays with the normal to the interface are inversely proportional to the indices of refraction of the media:

$$\frac{\sin\alpha_1}{\sin\alpha_2} \;=\; \frac{n_2}{n_1}.$$

Find the form of the light rays in the plane $(x, y) \in \mathbb{R}^2$ if the index of refraction is $n = n(y)$. Study the case $n(y) = 1/y$.

Remark: The half plane $\{y > 0\}$ with the index of refraction $n(y) = 1/y$ gives a model of Lobachevskian geometry.

Problem 60: [Arnol'd, (Russian) sample examination problems] To stop a boat at a dock, a rope is thrown from the boat which is then wound around a post attached to the dock. What is the breaking force on the boat if the rope makes 3 turns around the post, if the coefficient of friction of the rope around the post is 1/3, and if a dockworker pulls at the free end of the rope with a force of 100 N (approximately the force one needs to lift $22\frac{1}{2}$ pounds of potatoes)?

Problem 61: [Arnol'd, (Russian) sample examination problems] Draw the rays emanating in different directions from the origin in a plane with index of refraction $n = n(y) = y^4 - y^2 + 1$.

Remark: This explains the formation of a mirage: the air over a desert has its maximum of refraction in a certain finite height, due to more rarefied air in higher and lower layers.

Acoustic channels in the ocean are a similar phenomenon: the maximum of rarefaction is found at a depth of 500m-1000m.