VIII.6 Subharmonic bifurcation [corrections]

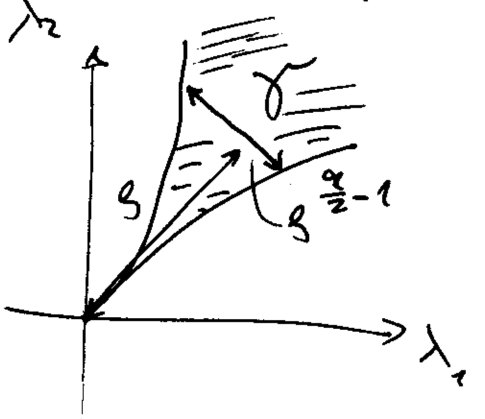
Consider (*) $\quad x_{n+1} = h(\lambda, x_n) \in \mathbb{R}^N; \qquad h(\lambda, 0) = 0.$
with linearization $A(\lambda) := D_x h(\lambda, 0)$.

<u>Theorem</u>: Let $h \in C^k$, $k \geq q-1$, for some fixed $q \geq 3$.
We assume
(i) "<u>root of unity</u>": $\mu_0 = e^{\pm 2\pi i p/q} \in \operatorname{spec} A(0)$,
algebraically simple, with p, q coprime
(i.e. μ_0 is a primitive q-th root of unity);
(ii) "<u>nonresonance</u>": $\mu_0^n \notin \operatorname{spec} A(0)$ for $n \not\equiv \pm 1 \pmod q$,
i.e. spec $A(0)$ does not contain other q-th roots of unity;
(iii) "<u>transversality</u>": $\lambda \in \mathbb{R}^2$ and the local continuation (of $\mu_0 = \mu(0)$)
$\lambda \mapsto \mu(\lambda) \in \mathbb{C} \cong \mathbb{R}^2$ is a local diffeo.

Then there exists a local C^{q-2} function
$$(r, \vartheta) \mapsto \lambda(r, \vartheta) = \lambda_0(r^2) + a e^{-iq\vartheta} r^{q-2} + o(r^{q-2})$$
with period $2\pi/q$ in ϑ such that (*) possesses a q-periodic orbit near $x = 0$ if, and only if, $\lambda = \lambda(r, \vartheta)$ for some small $r > 0$ and some $\vartheta \in [0, 2\pi)$.

<u>Remarks:</u>

1. As usual, the complex coordinate $z = re^{i\vartheta} \in \mathbb{C} \cong \mathbb{R}^2$ parametrizes the q-periodic orbits of $(*)$ in the real two-dimensional eigenspace of $\mu_0 \in \operatorname{spec} A(0)$. Values of ϑ which differ by $2\pi/q$ indicate different points x_n on the same q-periodic orbit, and hence provide the same λ.
2. For $q \geq 5$ and in the nondegenerate case $\lambda_0'(0) \neq 0$, $a \neq 0$ we obtain an open sector $\gamma \subseteq \mathbb{R}^2$ with tip of order $\rho^{\frac{q}{2}-1}$ at $\lambda = 0$ such that $(*)$ possesses exactly



$$\left.\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right\} \begin{array}{l} \text{periodic orbits } x_n \\ \text{with min. period} \\ q, \text{ near } x = 0, \text{ for} \end{array} \left\{\begin{array}{l} \lambda \notin \bar{\gamma} \setminus \{0\} \\ \lambda \in \partial\gamma \setminus \{0\} \\ \lambda \in \gamma \end{array}\right.$$

<u>Proof:</u> We have an equivalent formulation
$$f(\lambda, \underline{x}) = 0$$
for q-periodic orbits of $(*)$, with
$\underline{x} \in \underline{X} := \{(x_j)_{j\in\mathbb{Z}} \mid x_j \in \mathbb{R}^N,\ x_{j+q} = x_j \text{ for all } j\} \cong \mathbb{R}^{Nq}$

$f(\lambda, \underline{x})_j := (S\underline{x})_j - h(\lambda, x_j)$; $\quad (S\underline{x})_j := x_{j+1}$.
Moreover $\langle S \rangle \cong \mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ defines a $\mathbb{Z}_q$-action on $\underline{x} \in \underline{X}$ and f is $\mathbb{Z}_q$-equivariant.
Equivariant Lyapunov-Schmidt reduction provides an equivalent reduced equation
$$\Phi(\lambda, z) = 0$$
with ~~zeta~~ $z \in \mathbb{C} \cong \mathbb{R}^2$ parametrizing $\operatorname{Ker} \mathcal{L}$ for the linearization $\mathcal{L} := D_{\underline{x}} f(\lambda, 0)$.
Note
$$\operatorname{Ker} \mathcal{L} = \{\underline{y} \in \underline{X} \mid \mu_0^{-j} y_j = y_0 \in \operatorname{Ker}(\mu_0 - A(0))\}$$
with equivariant projection
$$\mu_0^{-j} (\underline{Q}\underline{y})_j := \frac{1}{q} \sum_{k=0}^{q-1} \mu_0^{-k} Q y_k, \quad Q = \text{eigenprojection of } \mu_0 .$$

In particular the action of S becomes multiplication by $\mu_0 = e^{2\pi i p/q}$ on $\mathrm{Ker}\,\mathcal{L}$. Because μ_0 is a primitive q-th root of unity, this implies the equivariance

$$\Phi(\lambda, e^{2\pi i/q} z) = e^{2\pi i/q}\,\Phi(\lambda, z).$$

We now expand $\Phi(\lambda, z) = 0$ and compare coefficients $a_{jk}(\lambda)$ in

$$\Phi(\lambda, z) = \sum_{j,k \geq 0} a_{jk}(\lambda)\, z^j \bar{z}^k + \dots,$$

to obtain

$$a_{jk} = 0 \qquad \text{for } j-k \not\equiv 1 \pmod q,$$

by $\mathbb{Z}_q$-equivariance of Φ. The ~~lowest~~ term of lowest order $j+k$ with $j \neq k+1$ which may not vanish is therefore

$$a_{0,q-1}(\lambda)\, \bar{z}^{q-1}$$

For $\Phi \in C^{q-1}$ (by $h \in C^k$, $k = q-1$) we therefore obtain the reduced equation

$$0 = \Phi(\lambda, z) = \alpha(\lambda, |z|^2)z + a_{0,q-1}\bar{z}^{q-1} + o(|z|^{q-1})$$

Here α collects terms with $j=k+1$.
With polar coordinates $z = re^{i\vartheta}$, as promised in our theorem, nontrivial solutions $r \neq 0$ become equivalent to

$$0 = \alpha(\lambda, r^2) + a_{0,q-1} e^{-iq\vartheta} r^{q-2} + o(r^{q-2}).$$

As we know from Hopf bifurcation, $\alpha_0(\lambda) := \alpha(\lambda, 0) = \mu_0 - \mu(\lambda) + o(\lambda)$ comes from the linearization $D_x f(\lambda, 0)$, restricted to $\ker L$. Indeed μ_0 is the action of S on that kernel, and $-\mu(\lambda)$ comes from $D_x h(\lambda, 0) = A(\lambda)$.
Because $\alpha_0(0) = 0$ and $\alpha_0'(0): \mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2$ is invertible, we can solve the above reduced equation for

$$\lambda = \lambda(r, \vartheta)$$

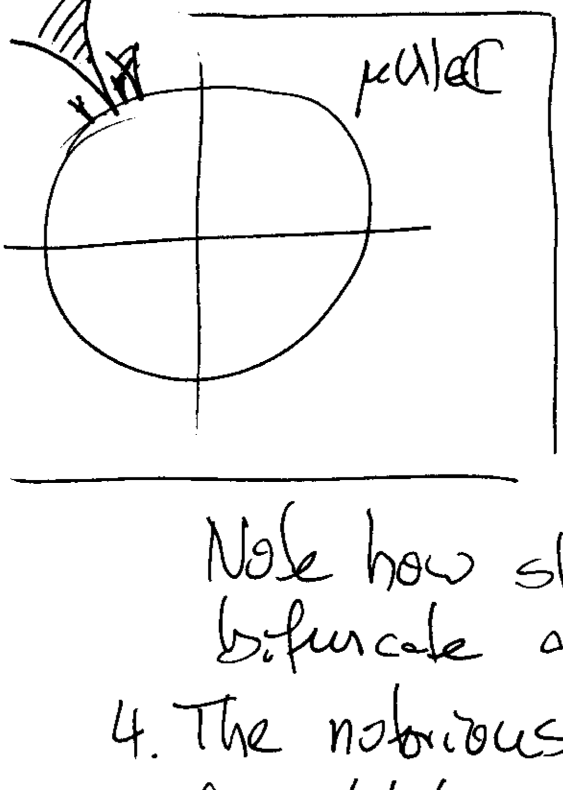
as claimed. Equivariance of

$\Phi(\lambda, z)$ under $\mathbb{Z}_q$ implies $2\pi/q$-periodicity in ϑ. Expansion with respect to r proves the remaining claims of the theorem.

∞

Remarks:

3. Using center manifolds and normal forms, e.g. as in [Marsden, McCracken], it can be proved that an invariant circle (in $\mathbb{R}^N$) bifurcates when $\mu(\lambda)$ crosses the unit circle in $\mathbb{C}$. The sectors γ then provide Arnold tongues for the rational plateaus of the rotation number on that circle.



Note how sharper and sharper tongues bifurcate at <u>each</u> rational angle $\exp(2\pi i p/q)$!

4. The notorious cases $q=3,4$ are still not completely understood in their total dynamics. See e.g. ch. 1, 4 in the Takens birthday volume [Broer, Krauskopf, Vegter] Global Analysis of Dynamical Systems, IOP 2001. [BF, 06/2008