AN ALGORITHM FOR THE COMPUTATION OF MELNIKOV FUNCTION OF A PLANAR OSCILLATOR

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INTRODUCTION

One of the central problems of nonlinear dynamics is the analysis of the global bifurcation, which occurs at homoclinic loops and in one parameter family of periodic orbits under small perturbations. The study of these special orbits comes from the fact that, homoclinic orbits serve as possible cause of complicated dynamics. It has become apparent that homoclinic and heteroclinic orbits are often the mechanism for the chaos and transient chaos numerically observed in physical systems in higher dimensions. Indeed, homoclinic and heteroclinic orbits are of great importance from an applied point of view .for instance, they form the profiles of traveling wave solutions in reactions in reaction-diffusion problems. Their existence can be a source of chaotic dynamics in three-dimensional systems. In static-dynamics analogies, a homoclinic orbit corresponds to a spatially localized post-buckling state (see, for example, [1] and references therein). From the abstract point of view, the theory of homoclinic bifurcation is fairly well understood. However this theory is not accessible to persons in the applications. Several papers have been devoted to discussion of specific examples, which illustrated some proprieties of homoclinic points. The purpose of this work is to give an explicit computation of the Melnikov function for any perturbed polynomial ordinary differential equation. This function plays a fundamental role in the theory of homoclinic bifurcation. Furthermore, this method is one of the few analytical methods for determining the threshold of homoclinic chaos.

MELNIKOV METHOD FOR PLANAR SYSTEM

In the simple pendulum, the homoclinic orbits separate merely two qualitatively distinct motions, namely, the librational motions inside the homoclinic loops and the rotational motions outside the homoclinic orbits. Recall that in the context of planar ordinary differential equations the name separatrix is often given to what we have called the homoclinic orbits. This because the one dimensional orbits separate the two planes into disjoints parts. In continuum mechanics homoclinic and heteroclinc orbits often arise as structures separating two distinct phases of the continua. More specifically, they may arise in the phase space of the Euler-Lagrange Equation associated with minimizing some type of functional energy of a system . For more information, see for instance [2] and references therein.
According to Peixoto's theorem [3], the bifurcations that take place in planar analytic systems are either local bifurcations that occur near a non-hyperbolic equilibrium point or periodic orbit of the system, or global bifurcation that occur near saddle-saddle connection or near one of the cycles in a continuous band of cycles.
The Melnikov method gives us an excellent tool for determining the parameter values for which a limit cycle bifurcates from a homoclinic (or heteroclinic) loop and for determining the number of limit cycles in a continuous band of cycles that are preserved under perturbations.
We establish the Melnikov theory for perturbed planar systems of the form

$$\dot{x} = f(x) + \varepsilon\, g(x, \dot{x}, \lambda) \qquad (1)$$

with $x \in R^2$ and $\lambda \in R^n$. For $\varepsilon = 0$, the system (1) has a homoclinic orbit $\Gamma_0 : x = \gamma_0(t), -\infty < t < +\infty$ at a hyperbolic saddle point x_0 and one parameter family of periodic orbits $\Gamma_\alpha : x = \gamma_\alpha(t)$ of period T_α on the interior of the separatrix of Γ_0 with $\frac{\partial \gamma_\alpha(0)}{\partial \alpha} \neq 0$

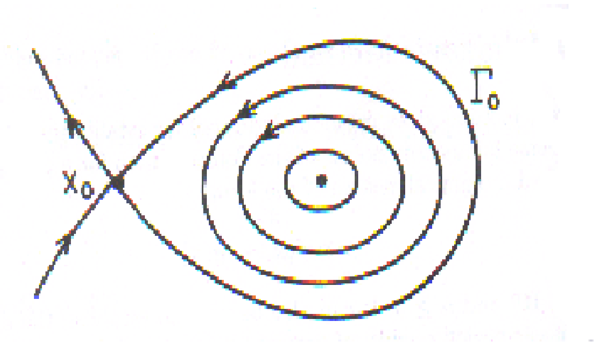

Figure 1: phase portrait of the system (1) for $\varepsilon = 0$.

The Melnikov [2] function for the equation (1) along a homoclinic or heteroclinic orbit is defined as

$$M(\lambda) = \int_{-\infty}^{+\infty} e^{-A(t)} f(\gamma_0(t)) \wedge g(\gamma_0(t), t + t_0, \lambda)\; dt \qquad (2)$$

with $A(t) = \int_{-t_0}^{t} \nabla . f(\gamma_s(s))\; ds.$

Note that the Melnikov function determine the relationship between the distance between the saddle separatrices of (1) along a normal line to homoclinic orbit $\gamma_0(t)$ above at the point $\gamma_0(0)$. This done [3] by integrating the first variation of the system (1) with respect to ε. The Melnikov distance is then proportional to the derivative of the Poincaré map with respect to the parameter ε in an interior of the neighborhood of the separatrix cycle.
For sufficiently small ε, there is analytic function $\lambda(\varepsilon) = \lambda_0 + o(\varepsilon)$ such that the system (1) has a unique homoclinic loop Γ_ε in an $o(\varepsilon)$ neighborhood of Γ_0. Moreover, if $M(\lambda) \neq 0$, then for all sufficiently small

ε and $|\lambda - \lambda_0|$, the system (1) has no separatrix cycle in an $o(\varepsilon)$ neighborhood of Γ_0

Let consider for the study of Melnikov function of a planar polynomial oscillator of pendulum type perturbed by an arbitrary non dissipative analytic function in the form

$$\ddot{x} + V'(x) = \varepsilon\, g(\lambda, x, \dot{x}) \qquad (3)$$

where the potential V(x) is polynomial function of degree n and $g(\lambda, x, \dot{x})$ is an analytic function in the form

$$g(\lambda, x, x) = \sum g_{mn}(\lambda) \quad x^m \ \dot{x}^n \qquad (4)$$

MAIN RESULTS

The unperturbed system of (1) is Hamiltonian, then $\nabla. f = 0$. Hence, the Melnikov function is given by

$$M(\lambda) = \int_{-\infty}^{+\infty} \dot{x}(t)\ g(\lambda, x(t), \dot{x}(t)) \quad dt \qquad (5)$$

We assume that the potential related to the system (3) for $\varepsilon = 0$ has a non degenerate local maximum at x=0 i.e., the system has an equilibrium of saddle type with the associated Hamiltonian $H(x, \dot{x}) \equiv E$ at this level.

Since 2E is non-degenerate local maximum of the potential $V(x)$ at x=0, we can write $2(E - V(x)) = x^2 \widetilde{V}(x)$. Notice that the perturbation $g(\lambda, x, \dot{x})$ is reversible in time. Since Melnikov function depends linearly on g, we only have to compute

$$M_m = \int_\gamma \frac{x^{m-1}}{\sqrt{\widetilde{V}(x)}} \quad dx \qquad (6)$$

where γ is a suitable path on the Riemann surface $\Re$ of $\sqrt{\widetilde{V}(x)}$. Note that all integral M_m live on the Riemann surface $\Re$ and they define meromorphic functions given by (6). Following an elementary exposition [4] about the elliptic integrals, we generalize it to the abelian integrals and we can write.

$$\begin{aligned}(m + \frac{n}{2} - 1)\, a_0\ M_{m+n-2}(\lambda) + \cdots \\ + (m - 1 + \frac{j}{2})\, a_j\ M_{m+j-2}(\lambda) + \ldots \\ + (m-1)\, a_n\ M_{m+n-2}(\lambda) = 0\end{aligned}$$

This is (n-1)-term recursion for $M_m(\lambda)$. We need to compute only $M_0, M_1, \ldots, M_{n-1}$ to evaluate all integrals. For example, for n=1 or 2, it is related to elementary function that are connected with the circular functions, for n=3 or 4 it is related to elliptic functions and for n > 4 it is related to hyperelliptic functions. We will glimpse into of Riemann surfaces to develop the last case. We study then compact Riemann surfaces $\Re$ of genus g, which are surfaces of the algebraic functions in the form:

$$w^2 = \prod_{i=1}^{2g+2} (z - e_j),\ e_j \neq e_k \text{ For } j \neq k \qquad (7)$$

where g is a topologically invariant property of a surface defined as the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it. For example, the Riemann sphere has genus zero. In the hyperelliptic case $g = (n-1)/2 \quad \geq 2$ with n is the degree of $\widetilde{V}(x)$. This is property of genus g allows to prove that the both cases n odd and even are equivalents.

So, the algorithm described above can be simplified.

Indeed, the integration of the problem then reduces the solution of Jacobi inversion problem [5] associated with the curve (7).

We will show in this frame how the Melnikov function is related to Theta function as a generalization of the Weierstrass elliptic function (for the degree of the potential n=3 and 4). The general theta function of the first order is denoted by $\theta(u; A)$ where the matrix $A = \alpha_{jk}$ is symmetric and depends on $\frac{1}{2} g\,(g+1)$ parameters α_{jk} $(j \leq k)$. The numbers α_{jk} are called the moduli of the theta function. The variables $u_1, \ldots, u_g$ are called the arguments of the theta function. Thus

$$\theta(u; A) = \sum_{m \in Z^g} \exp 2\pi\, i\, [{}^t m\, (Am + 2u)]$$

DISCUSSION AND CONCLUSION

The problem of homoclinic bifurcation of planar system (3) can be reduced to the study of three cases depending of the degree of the potential function.

We establish then the connection with the modern language of Mathematics of the so-called Inversion problem of Jacobi. Furthermore, the number of zeros of Melnikov function for planar ordinary differential equations gives the upper band of number of limit cycles. Therefore, the solution of the difficult problem in this theory.

This problem was posed in 1900 by David Hilbert as one of the problem in his famous list of outstanding mathematical problems at the turn of the century [3].

Our results may be can contribute to the solution of the weakness version of the 16^{th} Hilbert's problem.

The most groundbreaking results were that for a given planar polynomial ordinary differential system, it is possible to express the Melnikov functions with $\theta -$function or generalized Weierstrass functions

The rapid convergence of the $\theta -$function allows us to use easily the numerical simulations.

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