Basins of attraction in strongly damped coupled mechanical oscillators: a global example [1]

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1 Introduction

A mechanical oscillator with one degree of freedom and with a linear friction force can be modelled as a pendulum

$$\ddot{u} + \alpha\dot{u} + f(u) = 0. \tag{1.1}$$

For damping $\alpha = 0$, a strict local minimum u_0 of the potential energy $F(u) = \int^u f(s)ds$ is surrounded by periodic orbits in the phase plane $(u, \ddot{u}) \in X = I\!R^2$. For positive damping, $\alpha > 0$, the minimum u_0 becomes an attracting equilibrium $(u_0, \dot{u} \equiv 0)$ of (1.1) in the phase plane X. Its *basin of attraction*, $B(u_0)$, consists of all initial conditions $(u(0), \dot{u}(0))$ in X such that the solution of (1.1) satisfies

$$\lim_{t \to +\infty} (u(t), \dot{u}(t)) = (u_0, 0). \tag{1.2}$$

The basin of attraction depends on α and can, in fact, undergo quite intricate global changes as the Hamiltonian situation $\alpha = 0$ is approached by the gradient-like cases $\alpha > 0$ with Lyapounov function $\dot{u}^2/2 + F(u)$; see [Ter85].

In the limit of strong damping, $\alpha \to +\infty$, system (1.1) reduces to the gradient flow

$$\dot{u} = -f(u). \tag{1.3}$$

Indeed, we may rescale time $(t \to \alpha t)$ and rewrite (1.1) as a system

$$\begin{aligned} u' &= v \\ \alpha^{-2}v' &= -v - f(u) \end{aligned} \tag{1.4}$$

Standard geometric singular perturbation theory ([Fen79], [Wig94]) then identifies a slow manifold

$$v = -f(u) + \mathcal{O}(\alpha^{-2}). \tag{1.5}$$

in the phase plane $(u, v) \in X$ which is normally hyperbolic with exponential rate $\sim \alpha^2$ of attraction. Inside this "inertial manifold", being a graph over u, the flow is given by

$$u' = -f(u) + \mathcal{O}(\alpha^{-2}). \tag{1.6}$$

The perturbation is small of the order indicated, uniformly for derivatives up to the order of differentiability of the force $f(u)$. This justifies the limit (1.3).

A finite array of N mechanical oscillators, linearly coupled along a line with damping coefficients α_i, β_i and coupling strengths λ_i, can be modelled by

$$\begin{aligned} 0 = \ddot{u}_i + \alpha_i \dot{u}_i + f_i(u_i) \quad &+ \quad \beta_{i-1}(-\dot{u}_{i-1} + \dot{u}_i) + \beta_i(-\dot{u}_{i+1} + \dot{u}_i) + \\ &+ \quad \lambda_{i-1}(-u_{i-1} + u_i) + \lambda_i(-u_{i+1} + u_i), \end{aligned} \tag{1.7}$$

where $i = 1, \ldots, N$. Neumann boundary conditions amount to

$$u_0 := u_1 \ , \ u_{N+1} := u_N. \tag{1.8}$$

Dirichlet conditions, for example, would be $u_0 := 0, \ u_{N+1} := 0$.

Note that (1.7) can also be viewed as a finite difference semidiscretization in space of a hyperbolic partial differential equation

$$0 = u_{tt} + \alpha u_t + f(x,u) - (\beta u_{tx})_x - (\lambda u_x)_x$$

We call the effect of $\alpha_i > 0$ a *local damping*, whereas $\beta_i > 0$ indicate *neighboring damping*. Rewriting (1.7), (1.8) in vector form $u = (u_1, \ldots, u_N)$, we obtain

$$0 = \ddot{u} + \alpha A \dot{u} + Lu + f(u) \tag{1.9}$$

We henceforth assume $\alpha > 0$, L positive semidefinite, A strictly positive definite, and A, L symmetric. This is the case for $\lambda_i > 0$, $\alpha_i > 0$, $\beta_i \geq 0$. Note that $(f(u))_i = f_i(u_i)$.

In the following, we fix A, L and consider the limit of *strong damping*, $\alpha \to +\infty$. By the same arguments as in (1.3)-(1.6) above, the system 1.9 reduces to

$$A\dot{u} = -f(u) - Lu \tag{1.10}$$

on a slow manifold $v = -f(u) - Lu$, for $\alpha = +\infty$. This equation will be our main object of study, in the present paper.

System (1.10) is *gradient-like* with respect to the potential (or energy or Lyapounov) function

$$V(u) := F_1(u_1) + \ldots + F_N(u_N) + \frac{1}{2}u^T L u, \tag{1.11}$$

where F_i denote the primitives of f_i, as before. Indeed, (1.10) reads

$$A\dot{u} = -\mathrm{grad}_u V(u) \tag{1.12}$$

and therefore positivity of A implies

$$\frac{d}{dt}V(u) = -(\mathrm{grad}_u V(u))^T A^{-1} \mathrm{grad}_u V(u) < 0, \tag{1.13}$$

along solutions $u(t)$ of (1.10), unless $\mathrm{grad}_u V(u) = 0$. In particular, bounded solutions $u(t)$ tend to equilibrium for $t \to +\infty$, $t \to -\infty$, respectively.

Consider, more specifically an equilibrium $u = E_0$ which is asymptotically stable, that is, a strict local minimum of the potential $V(u)$. Note that E_0 remains an equilibrium, for all choices of the positive definite symmetric damping matrix A. In section 2, we prove that (linear) stability of E_0 is also independent of A. Therefore it makes sense to define its basin of attraction, $B(E_0)$, as in (1.2) above. Our main result, theorem 1 below, states that the basin of attraction $B(E_0)$ can depend on the precise form of the damping matrix A, even in the limit of large damping $\alpha = +\infty$ which we consider in (1.10).

To be completely specific, we consider (1.7) for the following example of $N = 2$ oscillators

$$\begin{aligned} f_1(u_1) &= -\tfrac{1}{2}u_1^3 - \tfrac{3}{2}u_1^2 - u_1 + 1 \\ f_2(u_2) &= -u_2^2 \\ \alpha_1 &= \alpha_2 = (1-\tau)\alpha \\ \beta_1 &= \tau\alpha \\ \lambda_1 &= \lambda \end{aligned} \tag{1.14}$$

under Neumann boundary conditions (1.8) and in the limit $\alpha \to +\infty$ of large damping. Here $\lambda > 0$, and the homotopy parameter $0 \leq \tau < 1$ indicates the relative size of the two types of strong damping: $\tau = 0$ indicates purely local damping and absence of neighbor friction, whereas $\tau \nearrow 1$ denotes the limit of pure neighbor damping of individually undamped mechanical oscillators. We now state our main result.

1.1 Theorem

Consider the four-dimensional system (1.7), (1.8) of coupled mechanical oscillators with $N = 2$ degrees of freedom and with specific nonlinearities and parameters (1.14). For $0 \leq \tau < 1$, $\alpha > 0$ and all λ sufficiently close to one, the following then holds.

System (1.7), (1.8) possesses precisely four equilibria $E_0, E_{1\pm}, E_2$. Each equilibrium is hyperbolic, that is, a nondegenerate critical point of the potential $V(u)$ defined in (1.11). The unstable dimensions, alias Morse indices, of these four equilibria are $0, 1, 1, 2$, respectively, as indicated by their numerical subscripts. In particular, E_0 is an attractor with basin $B(E_0)$.

For strong damping, that is, for some sufficiently a large α_0, and any fixed $\alpha \geq \alpha_0 > 0$, the basin $B(E_0)$ depends significantly on the relative size $\tau \in [0, 1)$ of neighbor versus local damping. More precisely, there exist parameters λ_0 near 1 and $\tau(\alpha) \in (0, 1)$, such that for $\lambda = \lambda_0$ and all $\tau \in (0, 1) \setminus \tau(\alpha)$ sufficiently close to $\tau(\alpha)$ the following dichotomy arises. For τ on one side of $\tau(\alpha)$, the repelling equilibrium E_2 lies in the boundary $\partial B(E_0)$ of the E_0-basin, whereas $E_2 \notin \partial B(E_0)$ for τ on the other side of $\tau(\alpha)$. In particular, the basin boundary experiences a finite nonzero jump as the relative damping τ increases through $\tau = \tau_0$. This effect is caused by a nongeneric, nontransverse saddle-saddle heteroclinic orbit from E_{1+} to E_{1-} at $\tau = \tau(\alpha)$.

We emphasize that our results are proved with mathematical rigor rather than just suggested by numerical simulation. For illustration of the geometry in the planar limiting system (1.10) of damping $\alpha = +\infty$, we refer to Figs. 5.1.a,b. For a general background of methodology we refer to [CH82], [GH83], [Wig94].

Altough the explicit statement of our theorem is planar, we emphasize that our methodology is not planar. In principle, it applies to any number of coupled oscillators. Our approach will carefully outline a general strategy to detect jumps in basin boundaries induced by a transition from local to neighboring friction. In this sense, our specific planar example serve as a minimalistic paradigm for the general situation.

Outline of the paper. In section 2, we investigate equilibria and the variational structure of the limiting system (1.10) at $\alpha = +\infty$, including issues like Morse indices, heteroclinic connections, and boundaries of basins of attraction. Section 3 is devoted to *nodal properties* of (1.10). This is a global structure of Sturm oscillation type, which is peculiar to the case $\tau = 0$ of purely local damping. In contrast to theorem 1.1, systems (1.10) with purely local damping are Morse-Smale, if all equilibria are hyperbolic. In particular, saddle-saddle heteroclinics as in theorem 1.1 cannot occur, see proposition 3.1.

In section 4, we address the limit $\tau \nearrow 1$ by methods of geometric singular perturbation theory. In particular, the degenerate saddle-saddle heteroclinic orbit from E_{1+} to E_{1-} will appear at coupling parameter $\lambda = 1$, in this limit. Throughout sections 2-4 we will develop our approach to coupled mechanical oscillators in reasonable generality, not just restricted to our particular example (1.14). Only in section 5, we condense our approach into a rigorous proof of theorem 1. We conclude, in section 6, with a discussion of our results.

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2 Variational structure

In the introduction we have seen how variational gradient systems (1.10) arise from coupled mechanical oscillators (1.7), (1.8) in the limit of strong damping. We now collect some facts on gradient systems, in their own right, for later use. We first observe that the unstable dimension (= Morse index) of equilibria does not depend on the type τ of damping, local versus neighbor; see proposition 2.1. Next, we give a brief account of structural stability of Morse-Smale systems, as introduced by Palis and Smale [PS70]; see also [PdM82]. We conclude, in proposition 2.3, with an investigation of isolated equilibria in the basin boundary of attractors.

To address hyperbolicity of equilibria, we consider the following slight generalization of (1.10), (1.12):

$$A(u)\dot{u} = -\mathrm{grad}_u V(u) =: -V_u(u) \tag{2.1}$$

where $A(u)$ is a C^1 function of strictly positive definite, symmetric matrices, and V is a C^2 scalar function of $u \in I\!R^N$. Note that (2.1) is gradient-like with respect to the Lyapounov functional V. For hyperbolic equilibria, alias critical points $u = E$ of V, we let $i(E)$ denote the *unstable dimension*, alias *Morse index* of E, that is, the number of eigenvalues of the linearization

$$-A(E)^{-1}V_{uu}(E) \tag{2.2}$$

with strictly positive real part, counting algebraic multiplicity.

2.1 Proposition

The Morse index $i(E)$ of the equilibrium E of (2.1) does not depend on the matrix function $A(u)$. In particular, E is hyperbolic iff it is nondegenerate as a critical point of E. The unstable dimension of E, in (2.1), equals the Morse index of E, as a critical point of V.

Proof:

By linear conjugation with the positive definite, symmetric square root $A(E)^{1/2}$, the matrix (2.2) transforms into

$$-A(E)^{-1/2}V_{uu}(E)A(E)^{-1/2}. \tag{2.3}$$

As a quadratic form, this latter matrix is equivalent to the Hessian $-V_{uu}(E)$ itself. In particular, the Morse indices coincide, and the proof is complete. □

So, the proof is by basic linear algebra. A slightly more abstract view point emphasizes that the notion of a gradient $\text{grad}_u V(u)$ depends on the underlying Riemannian metric and the associated direction $-\text{grad}_u V(u)$ of *steepest descent*. Thus, the metric $A(u)$ does not change the Morse index, or saddle type, of the nondegenerate critical point $u = E$. Nevertheless, a particular change in the damping metric $A(\tau)$ can change the global pattern of basins and heteroclinic orbits, in our context of coupled mechanical oscillators. This is our main result, theorem 1.1.

We now begin our investigation of heteroclinic orbits and basins of attraction of general gradient-like systems

$$\dot{u} = g(u), \tag{2.4}$$

$u \in I\!R^N$. By *gradient-like* we mean that there exists a Lyapounov function V such that $t \to V(u(t))$ decreases strictly along solution $u(t)$ except, of course, at equilibria, where $g(E) = 0$. Note that (1.10), (1.12) and, more generally, (2.1) are gradient-like systems. Similarly, the original system of coupled oscillators (1.7), (1.8) is gradient-like, provided all damping coefficients α_i are positive (but not necessarily large).

We recall that α- and ω-limit sets of bounded solutions $u(t)$ consist entirely of equilibria, for gradient-like systems, by LaSalle's invariance principle; see [Hal69]. Denote the (local) flow of (2.4) by $u(t) = u_0 \cdot t$; initial condition is $u(0) = u_0$. We call an equilibrium E_0 an *attractor*, if

$$\lim_{t \to +\infty} u_0 \cdot t = E_0 \tag{2.5}$$

for all initial conditions in a sufficiently small neighborhood of E_0. In particular, E_0 is a strict local minimum of V, and hence stable, and is isolated as an equilibrium. Its *basin of attraction* $B(E_0)$ is the set of all u_0, including E_0, for which (2.5) holds. Attractors have open basins. By $\partial B(E_0)$ we denote the topological basin boundary. Both B and ∂B are flow invariant.

We say that an equilibrium E_2 *connects to* another equilibrium E_1, symbolically $E_2 \rightsquigarrow E_1$, by a *heteroclinic orbit* $u_0 \cdot t$ if

$$\begin{aligned} \lim_{t\to-\infty} u_0 \cdot t &= E_2 \\ \lim_{t\to+\infty} u_0 \cdot t &= E_1 \end{aligned} \tag{2.6}$$

If both E_1 and E_2 are hyperbolic, then (2.6) is equivalent to a nonempty intersection of the respective stable and unstable manifolds:

$$u_0 \in W^u(E_2) \cap W^s(E_1). \tag{2.7}$$

We call a gradient-like system *Morse-Smale*, if all equilibria are hyperbolic and the intersections (2.7) are all transverse. *Structural stability* holds for Morse-Smale systems on compact manifolds: any C^1-close gradient-like flow can by conjugated to the original, unperturbed flow by a homeomorphism h which preserves time orbits and is C^0-close to identity [PS70], [PdM82]. In addition to this openness property, the Morse-Smale property is generic, in particular dense, in the class of all gradient-like flows. Similar results hold for gradient-like system on $I\!R^N$, replacing the compact manifold, if we require dissipativity: the set of equilibria is bounded, and

$$V(u) \to +\infty, \quad \text{for } |u| \to \infty. \tag{2.8}$$

2.2 Proposition

Consider a Morse-Smale system on a compact manifold or, alternatively, on $I\!R^N$ with dissipativity. Let E_0 be an attractor, and hence of zero Morse index $i(E_0)$, with basin $B(E_0)$ and basin boundary $\partial B(E_0)$.

Then $B(E_0), \partial B(E_0)$ change continuously, with respect to symmetric Hausdorff distance, under C^1 gradient-like, dissipative perturbations.

More precisely, the basin boundary $\partial B(E_0)$ is the union of stable manifolds $W^s(E)$ of all equilibria E which connect heteroclinically to E_0:

$$\partial B(E_0) = \bigcup_{E \rightsquigarrow E_0} W^s(E). \tag{2.9}$$

In particular, an equilibrium E lies in the basin boundary $\partial B(E_0)$ iff $E \rightsquigarrow E_0$.

Proof:

Continuity of $B(E_0) = W^s(E_0)$ and $\partial B(E_0)$ follow from the work of Smale and Palis. They also show that the relation $E \rightsquigarrow E'$ is transitive, for Morse-Smale systems. Now let $u_0 \in W^s(E)$ for some $E \rightsquigarrow E_0$. Then their arguments also yield $u_0 \in \partial B(E_0)$. Therefore $\partial B(E_0)$ contains the right hand side of (2.9).

Conversely, let $u_0 \in \partial B(E_0)$ be a limit of $u_0^n \in B(E_0) = W^s(E_0)$. Passing to subsequences, if necessary, the trajectories $u_0^n \cdot t,\ t \geq 0$, then converge to a finite sequence of equilibria $E = E_k, \ldots, E_1$ in $\partial B(E_0)$ of strictly decreasing Morse index such that

$$u_0 \cdot t \rightsquigarrow E, \quad \text{for} \quad t \to +\infty, \tag{2.10}$$

and to additional heteroclinic orbits

$$E = E_k \rightsquigarrow \ldots \rightsquigarrow E_1 \rightsquigarrow E_0 \tag{2.11}$$

In particular, $u_0 \in W^s(E)$ and, by transitivity of $\rightsquigarrow$, also $E \rightsquigarrow E_0$. This proves (2.9) and the proposition. □

3 Local damping and Sturm properties

In this section we consider gradient systems

$$\alpha_i^0 \dot{u}_i = -f_i(u_i) + \lambda_{i-1}(u_{i-1} - u_i) + \lambda_i(u_{i+1} - u_i), \tag{3.1}$$

$i = 1, ..., N$, with Neumann boundary conditions $u_0 := u_1, u_{N+1} := u_N$ and positive coupling constants λ_i as well as damping rates $\alpha_i^0 > 0$. This corresponds to large, purely local damping with neighbor damping $\beta_i = 0$. The system is of form (1.12) with diagonal damping matrix A. Abstractly, (3.1) can be rewritten as

$$\dot{u}_i = g_i(u_{i-1}, u_i, u_{i+1}). \tag{3.2}$$

The tridiagonal nonlinearity g_i has positive partial derivatives with respect to the off-diagonal entries $u_{i\pm1}$, because

$$\begin{aligned} \partial_{i-1} g_i &= \lambda_{i-1} > 0 \\ \partial_{i+1} g_i &= \lambda_i > 0 \end{aligned} \tag{3.3}$$

In other words, (3.2) is a *Jacobi system* [FO88], in the sense of Fusco and Oliva.

Jacobi systems possess a characteristic *nodal property* or *Sturm property*. For any vector $\eta = (\eta_1, ..., \eta_N) \in I\!R^N$ let $z(\eta)$, the *zero number* of η, denote the number of strict sign changes of the ordered sequence $\eta_1, ..., \eta_N$. Now consider any two solutions $u^1(t), u^2(t)$ of a Jacobi system (3.2). Then

$$t \mapsto z(u^2(t) - u^1(t)) \tag{3.4}$$

is nonincreasing with t. More precisely, z drops strictly at $t = t_0$ whenever the difference $\eta(t_0) := u^2(t_0) - u^1(t_0) \in I\!R^N$ possesses a "multiple zero":

$$\eta_i(t_0) = 0 \quad \text{and} \quad \eta_{i-1}(t_0) \cdot \eta_{i+1}(t_0) \geq 0.$$

(We use Neumann boundary conditions $\eta_0 = \eta_1,\ \eta_{N+1} = \eta_N$ at $i = 1, N$, here.) For linear partial differential equations, this structure was discovered by Sturm [Stu36] in 1836. For nonlinear PDEs, the subject was successfully revived by [Mat82]. For Jacobi systems, see [FO88].

3.1 Proposition [FO88]

Consider a Jacobi system (3.2) with a finite number of equilibria. If all equilibria are hyperbolic, then the Jacobi system is Morse-Smale. We note that an explicit

Lyapounov function for Jacobi systems was constructed in [FG97], identifying (3.2) to be gradient-like in general. In our special oscillator case (3.1), the gradient-like structure is inherited from the potential energy V, of course; see (1.11).

Fusco and Oliva prove the Morse-Smale property of Jacobi systems, in [FO88], by establishing the one missing ingredient: transversality of stable and unstable manifolds. The Sturm property (3.4) is the crucial tool in their proof. Note that the Morse-Smale property excludes, in particular, heteroclinic connections $E \rightsquigarrow E^{'}$ between hyperbolic equilibria of equal Morse index $i(E) = i(E^{'})$.

For dissipative Morse-Smale systems (3.2), Fiedler and Rocha have developed an explicit algorithm to determine which equilibria possess a heteroclinic connection, and which don't. The only required input information is the relative ordering of all equilibria at the left boundary, $i = 1$, versus the right boundary, $i = N$. See [FR96a], [FR96b] for details. In particular, the algorithm is able to explicitly identify all equilibria E which connect to a given attractor E_0 in our description (2.9) of the basin boundary $\partial B(E_0)$.

Our specific example (1.14) is not dissipative. It is possible to adapt the theory in [FR96a] to cover that case, confirming the numerical result in Fig. 6.1. Since this adaptation will not be used in our proof of theorem 1.1, though, we do not dwell on the necessary details here.

Because the Morse-Smale property is open (see section 2), the system (1.7) of coupled mechanical oscillators with $\beta_i = 0$ becomes Morse-Smale for $\alpha_i = \alpha_i^0 \cdot \alpha,\ \alpha_i^0 > 0$, and

$$\alpha \geq \alpha^0(\alpha_1^0, ..., \alpha_N^0) \tag{3.5}$$

Indeed, the bounded solutions are contained in the strongly attracting slow manifold, where the flow limits onto (3.1); see (1.9), (1.10), and section 4. Within the slow manifold, the Morse-Smale argument applies.

In particular, we conclude that saddle-saddle connections $E \rightsquigarrow E^{'}, i(E) = i(E')$ are impossible in the case of strong damping of purely local type.

4 Neighbor damping and singular perturbations

In this section, we consider the limit of strong damping of pure neighbor type. Specifically, as in (1.9), (1.10), we consider systems

$$A(\tau)\dot{u} = -f(u) - Lu \tag{4.1}$$

with diagonal nonlinearity $(f(u))_i = f_i(u_i)$ and linear nearest neighbor coupling L. The damping matrix $A(\tau)$ corresponds to choices

$$\begin{aligned} \alpha_i &= (1-\tau)\alpha \cdot \alpha_i^0 \\ \beta_i &= \tau\alpha \cdot \beta_i^0 \end{aligned} \tag{4.2}$$

$0 \le \tau < 1$, in the limit $\alpha \to +\infty$ of strong damping. In complete detail, $A(\tau) = (1-\tau)A_0 + \tau A_1$ with symmetric damping matrices

$$A_0 = \begin{pmatrix} \alpha_1^0 & & \\ & \ddots & \\ & & \alpha_N^0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \beta_1^0 & -\beta_1^0 & & \\ -\beta_1^0 & (\beta_1^0 + \beta_2^0) & \ddots & \\ & \ddots & \ddots & -\beta_N^0 \\ & & -\beta_N^0 & \beta_N^0 \end{pmatrix}. \tag{4.3}$$

In accordance with section 1, we assume $\alpha_i^0 > 0$, $\beta_i^0 > 0$, for $i = 1, ..., N$. Note that A_0 is strictly positive definite, whereas A_1 is positive semidefinite with one-dimensional Kernel given by $\mathbf{e} = (1, ..., 1)$:

$$\ker A_1 = \text{span } \mathbf{e} = \text{ co-ker } A_1 \tag{4.4}$$

By standard perturbation theory [Kat80], the orthogonal eigenprojection $P_0 = N^{-1}\mathbf{e}\mathbf{e}^T$ associated to the simple eigenvalue $\varepsilon = 0$ of the symmetric matrix $A_1 = A(\tau = 1)$ continues differentiably to a projection onto the eigenspace of the corresponding eigenvalue ε near zero of $A(\tau)$, for $\tau \in [0, 1)$ near 1. Note the expansion

$$\varepsilon = (1-\tau) \cdot (\frac{1}{N}\sum_{i=1}^{N} \alpha_i^0) + o(1-\tau). \tag{4.5}$$

This allows us to introduce the eigenvalue $\varepsilon > 0$ as a new, small parameter, replacing τ. With eigenprojections $P_\varepsilon, Q_\varepsilon := 1 - P_\varepsilon$, and the associated decomposition

$$u = v + w, \quad v := P_\varepsilon u \tag{4.6}$$

we obtain the singular perturbation form

$$\begin{aligned} \varepsilon\dot{v} &= -P_\varepsilon(f(v+w) + L(v+w)) \\ \dot{w} &= -A^\dagger(\varepsilon)Q_\varepsilon(f(v+w) + L(v+w)), \end{aligned} \tag{4.7}$$

where $A^\dagger(\varepsilon)$ denotes the inverse of $A(\tau)$ on the invariant subspace range Q_ε. Note that $A^\dagger(\varepsilon = 0)$ is the usual pseudo inverse $A_1^\dagger$ of A_1. In *fast time* $t' = t/\varepsilon$, we obtain the reduced *fast system*

$$\begin{aligned} v' &= -(\tfrac{1}{N}\sum_{i=1}^{N} f_i(v_i + w_i)) \cdot \mathbf{e} \\ w' &= 0 \end{aligned} \tag{4.8}$$

in the limit $\varepsilon \searrow 0$. Here $' = d/dt'$, and we have used $P_0 L = 0$. In *slow time* t, we obtain the corresponding *slow system*

$$\begin{aligned} 0 &= -(\tfrac{1}{N}\sum_{i=1}^{N} f_i(v_i + w_i)) \cdot \mathbf{e} \\ \dot{w} &= -A_1^\dagger Q_0(f(v+w) + Lw), \end{aligned} \tag{4.9}$$

formally, for $\varepsilon \searrow 0$. We have used $Q_0 L v = L Q_0 v = 0$ here. Finally we let

$$S_0 := \{u \in I\!R^N | \sum_{i=1}^{N} f_i(u_i) = 0\} \tag{4.10}$$

denote the (limiting) *slow manifold.* For S_0 to actually be a manifold we require 0 to be a regular value of the function

$$u \mapsto \sum_{i=1}^{N} f_i(u_i). \tag{4.11}$$

In other words, $\sum f_i(u_i) = 0$ implies $f_i'(u_i) \neq 0$, for some i. The slow manifold is the equilibrium set of the fast system (4.8) or, alternatively, the set where the slow system (4.9) makes sense. The manifold S_0 can be written as a graph

$$S_0 : \; v = \psi_0(w) \tag{4.12}$$

locally, by the implicit function theorem, near points $u \in S_0$ where

(4.13)
$$\sum_{i=1}^{N} f_i'(u_i) \neq 0.$$

We call these u *regular points* of S_0; all other points of S_0 are called *singular*.

4.1 Proposition

In compact regions of regular points, bounded away from the singular set, the slow manifold S_0 is uniformly normally hyperbolic under the fast flow (4.8). The slow manifold is exponentially attracting, if

(4.14)
$$\sum_{i=1}^{N} f_i'(u_i) > 0,$$

and exponentially repelling in case of negative sign. In either case, S_0 continues to a (nonunique) normally hyperbolic invariant manifold $S_\varepsilon = \text{graph } \psi_\varepsilon$, for $0 < \varepsilon \leq \varepsilon_0$, uniformly in the compact region.

Similarly, let E be a hyperbolic equilibrium of (4.1) at a regular point of S_0. Then the Morse index $i_S(E)$ of E within the slow flow on S_ε, where $v = \psi_\varepsilon(w)$, coincides with the ε-independent Morse index $i(E)$ in the full system (4.7), for $0 < \varepsilon \leq \varepsilon_0$, in the attracting case (4.14). Moreover, the local unstable manifolds $W^u_\varepsilon(E)$ converges to the local unstable manifold $W^u_0(E)$ of E for the slow flow (4.9) within S_0, in the C^1-topology. The local stable manifolds $W^s_\varepsilon(E)$, however, converge in C^1 to the product

(4.15)
$$W^s_0(E) \oplus \text{span}_{\text{loc}}\{\mathbf{e}\},$$

where span_{loc} *denotes a local span and $W^s_0(E)$ is again understood for the slow flow (4.9) in S_0.*

In the repelling case of a negative sign in (4.14), the roles of the stable and unstable manifolds are reversed, and in particular $i(E) = i_{S_\varepsilon}(E) + 1$. For a proof we refer to geometric singular perturbation theory [Fen79]; see also [Wig94]. The appearance of ε in (4.7) together with regularity condition (4.13) ensure, in fact, normal hyperbolicity in the sense of [HPS77] with exponential normal contraction/expansion rate of the order $\mathcal{O}(1/\varepsilon)$.

5 Proof of theorem 1.1

We give the proof, postponing three computational details. We postpone showing that the system (1.7), (1.8) of $N = 2$ coupled mechanical oscillators with nonlinearities and parameters (1.14) possesses precisely four equilibria $E_0, E_{1\pm}, E_2$ of the appropriate Morse indices, for coupling constant $\lambda = 1$. By proposition 2.1, the Morse indices do not depend on the damping coefficients $\alpha > 0$, $0 \leq \tau < 1$. Hyperbolicity persists for λ near 1, independently of α, τ. For $\tau = 0$, $\alpha = +\infty$, the system is Morse-Smale, by proposition 3.1, independently of λ, as long as hyperbolicity of equilibria persists. This structure persists for large damping $\alpha \geq \alpha^0$; see (3.5). In the singular limit $\tau = 1$, alias $\varepsilon = 0$, we obtain a degenerate formal saddle-saddle connection $E_{1+} \rightsquigarrow E_{1-}$. As λ increases through 1, the (formal) manifolds $W_0^u(E_{1+}), W_0^s(E_{1-})$ cross each other with nonvanishing speed, that is, transversely with respect to λ, as we postpone showing. By proposition 4.1, this fact persists, for $\tau \nearrow 1$, at a unique point

$$\lambda = \lambda(\tau) \tag{5.1}$$

By analyticity of separatrices, the function $\lambda(\tau)$ is analytic for $\tau < 1$. Note that $\lambda(\tau) \not\equiv 1$, because a saddle-saddle connection is impossible at $\tau = 0$, due to the Sturm structure there. Now fix values λ_0, τ_0 near one such that $\lambda(\tau_0) = \lambda_0$ and $\lambda'(\tau_0) \neq 0$. Then the $E_{1\pm}$ separatrices also cross transversely when the relative damping parameter τ increases through $\tau_0 < 1$.

Since values τ near τ_0 are in the singular perturbation regime, we obtain heteroclinic connections $E_2 \rightsquigarrow E_{1+}$ and $E_{1-} \rightsquigarrow E_0$, on regular pieces of the slow manifold S_0, and S_ε. Indeed, the singular points $S_{0,1}$ and $S_{0,2}$ on the slow manifold S_0 do not interfere with these slow heteroclinics, as we postpone showing. By transverse crossing of the $E_{1\pm}$ separatrices, as τ increases through τ_0, and by singular perturbations, we also obtain heteroclinics $E_2 \rightsquigarrow E_{1-}$ and $E_{1+} \rightsquigarrow E_0$, for τ on one side of τ_0, say "below". On the other side, these latter two connections do not exist. See Fig. 5.1 for illustrations of the broken saddle-saddle connection $E_{1+} \rightsquigarrow E_{1-}$. In either case, the dynamics is Morse-Smale. Again, this situation persists for $\alpha \geq \alpha^0$, as τ increases through

$$\tau = \tau(\alpha), \tag{5.2}$$

for fixed $\lambda = \lambda_0$; see section 2.

To conclude, we now invoke proposition 2.2. By transitivity of the relation $\rightsquigarrow$ in Morse-Smale systems, $E_2 \rightsquigarrow E_0$ holds for τ with $E_2 \rightsquigarrow E_{1+}$ and $E_{1+} \rightsquigarrow E_0$; these were the τ "below" $\tau(\alpha)$. In that case, $E_2 \in \partial B(E_0)$. On the other side of $\tau(\alpha)$, in contrast, E_2 cannot connect to E_0, heteroclinically. Rather, $\partial B(E_0) = W^s(E_{1-})$, which shields E_2 away. In particular, it will be proved that the basin boundary $\partial B(E_0)$ jumps at $\tau = \tau(\alpha)$, modulo the three postponed details.

We now give these details. To compute, first, all equilibria with their Morse indices we fix $\tau = 0,\ \alpha = 1,\ \lambda = 1$. Solving $\dot{u}_2 = 0$ for u_1 and inserting into $\dot{u}_1 = 0$ we obtain a polynomial in u_2 of order six. "Guessing" the two exact solutions

$$\begin{aligned} E_{1+} &= (0, 1), \\ E_{1-} &= (-2, -1) \end{aligned} \tag{5.3}$$

reduces the polynomial to order four. Evaluating explicitly, by symbolic computation, provides two additional real roots. Evaluating the symbolic expressions numerically, we find the remaining two solutions

$$\begin{aligned} E_2 &= (-3.26,\ 2.37) \\ E_0 &= (-1.75, -0.91), \end{aligned} \tag{5.4}$$

to two decimals. Computing the Morse indices is trivial.

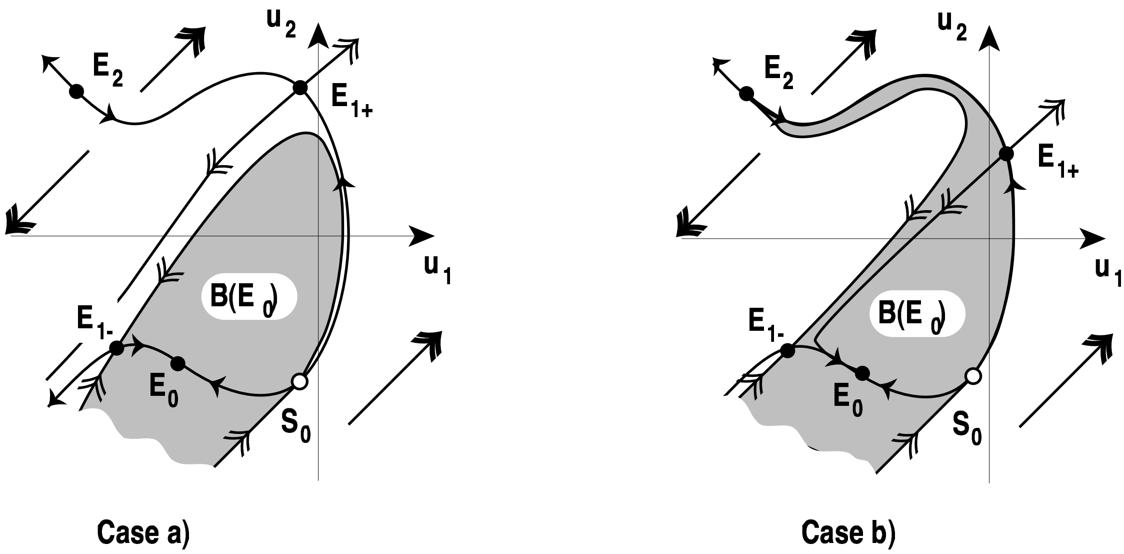

Figure 5.1: Breaking a singular saddle-saddle heteroclinic, $\tau \approx 1$. For $\lambda < 1$ see case a), for $\lambda > 1$ case b). Shaded: basin $B(E_0)$. Drawing is not to scale.

The second computational detail concerns the singular points $S_{0,1}$ and $S_{0,2}$ on the slow manifold S_0. By (4.10), (4.13), these points satisfy

$$
\begin{aligned}
f_1(u_1) + f_2(u_2) &= 0 \\
f_1'(u_1) + f_2'(u_2) &= 0
\end{aligned} \tag{5.5}
$$

Since $f_2'(u_2) = -2u_2$, we can eliminate u_2 from the second equation. The first equation then becomes a quartic polynomial in u_1

$$
f_1(u_1) - (\frac{1}{2} f_1'(u_1))^2 = 0. \tag{5.6}
$$

Evaluating explicitly, by symbolic computation, provides two real solutions. Evaluating the symbolic expressions numerically, we find

$$
\begin{aligned}
S_{0,1} &= (-2.46, -1.35) \\
S_{0,2} &= (0.21, -0.85)
\end{aligned} \tag{5.7}
$$

to two decimals.

The third computational detail concerns the transverse crossing of separatrices of $E_{1\pm}$, in the singular limit $\tau = 1$ alias $\varepsilon = 0$, as the coupling constant λ increases through $\lambda = 1$. The slow manifold S_0 does not depend on λ and is given explicitly by (4.10) to be

$$0 = -f_1 - f_2 = u_2^2 - 1 + \frac{1}{2}((u_1+1)^3 - (u_1+1)), \tag{5.8}$$

a very classical algebraic curve which, of course, contains all equilibria. The slow-fast decomposition $u = v + w$ is given explicitly by

$$A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad v = \frac{1}{2}(u_1 + u_2) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad w = \frac{1}{2}(u_1 - u_2) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.9}$$

Consequently, the fast system (4.8) leaves the lines

$$u_1 - u_2 = a \tag{5.10}$$

invariant, due to $w' = 0$. The sign of $-(f_1 + f_2)$ determines the direction of fast motion, as indicated in fig. 5.1. Similarly, the direction of the slow flow (4.9) on S_0 can be computed easily. Note the reversal of the flow direction on S_0 at the nondegenerate equilibria and at the singular points $S_{0,1}$ and $S_{0,2}$ where S_0 is tangent to the fast direction $\mathbf{e} = (1, 1)$. By the ordering of $w-$components of the equilibria and singular points, there exist heteroclinic connections $E_2 \rightsquigarrow E_{1+}$ and $E_{1-} \rightsquigarrow E_0$ in the regular pieces of the slow manifold S_0.

The components $a_\pm = a_\pm(\lambda)$ of the saddles $E_{1\pm} = (u_{1\pm}, u_{2\pm})$ in the slow w direction are given by $a_\pm = u_{1\pm} - u_{2\pm}$. An explicit calculation using the implicit function theorem yields the derivatives at $\lambda = 1$:

$$\begin{aligned} a'_+(1) &= +1 \\ a'_-(1) &= -1 \end{aligned} \tag{5.11}$$

Because $a'_+(1) \neq a'_-(1)$, the two fast separatrices (5.7) of $E_{1\pm}$ cross transversely with respect to λ, in the singular limit $\tau \nearrow 1$.

To complete the proof it remains to show that the fast line (5.10) through the saddles $E_{1\pm}$,

$$u_1 - u_2 = a = -1 \tag{5.12}$$

does not intersect the slow manifold S_0 between those saddles, for $\lambda = 1$. Inserting $u_2 = u_1 + 1$ in the defining equation (5.8) of S_0, we indeed obtain a cubic polynomial in u_1 with the three real roots $u_1 = -3, -2, 0$. This proves the existence of a saddle-saddle connection $E_{1+} \rightsquigarrow E_{1-}$ for $\lambda = 1$ in the singular limit $\tau \nearrow 1$. The separatrices cross transversely with respect to λ, and the proof of theorem 1.1 is complete.

6 Discussion

In addition to technical remarks concerning our result, we comment on possible generalizations and the applied question of basin design.

One technical point concerns the saddle-saddle heteroclinic $E_{1+} \rightsquigarrow E_{1-}$ which is responsible for the jump in the basin boundary $\partial B(E_0)$. For large damping α and relative damping $\tau < 1$ near the pure neighbor limit $\tau = 1$, the heteroclinic occurs at coupling constants

$$\lambda = \lambda(\alpha, \tau) \tag{6.1}$$

Rather than invoking analyticity, it should be possible to compute expansions for this jumping surface of $\partial B(E_0)$ in parameter space (α, τ, λ) by a Melnikov based method. Some difficulty is posed by the singular limit $\tau \nearrow 1$, alias $\varepsilon \searrow 0$. Periodic forcing of the coupled oscillators by an external vibration of period 2π, by the way, would lead to the additional complication of a rapid forcing of period $2\pi/\alpha$ in (1.10). For exponentially small splitting of separatrices with accordingly complicated basin boundaries in such situations see [FS96] and the references there.

Our basin jump occurs at $\tau = \tau_0$ near one: the basin $B = B(E_0)$ develops a tongue which reaches through all the way to the repeller E_2; see fig. 5.1,b). This tongue is very sharp and thin; in fact it follows the heteroclinic $E_2 \rightsquigarrow E_{1\pm}$ in the slow

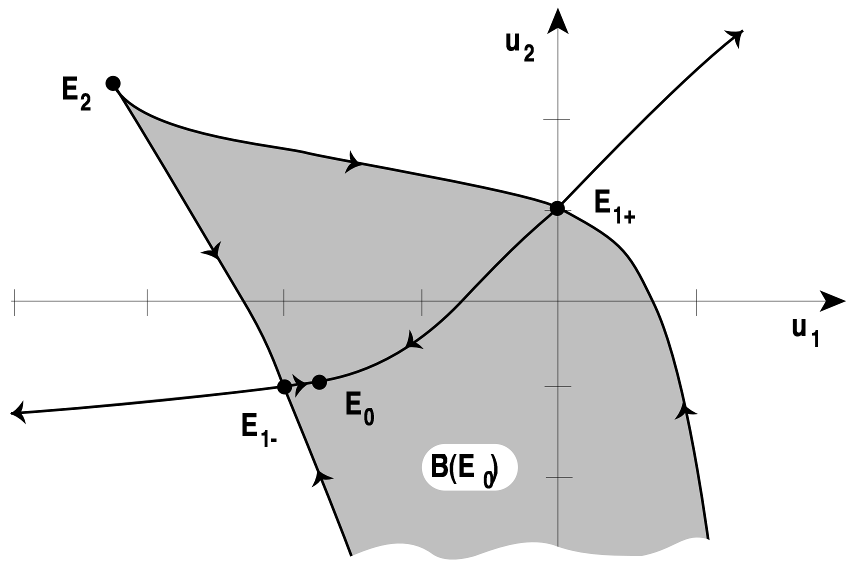

Figure 6.1: A numerical phase portrait of (1.10), (1.14) with $\alpha = +\infty,\ \tau = 0,\ \lambda = 1$.

manifold S_ε to within a distance of exponentially small order

$$\mathcal{O}(\exp(-c/\varepsilon)), \tag{6.2}$$

uniformly in regions bounded away from E_{1+}. Although the singular limit facilitates analysis, it may therefore be virtually impossible to detect this tongue by mechanical (or even by numerical) experiments. Nevertheless, we expect the heteroclinic surface (6.1) to extend globally, reaching well into moderate regions of τ, where the jump in the basin boundary becomes quite visible. For a numerical phase portrait at $\tau = 0,\ \lambda = 1,\ \alpha = +\infty$, see fig. 6.1. By the Sturm and Morse-Smale property, section 3, the surface (6.1) cannot extend down to $\tau = 0$ and hence cannot be globally parametrized over τ, of course.

Saddle-saddle connections $E_{1+} \rightsquigarrow E_{1-}$ in the singular limit $\tau = 1$ may also run through one or several singular points of the slow manifold S_0. For the simplest case, see fig. 6.2. In that case, it is the λ-dependent vanishing of the difference of

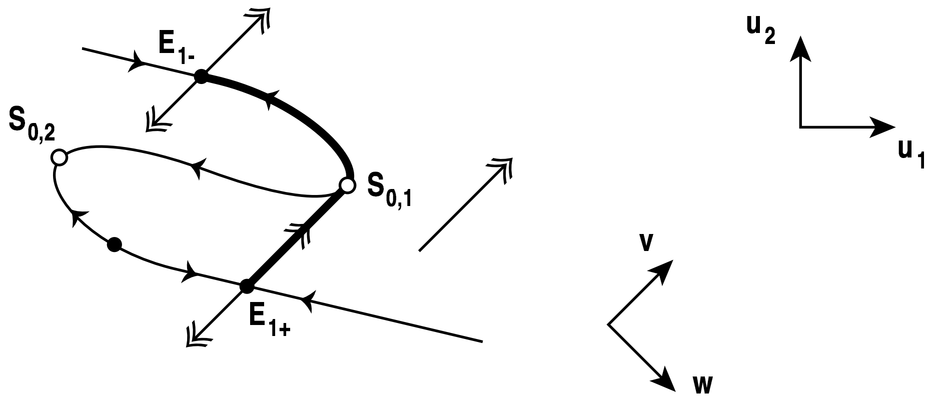

Figure 6.2: A saddle-saddle heteroclinic orbit running through a singular point of the slow manifold S_0.

the w-components of E_{1+} and the singular point $S_{0,1}$ which determines transverse crossing of the separatrices with respect to the parameter λ. In other geometric situations, it may also be the w-components of two singular points $S_{0,1}$ and $S_{0,2}$ which cross each other transversely with respect to λ and thus cause saddle-saddle heteroclinics and jumping basin boundaries.

For $N > 2$ degrees of freedom, the singular perturbation geometry in $\mathbb{R}^N$ becomes more intricate. The slow manifold S_0 still has codimension 1. The projection Q onto w along the fast v-direction allows us a superimposed view of the slow flows (4.9) in v-adjacent regular pieces.

$$V = \psi_{\pm}(w) \tag{6.3}$$

of the slow manifold S_0; see also (4.12). For example, suppose the ψ_+ sheet is repelling whereas ψ_- attracts in the fast v-direction. Consider equilibria $E_{k\pm}$ of Morse index $i(E_{k\pm}) = k$, in the respective sheets. By proposition 4.1, the Morse indices i_S of $E_{k\pm}$, viewed as equilibria of the slow flows are given by

$$\begin{aligned} i_S(E_{k+}) &= k-1, \\ i_S(E_{k-}) &= k \end{aligned} \tag{6.4}$$

In w-coordinates, the superimposed local submanifolds $W^u(E_{k+}) \cap S$ and $W^s(E_{k-}) \cap S$ can therefore cross transversely, at a distinguished parameter value $\lambda = \lambda_0 \in \mathbb{R}$. Indeed,

$$\begin{aligned} & \dim(W^u(E_{k+}) \cap S) + \dim(W^s(E_{k-}) \cap S) = \\ = \;& i_S(E_{k+}) + (\dim S - i_S(E_{k-})) = \\ = \;& \dim S \;-1 \end{aligned} \tag{6.5}$$

add up right. In the singular limit $\tau = 1$, we therefore can have a λ-transverse heteroclinic $E_{k+} \rightsquigarrow E_{k-}$ at $\lambda = \lambda_0$.

In theorem 1.1 we have investigated the simplest variant $N = 2$, $\dim S = 1, k = 1$ of this general construction. It was our principal goal to establish the effect of jumping basins of attraction, as induced by relative changes of damping, even in this most elementary case. The planar effects described, including λ-transverse heteroclinics via one or several singular points, can of course be expected to also occur for $N > 2$ degrees of freedom.

We have noted the Sturm property, at purely local damping $\tau = 0$, by which hyperbolicity of equilibria implies the system to be Morse-Smale (proposition 3.1) and therefore structurally stable. In absence of degenerate heteroclinics, the Morse-Smale property also holds for pure neighbor damping $\tau = 1$. In the dissipative case, a detailed enumeration of the connection patterns of heteroclinics $E \rightsquigarrow E'$ is available for the Sturm case $\tau = 0$; see [FR96a] and the references there. For $\tau = 1$, an analogous classification is missing. For example, it is not clear whether or not Morse-Smale attractors can arise, for τ near one, which are not conjugate to any Morse-Smale attractor of Sturm type as classified in [FR96a].

A related question, for all τ, is the *design of basins of attraction* by suitable choices of damping $A(\tau)$ and coupling L. We have seen how basin boundaries can change drastically by adjusting $A(\tau)$ alone, even in the planar case $N = 2$. Optimality issues arise, naturally, when strongly damped systems of coupled mechanical oscillators are considered in practical applications. For example, one might want to enlarge the basin $B(E_0)$ of a desirable stable equilibrium state E_0. With our paper, we hope to have achieved a first small step in such directions.

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