Plane Kolmogorov flows and Takens-Bogdanov bifurcation without parameters: The doubly reversible case.

Dedicated to Klaus Kirchgässner

Andrei Afendikov * Bernold Fiedler † Stefan Liebscher ‡

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Abstract

We consider the Kolmogorov problem of viscous incompressible planar fluid flow under external spatially periodic forcing. Looking for time-independent bounded solutions near the critical Reynolds number, we use the Kirchgässner reduction to obtain a spatial dynamical system on a 6-dimensional center manifold. The dynamics is generated by translations in the unbounded spatial direction. Reduction by first integrals yields a 3-dimensional reversible system with a line of equilibria. This line of equilibria is neither induced by symmetries, nor by first integrals. At isolated points, normal hyperbolicity of the line fails due to a transverse double eigenvalue zero. In particular we describe the complete set $\mathcal{B}$ of all small bounded solutions. In the classical Kolmogorov case, $\mathcal{B}$ consists of periodic profiles, homoclinic pulses and a heteroclinic front-back pair. This is a consequence of the symmetry of the external force.

*andre@spp.keldysh.ru, Keldysh Institute of Applied Mathematics, Moscow, Russia.

†fiedler@math.fu-berlin.de, liebsch@math.fu-berlin.de, Free University of Berlin, Institute of Mathematics I, Arnimallee 2-6, D–14195 Berlin, Germany, http://dynamics.mi.fu-berlin.de/.

1 Introduction

Let us consider a viscous incompressible planar fluid flow generated by the action of a body force $\sigma\mathbf{F}$. The governing Navier-Stokes equations in $\mathbb{R}^2$ are

$$\begin{aligned} (\partial_t + \mathbf{u}\cdot\nabla)\,\mathbf{u} &= -\tfrac{1}{\varrho}\nabla p + \nu\Delta\mathbf{u} + \sigma\mathbf{F}, \\ \nabla\cdot\mathbf{u} &= 0. \end{aligned} \tag{1.1}$$

In balance-of-momentum form with the summation convention they read

$$\begin{aligned} \frac{\partial \varrho u_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} &= \varrho\sigma F_i, \\ \frac{\partial u_j}{\partial x_j} &= 0, \end{aligned} \tag{1.2}$$

with the tensor of momentum flux density

$$\Pi_{ij} \;=\; \varrho u_i u_j + p\delta_{ij} - \varrho\nu(\partial_{x_j} u_i + \partial_{x_i} u_j).$$

Here $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ denotes the velocity field, p pressure, ϱ density, and ν viscosity.

We are not going to discuss the problem in such a general setting. Instead we follow A.N. Kolmogorov [MS61] and consider the fluid flow in a cylindrical channel $(x_1, x_2) \in K = \mathbb{R}\times S^1$ under the action of a horizontal external (body) force

$$\mathbf{F}(x_1, x_2) \;=\; \begin{pmatrix} F(x_2) \\ 0 \end{pmatrix}, \qquad F(x_2) \;=\; F(x_2 + 2\pi), \tag{1.3}$$

which depends only on the vertical cross-section coordinate x_2. Here $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and vector (u, p) is equipped with periodic boundary conditions,

$$\mathbf{u}(t, x_1, x_2) \;=\; \mathbf{u}(t, x_1, x_2 + 2\pi), \qquad p(t, x_1, x_2) \;=\; p(t, x_1, x_2 + 2\pi) \tag{1.4}$$

instead of the usual no-slip boundary conditions. Originally, Kolmogorov suggested to investigate the effects of decreasing viscosity on the dynamics of the problem (1.1–1.4) with

$$\mathbf{F}(x_1, x_2) \;=\; \begin{pmatrix} \sqrt{2}\sin x_2 \\ 0 \end{pmatrix}. \tag{1.5}$$

There are various reformulations of the Kolmogorov question, see for example [AK98b, Sma91]. Even under an additional periodicity condition in the unbounded horizontal direction x_1

$$\mathbf{u}(t, x_1, x_2) = \mathbf{u}(t, x_1 + 2\pi/\alpha, x_2), \qquad \alpha \in (0, 1) \tag{1.6}$$

and for vanishing mean flux

$$\mathbf{Q} := \frac{\alpha}{(2\pi)^2}\int_0^{2\pi}\!\!\int_0^{2\pi/\alpha} \mathbf{u}\,\mathrm{d}x_1\,\mathrm{d}x_2 \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.7}$$

it is still unclear whether Navier-Stokes solutions can be dynamically nontrivial for the Kolmogorov forcing $F(x_2) = \sqrt{2}\sin x_2$.

Stability and bifurcation of the basic steady state $\mathbf{u}_*{=}(U(x_2), 0)^{\mathrm{T}}$, $p_*{=}\,const.$ of the problem (1.1)–(1.7) depends on the Reynolds number $R = \nu^{-2}\sigma L$ where L is the length unit, and were studied in [MS61, Yud65, Yud66, AB86, Afe95]. It was shown in [MS61, Yud65] that the curve $R(\alpha)$ of neutral stability is monotone for $F(x_2) = \sqrt{2}\sin x_2$ and that the minimal critical Reynolds number $R_0 = 1$ corresponds to $\alpha = 0$ with stability exponent $\lambda = 0$. There is numerical evidence that for small enough α a similar statement is true for general forcing $F(x_2)$, [Afe95]. Therefore the loss of stability and the bifurcation problem for $\alpha = 0$, i.e. the bifurcation problem in the unbounded cylinder K, are of particular interest.

It is therefore our goal, in the present paper, to provide a detailed study of time-independent fluid-flow profiles (u_1, u_2) near $\mathbf{u}_* = (U, 0)^{\mathrm{T}}$ without the periodicity requirement (1.6) in the unbounded spatial variable x_1. We thus consider only the time-independent Navier-Stokes system

$$\begin{aligned} 0 &= \nu\Delta u_1 - (u_1\partial_{x_1} + u_2\partial_{x_2})\, u_1 - \varrho^{-1}\partial_{x_1}p + \sigma F(x_2), \\ 0 &= \nu\Delta u_2 - (u_1\partial_{x_1} + u_2\partial_{x_2})\, u_2 - \varrho^{-1}\partial_{x_2}p, \\ 0 &= \partial_{x_1}u_1 + \partial_{x_2}u_2. \end{aligned} \tag{1.8}$$

The fundamental tool for reaching our goal will be the *Kirchgässner reduction* introduced in [Kir82], and extended to the analysis of viscous fluid flows in [IMD89]. In Kolmogorov problem the space of all bounded solutions of elliptic problem (1.8) near the x_1-independent Kolmogorov solution $\mathbf{u}_*$ fits into a 6-dimensional center manifold of spatial profiles. Even though the initial value problem for elliptic equations is ill-posed, translation by x_1 induces an autonomous flow on this manifold. See Section 2 for details. Our study of small bounded solutions of this reduced *spatial-dynamics flow*, where x_1-translation plays the role of a “time” action, will account for new homoclinic pulse-type and heteroclinic multi-pulse solutions to the Kolmogorov problem, near Kolmogorov's instability threshold.

Going beyond Kolmogorov's original choice (1.5), we admit more general horizontal forcing functions $F(x_2)$ which are periodic in the cross-sectional vertical coordinate x_2 and of zero average, but are still independent of the horizontal coordinate x_1. A simple prototype is given by

$$F(x_2) = \sqrt{2}\sin x_2 + \omega \sin 3x_2. \tag{1.9}$$

Such higher harmonics don't increase the complexity of the bifurcation diagram near the instability threshold.

Let $\langle u(x_1,\cdot)\rangle = \frac{1}{2\pi}\int_0^{2\pi} u(x_1,x_2)\mathrm{d}x_2$ denote the cross-sectional average. By incompressibility the mean flux $\langle u_1(x_1,\cdot)\rangle$ does not depend on x_1. In fact $\partial_{x_1}\langle u_1(x_1,\cdot)\rangle = -\langle \partial_{x_2} u_2(x_1,\cdot)\rangle$ vanishes. The same is also true for Couette and Poiseuille plane-channel flows with no-slip boundary conditions. Periodicity (1.4) and vanishing mean value of the forcing, $\langle F\rangle = 0$, imply the existence of two additional conserved quantities

$$\frac{\partial}{\partial x_1}\langle \Pi_{11}(x_1,\cdot)\rangle = 0 \qquad \text{and} \qquad \frac{\partial}{\partial x_1}\langle \Pi_{21}(x_1,\cdot)\rangle = 0. \tag{1.10}$$

This property strongly relies on the absence of stress at the boundaries and is not true for Couette and Poiseuille flows.

The existence of these conserved quantities implies the presence of three nontrivial first integrals I_1, I_2, I_3 of the dynamical system generated by x_1-translations on the spatial center manifold. This fact significantly facilitates our analysis of the reduced equations. Moreover, a three dimensional manifold Ψ_* of x_1-independent solutions arises. For most values of the vector $\mathbf{I} = (I_1, I_2, I_3) \in \mathbb{R}$ of first integrals, the three-dimensional level set $\mathbf{I} = \text{const.}$ is transverse to the equilibrium manifold Ψ_*, in the 6-dimensional spatial-dynamics flow after Kirchgässner reduction. This transversality degenerates precisely in one fiber $I_1 = 0$. In the critical three-dimensional level surface $\mathbf{I} = 0$, a one-dimensional family of equilibria remains (see Section 2 for details).

Vector fields with one- and two-dimensional families of equilibria have been studied by two of the present authors, in a series of papers. The normally hyperbolic case is well-known; see for example [Fen77, Arn88, Sho75, Aul84], and the references there. When normal hyperbolicity fails, the situation resembles bifurcation theory. While the familiar foliation of the flow by a constant bifurcation parameter is absent, a manifold of trivial solutions persists. We call this situation *bifurcation without parameters*; see [FLA00a, FL00, FLA00b, FL01, FL02]. Applications include coupled-oscillator dynamics, oscillatory viscous shock profiles of nonlinear systems of hyperbolic conservation laws with source terms, and binary oscillations of certain discretizations of systems of conservation laws. For an early example involving competition models in population biology see [Far84].

The critical equilibrium in the level surface $\mathbf{I} = 0$ of the Kolmogorov problem is characterized by a triple zero eigenvalue. Transversely to the equilibrium line, the zero eigenvalue is double. In standard bifurcation theory, with parameters, this linearization would correspond to a Takens-Bogdanov bifurcation; see for example [Arn72, Tak74, Bog81a, Bog81b, GH82], and also [Gel99] for the time discrete case. Takens-Bogdanov bifurcation without parameters has been studied in [FL01].

Reversibility of the reduced, spatial dynamics, however, is an additional feature of the plane Kolmogorov flow which substantially changes the Takens-Bogdanov dynamics of [FL01]. Two types of reversibilities, S_1 and S_2, arise, depending on the symmetry properties of the spatially 2π-periodic forcing $F(x_2)$ in the Navier-Stokes system (1.1). Specifically, these reversibilities are generated by the following symmetries

$$\begin{array}{llcl} S_1: & F(-x_2) & = & -F(x_2), \\ S_2: & F(x_2+\pi/s) & = & -F(x_2). \end{array} \tag{1.11}$$

The dimension of the fixed-point subspaces $\mathrm{Fix}(S_j)$ of S_j-fixed vectors will turn out to be j, in the three-dimensional reduced spatial dynamics within $\mathbf{I} = 0$. See Lemma 2.1 and section 3. Note that $F(x_2) = (\sqrt{2}\sin x_2, 0)^T$, as chosen by Kolmogorov, (1.5), with $s = 1$ satisfies both reversibilities. The choice

$$F(x_2) = \sqrt{2}\sin x_2 + \omega \sin 2x_2 \tag{1.12}$$

in contrast, satisfies only S_1, but not S_2, for $\omega \neq 0$.

In section 3, we derive a local normal form for Takens-Bogdanov bifurcations without parameters in the presence of only reversibility S_1.

The easier case of double reversibility, S_1 and S_2, is studied in section 4. See in particular Figure 4.2 for the set of all (small) bounded solutions. The x_1-periodic solutions were known to Yudovich [Yud65] already. In addition, we find homoclinic or pulse type solutions, as well as heteroclinics of front type. Parenthetically we note that similar results can be derived when only the reversibility S_2 is present. It is in fact the two-dimensional fixed-point subspace $\mathrm{Fix}(S_2)$, which greatly facilitates the analysis. Such reversible system appeared in the paper [Ioo00], where travelling waves of the Hamiltonian Fermi-Pasta-Ulam model were studied.

The more intricate case of only the single reversibility S_1, as exemplified by forcing (1.12), will be treated in [AFL08].

We now give a sample statement on the existence of bounded uniformly continuous solutions to Navier-Stokes problem (1.4)-(1.5), (1.8) near the Kolmogorov solution $\mathbf{u}_* = (\sqrt{2}\sin x_2, 0)^{\mathrm{T}}$ and near the instability threshold $R_0 = 1$, where $\varepsilon^2 = R - R_0$ is small.

Theorem 1.1 *To each bounded solution $y(t)$, $t \in \mathbb{R}$ of equation (4.1) corresponds a unique smooth solution $\mathbf{u}(R, x_1, x_2)$ to problem (1.4–1.5, 1.8) which satisfies the asymptotic relation*

$$\begin{aligned} \mathbf{u}(x_1,x_2) &= \begin{pmatrix} U(x_2) \\ 0 \end{pmatrix} + \varepsilon\sqrt{3}y(\sqrt{2/3}\varepsilon x_1)\begin{pmatrix} \sqrt{2}\cos x_2 \\ 1 \end{pmatrix} + \mathcal{O}(\varepsilon^2), \\ p(x_1,x_2) &= \mathcal{O}(\varepsilon^2). \end{aligned}$$

Here, $y \in \{y_{per}^{\Theta,H}, y_{hom}^{\Theta,H}, y_{het}^{0,\frac{1}{4}}\}$ is a bounded orbit of Duffing equation $\ddot{y} + y - y^3 = \Theta$.

These solutions can be readily expressed in terms of special functions. For instance $y_{\mathrm{het}}^{0,\frac{1}{4}}(t) \;=\; \pm\sqrt{2}\tanh(t/\sqrt{2})$.

Notice that in the present article Kirchgässner reduction was performed analytically without using computers in contrast to [IMD89, AM95] where numerical information was exploited in the study of Couette-Taylor and Poiseuille problems.

To conclude this introduction, we remark that the horizontal forcing function $F(x_2)$ in (1.1), a priori, does not have to satisfy any of the reversibility constraints (1.11). Without any reversibilities, however, the investigation of only stationary solutions is not adequate to the hydrodynamical problem since a variety of time periodic solutions close to the Kolmogorov flow can appear [AB86].

Moreover, we do not address the formidable task of determining the PDE stability of our nonlinear Kolmogorov flow profiles, under the time dependent Navier-Stokes system, in two or three space dimensions. Even the global existence of $L^\infty(K)$-solutions of the nonstationary Navier-Stokes system in the Kolmogorov problem has been addressed only recently, see [AM05]. In this sense, our analysis presents only another naive step stumbling into such largely unexplored territories.

2 Basic equations, Kirchgässner reduction, reversibilities and conserved quantities

We consider the Navier-Stokes system (1.1–1.4) in the cylindrical domain $K = \mathbb{R} \times S^1$. We drop the periodicity condition (1.6) in the unbounded horizontal direction x_1, as well as the condition of zero mean flux. The basic steady state is $\mathbf{u}_* = (U(x_2), 0)$, $p_* = \text{const.}$.

We introduce the Reynolds number $R = \sigma\nu^{-2}L$, where $L = 1$ is the unit of length and we take σ/ν as the velocity unit.

In the thus rescaled equation, the first component $U = U(x_2)$ of the basic steady state $\mathbf{u}_*$ is the 2π-periodic solution of the equation

$$U'' + F \;=\; 0 \tag{2.1}$$

with vanishing mean value. Looking for perturbations $\mathbf{v} = (v_1, v_2)^{\mathrm{T}}$ of the Kolmogorov flow in the scaled form $\mathbf{u} = \mathbf{u}_* + R^{-1}\mathbf{v}$, we arrive at

$$\begin{aligned}
\partial_\tau v_1 + R\,(U\partial_{x_1}v_1 + U'v_2) + \partial_{x_1}p &= \Delta v_1 - v_1\partial_{x_1}v_1 - v_2\partial_{x_2}v_1,\\
\partial_\tau v_2 + R\;\; U\partial_{x_1}v_2 \qquad\qquad + \partial_{x_2}p &= \Delta v_2 - v_1\partial_{x_1}v_2 - v_2\partial_{x_2}v_2,\\
\partial_{x_1}v_1 + \partial_{x_2}v_2 &= 0,
\end{aligned} \tag{2.2}$$

with periodic boundary condition in the cross section x_2:

$$\mathbf{v}(\tau, x_1, x_2) = \mathbf{v}(\tau, x_1, x_2 + 2\pi). \tag{2.3}$$

To study the loss of linear stability of the Kolmogorov flow, we temporarily reintroduce the artificial periodicity condition (1.6) in x_1 along the channel, and fix the vector of mean flux $\mathbf{Q} = (0, 0)^{\mathrm{T}}$. The eigenvalue problem then reduces in a standard way (see e.g. [AM95]) to Orr-Sommerfeld equation

$$\lambda\ell_\alpha\chi + \mathrm{i}\alpha(U\ell_\alpha\chi - U''\chi) = R^{-1}\ell_\alpha^2\chi, \tag{2.4}$$

with $' = \frac{\mathrm{d}}{\mathrm{d}x_2}$, $\ell_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}x_2^2} - \alpha^2$, and

$$\chi(x_2) = \chi(x_2 + 2\pi). \tag{2.5}$$

General properties of the Orr-Sommerfeld problem (2.4, 2.5), are well known, see e.g. [DH69]. For instance the spectrum of the problem is discrete. Let $\lambda_0(\alpha, R)$ denote the eigenvalue with maximal real part. Then the condition $\Re\mathrm{e}\,\lambda_0(\alpha, R) = 0$ defines the curve $R = R_0(\alpha)$ of neutral stability in the plane (α, R). For small α this neutral curve is monotone and the minimal Reynolds number corresponds to $\alpha = 0$.

Each of the following two conditions is sufficient to prove that $\lambda_0(R, \alpha)$ is real:

Condition (A) *There exists a shift $x_2 \to x_2 + \varsigma$ such that $U(x_2 + \varsigma)$ is odd,*

$$U(x_2 + \varsigma) = -U(-x_2 + \varsigma). \tag{2.6}$$

Without loss of generality, we use $\varsigma = 0$ throughout the remaining paper, whenever referring to condition (A).

Condition (B) *There exists $s \in \mathbb{N}$ such that*

$$U(x_2) = -U(x_2 + \pi/s). \tag{2.7}$$

Since we know that the minimal critical Reynolds number corresponds to the limit $\alpha = 0$ of unbounded x_1-periods it is necessary to study solutions of the perturbation system (2.2) on the unbounded domain K with $x_1 \in \mathbb{R}$. The method of choice is the Kirchgässner reduction [Kir82], extended to the analysis of viscous fluid flows in [IMD89, AM95, AM01].

Lemma 2.1 *Under condition* (A), the Navier-Stokes system (2.2) is equivariant under the reflection $\widetilde{S}_1 : (x_1, x_2, v_1, v_2) \mapsto (-x_1, -x_2, -v_1, -v_2)$. Under condition (B), system (2.2) is equivariant under the transformation

$$\widetilde{S}_2 : (x_1, x_2, v_1, v_2) \mapsto (-x_1, x_2 + \pi/s, -v_1, v_2).$$

By equivariance we mean that the transformed quantities are solutions whenever the original quantities are. The proof of the lemma is therefore an obvious calculation; see also (2.13, 2.14) below.

Kirchgässner's idea is to rewrite time independent system (2.2) in the form of a dynamical system with respect to spatial "time" x_1 and look for solutions which are uniformly small in x_1. In this setting we consider perturbations of the basic Kolmogorov solution. The Kirchgässner reduction amounts to a center manifold reduction which captures all solutions (v, p) which remain uniformly small for both positive and negative spatial "times" x_1.

Following [IMD89] denote

$$w_1 = -p + \partial_{x_1} v_1 \qquad \text{and} \qquad w_2 = \partial_{x_1} v_2. \tag{2.8}$$

Then the stationary τ-independent Navier-Stokes problem (2.2) takes the form

$$\begin{aligned}
\partial_{x_1} v_1 &= -v_2', \\
\partial_{x_1} v_2 &= w_2, \\
\partial_{x_1} w_1 &= -v_1'' + R(-U v_2' + U' v_2) - v_1 v_2' + v_2 v_1', \\
\partial_{x_1} w_2 &= R U w_2 - 2 v_2'' - w_1' + v_1 w_2 + v_2 v_2'.
\end{aligned} \tag{2.9}$$

Here, $'$ denotes differentiation with respect to x_2. With $\psi = (v_1, v_2, w_1, w_2)^{\mathrm{T}}$, equation (2.9) can be written as

$$\partial_{x_1}\psi = \mathcal{A}_R\psi + \mathcal{B}(\psi,\psi) \tag{2.10}$$

where $\mathcal{A}_R = \mathcal{A}_0 + R\mathcal{A}_1$ and

$$\mathcal{A}_0\psi = -\begin{pmatrix} v_2' \\ -w_2 \\ v_1'' \\ 2v_2'' + w_1' \end{pmatrix}, \mathcal{A}_1\psi = \begin{pmatrix} 0 \\ 0 \\ U'v_2 - Uv_2' \\ Uw_2 \end{pmatrix}, \mathcal{B}(\psi,\psi) = \begin{pmatrix} 0 \\ 0 \\ v_1'v_2 - v_1v_2' \\ v_1w_2 + v_2v_2' \end{pmatrix}. \tag{2.11}$$

The periodic boundary condition is

$$\psi(x_2) = \psi(x_2 + 2\pi). \tag{2.12}$$

The evolution problem (2.10) is reversible with respect to spatial "time" x_1. In the present context, this means that under conditions (A), (B) the relations

$$\mathcal{A}_R \circ S_j = -S_j \circ \mathcal{A}_R, \qquad \mathcal{B} \circ S_j = -S_j \circ \mathcal{B} \tag{2.13}$$

hold for $j = 1, 2$. Here

$$S_1\begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix}(x_2) = \begin{pmatrix} -v_1(-x_2) \\ -v_2(-x_2) \\ w_1(-x_2) \\ w_2(-x_2) \end{pmatrix}, \quad S_2\begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \end{pmatrix}(x_2) = \begin{pmatrix} -v_1(x_2 + \pi/s) \\ v_2(x_2 + \pi/s) \\ w_1(x_2 + \pi/s) \\ -w_2(x_2 + \pi/s) \end{pmatrix}. \tag{2.14}$$

If $\psi(x_1, x_2)$ is a solution of (2.10, 2.12), then in fact $S_j(\psi(-x_1, x_2))$ is also a solution. This is a precise form of the equivariance statement of lemma 2.1, in the spatial dynamics setting.

Equation (2.10) possesses a family of x_1-independent solutions

$$\Psi_* = (\beta_1 + V_{\beta_2}(x_2), \beta_2, \beta_3, 0)^{\mathrm{T}} \tag{2.15}$$

with three independent parameters $\beta_1, \beta_2, \beta_3$. Here β_1 corresponds to the action of the one-parameter subgroup of the symmetry group of the Navier-Stokes system (1.1) generated by $t\partial_{x_1} + \partial_{u_1}$. This is Galilean invariance along the cylinder domain K. Parameter β_3 corresponds to the action of the subgroup generated by ∂_p. Indeed, the pressure in an incompressible fluid is determined only up to a constant. If $F(x_2) = 0$, then β_2 also corresponds to Galilean invariance. In the Kolmogorov problem, however, the Galilean invariance is violated in the cross-sectional

x_2-direction of cylinder due to the nontrivial forcing. Instead there is a curve of solutions parametrized by β_2, such that $V_{\beta_2}(x_2)$ satisfies

$$V''_{\beta_2} - \beta_2 V'_{\beta_2} - R\beta_2 U' = 0, \tag{2.16}$$

with x_2-average $\langle V_{\beta_2}\rangle = 0$. We discuss conserved quantities associated with this problem, next.

Lemma 2.2 *System (2.9) possesses three conserved quantities:*

$$\widetilde{I}_1(\psi) = \langle v_1\rangle, \tag{2.17}$$

$$\widetilde{I}_2(\psi) = \langle(\partial_{x_1} v_1 - p)\rangle - 2R\langle U v_1\rangle - \langle v_1^2\rangle, \tag{2.18}$$

$$\widetilde{I}_3(\psi) = \langle\partial_{x_1} v_2\rangle - R\langle U v_2\rangle - \langle v_1 v_2\rangle, \text{ and } \widetilde{I}_3(S_j\psi) = (-1)^{j+1}\widetilde{I}_3(\psi). \tag{2.19}$$

Along the family (2.15) of x_1-independent solutions Ψ_*, the map $\widetilde{I} : \mathbb{R}^3 \to \mathbb{R}^3$, $(\beta_1, \beta_2, \beta_3) \mapsto (\widetilde{I}_1(\Psi_*), \widetilde{I}_2(\Psi_*), \widetilde{I}_3(\Psi_*))$ is locally invertible for $\beta_1 \neq 0$.

Proof. Invariance claim (2.17), as already discussed in the introduction, is a general consequence of incompressibility.

We prove invariance of (2.18) to be a consequence of the x_1-independence (or x_1-conservation) of the averaged component Π_{11} of the momentum flux density tensor. Indeed, by (2.9) the x_2-average of w_1 satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\langle w_1\rangle = R\langle U' v_2 - U v_2'\rangle + \langle v_1' v_2\rangle - \langle v_1 v_2'\rangle.$$

Integration by parts and incompressibility implies $\langle v_1' v_2\rangle = -\langle v_1 v_2'\rangle = \langle v_1 \partial_{x_1} v_1\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x_1}\langle v_1^2\rangle$. Similarly $\langle U' v_2\rangle = -\langle U v_2'\rangle = \langle U \partial_{x_1} v_1\rangle = \frac{\mathrm{d}}{\mathrm{d}x_1}\langle U v_1\rangle$, etc. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\langle\partial_{x_1} v_1 - p\rangle := \frac{\mathrm{d}}{\mathrm{d}x_1}\langle w_1\rangle = \frac{\mathrm{d}}{\mathrm{d}x_1}\left(2R\langle U v_1\rangle + \langle v_1^2\rangle\right).$$

Invariance of (2.19) can be established in a similar way and the symmetry relations follow from conditions (A) and (B) respectively. The last statement of the lemma follows from the relation

$$\det\left(\left.\frac{\partial(\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3)}{\partial(\beta_1, \beta_2, \beta_3)}\right|_{\psi=\Psi_*}\right) = \beta_1.$$

$\bowtie$

Let us briefly introduce the functional-analysis setting of problem (2.9). In the space P of real-valued trigonometric polynomials, i.e., of finite sums

$$u(x_2) = \sum_k u_k \mathrm{e}^{\mathrm{i}kx_2}, \quad u_{-k} = \overline{u}_k,$$

consider the scalar product

$$(u, \widetilde{u})_\alpha = \sum_k (1 + k^2)^\alpha u_k \widetilde{u}_{-k}, \tag{2.20}$$

for $\alpha \geq 0$. The closure of P in the induced norm $\| \cdot \|_\alpha$ is denoted H^α.

Consider the operator $\mathcal{A}_R : D(\mathcal{A}_R) \to X$, where

$$X{=}(H^1)^2 \times (H^0)^2 \text{ and } D(\mathcal{A}_R){=}\{(v_1, v_2, w_1, w_2) \in (H^2)^2 \times (H^1)^2\}$$

Then $\mathcal{A}_R$ has compact resolvent, with resolvent estimate

$$\|(\mathcal{A}_0 - \mathrm{i}\theta\, \mathrm{id})^{-1}\|_{L(X,X)} \leq \frac{c}{1 + |\theta|}, \qquad \theta \in \mathbb{R}. \tag{2.21}$$

See [IMD89], where the more complicated case of no-slip boundary conditions is considered.

The operator $\mathcal{B}(\psi, \psi)$ is bilinear in $\Psi = (v_1, v_2, w_1, w_2) \in (H^2)^2 \times (H^1)^2$. The Sobolev embedding theorem $H^2 \subset C^1_{\mathrm{per}}[0, 2\pi]$ implies that the operator $\mathcal{B}(\psi, \psi) : D(\mathcal{A}_R) \times D(\mathcal{A}_R) \to X$ is bounded. Since it is also bilinear, it is analytic.

As $\mathcal{A}_R$ possesses compact resolvent, its spectrum consists of eigenvalues of finite multiplicity.

Consider small $|R - R_0|$. For $R < R_0$ there is a pair of real eigenvalues which become purely imaginary for $R > R_0$. This is a direct consequence of the equivalence of the eigenvalue problem $\mathcal{A}_R\psi = \mathrm{i}\alpha\psi$ with the Orr-Sommerfeld equation (2.4) equipped with x_1-periodicity of $2\pi/\alpha$. Perturbation theory [Kat66] implies that there is a multiple eigenvalue $\lambda_0 = 0$ at $R = R_0(0)$. All other eigenvalues are at finite distance from the imaginary axis.

As a first step in the Kirchgässner reduction we determine the eigenspace of $\mathcal{A}_R$ corresponding to the eigenvalue $\lambda_0 = 0$. Our results will be summarized in lemmas 2.3 and 2.4 below. Since operator $\mathcal{A}_R : D(\mathcal{A}_R) \to X$ possesses compact resolvent, the range $\mathrm{rg}\, \mathcal{A}_R$ is closed in X, and $\mathcal{A}_R$ is a Fredholm operator. By [BJS64, Sch59], $\mathrm{rg}\, \mathcal{A}_R$ is orthogonal in $(L^2)^4$ to $\ker \mathcal{A}^*_R$, where $\ker \mathcal{A}^*_R$ is the kernel of the unbounded formal adjoint operator

$$\mathcal{A}^*_R : (L^2_{\mathrm{per}}[0, 2\pi])^4 \to (L^2_{\mathrm{per}}[0, 2\pi])^4, \tag{2.22}$$

$$\mathcal{A}_R^* = \begin{pmatrix} 0 & 0 & -\dfrac{\mathrm{d}^2}{\mathrm{d}x_2^2} & 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}x_2} & 0 & R(2U' + U\dfrac{\mathrm{d}}{\mathrm{d}x_2}) & -2\dfrac{\mathrm{d}^2}{\mathrm{d}x_2^2} \\ 0 & 0 & 0 & \dfrac{\mathrm{d}}{\mathrm{d}x_2} \\ 0 & 1 & 0 & RU \end{pmatrix}. \tag{2.23}$$

Straightforward calculations demonstrate, that $\ker \mathcal{A}_R^*$ is 3-dimensional and is spanned by

$$\xi_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi_{-2} = \begin{pmatrix} -2RU \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \xi_1 = \begin{pmatrix} 0 \\ -RU \\ 0 \\ 1 \end{pmatrix}. \tag{2.24}$$

The next step is to determine the generalized eigenspace of $\mathcal{A}_R$ associated with $\lambda = 0$. Define the integral operator $\mathrm{r} : H^0 \to H^1$ by

$$\mathrm{r}\phi = \Phi \quad \Longleftrightarrow \quad \langle \Phi \rangle = 0, \text{ and } \phi = \Phi'. \tag{2.25}$$

It is straight forward that the kernel $\ker \mathcal{A}_R$ is spanned by the tangent vectors κ_j at $\beta = 0$ to the family Ψ_* of x_1-independent solutions defined in (2.15):

$$\kappa_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \kappa_{-2} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \kappa_1 = \begin{pmatrix} R\mathrm{r}U \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.26}$$

The eigenfunctions κ_{-1} and κ_{-2} correspond to the actions of generators of the symmetry group. They do not give rise to a generalized eigenspace; but κ_1 does. Indeed $\kappa_1 \in \mathrm{rg}\, \mathcal{A}_R$, i.e.

$$\begin{aligned} R\mathrm{r}U &= -v_2', \\ 1 &= w_2, \\ 0 &= -v_1'' + R(U'v_2 - Uv_2'), \\ 0 &= RUw_2 - 2v_2'' - w_1', \end{aligned} \tag{2.27}$$

is solvable. The solution of (2.27) which is orthogonal to ξ_{-1}, ξ_{-2} is given explicitly by

$$\kappa_2 = \begin{pmatrix} R^2\mathrm{r}^2(U\mathrm{r}U - U'\mathrm{r}^2U) \\ -R\mathrm{r}^2U \\ 3R\mathrm{r}U + R^3D_0 \\ 1 \end{pmatrix}. \tag{2.28}$$

The L^2-orthogonality condition $(\kappa_2, \xi_{-2}) = 0$ implies $D_0 = \langle 2U'\mathrm{r}^2 h\rangle$, where

$$h = \mathrm{r}^2(U''\mathrm{r}^2U - U^2) = \mathrm{r}(U'\mathrm{r}^2U - U\mathrm{r}U). \tag{2.29}$$

Notice that condition (B) implies $D_0 = 0$.

In fact, also $\kappa_2 \notin \operatorname{rg}\mathcal{A}$, except at a unique value of the Reynolds number. Indeed, the orthogonality condition

$$(\kappa_2, \xi_1) = 0 \tag{2.30}$$

for the solvability of equation

$$\kappa_2 = \mathcal{A}_R\kappa_3 \tag{2.31}$$

provides us with the critical Reynolds number

$$R_0 = \langle(\mathrm{r}U)^2\rangle^{-1/2}. \tag{2.32}$$

For $R = R_0$ a Jordan block of length at least 3 arises. In the correspondence of the general theory [AM95] and calculations of [Afe95] this critical Reynolds number corresponds to $R_0(0)$ as defined from Orr-Sommerfeld equation (2.4). Define

$$g = \mathrm{r}\left(U\mathrm{r}h - U'\mathrm{r}^2h + 2D_0\right). \tag{2.33}$$

Then the solution orthogonal to $\xi_{-1}, \xi_{-2}, \xi_1$ is

$$\tilde{\kappa}_3 = \begin{pmatrix} -3R_0\mathrm{r}^3U - R_0^3\mathrm{r}g \\ R_0^2\mathrm{r}^2h \\ -\mathrm{r}(1 + R_0^2U\mathrm{r}^2U) - 2R_0^2\mathrm{r}h - D_1 \\ -R_0\mathrm{r}^2U \end{pmatrix} \tag{2.34}$$

with

$$D_1 = 2R_0^2\langle 3U\mathrm{r}^3U + R_0^2U\mathrm{r}g\rangle. \tag{2.35}$$

Condition (A) implies $D_1 = 0$. Setting $U = \sqrt{2}\sin x_2$ in (1.5), for instance, we have $R_0 = 1$, $h = 0$, $D_0 = 0$, $\mathrm{r}^3U = \sqrt{2}\cos x_2$ and therefore $g = 0$, $D_1 = 0$.

The next compatibility condition

$$(\tilde{\kappa}_3, \xi_1) = 0$$

is equivalent to

$$\langle U\mathrm{r}^2h\rangle = 0. \tag{2.36}$$

Obviously, this holds true in any of the cases (A), (B) under consideration. We obtain

$$\tilde{\kappa}_4 = \begin{pmatrix} \kappa_{4,1} \\ 3R_0 \mathrm{r}^4 U + R_0^3 \mathrm{r}^2 g \\ R_0^3 \mathrm{r}(U\mathrm{r}^2 h) - 5R_0 \mathrm{r}^3 U - 2R_0^3 \mathrm{r} g \\ R_0^2 \mathrm{r}^2 h \end{pmatrix}, \tag{2.37}$$

where $\kappa_{4,1} = \mathrm{r}^2 \left(3R_0^2(U'\mathrm{r}^4 U - U\mathrm{r}^3 U) + R_0^4(U'\mathrm{r}^2 g - U\mathrm{r} g) + D_1\right) + 2R_0^2 \mathrm{r}^3 h + \mathrm{r}^3 \left(1 + R_0^2 U\mathrm{r}^2 U\right)$. The compatibility condition $(\tilde{\kappa}_4, \xi_1) = 0$ for the existence of a fifth generalized eigenvector yields

$$K_0 := \; -3R_0^2 \langle U\mathrm{r}^4 U\rangle - R_0^4 \langle U\mathrm{r}^2 g\rangle = 0, \tag{2.38}$$

In general, we expect $K_0 \neq 0$ and therefore a 4-dimensional Jordan block. However, we are unable to prove that $K_0 \neq 0$ for all $U(x_2)$. This condition, however, can easily be checked for specific $U(x_2)$. For the original Kolmogorov setting $U = \sqrt{2}\sin x_2$, we have $\mathrm{r}^4 U = U$, $g = 0$, and $K_0 = -3$; hence $K_0 \neq 0$ for any small perturbations of Kolmogorov's velocity profile. Therefore the following lemma is proved.

Lemma 2.3 *We assume any one of the conditions* (A), (B), *as well as* $K_0 \neq 0$ *in* (2.38). *Then the generalized eigenspace* $M = M(R)$ *associated with the eigenvalue* $\lambda = 0$ *of the operator* $\mathcal{A}_R$ *for the stationary Navier-Stokes problem* (2.10) *possesses* $\dim M(R) = 4$, *if* $R \neq R_0$, *and* $\dim M(R) = 6$, *if* $R = R_0$. *In the latter case the vectors* $\{\kappa_{-2}, \kappa_{-1}, \kappa_1, \kappa_2, \tilde{\kappa}_3, \tilde{\kappa}_4\}$ *defined above form a basis of* $M(R_0)$.

In particular for $U = \sqrt{2}\sin x_2$, we get $R_0 = 1$, $h = g = 0$, $K_0 = -3$ and

$$\kappa_1 = \begin{pmatrix} -\sqrt{2}\cos x_2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \kappa_2 = \begin{pmatrix} 0 \\ \sqrt{2}\sin x_2 \\ -3\sqrt{2}\cos x_2 \\ 1 \end{pmatrix}, \quad \tilde{\kappa}_3 = \begin{pmatrix} -3\sqrt{2}\cos x_2 \\ 0 \\ -1/2\sin 2x_2 \\ \sqrt{2}\sin x_2 \end{pmatrix},$$

$$\tilde{\kappa}_4 = \begin{pmatrix} -1/8\sin 2x_2 \\ 3\sqrt{2}\sin x_2 \\ -5\sqrt{2}\cos x_2 \\ 0 \end{pmatrix}.$$

In a same way the following statement for the adjoint problem holds true.

Lemma 2.4 *Let the assumptions of Lemma* 2.3 *hold. Then the generalized eigenspace $M^*(R_0)$ associated with the eigenvalue $\lambda = 0$ of the formal adjoint operator $\mathcal{A}_R^* : (L^2_{\rm per})^4 \to (L^2_{\rm per})^4$ is spanned by $\{\xi_{-2}, \xi_{-1}, \xi_1\}$ defined in (2.24), and by $\{\xi_2, \xi_3, \xi_4\}$, where*

$$\xi_2 = \begin{pmatrix} -R_0 \mathrm{r} U \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} -R_0^2 U \mathrm{r}^3 U + \mathrm{r}(1 - R_0^2 U' \mathrm{r}^3 U) \\ 0 \\ R_0 \mathrm{r}^3 U \\ 0 \end{pmatrix}, \tag{2.39}$$

$$\xi_4 = \begin{pmatrix} \xi_{4,1} \\ -R_0^2 U \mathrm{r}^4 U \\ R_0^2 \mathrm{r}^2 (U \mathrm{r}^3 U) - \mathrm{r}^3 (1 - R_0^2 U' \mathrm{r}^3 U) \\ R_0 \mathrm{r}^4 U \end{pmatrix},$$

where $\xi_{4,1} = \widetilde{\xi_{4,1}} - \langle \widetilde{\xi_{4,1}} \rangle$,

$$\widetilde{\xi_{4,1}} = R_0 \Big(2\mathrm{r}^3 U - R_0^2 U \mathrm{r}^2 (U \mathrm{r}^3 U) + U \mathrm{r}^3 (1 - R_0^2 U' \mathrm{r}^3 U) -$$

$$\mathrm{r} \left(U' \left(R_0^2 \mathrm{r}^2 (U \mathrm{r}^3 U) - \mathrm{r}^3 (1 - R_0^2 U' \mathrm{r}^3 U) \right) \right) \Big).$$

Note that $\mathrm{span}\{\xi_1, \xi_2, \xi_3\}$ is orthogonal to $\mathrm{span}\{\kappa_{-2}, \kappa_{-1}, \kappa_1\}$ by the construction.

Remark 2.5 . *For the Kolmogorov forcing, $U = \sqrt{2} \sin x_2$, we have $R_0 = 1$ and*

$$\xi_1 = \begin{pmatrix} 0 \\ -\sqrt{2} \sin x_2 \\ 0 \\ 1 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} \sqrt{2} \cos x_2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \xi_3 = \begin{pmatrix} -3/2 \sin 2x_2 \\ 0 \\ \sqrt{2} \cos x_2 \\ 0 \end{pmatrix},$$

$$\xi_4 = \begin{pmatrix} 2\sqrt{2} \cos x_2 - \frac{1}{4} \sqrt{2} \cos 3x_2) \\ -2 \sin^2 x_2 \\ -3/8 \sin 2x_2 \\ \sqrt{2} \sin x_2 \end{pmatrix}.$$

Denote $m_0 = K_0^{-1}(\tilde{\kappa}_3, \xi_4)$ and $\kappa_3 = \tilde{\kappa}_3 - m_0 \kappa_1$, $\kappa_4 = \tilde{\kappa}_4 - m_0 \kappa_2$. For the forcing $F_1 = \sqrt{2} \sin x_2$ we get $m_0 = 157/96$.

Lemma 2.6

The vectors $\{\kappa_{-2}, \kappa_{-1}, \kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ and $\{\xi_{-1}, \xi_{-2}, \frac{1}{K_0}\xi_4, \frac{1}{K_0}\xi_3, \frac{1}{K_0}\xi_2, \frac{1}{K_0}\xi_1\}$ are bi-orthogonal.

Proof. The orthogonality of span$\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ vs. span$\{\xi_{-1}, \xi_{-2}\}$, span$\{\xi_1, \xi_2, \xi_3, \xi_4\}$ vs. span$\{\kappa_{-2}, \kappa_{-1}\}$, and the orthogonality conditions span$\{\kappa_1, \kappa_2, \kappa_3\}\perp\xi_1$ as well as span$\{\xi_1, \xi_2, \xi_3\}\perp\kappa_1$ holds by construction. For $1 \leq l, m \leq 4$ we note the reversibility action

$$S_j\kappa_l = (-1)^{j+l}\kappa_l \quad \text{and} \quad S_j\xi_m = (-1)^{j+m+1}\xi_m. \tag{2.40}$$

From the relations

$$(\kappa_l, \xi_m) = (S_j\kappa_l, S_j\xi_m) = (-1)^{2j+m+l+1}(\kappa_l, \xi_m),$$

follows that $(\kappa_l, \xi_m) = 0$ for $m + l = 0 \bmod 2$. Since for $l + m \leq 5$

$$(\kappa_l, \xi_m) = (\mathcal{A}_{R_0}^{m-1}\kappa_{l+m-1}, \xi_m) = (\kappa_{l+m-1}, \xi_1)$$

we have that $(\kappa_{5-l}, \xi_l) = 1$ for $l \leq 4$ and $(\kappa_l, \xi_m) = 0$ for $l + m - 1 \leq 5$. By construction $\{\xi_{-1}, \xi_{-2}\}$ and $\{\kappa_{-2}, \kappa_{-1}\}$ are bi-orthogonal and since $(\kappa_3, \xi_4) = (\kappa_4, \xi_3) = 0$ the proof is complete.

$\bowtie$

3 Equations on a spatial center manifold

Since the generalized eigenspace of the operator $\mathcal{A}_R$ at $R = R_0$ is 6-dimensional and the resolvent estimate (2.21) holds, the spatial center manifold theorem in the form of [Mie88] can be used to derive the 6-dimensional ODE system that depends on small parameter $\gamma_0 = R - R_0$ and describes the nonlinear spatial dynamics in a vicinity of the equilibria. This theorem is the basis of the Kirchgässner reduction and covers the case of spatial dynamics generated by elliptic problems in cylindrical domains with an infinite number of eigenvalues of the associated linearized problem in both the left and right half of the complex plane.

As a consequence of the symmetries and the related conserved quantities $\widetilde{I}_1(\psi), \widetilde{I}_2(\psi)$ described in Lemma 2.2, the 6-dimensional ODE system on the center manifold can be further reduced to a 4-dimensional system. Indeed, $\widetilde{I}_j(\psi)$, $j = 1, 2$ generate functionally independent first integrals $I_j(\cdot)$ of the system on the center manifold. The resulting 4-dimensional ODE problem inherits the reversibility of the original problem and possesses a line of equilibria which corresponds to the family of solutions Ψ_*. To study the dynamics in a vicinity of this line we will use the first integral $I_3(\cdot)$ generated by $\widetilde{I}_3(\psi)$ and blow up the singularity on the level set $I_3 = 0$. This step requires lengthy calculations of the coefficients of the corresponding Taylor expansions which will be done explicitly.

To derive the nonlinear spatial dynamics of (2.9), (2.12) in the vicinity of the origin on the spatial center manifold M_R, we choose the projector $\mathcal{P}_0 : X \to M(R_0)$ onto the generalized eigenspace $M(R_0)$ of the eigenvalue $\lambda_0 = 0$ to be

$$\mathcal{P}_0\psi = (\psi, \xi_{-2})\kappa_{-1} + (\psi, \xi_{-1})\kappa_{-2} + \sum_{j=1}^{4}(\psi, \xi_{5-j})\kappa_j \tag{3.1}$$

and denote $\mathcal{Q}_0 = (\mathrm{id} - \mathcal{P}_0)$. See (2.24), (2.26), (2.28), (2.34), (2.37) and Lemma 2.4. By construction, the spectral projectors $\mathcal{P}_0$ and $\mathcal{Q}_0$ commute with $\mathcal{A}_{R_0}$. If we decompose

$$\psi = \psi_{\mathrm{P}} + \psi_{\mathrm{Q}}, \qquad \psi_{\mathrm{P}} \in \mathcal{P}_0 X, \quad \psi_{\mathrm{Q}} \in \mathcal{Q}_0 X, \tag{3.2}$$

accordingly, then the stationary Navier-Stokes problem (2.10, 2.12) takes the form

$$\begin{aligned} \partial_{x_1}\psi_{\mathrm{P}} &= \mathcal{A}_{R_0}\psi_{\mathrm{P}} + \mathcal{P}_0(\mathcal{B}(\psi_{\mathrm{P}} + \psi_{\mathrm{Q}}, \psi_{\mathrm{P}} + \psi_{\mathrm{Q}}) + \\ &\quad +(R - R_0)\mathcal{P}_0\mathcal{A}_1(\psi_P + \psi_Q)), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \partial_{x_1}\psi_{\mathrm{Q}} &= \mathcal{A}_{R_0}\psi_{\mathrm{Q}} + \mathcal{Q}_0(\mathcal{B}(\psi_{\mathrm{P}} + \psi_{\mathrm{Q}}, \psi_{\mathrm{P}} + \psi_{\mathrm{Q}}) + \\ &\quad (R - R_0)\mathcal{Q}_0\mathcal{A}_1(\psi_P + \psi_Q)). \end{aligned} \tag{3.4}$$

For Reynolds numbers R in a small neighborhood J_{R_0} of R_0, the center manifold theorem asserts the existence of neighborhoods $Y_{\mathrm{P}} \subset M(R_0)$, $Y_{\mathrm{Q}} \subset \mathcal{Q}_0(X)$ of the origins in the generalized eigenspace $M(R_0)$ and its orthogonal complement $\mathcal{Q}_0(X)$ and a smooth function

$$\Phi : J \times Y_{\mathrm{P}} \longrightarrow Y_{\mathrm{Q}} \cap \mathcal{D}(\mathcal{A}_{R_0})$$

with the following properties:

(i) $\Phi(R_0, 0) = 0$, and $\partial_{\psi_{\mathrm{P}}}\Phi(R_0, 0) = 0$;

(ii) $M_R = \operatorname{graph}\Phi(R, \cdot)$ is a local invariant manifold of (3.3), (3.4) – the center manifold;

(iii) Every solution of (3.3), (3.4) that remains in the neighborhood,

$$(\psi_{\mathrm{P}}(x_1), \psi_{\mathrm{Q}}(x_1)) \in Y_{\mathrm{P}} \times Y_{\mathrm{Q}}$$

for all $x_1 \in \mathbb{R}$, lies on M_R.

This theorem reduces the problem of the local spatial dynamics of system (2.9), (2.12) (i.e. the spatial structure of solutions of the Kolmogorov problem which are uniformly small in x_1) to the 6-dimensional ODE

$$\partial_{x_1}\psi_{\mathrm{P}} = \mathcal{A}_{R_0}\psi_{\mathrm{P}} + \mathcal{P}_0\left(\mathcal{B}\left(\psi_{\mathrm{P}} + \Phi(R, \psi_{\mathrm{P}}), \psi_{\mathrm{P}} + \Phi(R, \psi_{\mathrm{P}})\right) + \gamma_0\mathcal{A}_1(\psi_P + \Phi(R, \psi_P))\right), \tag{3.5}$$

where $\gamma_0 \stackrel{\mathrm{def}}{=} \varepsilon^2 = R - R_0$. In local coordinates $\widehat{\gamma} = (\gamma_0, \gamma_{-2}, \gamma_{-1}, \gamma_1, \ldots, \gamma_4)$ defined by

$$\psi_{\mathrm{P}} \quad = \quad \gamma_{-2}\kappa_{-2} + \gamma_{-1}\kappa_{-1} + \textstyle\sum_{j=1}^{4} \gamma_j \kappa_j, \tag{3.6}$$

equation (3.5) can be written as 7-dimensional ODE system

$$\dot{\gamma_j} = \mathcal{F}_j(\widehat{\gamma}), \quad j = 0, -2, -1, 1, 2, 3, 4; \qquad \mathcal{F}_j : \mathbb{R}^7 \mapsto \mathbb{R} \tag{3.7}$$

with $\dot{\gamma_0} = 0$. The coefficients of the expansion of $\mathcal{F}_j(\widehat{\gamma})$ up to second order does not depend on $\Phi(R, \psi_{\mathrm{P}})$. The higher order terms $\mathcal{F}_{j,\mathbf{k}}\widehat{\gamma}^{\mathbf{k}}$, $|\mathbf{k}| > 2$ and the coefficients of the expansion of $\Phi(R, \psi_{\mathrm{P}})$ can be derived recursively. Expand

$$\widehat{\Phi}(\widehat{\gamma}) = \sum_{|\mathbf{k}| \le n_0} \widehat{\Phi}_{\mathbf{k}} \widehat{\gamma}^{\mathbf{k}} + \mathcal{O}(|\gamma|^{n_0}), \quad \text{where} \quad \widehat{\Phi}(\widehat{\gamma}) := \Phi(R_0 + \gamma_0, \psi_{\mathrm{P}}). \tag{3.8}$$

We substitute the expansion (3.8) into equation (3.4),

$$\frac{d}{dx_1}\widehat{\Phi}(\widehat{\gamma}) = \mathcal{A}_{R_0}(\psi_{\mathrm{P}}) + \mathcal{Q}_0\left(\mathcal{B}\left(\psi_{\mathrm{P}} + \widehat{\Phi}(\widehat{\gamma}), \psi_{\mathrm{P}} + \widehat{\Phi}(\widehat{\gamma})\right) + \gamma_0 \mathcal{A}_1(\psi_{\mathrm{P}} + \widehat{\Phi}(\widehat{\gamma}))\right) \tag{3.9}$$

and express $\dot{\gamma}_j$ from (3.3). Comparing coefficients we obtain an infinite set of equations. For instance,

$$\begin{aligned}
0 &= \mathcal{A}_{R_0}\widehat{\Phi}_{1001000} + \mathcal{Q}_0\mathcal{A}_1(\kappa_1), \\
0 &= \mathcal{A}_{R_0}\widehat{\Phi}_{0002000} + \mathcal{Q}_0\mathcal{B}(\kappa_1, \kappa_1), \\
2\widehat{\Phi}_{0002000} &= \mathcal{A}_{R_0}\widehat{\Phi}_{0001100} + \mathcal{Q}_0\left(\mathcal{B}(\kappa_1, \kappa_2) + \mathcal{B}(\kappa_2, \kappa_1)\right), \\
\widehat{\Phi}_{0001100} &= \mathcal{A}_{R_0}\widehat{\Phi}_{0000200} + \mathcal{Q}_0\mathcal{B}(\kappa_2, \kappa_2).
\end{aligned}$$

Solving these equations, we obtain:

$$\begin{aligned}
\widehat{\Phi}_{0002000} &= \begin{pmatrix} R_0 \mathrm{r}^2 U \\ 0 \\ -2 \\ 0 \end{pmatrix}, \widehat{\Phi}_{0001100} = \begin{pmatrix} -2R_0^2 \mathrm{r}^2 h + 2R_0^2 \mathrm{r}^2 (U\mathrm{r}^2 U - U'\mathrm{r}^3 U + 2/R_0^2) \\ -2R_0 \mathrm{r}^3 U \\ 4R_0 \mathrm{r}^2 U \\ 0 \end{pmatrix} \\
\widehat{\Phi}_{0000200} &= \begin{pmatrix} \widehat{\Phi}^1_{0000200} \\ 2R_0^2 \mathrm{r}^3 h - 2R_0^2 \mathrm{r}^3 (U\mathrm{r}^2 U - U'\mathrm{r}^3 U + 2/R_0^2) \\ \widehat{\Phi}^3_{0000200} \\ -2R_0 \mathrm{r}^3 U \end{pmatrix}.
\end{aligned} \tag{3.10}$$

where $\widehat{\Phi}^1_{0000200}, \widehat{\Phi}^3_{0000200}$ can readily be found. The following lemma summarizes several properties of system (3.7).

Lemma 3.1

(i) Parameter dependent system (3.7) is reversible with respect to involutions

$$\widehat{S}_1 \stackrel{\text{def}}{=} \mathrm{diag}(1,-1,1,-1,1,-1,1), \quad \widehat{S}_2 \stackrel{\text{def}}{=} \mathrm{diag}(1,-1,1,1,-1,1,-1)$$

for cases (A), (B)*, respectively.*

(ii) System (3.7) *is invariant under shifts* $\gamma_{-1} \to \gamma_{-1} + \mathrm{const.}$, *i.e.* $\mathcal{F}(\cdot)$ *doesn't depend on* γ_{-1}.

(iii) There are three first integrals

(a) $I_1(\widehat{\gamma}) = \gamma_{-2}$

(b) $I_2(\widehat{\gamma}) = \gamma_{-1} + \varphi_2(\widehat{\gamma})$, *with function* φ_2 *satisfying* $\partial_{\gamma_{-1}}\varphi_2|_{\widehat{\gamma}=0} = 0$

(c) $I_3(\widehat{\gamma}) = K_0\gamma_4 + \varphi_3(\widehat{\gamma})$, *with* $\partial_{\gamma_4}\varphi_3|_{\widehat{\gamma}=0} = 0$.

(iv) $\mathcal{F}_3(\widehat{\gamma}) = \gamma_4$.

(v) Family (2.15) *of* x_1*-independent solutions* $\Psi_{\beta_2} = \Psi_*|_{\beta_1=\beta_3=0} = (V_{\beta_2}, \beta_2, 0, 0)$ *of the Lemma* 2.2*, is contained in the level set* $I_3(\widehat{\gamma}) = 0$*. For small* β_2 *the family is given by* $\gamma_{-2} = \gamma_{-1} = \gamma_4 = 0$, $\gamma_1 = \beta_2 + \mathcal{O}(\beta_2)$, $\gamma_2 = 0$, *and* $\gamma_3 = -(R_0K_0)^{-1}\beta_2\gamma_0 + c\beta_2^3 + \mathcal{O}(\beta_2^3)$.

(vi) $I_3(\widehat{S}_j\widehat{\gamma}) = (-1)^{j+1}I_3(\widehat{\gamma})$

Proof. Reversibility with respect to $\widehat{S}_1$ or $\widehat{S}_2$ follows from the general statement of [Mie88] since $\widetilde{S}_1(\kappa_{-1}) = \widetilde{S}_2(\kappa_{-1}) = \kappa_{-1}$, $\widetilde{S}_1(\kappa_{-2}) = \widetilde{S}_2(\kappa_{-2}) = -\kappa_{-2}$, and $\widetilde{S}_i(\kappa_j) = (-1)^{i+j+1}\kappa_j$ for $j = 1, \cdots, 4$. The second statement does not follow directly from [IA92, Mie88] since the symmetry group $\kappa_{-1} \to \kappa_{-1} + \mathrm{const.}$ of (2.9) is not compact. It is not difficult, however, to adjust the proof to the present case and to see that the function $\Phi(\gamma_0, \psi_{\mathrm{P}})$ can be chosen to be independent of γ_{-1}.

Existence and symmetry properties of the first integrals I_1, I_2, I_3 of system (3.7) on any center manifold are inherited from the corresponding conserved quantities $\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3$ of lemma 2.2. Claims (iii) follow from the explicit calculations.

Since $(\mathcal{B}(\cdot,\cdot), \xi_2)_{L_2} = 0$ claim (iv) follows from (3.3) and direct calculations, see (2.11).

Statement (v) follows from the relation $\widetilde{I}_3(\Psi_{\beta_2}(x_2)) = 0$ and normal hyperbolicity of the center manifold. For small β_2 the solution Ψ_{β_2} can be expanded into the power series

$$\Psi_{\beta_2} = \begin{pmatrix} (R_0+\gamma_0)\sum_{j=0}^{\infty} \mathrm{r}^{j+1} U {\beta_2}^{j+1} \\ \beta_2 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore

$$\begin{array}{rclrcl} (\Psi_{\beta_2},\xi_{-1}) & = & 0, & (\Psi_{\beta_2},\xi_{-2}) & = & 0, \\ (\Psi_{\beta_2},\xi_4) & = & \beta_2 + \mathcal{O}(\beta_2), & (\Psi_{\beta_2},\xi_3) & = & 0, \\ (\Psi_{\beta_2},\xi_2) & = & -\frac{\gamma_0\beta_2}{R_0K_0} + c\beta_2^3 + \mathcal{O}(\beta_2^5) + \mathcal{O}(\gamma_0\beta_2^2), & (\Psi_{\beta_2},\xi_1) & = & 0, \end{array} \tag{3.11}$$

where

$$c = K_0^{-1} R_0^2 \langle (\mathrm{r}^2 U)^2 \rangle, \tag{3.12}$$

and $(\Psi_{\beta_2},\xi_3) = 0$ due to the condition (B).

$\bowtie$

In this article only the case $\beta_1 = \gamma_{-2} = 0$ of vanishing first integral I_1 and degenerate family I_1, I_2, I_3 will be considered. For $\gamma_{-2} \neq 0$ the family of solutions (3.11) can be parametrized by the values of the first integrals and the phenomenon of bifurcation without parameters disappears.

Remark 3.2
In fact, going beyond our present setting, the general situation $\gamma_{-2} \neq 0$ should be considered in the time dependent frame and with full use of the Galilean invariance of problem (1.1).

The equation for $\gamma_{-1}(x)$ obviously decouples, but $\gamma_{-1}(x)$ can be also determined from the first integral $I_2 = 0$ of lemma 3.1. Therefore problem (3.7) reduces to the 5-dimensional ODE system

$$\dot{\gamma}_j = f_j(\gamma), \quad j = 0, 1, \cdots, 4; \qquad \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_4) \in \mathbb{R}^5, \tag{3.13}$$

where $f_j(\gamma) = \mathcal{F}_j(\gamma_0, 0, 0, \gamma_1, \ldots, \gamma_4)$. Recall that γ_0 is a parameter and $\dot{\gamma}_0 = 0$.

Remark 3.3
System (3.13) is $\check{S}_j$-reversible with $\check{S}_1 \stackrel{\text{def}}{=} \mathrm{diag}(1,-1,1,-1,1)$ or $\check{S}_2 \stackrel{\text{def}}{=} \mathrm{diag}(1,1,-1,1,-1)$ for cases (A), (B), respectively.

Remark 3.4

We do not give the full description of the local dynamics of system (3.13) *and our study is restricted to the problem on the integral manifold*

$$\mathcal{M} = \{\gamma : I_3(\gamma) = 0\},$$

where $I_3(\gamma) := I_3(\gamma_0, 0, 0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *is a first integral. Motivated by the normal form theory (see* [IA92] *and section 5) it is expected that in appropriate coordinates the problem yields a 3D reversible integrable equation. Such kind of reduction appeared in the paper* [Ioo00], *where travelling waves of the Hamiltonian Fermi-Pasta-Ulam model were studied.*

However we prefer direct and elementary way of the study and avoid using a normal form transform. To find the reduced system we express γ_4 from the relation $I_3(\gamma) = 0$ via implicit function theorem (see lemma (3.1) item 3) and then substitute it into the first three equations of (3.13).

Since the one-parameter family of solutions Ψ_{β_2} lies in $\mathcal{M}$ we arrive at a problem which is strongly degenerate. This leads to specific difficulties, including the analysis of bifurcations without parameter as discussed in [FLA00a, FL01, FL02]. There is an essential difference between our problem and those discussed in these papers since system (3.13) is reversible and depends on the additional parameter γ_0.

We denote the restricted problem as

$$\dot{\gamma}^{\mathcal{M}} = f^{\mathcal{M}}(\gamma^{\mathcal{M}}), \qquad \gamma^{\mathcal{M}} = (\gamma_0, \gamma_1, \gamma_2, \gamma_3), \tag{3.14}$$

a 3-dimensional ODE system with small parameter γ_0. The following lemma is a direct consequence of lemma 3.1 and the equivariant implicit function theorem.

Lemma 3.5 *The integral manifold* $\mathcal{M}$ *contains the line of equilibria* Ψ_{β_2}. *Furthermore, system* (3.14) *with parameter* γ_0 *is defined in a neighborhood of* $0 \in \mathbb{R}^4$ *and is reversible under the actions of* $S_1 \stackrel{\text{def}}{=} \text{diag}\{-1, 1, -1\}$ *and* $S_2 \stackrel{\text{def}}{=} \text{diag}\{1, -1, 1\}$ *for cases* (A) *and* (B)*, respectively.*

Note that the trivial action of the reversor on parameter γ_0 is not included in this notation.

Our next aim is to simplify problem (3.14) in order to facilitate a description of solutions which remain close to the line of equilibria. The structure of the linear part of (3.14) and the explicit expression for the line of equilibria enables us to find local coordinates, such that the dynamics of (3.14) is described by a single equation of third order, and the equilibrium curve coincides with the coordinate line $\gamma_1 = \text{const}$.

To find the leading terms of system (3.14) we have to start from the restriction of system (3.13) to the integral manifold $I_3(\gamma) = 0$. This yields

$$\gamma_4 = \gamma_4^{\mathcal{M}}(\gamma^{\mathcal{M}}) = \sum_{|\mathbf{k}|\le n_0} \gamma_{4,\mathbf{k}}^{\mathcal{M}} \gamma^{\mathbf{k}} + \mathcal{O}(\|\gamma^{\mathcal{M}}\|^{n_0})$$

via the equivariant implicit function theorem. For $I_3(\gamma)$ we have the expansion

$$\left| I_3(\gamma) - \sum_{|\mathbf{k}|\le n_0} I_{3,\mathbf{k}} \gamma^{\mathbf{k}} \right| = \mathcal{O}(\|\gamma\|^{n_0}). \tag{3.15}$$

We now introduce the ordering in the space of power exponents $\mathbf{k} \in \mathbb{N}^5$ which singles out the "lower order" monomials $I_{3,\mathbf{k}}\gamma^{\mathbf{k}}$ responsible for the dynamics in a vicinity of the singularity.

For a fixed weight $\mathbf{s} \in (\mathbb{R}_+)^5$ we consider the group action

$$g_\theta^{\mathbf{s}}(\gamma) = (\theta^{s_0}\gamma_0, \theta^{s_1}\gamma_1, \cdots, \theta^{s_4}\gamma_4), \tag{3.16}$$

with group parameter $\theta > 0$. Recall that function $\phi(\gamma)$ is called *(quasi) homogeneous* with respect to the group $g_\theta^{\mathbf{s}}$ (or short: $\mathbf{s}$-*homogeneous*) if

$$\phi(g_\theta^{\mathbf{s}}(\gamma)) = \theta^\sigma \phi(\gamma), \tag{3.17}$$

for all $\theta > 0$ and some scaling exponent $\sigma \ge 0$. The exponent σ is called the *order* of $\mathbf{s}$-homogeneity (shortly s-order) of the function ϕ and is denoted as $\mathrm{ord}_{\mathbf{s}}\phi$.

In a same way ODE system $\dot\gamma_j = f_j(\gamma)$ is called $\mathbf{s}$-*homogeneous* if it is invariant under the group action $g_\theta^{\mathbf{s}}$, i.e. functions

$$f_j(\gamma), \quad j = 0, 1, \cdots, 4$$

are $\mathbf{s}$-homogeneous of the order $\sigma + s_j$.

Suppose now that $\phi \in \mathcal{C}^{n_0+1}(\mathbb{R}^5)$ and

$$\phi(\gamma) = \sum_{|\mathbf{k}|\le n_0} \phi_{\mathbf{k}} \gamma^{\mathbf{k}} + \widetilde{\phi}(\gamma), \qquad \text{where} \quad \widetilde{\phi}(\gamma) = \mathcal{O}(|\gamma|^{n_0}). \tag{3.18}$$

For any weight $\mathbf{s}$ we define the $\mathbf{s}$-homogeneous truncation of a function $\phi(\gamma)$ as follows. First the order of the truncation is defined as $\sigma_0 \stackrel{\mathrm{def}}{=} \min\{(\mathbf{s}, \mathbf{k}) \mid \phi_k \ne 0\}$. Next the sum

$$\mathrm{tr}_{\mathbf{s}}^{\sigma}\phi(\gamma) \stackrel{\mathrm{def}}{=} \sum_{(\mathbf{s},\mathbf{k})=\sigma_0} \phi_{\mathbf{k}} \gamma^{\mathbf{k}} \tag{3.19}$$

that corresponds to the summation of all **s**-homogeneous monomials with **s**-order σ_0 is called the **s**-*homogeneous truncation* of the function $\phi(\gamma)$ if for some $\mu > 0$,

$$\widetilde{\phi}(g^{\mathbf{s}}_{\theta}\gamma) = \theta^{\sigma_0}\mathcal{O}(\theta^{\mu}). \tag{3.20}$$

Similarly **s**-*homogeneous truncation* of the smooth vector field $\sum f_j(\gamma)\partial_{\gamma_j}$ can be defined as

$$\dot{\gamma}_j = \mathrm{tr}_{\mathbf{s}}^{\sigma+s_j} f_j(\gamma) \qquad j = 0, 1, \cdots, 4 \tag{3.21}$$

if the corresponding truncations of $f_j(\gamma)$ are defined.

Lemma 3.6 *The* **s**-*homogeneous truncation* $\widehat{I}_3(\gamma)$ *of* $I_3(\gamma)$ *with* $\mathbf{s} = (2, 1, 2, 3, 4)$ *is given by*

$$\widehat{I}_3(\gamma) = K_0(\gamma_4 - \widehat{\gamma}_4^{\mathcal{M}}(\gamma^{\mathcal{M}})),$$

where

$$\begin{aligned} \widehat{\gamma}_4^{\mathcal{M}}(\gamma^{\mathcal{M}}) &:= \mathrm{tr}^4_{\mathbf{s}^{\#}}\gamma_4^{\mathcal{M}}(\gamma^{\mathcal{M}}) \\ &= \gamma^{\mathcal{M}}_{4,(1,0,1,0)}\gamma_0\gamma_2 + \gamma^{\mathcal{M}}_{4,(0,2,1,0)}\gamma_1^2\gamma_2. \end{aligned} \tag{3.22}$$

with $\mathbf{s}^{\#} = (2, 1, 2, 3)$.

Proof. For $\psi \in X$ let $[\psi]_j$ denote the j-th component of the vector ψ. Then for $\gamma_{-1} = \gamma_{-2} = 0$ we have

$$\begin{aligned} I_3(\gamma) &:= \langle[\textstyle\sum_j \gamma_j\kappa_j + \widehat{\Phi}(\gamma)]_4\rangle - (R_0 + \gamma_0)\langle[U[\sum_j \gamma_j\kappa_j + \widehat{\Phi}(\gamma)]_2\rangle \\ &- \langle[\textstyle\sum_j \gamma_j\kappa_j + \widehat{\Phi}(\gamma)]_1[\sum_j \gamma_j\kappa_j + \widehat{\Phi}(\gamma)]_2\rangle. \end{aligned} \tag{3.23}$$

Since $\gamma_2 + R_0^2\langle U\mathrm{r}^2U\rangle\gamma_2 = 0$ we get

$$I_3(\gamma) = R_0\langle U\mathrm{r}^2U\rangle\gamma_0\gamma_2 + K_0\gamma_4 - K_0\gamma^{\mathcal{M}}_{4,(0,2,1,0)}\gamma_1^2\gamma_2 + \phi^{rest}(\gamma) \tag{3.24}$$

and hence

$$\gamma^{\mathcal{M}}_{4,(1,0,1,0)} = (K_0R_0)^{-1}. \tag{3.25}$$

To determine the coefficients of $\gamma^{\mathcal{M}}_{4,(0,2,1,0)}$, we substitute the expressions for $\dot{\gamma}_1, \dot{\gamma}_2, \dot{\gamma}_3$ from (3.7) into $\dot{\gamma}_4 = \sum_{j=1}^{3}\frac{\partial\gamma_4^{\mathcal{M}}(\gamma^{\mathcal{M}})}{\partial\gamma_j}\dot{\gamma}_j$ and compare the result with the expansion of $f_4(\gamma)$. It follows that

$$\gamma^{\mathcal{M}}_{4,(0,2,1,0)} = \tfrac{1}{2}f_{4,(0,1,2,0,0)}. \tag{3.26}$$

Denote $\mathcal{B}^0(\phi_1, \phi_2) = \mathcal{B}(\phi_1 + \phi_2, \phi_1 + \phi_2) - \mathcal{B}(\phi_1, \phi_1) - \mathcal{B}(\phi_2, \phi_2)$. From (3.5), (3.13) follows that

$$\begin{aligned} f_{4,(0,1,2,0,0)} &= \Big(\mathcal{B}^0(\kappa_1, \widehat{\Phi}_{0000200}) + \mathcal{B}^0(\kappa_2, \widehat{\Phi}_{0001100}), \xi_1\Big) \\ &= \langle\Big[\mathcal{B}^0(\kappa_1, \widehat{\Phi}_{0000200}) + \mathcal{B}^0(\kappa_2, \widehat{\Phi}_{0001100}\Big]_4\rangle. \end{aligned}$$

and explicit calculations with the use of (3.10) yield

$$\gamma^{\mathcal{M}}_{4,(0,2,1,0)} \quad = \quad c, \tag{3.27}$$

where $c = K_0^{-1} R_0^2 \langle (\mathrm{r}^2 U)^2 \rangle$ is defined in (3.12). Notice that from

$$I_3(S_j \gamma) = (-1)^{j+1} I_3(\gamma) \tag{3.28}$$

follows that

$$\phi^{rest}(g_\theta^{\mathbf{s}^\#}\gamma) = \theta^4 \mathcal{O}(\theta^\mu), \quad \text{for some} \quad \mu > 0$$

and hence the proof is finished. ⋈

It is left to notice that $f^{\mathcal{M}}_{2,(1,1,0,0)}$ and $f^{\mathcal{M}}_{2,(0,3,0,0)}$ are defined from (3.11). As a consequence we have that polynomial system

$$\begin{aligned}
\dot{\gamma_0} &= 0, \\
\dot{\gamma_1} &= \gamma_2, \\
\dot{\gamma_2} &= \gamma_3 + (K_0 R_0)^{-1}\gamma_0\gamma_1 - c\gamma_1^3, \\
\dot{\gamma_3} &= (K_0 R_0)^{-1}\gamma_0\gamma_2 + c\gamma_1^2\gamma_2
\end{aligned} \tag{3.29}$$

is the $\mathbf{s}$-homogeneous truncation of system (3.14) for $s = (2, 1, 2, 3)$.

System (3.29) is equivalent to a single equation of third order

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\ddot{\gamma}_1 - 2\frac{1}{R_0 K_0}\gamma_0\gamma_1 + \frac{2}{3}c\gamma_1^3\right) = 0. \tag{3.30}$$

Note that (3.30) can be considered as the $\mathbf{s}^\#$-homogeneous truncations of the equation

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\ddot{\gamma}_1 - 2\frac{1}{R_0 K_0}\gamma_0\gamma_1 + \frac{2}{3}c\gamma_1^3\right) = Q\left(\gamma_0, \gamma_1, \frac{\mathrm{d}}{\mathrm{d}x_1}\gamma_1, \frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\gamma_1\right). \tag{3.31}$$

which can be derived via the implicit function theorem from the restriction of (3.14) to the integral manifold $I_3(\gamma) = 0$. As we indicate in section 5 below, equation (3.30), which has been obtained by a subsequent quasihomogeneous truncation so far, is also a reversible normal form in the sense of [IA92],[Ioo00] and of [FLA00a, FL01]. For $U(x_2) = \sqrt{2}\sin x_2$ our calculations give $c = -1/3$ and

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(\ddot{\gamma}_1 + \frac{2}{3}\gamma_0\gamma_1 - \frac{2}{9}\gamma_1^3\right) = 0. \tag{3.32}$$

Finally we obtain the following statement.

Lemma 3.7 *If condition* (A) *is fulfilled then*

$$Q(S_1^{\#}(\gamma^{\#})) = Q(\gamma^{\#}),$$

and $Q(\gamma^{\#})(x_1)$ *is even if* $\gamma_1(x_1)$ *is odd. Under the condition* (B)

$$Q(S_2^{\#}(\gamma^{\#})) = -Q(\gamma^{\#})$$

and $Q(\gamma^{\#})(x_1)$ *is odd if* $\gamma_1(x_1)$ *is even.*

The proof uses the equivariant implicit function theorem.

Remark 3.8 *For the Kolmogorov forcing* (1.5)*, where* $U = \sqrt{2}\sin x_2$*, problem* (3.31) *is reversible with respect to* S_1 *and invariant under* $-I = S_1S_2$*. Hence* $Q \circ (-I) = Q$ *and there exists a reversor,* S_2*, with 2-dimensional fixed-point set. For* $F(x_2) = \sqrt{2}\sin x_2 + \omega \sin 2x_2$ *the only reversor,* S_1*, has a 1-dimensional fixed-point set. This case is much more complicated and will be discussed in* [AFL08].

According to the definition of **s**-homogeneous truncation after the scaling

$$\tau = (\gamma_0)^{1/2} x_1, \qquad (\gamma_0)^{1/2}\Gamma = \gamma_1 \tag{3.33}$$

we arrive at the equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\Gamma - 2\frac{1}{R_0K_0}\Gamma + \frac{2}{3}c\Gamma^3\right) = \mathcal{O}(\gamma_0^{1/2}). \tag{3.34}$$

and therefore the term $Q(\gamma^{\#})$ can be considered as a perturbation for small γ_0.

4 Small solutions close to Kolmogorov flow

With the results of the previous section at hand we are able to study small solutions that are uniformly close in x_1 to the Kolmogorov flow with forcing term (1.5).

After some normalization, the truncated equation (3.32), reads

$$\dddot{y} + \dot{y} - 3y^2\dot{y} = 0. \tag{4.1}$$

Higher order terms have to respect both reversibilities $S_1, S_2 : \mathbb{R}^3 \to \mathbb{R}^3 = \{(\ddot{y}, \dot{y}, y)\}$. The complete discussion of such problem can be extracted from [Ioo00], pp.854-864, where travelling waves of the Hamiltonian Fermi-Pasta-Ulam model were studied.

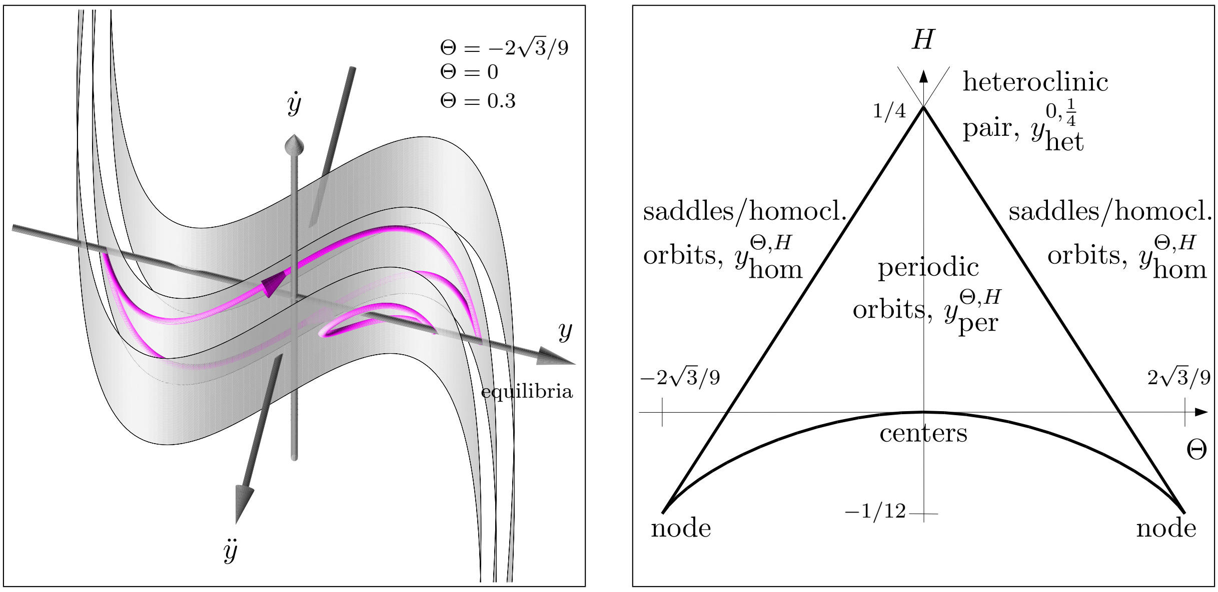

Figure 4.1: Flow (4.1) with Θ-foliation (left) and triangle of bounded solutions in the Θ-H-plane (right).

The simple exhaustive geometrical presentation is given below to fix notations and for the convenience of the reader. We describe first the set of bounded solutions of equation (4.1). Integrating (4.1) once, we obtain well known Duffing equation which is the integrable Hamiltonian system

$$\ddot y + y - y^3 = \Theta, \tag{4.2}$$

with energy

$$\begin{aligned} H &= \tfrac12 \dot y^2 - \tfrac14 y^4 + \tfrac12 y^2 - \Theta y \\ &= -\ddot y y + \tfrac12 \dot y^2 + \tfrac34 y^4 - \tfrac12 y^2 \end{aligned} \tag{4.3}$$

on any fiber $\Theta \equiv$ const.; see Figure 4.1.

The equilibria of (4.2) are exactly the trivial equilibria, i.e. the intersections of the Θ-fibers with the y-axis, $y^3 - y + \Theta = 0$. We encounter saddles at points $|y| > \sqrt{3}/3$, and centers for $|y| < \sqrt{3}/3$. Centers exist on fibers $|\Theta| < 2\sqrt{3}/9$ and are accompanied by two saddles, in each fiber. Nontrivial bounded solutions of (4.2) exist on fibers $\{|\Theta| < 2\sqrt{3}/9\}$; only these fibers have more than one intersection with the equilibrium line. The values of Θ and H corresponding to the equilibrium line form an algebraic curve $\Theta = y - y^3$, $H = \frac34 y^4 - \frac12 y^2$, in the (Θ, H)-plane. They bound a curved triangle with two cusp points and one crossing; see Figure 4.1. The cusp points correspond to the degenerate equilibria at $y = \pm\sqrt{3}/3$. The crossing point corresponds

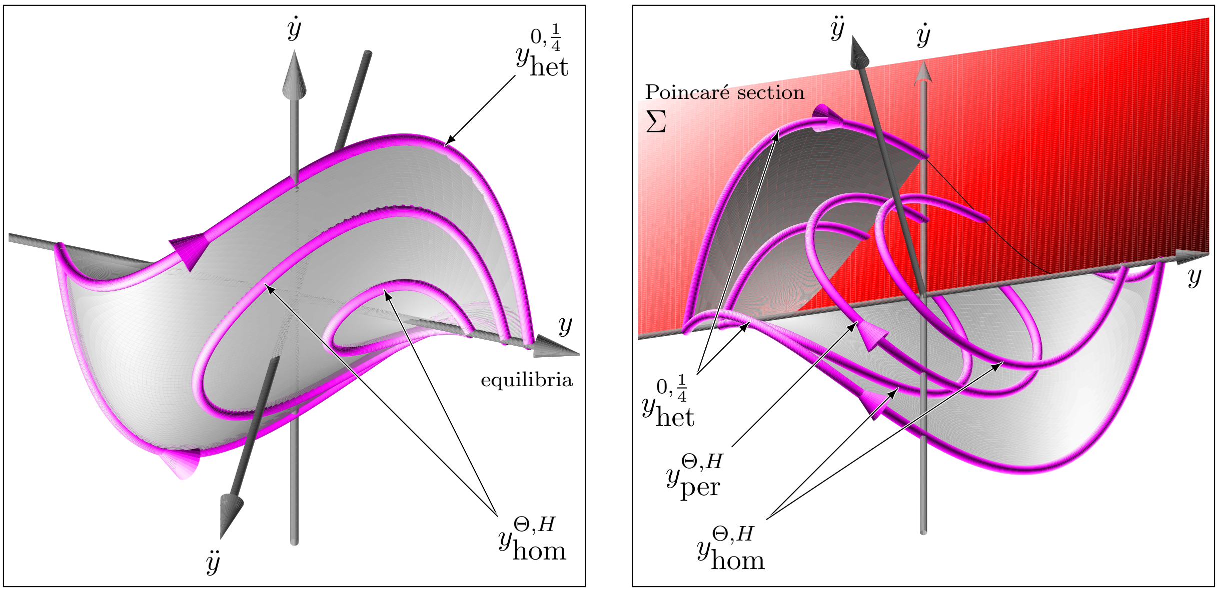

Figure 4.2: "Periodic bubble": set of bounded solution of equation (4.1) (left) and Poincaré section (right).

to the equilibria at $y = \pm 1$.

We consider the set of all bounded solutions of (4.1), next. Here boundedness refers to both positive and negative time. We first observe that the set of all bounded non-equilibrium trajectories is itself bounded. Its interior consists of the centers $\{|y| < \sqrt{3}/3, \dot{y} = 0, \ddot{y} = 0\}$ and of periodic orbits $y_{\mathrm{per}}^{\Theta,H}$ around them. Its boundary is provided by the line of equilibria and the homoclinic orbits $y_{\mathrm{hom}}^{\Theta,H}$ to the saddles $\{\sqrt{3}/3 < |y| < 1, \dot{y} = 0, \ddot{y} = 0\}$. The two sets of homoclinic orbits meet at the pair of heteroclinics $y_{\mathrm{het}}^{0,\frac{1}{4}}$ to the saddles $y = \pm 1$ in the fiber $\Theta = 0$. This periodic bubble is shown in Figure 4.2.

4.1 Persistence

Note that all bounded non-equilibrium trajectories, except the heteroclinic pair, intersect the $(y, \ddot{y})$-plane, i.e. the two-dimensional fix space $\{\dot{y} = 0\}$ of the reversibility S_2. At the intersection, trajectories are necessarily perpendicular to the fix space. In particular, the intersections are transverse.

The heteroclinic pair intersects the one-dimensional fix space $\mathrm{Fix}(S_1) = \{y = 0, \ddot{y} = 0\}$ of the reversibility S_1, i.e. the $\dot{y}$-axis. The heteroclinics are images of each other under the oddness involution symmetry $S_1 S_2 = -\,\mathrm{id}$. Transversality is achieved by the intersection of the one-dimensional fix space $\{y = 0, \ddot{y} = 0\}$ with the two-dimensional strong-stable manifold of an interval of saddles $W^{\mathrm{ss}}(\{y \in (-1-\sigma, -1+\sigma), \dot{y} = 0, \ddot{y} = 0\})$, σ small. This follows easily by tracking the values of H and Θ along the saddles and along $\mathrm{Fix}(S_1)$, respectively. The same holds true for the saddles near $y = +1$ and their strong unstable manifolds.

With these preparations we are now able to investigate the influence of perturbation which respect both reversibilities.

Let us first discuss the flow outside a small neighborhood of the cusps and the crossing, in the (H, Θ) triangle. This excises the critical equilibria $|y| = \sqrt{3}/3$ with double zero eigenvalue, and the pair of heteroclinics connecting the equilibria at $y = \pm 1$. Under higher order perturbations, all trivial equilibria then remain in $\mathrm{Fix}(S_2)$ and retain their nature as saddles or centers. The homoclinic orbits are preserved because the transverse intersection of the strong unstable manifolds of the saddles with $\mathrm{Fix}(S_2)$ is structurally stable. The periodic orbits persist because they intersect $\mathrm{Fix}(S_2)$ transversely in two different points; this fact is preserved for large periodic orbits by the structural stability of transverse intersections and for small periodic orbits near the centers by the aforementioned persistence of the centers.

Transverse intersection of the two-dimensional strong-stable manifold of the saddles $W^{\mathrm{ss}}(\{y \in (-1-\sigma, -1+\sigma), \dot{y} = 0, \ddot{y} = 0\})$ with $\mathrm{Fix}(S_1)$ is also preserved, under small perturbations. Moreover, W^{ss} itself is foliated by the one-dimensional strong stable manifolds of the individual saddles. Therefore there exists a saddle $y = y_{\mathrm{het}} \approx -1$, such that its strong-stable manifold $W^{\mathrm{ss}}(y_{\mathrm{het}})$ intersects $\mathrm{Fix}(S_1)$. By the reversibility S_1 it consequently also intersects the strong-unstable manifold $W^{\mathrm{uu}}(-y_{\mathrm{het}})$ of $y = -y_{\mathrm{het}} \approx +1$. By oddness symmetry $S_1 S_2 = -\,\mathrm{id}$ we have a corresponding heteroclinic orbit between the same equilibria in the opposite direction. The heteroclinic pair therefore persists.

Basically, the periodic bubble persists in this case due to the very strong structure that is provided by the reversibility S_2 with a two-dimensional fix space. In fact, the described picture remains valid in the case of only one reversibility S_2 with 2-dimensional fix space. However, the heteroclinic pair need not be symmetric any more. Existence is nevertheless guaranteed by the transverse intersection of the strong stable and strong unstable manifolds, $W^{\mathrm{ss}}(\{y \in (-1-\sigma, -1+\sigma), \dot{y} = 0, \ddot{y} = 0\})$ and $W^{\mathrm{uu}}(\{y \in (1-\sigma, 1+\sigma), \dot{y} = 0, \ddot{y} = 0\})$, of the saddles near the unperturbed heteroclinic pair.

With the results of this section the proof of Theorem 1.1 is finished.

Let us recall that for fixed spatial period $2\pi/\alpha$ Kolmogorov problem was studied in [Yud65, Yud66, AB86, Afe95]. However the limit $\alpha \to 0$ in these papers is singular and Theorem 1.1 explains that in fact this singularity is artificial and depends on the method of analysis. The persistence of homoclinic solutions was established first in [AK98a] with analytical arguments. The proof of persistence of heteroclinic solutions was given in [Ioo00], pp. 862-863.

5 Comparison with the abstract normal forms

In the section 3 we obtained the reduced system (3.34) after a center-manifold reduction, and a suitable **s**-weighted rescaling. In this section, we discuss the normal form of a reversible Takens-Bogdanov point along a line of equilibria from an abstract point of view and compare it with the reversible nilpotent normal form in the sense of Belitskii, Iooss and others. As it turns out, for systems with both S_1 and S_2 reversor both normal forms yield a rescaled system of the same structure.

Consider any sufficiently smooth system

$$\dot{\mathbf{z}} \;=\; f(\mathbf{z}, \mu), \qquad \mathbf{z} \in \mathbb{R}^3, \mu \in \mathbb{R}, \tag{5.1}$$

with a trivial line of equilibria

$$0 \;=\; f(0, 0, z_3, \mu), \qquad \text{for all } z_3 \in \mathbb{R}, \mu \in \mathbb{R} \tag{5.2}$$

and with a nilpotent linearization

$$A \;:=\; \mathrm{D}f(0, 0) \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.3}$$

(In the Kolmogorov problem, $\mathbf{z} = (\gamma_1, \gamma_2, \gamma_3)$ parametrizes the resulting dynamics on the restricted center manifold $\mathcal{M}$, and $\mu = \gamma_0$ is the parameter. The line of equilibria corresponds to the x_1-equilibrium family Ψ_{β_2}.) Additionally, we require reversibility with respect to reflection through the z_2-axis:

$$f(S_1\mathbf{z}, \mu) \;=\; -S_1 f(\mathbf{z}, \mu), \qquad \text{with} \quad S_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.4}$$

(This symmetry with a one-dimensional fixspace corresponds to the symmetry S_1 of the Kolmogorov flow.) The case of a second reversibility

$$f(S_2\mathbf{z},\mu) \;=\; -S_2 f(\mathbf{z},\mu), \qquad \text{with} \quad S_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.5}$$

is then equivalent to oddness of f,

$$f(-\mathbf{z},\mu) \;=\; -f(\mathbf{z},\mu), \tag{5.6}$$

by $S_1 S_2 = -\,\mathrm{id}$.

Transversely to the plane $\{(0,0,z_3,\mu) \mid z_3,\mu \in \mathbb{R}\}$ of equilibria, the linearization of the flow possesses a geometrically simple and algebraically double eigenvalue at $z_3 = \mu = 0$. In classical bifurcation theory this is called a Takens-Bogdanov bifurcation. In our case, however, z_3 is not a parameter. In particular, there is no flow-invariant foliation transversely to the plane of equilibria through the singularity. Therefore, we call (5.1–5.3) a reversible *Takens-Bogdanov bifurcation without parameters.*

In [FL01] this type of bifurcation has been studied *without* the additional reversibility. In fact, a 4-dimensional system with a *plane* of equilibria and a nilpotent linearization with a (3×3)-Jordan block, but without additional parameters, was considered. Alternatively, a 3-dim system with one distinguished parameter and a *line* of equilibria has been studied there, under the same transverse linearization requirements. It was shown that both viewpoints coincide in the rescaled normal form system, to leading order in the rescaling parameter. In particular, a normal form was calculated under the constraint that the normal form transformation must preserve the plane of equilibria. Adjusting the calculations of [FL01] to additionally preserve the reversibility S_1, we obtain the following normal form.

Lemma 5.1 *There exist polynomial coordinate transformations which preserve the plane of equilibria and the reversibility S_1 such that system* (5.1–5.4) *takes the form*

$$\begin{aligned} \dot z_1 &= z_1 z_3 h_1(2z_1z_3 - z_2^2, z_3^2, \mu) + z_2 h_2(2z_1z_3 - z_2^2, z_3^2, \mu) + z_2^2 h_3(2z_1z_3 - z_2^2, z_3^2, \mu), \\ \dot z_2 &= z_1, \\ \dot z_3 &= z_2. \\ \dot\mu &= 0. \end{aligned} \tag{5.7}$$

with suitable formal Taylor series h_1, h_2, h_3, up to any finite order. (Note the restriction $h_2(0,0,0) = 0$ due to the prescribed linearization.) In case of the additional oddness (5.6), *h_1 and h_3 vanish identically.*

Looking for solutions in a small neighborhood of the origin, we impose the following scaling by small $0 < \sigma < \sigma_0$:

$$\begin{aligned} z_1 &= \sigma^3 \tilde{z}_1, & z_3 &= \sigma \tilde{z}_3, \\ z_2 &= \sigma^2 \tilde{z}_2, & \mu &= \sigma^2 \tilde{\mu}, \end{aligned} \tag{5.8}$$

and $\sigma t = \tilde{t}$. (In fact, this is the same scaling as in lemma 3.6.) Inserting into the normal form (5.7) and omitting tildes, as well as terms of order σ^2 and beyond yields

$$\begin{aligned} \dot{z}_1 &= z_1 z_3 h_1(0,0,0) + z_2 \mu \partial_3 h_2(0,0,0) + z_2 z_3^2 \partial_2 h_2(0,0,0) + z_2^2 h_3(0,0,0), \\ \dot{z}_2 &= z_1, \\ \dot{z}_3 &= z_2. \\ \dot{\mu} &= 0. \end{aligned} \tag{5.9}$$

Setting $y := z_3$ and normalizing coefficients yields

$$\dddot{y} + \mu_1 \dot{y} + 3\mu_2 y^2 \dot{y} \;=\; \tilde{a} y \ddot{y} + \tilde{b} \dot{y}^2, \tag{5.10}$$

with $\mu_1, \mu_2 \in \{-1, 1\}$, generically. In case of the additional oddness symmetry (5.6), both $\tilde{a}$ and $\tilde{b}$ vanish.

Note that for three-dimensional systems with a reversibility of 2-dimensional fix space, equilibria in this fix space generically form curves. See for example [FLA00a] p. 25. Therefore for 3D systems with additional S_2 reversor our normal form with no surprise coincides in the leading terms with the nilpotent normal form as given in [IA92] (see I12,I18). Additional oddness appears due to the symmetry relation $S_1 S_2 = -I$. The nilpotent normal form for 3D systems with S_2 reversor was used in [Ioo00], where such ODE problem is encountered by investigating traveling waves of the Hamiltonian Fermi-Pasta-Ulam model.

In near-symmetric cases we may assume $\tilde{a}$ and $\tilde{b}$ to be small:

$$\tilde{a} \;=\; \varepsilon a, \qquad \tilde{b} \;=\; \varepsilon b, \qquad \text{where} \quad 0 < \varepsilon \ll 1. \tag{5.11}$$

Notice that this case is not covered by [IA92, Ioo00] theory since the existence of curves of equilibria is not generic for systems with S_1 reversor.

Now it is time to compare the abstract calculation with the Kolmogorov-flow example (3.34). Note that $R_0 = 1$, $K_0 = -3$, $c = -1/3$ for the original Kolmogorov flow, see (3.32).

Remark 5.2 *System* (5.10) *has the same structure as the reduced near-Kolmogorov flow* (3.34)*. Specifically, for the near-Kolmogorov forcing* $F(x_2) = \sqrt{2}\sin x_2 + \omega\cos 3x_2$ *as in* (1.12)*, we have* $\mu_1 = +1$*,* $\mu_2 = -1$*.*

This proves that the calculations of section 3 in fact yield a normal form in the rescaled equation. The remaining terms of lowest order in the rescaling parameter are in fact determined by the third-order structure of the linearization and the symmetries alone. Further simplifications due to the complete normal-form procedure occur only in higher-order terms which are not needed in our analysis.

The cases with only one reversor S_1

$$\ddot{y} \pm \dot{y} - 3y^2\dot{y} \;=\; \varepsilon a y\ddot{y} + \varepsilon b\dot{y}^2.$$

possess unbounded sets of bounded orbits and will be addressed in [AFL08].

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