# Heteroclinic orbits of semilinear parabolic equations

Bernold Fiedler
Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
D-14195 Berlin
GERMANY

Carlos Rocha
Dept. de Matematica
Instituto Superior Tecnico
Avenida Rovisco Pais
1096 Lisboa Codex
PORTUGAL

version of December 6, 1995

# Contents

# 1 Introduction

The principal object of study in this paper is the global attractor $\mathcal{A}$ of scalar semilinear parabolic equations with Neumann boundary conditions

$$(1.1) \qquad \begin{array}{ll} u_t = a(x)u_{xx} + f(x,u,u_x) & , \ 0 < x < 1, \\ u_x = 0 & , \ x = 0 \text{ or } 1. \end{array}$$

We assume the diffusion coefficient $a(x) \geq c_0 > 0$ to be twice continuously differentiable and uniformly positive. The nonlinearity $f$ is also assumed to be twice continuously differentiable. In particular (1.1) defines a local $C^1-$ semiflow

$$(t, u_0) \mapsto u(t) = u(t, \cdot) \in X$$

on the Sobolev space $X \subset W^{2,2}([0,1], \mathrm{IR})$ of function $v$ with square integrable second $x$-derivative $v_{xx}$ and vanishing $v_x$ at $x = 0, 1$; see [Hen81]. Under suitable sign conditions on $f$, the semiflow is (global and) dissipative: there exists a (large) ball $B$ in $X$ which eventually absorbs any individual solution $u(t)$ for $t \geq t_0(u_0)$. Explicit sufficient conditions on $f$ are, for example,

$$(1.2) \qquad f(x, u, 0) \cdot u < 0$$

for $|u|$ large enough, uniformly in $x$, and moreover

$$(1.3) \qquad (\partial_x f(x,u,p) + \partial_u f(x,u,p) \cdot p) \cdot p < 0$$

for $|p|$ large enough, uniformly in $x$ and $u$. We also note that the semiflow is compact as a map on $X$, for any fixed $t > 0$.

By dissipativeness, the global attractor $\mathcal{A}$ of (1.1) is nonempty, compact, connected and invariant. It is the maximal compact invariant set. It attracts all bounded sets. It consists of all trajectories $u(t) \in X$ which are defined and uniformly bounded for all real $t$, both positive and negative. In general,

$\mathcal{A}$ also is of finite Hausdorff dimension. See [BV89], [Hal88], [Lad91] for surveys. These results hold in very broad generality.

For our specific equation (1.1) we aim at a specific description of the global attractor $\mathcal{A}$. Two known properties are going to provide us with efficient tools for such a description.

**Proposition [Zel68], [Mat78] 1.1** *Any trajectory $u(t), t \geq 0$, of (1.1) which is bounded in $X$ tends to some equilibrium (i.e. a time independent solution).*

Following Zelenyak, this result is conceptually most easily understood as a consequence of the existence of a Ljapunov functional

$$V(u) := \int_0^1 g(x, u, u_x) dx \tag{1.4}$$

on $X$ with the property that

$$\frac{d}{dt} V(u(t)) < 0,$$

along solutions $u(t) \in X$ of (1.1) except of course at equilibria.

We note that proposition 1.1 remains valid in backward time $t \leq 0$, for $t$ tending to $-\infty$, provided the trajectory is defined there. In particular the global attractor $\mathcal{A}$, which consists of globally defined trajectories, is composed of

a) the set $E$ of equilibria, and

b) the set of orbits connecting equilibria.

Moreover, the flow on $\mathcal{A}$ is gradient-like with an associated Morse decomposition. In fact, describing $\mathcal{A}$ is a problem in the calculus of variations.

Given equilibria $v, w$, let $C(v, w)$ denote the set of trajectories $u(t)$ which are defined for all $t \in \mathbb{R}$ and which converge to $v \in E$ for $t \to -\infty$, and to

$w \in E$ for $t \to +\infty$. By the Morse structure, $v \neq w$ and therefore $C(v,w)$ contains the *heteroclinic orbits* running from $v$ to $w$. We also call these orbits *connecting orbits*. We say that $v$ *connects* to $w$ if $C(v,w)$ is nonempty; we then also use the notation

$$v \searrow w. \tag{1.5}$$

At this stage, our description of the global attractor $\mathcal{A}$ reads

$$\mathcal{A} = E \cup \bigcup_{v,w \in E} C(v,w). \tag{1.6}$$

Determining the set $E$ of equilibria involves only ODE information: we have to solve the second-order boundary value problem

$$\begin{aligned} 0 &= a(x)v_{xx} + f(x,v,v_x) \quad , \ 0 < x < 1 \\ 0 &= v_x \qquad\qquad\qquad\qquad\quad , \ x = 0 \text{ or } 1. \end{aligned} \tag{1.7}$$

The PDE information is contained in the connecting orbits $C(v,w)$. It is our objective to determine precisely which equilibria are connected, using only information on $E$.

To guarantee finiteness and nondegeneracy of the equilibria $v \in E$, we will assume all equilibria $v$ to be *hyperbolic*, that is, $\lambda = 0$ is not an eigenvalue of the linearization at $v$ (alias the Sturm-Liouville problem)

$$\lambda u = a(x)u_{xx} + b(x)u_x + c(x)u \tag{1.8}$$

with Neumann boundary conditions. Here

$$\begin{aligned} b(x) &:= \partial_p f(x, v(x), v_x(x)), \\ c(x) &:= \partial_u f(x, v(x), v_x(x)). \end{aligned}$$

In particular, the set $E$ is finite. Note that all eigenvalues of (1.8) are necessarily real and algebraically simple. We call the number $i(v)$ of strictly positive eigenvalues the *unstable dimension* or the *Morse index* of the equilibrium $v$.

We encode the required ODE information on $E$ in a permutation $\pi$ of $|E|$ elements. Let $v_1, \ldots, v_\kappa$ enumerate the equilibria in $E$ such that

$$v_1 < v_2 < \ldots < v_\kappa \quad \text{at } x = 0. \tag{1.9}$$

By uniqueness of the initial value problem for the ODE (1.7), these boundary values are indeed distinct. At the other boundary point, $x = 1$, that order may have changed. This defines a permutation $\pi \in S_\kappa$ such that

$$v_{\pi(1)} < v_{\pi(2)} < \ldots < v_{\pi(\kappa)} \quad \text{at } x = 1. \tag{1.10}$$

Given a plot of the equilibrium profiles $v_j(x)$, $0 \leq x \leq 1$, it is very easy to actually determine $\pi$. For example, $\pi$ associated to fig. 1.1 is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 8 & 7 & 2 & 3 & 6 & 9 & 10 & 5 & 4 & 11 \end{pmatrix}$$

We are now ready to state our main result.

**Theorem 1.2** *Let the semiflow (1.1) be $C^1$ and dissipative on $X$. Assume all equilibria are hyperbolic and let $\pi$ denote the associated permutation given by (1.9),(1.10).*

*Then $\pi$ determines, in an explicit constructive process, which equilibria are connected and which are not. In other words, $\pi$ determines precisely which of the sets $C(v, w)$ in the decomposition (1.6) of the global attractor $\mathcal{A}$ are nonempty.*

In the remaining part of this introduction we indicate how the proof of theorem 1.2 breaks up into three basic aspects, formulated as lemmas 1.4, 1.5,

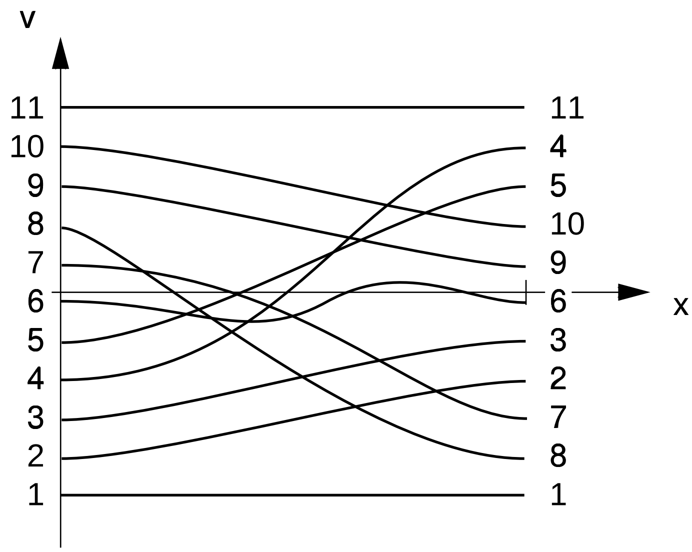



Figure 1.1: A sample of equilibrium profiles.

and 1.7 below. We also give an alternative formulation of our result, in theorem 1.8, which is more reminiscent of earlier results by Brunovský & Fiedler on the case where $f = f(u)$ is independent of $x, u_x$; see [BF88], [BF89]. Our proof, like theirs, hinges crucially on nodal properties of solutions of (1.1). These properties constitute the second tool in our analysis. For any $\tilde{u} \in X$ let $z(\tilde{u})$, the *zero number* of $\tilde{u}$, denote the number of strict sign changes of $x \mapsto \tilde{u}(x)$. Note that $X = W^{2,2}$ embeds into $C^1$; in particular $z$ is well defined.

**Proposition [Ang88] 1.3** *Let $\tilde{u}(t) = \tilde{u}(t, \cdot) \in X, t \geq 0$, be a solution of the linear nonautonomous equation*

$$\tilde{u}_t = a(x)\tilde{u}_{xx} + b(t,x)\tilde{u} + c(t,x)\tilde{u} \tag{1.11}$$

*with Neumann boundary conditions. Assume $a, b, c$ are continuously differentiable. Then the following holds:*

(i) $z(\tilde{u}(t,\cdot))$ *is finite, for any* $t > 0$,

(ii) *if* $\tilde{u}(t_0, x_0) = \tilde{u}_x(t_0, x_0) = 0$ *for some* $t_0 > 0, 0 \leq x_0 \leq 1$, *then* $t \mapsto z(\tilde{u}(t,\cdot))$ *drops strictly at* $t = t_0$, *or else* $\tilde{u}(t_0, x) = 0$ *for all* $x$.

In short, $z(\tilde{u}(t,\cdot))$ becomes finite immediately and drops if, and only if, the $x$-profile of $\tilde{u}(t,\cdot)$ possesses a multiple zero.

Proposition 1.3 is relevant in two ways. In a local spirit, (1.11) may arise as a variational equation along a solution $u(t,x)$ of the nonlinear equation (1.1). In particular, we may choose $\tilde{u} = u_t$, or we may use (1.11) to transport tangent vectors to manifolds which are invariant under the nonlinear semiflow (1.1). In a more global spirit, we may choose

$$\tilde{u} := u_1 - u_2$$

to be the difference of any two solutions $u_1, u_2$ of (1.1).

Remarkably, the "local" version has been applied to prove the Morse-Smale property of the nonlinear semiflow (1.1); see [Hen85], [Ang86]. In fact, under our hyperbolicity assumption on $E$, the unstable and stable manifolds of equilibria $v$ and $w$ intersect transversely, without further assumptions on $f$:

$$W^u(v) \pitchfork W^s(w). \tag{1.12}$$

In particular,

$$C(v,w) = W^u(v) \cap W^s(w) \tag{1.13}$$

is an embedded submanifold of dimension

$$\dim C(v,w) = i(v) - i(w). \tag{1.14}$$

The "global" variant has been used for $f = f(u)$ to describe the global attractor in the spirit of theorem 1.7, as was mentioned above. It also has provided a theorem of Poincaré-Bendixson type for (1.1) under periodic boundary conditions [FMP89]. In a similar way, proposition 1.1 can in fact be recovered without appealing to a gradient-like structure. For a summary of these results we refer our reader to the survey [Fie89].

We can now outline the three basic steps in the proof of theorem 1.2. Throughout, let the assumptions of theorem 1.2 hold.

**Lemma 1.4** *The permutation $\pi$ associated to the set $E$ of equilibria determines, constructively and explicitly, the Morse indices $i(v)$ and the numbers of sign changes $z(v-w)$, for all equilibria $v, w \in E$.*

The proof of this first step is based on earlier work by Fusco and Rocha; see [FR91]. The arguments are basically of ODE type, involving phase space analysis and some Sturm-Liouville comparison. We give a brief account in section 2. For explicit expressions of $i$ and $z$ in terms of $\pi$, see proposition 2.1.

The second step is a *cascading principle* which is peculiar to the nonlinear semiflow (1.1) and which does not hold for general variational problems. For example, consider the negative gradient (down-hill) flow on the unit $n$-sphere $S^n$ in $\mathbb{R}^{n+1}$ with respect to the height function given by the coordinate $x_{n+1}$. The north and south poles $x_{n+1} = \pm 1$ are the only equilibria. Their Morse indices are $n$ and 0, respectively. The remainder of $S^n$ consists of heteroclinic orbits connecting the north pole directly to the south pole. For the semiflow (1.1) the situation is quite different.

**Lemma (cascading) 1.5** *Under the assumption of theorem 1.2, let $v, w$ be hyperbolic equilibria of (1.1) with respective Morse indices $i(v), i(w)$.*

*Then $v$ connects to $w$ if, and only if, there exists a sequence (cascade)*

$$w = e_0, \quad e_1, \ \ldots, \quad e_n = v$$

*of equilibria, $n = i(v) - i(w) > 0$, such that for all $0 \leq k < n$ we have*

*(i)* $i(e_{k+1}) = i(e_k) + 1$*, and*

*(ii)* $e_{k+1}$ *connects to* $e_k$.

Clearly, the lemma does not hold for the down-hill flow on $S^n$. The "if"-part is a special case of a general transitivity principle and holds for a broad class of Morse-Smale systems; see [Oli92], [PS70]. In fact, suppose $v_1, v_2, v_3$ are equilibria such that $v_1$ connects to $v_2$, and $v_2$ connects to $v_3$. Then $v_1$ connects to $v_3$.

Conversely, the "only if"-part is peculiar to the semiflow (1.1), as our $S^n$ counterexample shows. As we will show in section 3, this property hinges critically on the nodal properties of proposition 1.3. A special case of this general cascading principle was already proved in [BF89] lemma 3.8.

By our cascading lemma 1.5, it is sufficient to determine whether or not $v$ connects to $w$ in the special case

$$i(v) = i(w) + 1. \tag{1.15}$$

Proposition 1.3 leads to two conditions, each of which excludes connections from $v$ to $w$.

**Definition 1.6** *Let $v, w$ be hyperbolic equilibria of (1.1) with respective Morse indices $i(v) = i(w) + 1$. We say that connections from $v$ to $w$ are* blocked *if one of the following* blocking principles *holds.*

*(i)* $z(v - w) \neq i(w)$*, or*

*(ii) there exists a third equilibrium $\bar{w}$ such that $z(v - \bar{w}) = z(w - \bar{w}) = z(v - w)$ and the value $\bar{w}(x)$ lies in between $v(x)$ and $w(x)$, at $x = 0$.*

*We call case (i) a* Morse blocking *and case (ii) a* zero number blocking.

It was observed in [BF89] that blocking prevents connections. Indeed, suppose $v$ connects to $w$. Then $C(v,w)$ is nonempty, containing a trajectory $u(t) \in X, t \in \mathbb{R}$. To address the case of zero number blocking, we now proceed indirectly. Suppose blocking of type (ii) occurs. Since $z(v-\bar{w}), z(w-\bar{w})$ are time independent, the differences $v-\bar{w}, w-\bar{w}$ can only possess simple zeros. Since convergence in $X \subseteq W^{2,2}$, by Sobolev embedding, implies $C^1$-convergence we can conclude for all sufficiently large $t > 0$ that

$$z(u(-t)-\bar{w}) = z(v-\bar{w}) = z(w-\bar{w}) = z(u(t)-\bar{w}). \tag{1.16}$$

On the other hand, we may pick $\tilde{u}(t,x) := u(t,x) - \bar{w}(x)$ in proposition 1.3. Since $\bar{w}$ lies between $v$ and $w$, at the boundary point $x = 0$, Neumann boundary conditions imply that

$$\tilde{u}(\tau,x) = u(\tau,x) - \bar{w}(x)$$

possesses a multiple zero with respect to $x$, at $x = 0$ and for some time $\tau$. In view of proposition 1.3, this implies that $z(\tilde{u}(t,\cdot))$ must drop strictly at $t = \tau$ and therefore

$$z(u(-t)-\bar{w}) > z(u(t)-\bar{w}),$$

for large enough $t > 0$. This contradicts (1.16). Therefore, zero number blocking indeed prevents connections.

The case of Morse blocking, (i), is similar. Again suppose $v$ connects to $w$, $i(v) = i(w) + 1$. Then we claim that

$$z(v-w) = i(w). \tag{1.17}$$

Indeed $z(u-w) \geq i(w)$, for $u \in W^s(w)\backslash\{w\}$; see [BF86]. Likewise $z(u-v) < i(v)$, for $u \in W^u(v)\backslash\{v\}$. Again consider a heteroclinic orbit $u(t) \in C(v,w) =$

$W^u(v) \cap W^s(w)$. For large $t > 0$ we obtain

$$\begin{array}{rcl} z(v-w) = z(u(-t)-w) & \geq & i(w), \\ z(v-w) = z(v-u(t)) & < & i(v) = i(w)+1. \end{array}$$

This proves (1.17). Therefore blocking of any type indeed prevents connections.

Some asymmetry in the statement of the zero number blocking arises by the apparent preference of the boundary values at $x = 0$ over those at $x = 1$. But the statement is in fact symmetric with respect to the boundary values. Indeed, $z(v - \bar{w}) = z(w - \bar{w})$ implies that $\bar{w}(x)$ lies in between $v(x)$ and $w(x)$ at $x = 1$ if, and only if, it does at $x = 0$. The orders of $v(x), \bar{w}(x), w(x)$ at the respective boundaries agree, if $z$ is even, and they are reverses of each other for odd $z$.

Our final ingredient to the proof of theorem 1.2 is an element of liberalism: any one-dimensional connections which are not explicitly forbidden by the blocking law actually do exist.

**Lemma (liberalism) 1.7** *In addition to the assumptions of theorem 1.2 suppose $v, w$ are hyperbolic equilibria such that $i(v) = i(w) + 1$ and connections from $v$ to $w$ are not blocked.*

*Then $v$ connects to $w$.*

In section 4, we will prove this lemma by a combination of proposition 1.3 and Conley index arguments for suitably chosen homotopies of the nonlinearity $f$.

**Proof of theorem 1.2:**

With lemmas 1.4, 1.5, and 1.7 at hand, we can now prove theorem 1.2. Given hyperbolic equilibria $v, w$ we have to determine, constructively from the permutation $\pi$, whether or not the set $C(v, w)$ of heteroclinic connections from $v$ to $w$ is nonempty.

By lemma 1.5, it is sufficient to consider the special case

$$i(v) = i(w) + 1.$$

By lemma 1.7, the set $C(v, w)$ is then empty if, and only if, connections from $v$ to $w$ are blocked in the sense of definition 1.6. By definition 1.6, blocking only requires information about Morse indices $i$ and zero numbers $z$ on the set $E$ of equilibria. By lemma 1.4, these quantities can be determined, constructively and explicitly from the permutation $\pi$ associated to $E$. This proves theorem 1.2. $\square$

In section 5, we illustrate our results by examples. We conclude the paper, in section 6, with an extensive discussion of earlier results and neighboring problems.

**Acknowledgements**

We are much indebted to Jack Hale and Palo Brunovský for introducing us to the connecting orbits problem more than ten years ago. Giorgio Fusco, Hiroshi Matano and Ignac Tereščák generously shared numerous valuable hints with us. The hospitality of Instituto Superior Tecnico is gratefully acknowledged by Bernold Fiedler. And a wonderful typesetting and layout job was carefully undertaken by Regina Löhr and Suzanne Sanabria.

## 2 Permutations, Morse indices, and zero numbers

In this section we prove lemma 1.4. In fact, we give explicit expressions for the Morse indices $i(v_n)$ and the zero numbers $z(v_n - v_m)$, for all equilibria $v_m, v_n$ in

$$E = \{v_1, \ \dots, \ v_\kappa\},$$

in terms of the permutation $\pi$ of the indices $\{1, \ \dots, \ \kappa\}$; cf. (1.10). The proofs of these expressions are just a paraphrase of earlier work by Fusco and Rocha; see [FR91], [Roc92]. Therefore we only sketch the basic idea here.

**Proposition 2.1** *Let the assumptions of theorem 1.2 hold. For any* $1 \leq m \leq \kappa$*, the Morse index* $i(v_m)$ *of the equilibrium* $v_m$ *is then given by*

$$(2.1) \qquad i(v_m) = \sum_{j=1}^{m-1} (-1)^{j+1} \text{ sign } (\pi^{-1}(j+1) - \pi^{-1}(j)).$$

*(As usual, empty sums denote zero.) For any* $1 \leq m < n \leq \kappa$*, the zero number* $z(v_n - v_m)$ *is given by*

$$(2.2) \qquad \begin{aligned} z(v_n - v_m) \ = \ i(v_m) \ &+ \tfrac{1}{2}[(-1)^n \text{ sign } (\pi^{-1}(n) - \pi^{-1}(m)) - 1] \\ &+ \textstyle\sum_{j=m+1}^{n-1} (-1)^j \text{ sign } (\pi^{-1}(j) - \pi^{-1}(m)). \end{aligned}$$

*For practical computations, the following recursions are useful.*

$$(2.3) \qquad i(v_{m+1}) = i(v_m) + (-1)^{m+1} \text{ sign } (\pi^{-1}(m+1) - \pi^{-1}(m)),$$

$$(2.4) \qquad i(v_1) = i(v_\kappa) = 0,$$

$$(2.5) \qquad z(v_{m+1} - v_m) = \min\{i(v_m), i(v_{m+1})\},$$

$$z(v_{m+1}-v_m) = z(v_n - v_m) + \tag{2.6}$$
$$+\tfrac{1}{2}[(-1)^{n+1}\,\text{sign}\,(\pi^{-1}(n+1)-\pi^{-1}(m))+$$
$$+(-1)^n\,\text{sign}\,(\pi^{-1}(n)-\pi^{-1}(m))],$$

$$z(v_n - v_1) = z(v_\kappa - v_m) = 0. \tag{2.7}$$

*These relations hold for all $m, n$ for which the expressions make sense.*

**Proof:**

The proof is based on a detailed analysis of the *shooting surface* $M$ and the *shooting curve* $S$. To define $M$, consider the differential equation of equilibria as an initial value problem for the system

$$\begin{aligned} v' &= w \\ w' &= -f(x,v,w) \\ x' &= 1 \end{aligned} \tag{2.8}$$

with initial conditions

$$v = \alpha,\ w = 0,\ x = 0. \tag{2.9}$$

This line of initial conditions defines a surface $M$, following the solutions $v = v(\alpha, x),\ w = w(\alpha, x)$:

$$M := \{(x, v(\alpha,x), w(\alpha,x)) | \alpha \in \mathbb{R}, 0 \le x \le 1\}. \tag{2.10}$$

Similarly, the shooting curve $S$ is defined as the $x = 1$ section of $M$,

$$S := \{(v(\alpha,x), w(\alpha,x)) | \alpha \in \mathbb{R}, x = 1\}. \tag{2.11}$$

Clearly, solutions $v_m(x)$ of the boundary value problem, alias equilibria of (1.1), corresponds to the intersection points of $S$ with the $v$-axis:

$$w(\alpha, 1) = 0 \quad \Leftrightarrow \quad \alpha = \alpha_m := v_m(0) \text{ for some } 1 \le m \le \kappa. \tag{2.12}$$

In that case, $v(\alpha, 1) = v_m(1)$.

To understand (2.1), consider the nonzero tangent vector

$$\underline{\vartheta}(\alpha, x) := (v_\alpha(\alpha, x), w_\alpha(\alpha, x)) \tag{2.13}$$

to $M$ in an $x$=const. section next. Note that $\underline{\vartheta}(\alpha, \cdot)$ solves the linearized differential equation along the solution $(v(\alpha, \cdot), w(\alpha, \cdot))$. By Sturm-Liouville theory, the number of completed clockwise half-windings of $\underline{\vartheta}(\alpha, \cdot)$ around $0 \in \mathbb{R}^2$, increased by 1, equals the Morse index $i(v(\alpha, \cdot))$, provided $v(\alpha, \cdot) = v_m(\cdot)$ is an equilibrium; see [Roc92]. This accounts for the left hand side of (2.1).

To interpret the equality in (2.1), we indicate why

$$i(v_j) + (-1)^{j+1} \text{ sign } (\pi^{-1}(j+1) - \pi^{-1}(j)) - i(v_{j+1}) = 0 \tag{2.14}$$

holds, for any $1 \leq j < \kappa$. (Note that (2.14) is equivalent to claim (2.3).) Indeed, the total winding of $\underline{\vartheta}(\alpha, x)$ is zero, if we move on the surface $M$ first along $v_j$, then on $S$ from $v_j(1)$ to $v_{j+1}(1)$, and finally back to $x = 0$ along $v_{j+1}$. Just note that the corresponding loop in $(\alpha, x)-$space is contractible. Counting completed clockwise half-turns, the part along the $S$-curve contributes the sign term. Here we use that

$$\text{sign } (v_{j+1}(1) - v_j(1)) = \text{ sign } (\pi^{-1}(j+1) - \pi^{-1}(j)). \tag{2.15}$$

This follows from definition (1.10) of $\pi$. We also use the alternation rule

$$(-1)^{j+1} w_\alpha(\alpha_j,\ x = 1) > 0 \tag{2.16}$$

for the $w$-components of $\underline{\vartheta}$ at the intersection points of $S$ with the $v$-axis. Alternation holds by hyperbolicity of the equilibria. For the first (lowest) equilibrium, $j = 1$, dissipativeness condition (1.2) on $f$ implies $w_\alpha(\alpha_1,\ x = 1) > 0$, because

$$w(\alpha,\ x = 1) \cdot \alpha > 0,$$

for all sufficiently large $|\alpha|$. This proves (2.16), (2.14) and (2.3).

To prove (2.1), we just sum (2.14) over $1 \leq j < n$. Note here that

$$i(v_1) = i(v_\kappa) = 0 \tag{2.17}$$

holds, as claimed in (2.4), again by dissipativeness. Indeed, $v_1$ and $v_\kappa$ denote the minimal (resp. maximal) stationary solution. These are stable by hyperbolicity and by monotonicity of the semiflow. Monotonicity also implies $z(v_n - v_1) = z(v_\kappa - v_m) = 0$, proving (2.7). In particular,

$$\pi(1) = 1, \;\; \pi(\kappa) = \kappa \tag{2.18}$$

are always fixed under $\pi$.

In summary, the left hand side of (2.1) accounts for clockwise winding of $\underline{\vartheta}$ along $v_m$, and the right hand side for clockwise winding of $\underline{\vartheta}$ along $S$, from $v_1(x = 1)$ to $v_m(x = 1)$. Since no winding occurs along $v_1$ or $x = 0$, these two numbers are equal.

We describe (2.2) in a similarly contracted fashion. The first expression, $z(v_n - v_m)$, describes the clockwise winding of the curve

$$x \mapsto (v(\alpha_n, x) - v(\alpha_m, x), \;\; w(\alpha_n, x) - w(\alpha_m, x)) \in \mathbb{R}^2 \setminus \{0\} \tag{2.19}$$

around zero for $0 \leq x \leq 1$. The second expression, $i(v_m)$, describes the clockwise winding of

$$x \mapsto \underline{\vartheta}(\alpha_m, x) = (v_\alpha(\alpha_m, x), \;\; w_\alpha(\alpha_m, x)) \tag{2.20}$$

for $0 \leq x \leq 1$, as above. Again by contracting paths on the surface $M$, the difference of these two winding numbers is given by the winding of the secant vectors to the shooting curve $S$ at $x = 1$,

$$\alpha \mapsto \frac{1}{\alpha - \alpha_m}(v(\alpha, 1) - v(\alpha_m, 1), \;\; w(\alpha, 1) - w(\alpha_m, 1)) \tag{2.21}$$

for $\alpha_m < \alpha \leq \alpha_n$. In other words, this is the winding of the $S$-curve around the point

$$(v_m(1), 0) \in S.$$

Again, clockwise halfturns are counted. The remaining expressions in (2.2) account for this secant winding; see [Roc92]. Thus considering curves (2.19) – (2.21) proves (2.2). Admittedly, we skip some book-keeping details here, which can be found in the above mentioned references.

The remaining recursion claims (2.5), (2.6) follow easily from the explicit formulas (2.1) and (2.2). This completes the proof of proposition 2.1, and of lemma 1.4. □

For explicit examples it will be convenient to apply proposition 2.1 to the concept of blocking; see section 5.

**Corollary (adjacency) 2.2** *Consider two equilibria $v_n$, $v_m$, numbered such that $i(v_n) \geq i(v_m)$. Assume $v_n$ and $v_m$ are* adjacent*, that is*

*(i)* $|n - m| = 1$, *or*

*(ii)* $|\pi^{-1}(n) - \pi^{-1}(m)| = 1$.

*Then*

$$i(v_n) = i(v_m) + 1 \tag{2.22}$$

*and connections from $v_n$ to $v_m$ are not blocked. In view of liberalism lemma 1.7, this implies that $v_n$ connects to $v_m$.*

**Proof:**

By definition 1.6 (ii) and the asymmetry remark preceding lemma 1.7, zero number blocking cannot occur between adjacent equilibria. To prove the corollary, we have to show (2.22) and

$$z(v_n - v_m) = i(v_m). \tag{2.23}$$

First assume that (i) holds; $n = m \pm 1$. Replacing $v$ by $-v$, if necessary, we may assume that $n = m + 1$. Then proposition 2.1 implies

$$i(v_n) = i(v_m) + 1,$$

by (2.3) and $i(v_n) \geq i(v_m)$. This proves (2.22). To prove (2.23) we use (2.5) and (2.22):

$$z(v_n - v_m) = \min\{i(v_m),\ i(v_n)\} = i(v_m).$$

This proves the corollary, if we assume (i) holds.

If (ii) holds, replace $x$ by $1 - x$. Then the equilibria have to be renumbered, according to (1.9), (1.10), and (i) holds after renumbering. This proves the corollary. □

# 3 Cascading

Throughout, let the assumptions of theorem 1.2 hold. We prove the cascading lemma 1.5 in this section. As we have mentioned in section 1, the "if"-part of lemma 1.5 follows from general transitivity results on the relation "connects to" for equilibria of Morse-Smale systems; see [Oli92], [PS70].

It remains to prove the "only if"-part. Let $v, w$ be two given hyperbolic equilibria with Morse indices $i(v), i(w)$ such that $v$ connects to $w$ by a nonempty set of trajectories $C := C(v, w)$. As we have seen in section 1, $C$ is a manifold of dimension

$$n := \dim C = i(v) - i(w) \geq 1 \tag{3.1}$$

embedded in $X$, cf. (1.12)–(1.14) and (1.16).

Our proof proceeds by induction on $n$. If $n = 1$, nothing has to be proved, since $e_0 = w, e_1 = v, i(v) - i(w) = 1$ and $v$ connects to $w$ by assumption.

Next suppose $n \geq 2$ and the lemma holds for integers $1, \ldots, n-1$ replacing $n$. Define the boundary $\partial C$ of $C$ by

$$\partial C := (\operatorname{clos} C) \setminus C; \tag{3.2}$$

closure is understood in the $W^{2,2}$-topology of $X$. Note that $v, w \in \partial C$. Moreover, $\partial C$ is closed and, like $C$ and $\operatorname{clos} C$, also invariant. The proof now reduces to the following two lemmas.

**Lemma 3.1** *Since $n = i(v) - i(w) \geq 2$, the boundary $\partial C$ of $C(v, w)$ must contain a third equilibrium $e$, besides $v$ and $w$.*

**Lemma 3.2** *Suppose $\partial C$ contains a third equilibrium $e$, besides $v$ and $w$. Then $v$ connects to $e$ and $e$ connects to $w$.*

Once these two lemmas are proved, we can indeed complete the induction as follows. The Morse indices satisfy

$$i(v) > i(e) > i(w), \tag{3.3}$$

since both $v$ connects to $e$, and $e$ to $w$. Therefore

$$i(v) - i(e) < n, \qquad i(e) - i(w) < n. \tag{3.4}$$

The induction hypothesis provides us with cascades of connected equilibria from $v$ to $e$ and from $e$ to $w$, as in lemma 1.5. Joining the two cascades completes the induction and, thereby, proves lemma 1.5.

We remark that lemma 3.2 holds in general for Morse-Smale systems. In contrast, lemma 3.1 uses the zero number structure of proposition 1.3. Clearly lemma 3.1 fails for the down-hill flow on the standard $n$-sphere. For reasons of presentation, we prove lemma 3.2 first.

**Proof of lemma 3.2**

We show that any equilibrium $e$ in $\partial C$ connects to $w$. The case of connections from $v$ to $e$ is analogous and will be omitted. We proceed inductively by increasing values of the Morse function (alias Ljapunov functional $V$, see (1.4)). The case that $e$ corresponds to the minimum on $\partial C$, that is $e = w$, is trivial. Now assume $e \neq w$ and all equilibria $e'$ in $\partial C$ below $e$ connect to $w$. By transitivity, we only have to show that $e$ connects to some $e'$ in $\partial C$. Note that $V(e')$ is then automatically below $V(e)$. Since $e \in \partial C$, we can find a sequence $u_0^m$ in $C$ converging to $e$. Since $u_0^m = u^m(0)$ are on trajectories $u^m(t)$ connecting $v$ to $w \neq e$, we can find positive times $t_m$ such that

$$V(u^m(t_m)) = V(e) - \varepsilon. \tag{3.5}$$

Here $\varepsilon > 0$ is chosen small enough such that the interval $[V(e) - \varepsilon, V(e))$ does not contain $V$-values of equilibria. Choosing a convergent subsequence, if necessary, we may assume that

$$u_0 := \lim_{m \to \infty} u^m(t_m) \in \partial C \tag{3.6}$$

exists. By hyperbolicity of the equilibrium $e$, the construction of $u_0$ implies

$$u_0 \in W^u(v). \tag{3.7}$$

The $\omega$-limit set of $u_0$, on the other hand, is given by an equilibrium $e'$. By (3.7), $e$ connects to $e'$. Since $u_0 \in \partial C$ and since $\partial C$ is closed and invariant, $e'$ also lies in $\partial C$. This completes our induction, and the proof of lemma 3.2.

□

**Proof of lemma 3.1**

Here is an outline of our indirect proof. Suppose $\partial C$ does not contain a third equilibrium $e$. We will first prove that then

$$\partial C = \{v, w\} \tag{3.8}$$

does not contain any points besides $v$ and $w$. Using this fact, we will then prove that there exist $u_0 \in C$ arbitrarily close to $v$ such that

$$z(u_0 - v) = i(w), \text{ and} \tag{3.9}$$

$$\text{sign}(u_0(x) - v(x)) \neq \text{sign}(w(x) - v(x)) \text{ at } x = 0 \tag{3.10}$$

both hold. In the construction of $u_0$ we will use $n = i(v) - i(w) \geq 2$ and, of course, the nodal properties of proposition 1.3. As a final ingredient we observe

$$z(u - w) \geq i(w) \tag{3.11}$$

for all $u \in W^s(w) \setminus \{w\}$; see [BF86].

Using (3.9)-(3.11) we obtain a contradiction as follows. Consider the trajectory $u(t) = u(t, \cdot)$ in $C$ through $u(0, \cdot) = u_0$ and let $t \to +\infty$. Since $u_0 \in C$, the trajectory $u(t, \cdot)$ converges to $w$ in $X$. Since $u, v$ both satisfy Neumann conditions as $x = 0$, fact (3.10) enforces a multiple zero of

$$\tilde{u}(t_0, \cdot) := u(t_0, \cdot) - v(\cdot)$$

to occur at $x = 0$, for some $t_0 > 0$. Choosing $t > t_0$, proposition 1.3 therefore implies that

$$z(u(t, \cdot) - v) < z(u_0 - v).$$

This clearly contradicts (3.9), (3.11) by which

$$z(u(t, \cdot) - v) \geq i(w) = z(u_0 - v).$$

Therefore, the proof of lemma 3.1 reduces to validating claims (3.8)-(3.10).

The proof of claim (3.8) is easy, but indirect. Suppose there exists any

$$u_0 \in \partial C \setminus \{v, w\} \subseteq \mathcal{A} \tag{3.12}$$

and consider the global trajectory $u(t), t \in \mathbb{R}$, through $u_0 = u(0)$. Its $\alpha$- and $\omega$-limit sets are single equilibria, respectively. These equilibria lie in $\partial C$, since $\partial C$ is closed and invariant. Since $v$ and $w$ are the only equilibria in $\partial C$, by assumption, $u(t)$ must connect these two equilibria. In particular $u_0 \in C$. This contradicts (3.12) and therefore proves claim (3.8).

The proof of claims (3.9), (3.10) is a little more involved. Let $\varphi_0, \varphi_1, \ldots$ denote the Sturm-Liouville eigenfunctions of the linearization at the equilibrium $v$ with associated nonzero real eigenvalues

$$\lambda_0 > \lambda_1 > \ldots > \lambda_{i(v)-1} > 0 > \lambda_{i(v)} > \ldots \tag{3.13}$$

Note that $z(\varphi_k) = k$. Let $T_{v,w}$ denote the subspace spanned by $\varphi_{i(w)}, \ldots, \varphi_{i(v)-1}$. By $T_u C$ we denote the $n$-dimensional tangent space to the manifold $C$ at

$u \in C$. We claim, and prove below, that

$$T_u C \longrightarrow T_{v,w} \tag{3.14}$$

converges uniformly for $u \to v$. Convergence is understood in the Grassmannian manifold

$$G(i(v), n) \tag{3.15}$$

of $n$-planes in $\mathbb{R}^{i(v)}$. Identification with $\mathbb{R}^{i(v)}$ is possible, near $v$, because $C$ is contained in the unstable manifold $W^u(v)$ of dimension $i(v)$. Note that

$$T_v W^u(v) = \operatorname{span}\{\varphi_0, \ldots, \varphi_{i(v)-1}\}. \tag{3.16}$$

Before proving convergence claim (3.14), we show how (3.14) implies claims (3.9), (3.10). Let $P$ denote the eigenprojection onto $T_{v,w}$. By convergence (3.14) and the implicit function theorem,

$$P: \quad C \to T_{v,w}$$

is a local diffeomorphism onto its respective image, at all points of $C$ in an a-priori fixed closed $\varepsilon_0$-neighborhood $\mathcal{N}_{\varepsilon_0}$ of $v$. For notational convenience we have shifted $v$ to zero here. Moreover,

$$\partial P(C) \subseteq P(\partial C) = \{v\}$$

holds, locally near $v$, because $C$ is relatively compact. Because $n = \dim T_{v,w} \geq 2$, any punctured disc around $v$ in $T_{v,w} \setminus \{v\}$ is connected. Therefore

$$P(C \cap \mathcal{N}_{\varepsilon_0})$$

still covers such a punctured disc, e.g. of radius $\varepsilon_0/2$. Now define $u_0 \in C \cap \mathcal{N}_{\varepsilon_0}$ such that

$$Pu_0 = v + \varepsilon\varphi_{i(w)},$$

for suitably small $0 < \varepsilon < \varepsilon_0/2$. Here we may assume that

$$\text{sign } \varphi_{i(w)}(x) \neq \text{sign}(w(x) - v(x)), \text{ at } x = 0.$$

Because $T_{u_0}C \to T_{v,w}$ for $\varepsilon \to 0$ by (3.14) and because $\varphi_{i(w)} \in T_{v,w}$ has only simple zeros we can guarantee

$$z(u_0 - v) = z(\varepsilon\varphi_{i(w)}) = i(w),$$

for sufficiently small $\varepsilon$. Similarly,

$$\begin{aligned} \text{sign}(u_0(x) - v(x)) = \text{sign}(\varepsilon\varphi_{i(w)}(x)) \neq \\ \neq \text{sign}(w(x) - v(x)) \end{aligned}$$

will hold, at $x = 0$. This proves the claims (3.9) and (3.10).

To complete the proof of the lemma, it remains to prove the "Grassmannian" convergence claim (3.14). We first prove convergence along a single trajectory $u(t), t \to -\infty$ in $C \subseteq W^u(v)$, postponing uniformity. The $C^1$-flow on $W^u(v)$ induces a $C^0$-flow on the product

$$W^u(v) \times G(i(v), n), \tag{3.17}$$

transporting $n$-frames by the linearized flow. Note that the Grassmannian

$$G(i(v), n) \;\cong\; SO(i(v)) \;/\; (SO(n) \times SO(i(v) - n)) \tag{3.18}$$

is compact, being diffeomorphic to a coset space of orthogonal groups. Consider the $\alpha$-limit set corresponding to an initial condition

$$(u(0), T_{u(0)}C)$$

in $W^u(v) \times G(i(v), n)$. The set is an invariant, chain recurrent subset of

$$\{v\} \times G(i(v), n).$$

The flow on that latter set is induced by the linearization at the equilibrium $v$. In components $y_0, y_1, \ldots$ with respect to $\varphi_0, \varphi_1, \ldots$, the linearized flow at $v$ is given by

$$\dot{y}_k = \lambda_k y_k, \quad 0 \leq k < i(v) \tag{3.19}$$

where $\lambda_0 > \ldots > \lambda_{i(v)-1} > 0$ are again the eigenvalues at $v$. The induced flow on $G(i(v), n)$ is gradient-like and, in fact, Morse-Smale. The same holds true for the flow on $W^u(v) \times G(i(v), n)$. These claims can be checked explicitly. For example, the $\binom{i(v)}{n}$ equilibria in $G(i(v), n)$, according to (3.19), are given explicitly by the $n$-planes

$$T_\kappa := \{(y_0, \ldots, y_{i(v)-1}) \mid y_k = 0 \text{ for } k \in \kappa\}, \tag{3.20}$$

where $\kappa$ ranges over all $n$-element subsets of $\{0, \ldots, i(v)-1\}$. As an aside, we note that the respective Morse indices of these equilibria are given by

$$i(T_\kappa) := -n(n-1)/2 + \sum_{k \in \kappa} k \tag{3.21}$$

in $G(i(v), n)$, and by

$$i((v, T_\kappa)) := i(T_\kappa) + i(v) \tag{3.22}$$

for the flow in $W^u(v) \times G(i(v), n)$. By the Morse property on $G(i(v), n)$,

$$T_\kappa := \lim_{t \to -\infty} T_{u(t)} C \tag{3.23}$$

exists and is a single equilibrium.

Still aiming at our convergence claim (3.14), we prove next that

$$\kappa = \{i(w), \ldots, i(v) - 1\} \tag{3.24}$$

in (3.23) does not depend on $u(0) \in C$. In particular

$$T_{u(t)} C \longrightarrow T_{v,w}$$

for $t \to -\infty$ and any $u(0) \in C$. In fact it is sufficient to show that

$$i(w) \leq z(\psi_0) < i(v), \tag{3.25}$$

for any $\psi_0 \in T_{u_0}C$, and any $u_0 \in C$. To prove (3.25) note that

$$T_{u_0}C = T_{u_0}W^u(v) \cap T_{u_0}W^s(w).$$

Now the left inequality of (3.25) holds on $T_{u_0}W^s(w)$ whereas the right one holds on $T_{u_0}W^u(v)$; see [BF86] or [Hen85] for details. Actually, evolving $\psi_0 = \psi(0)$ by the linearized flow, the normalized tangent vector

$$\psi(t)/|\psi(t)|_X$$

converges to a unit tangent vector of $W^u(v)$ at $v$, for $t \to -\infty$, and of $W^s(w)$ at $w$, for $t \to +\infty$, respectively. Choosing $\tilde{u} = \psi$, in proposition 1.3, claim (3.25) then follows from monotonicity of

$$t \mapsto z(\psi(t)) = z(\psi(t)/|\psi(t)|)$$

This proves (3.25), and therefore claim (3.24).

To complete the proof of convergence claim (3.14), we have to address uniformity of the convergence with respect to $u$. To this end choose the set of pairs $(u, T_uC)$ as initial conditions, where $u$ ranges over a small sphere around $v$ in $C$. Consider the $\alpha$-limit set $\alpha$ of that set. Any equilibrium in $\alpha \subseteq \{v\} \times G(i(v), n)$ must also arise as the $\alpha$-limit set of an individual trajectory $(u(t), T_{u(t)}C)$, by the Morse-Smale property. By (3.23), (3.24) there is only one such equilibrium. Again by the Morse property in $G(i(v), n)$, we conclude

$$\alpha = \{(v, T_{v,w})\},$$

because $\alpha$ is closed and invariant. This proves uniformity in (3.14), and therefore completes the proof of claims (3.9), (3.10), the proof of lemma 3.1, and the proof of our cascading lemma 1.5. $\square$

# 4 Liberalism

In this section we prove lemma 1.7 on "liberalism". Throughout, we assume that $v, w$ are hyperbolic equilibria of (1.1) such that $i(v) = i(w) + 1$ and connections from $v$ to $w$ are not blocked, in the sense of definition 1.6. In particular this implies

$$z(v - w) = n := i(w). \tag{4.1}$$

Without loss of generality, we assume $v(x) > w(x)$ at $x = 0$. Also, we may assume $a,\ f \in C^3$; since $C^2$-small perturbations do not destroy the Morse-Smale property (1.12).

We have to prove that $v$ connects to $w$. Our main tool will be the Conley index, as introduced in [Con78]. For an infinite dimensional adaptation see [Ryb82]. For our purposes, we briefly recall the definition and a few basic properties of the Conley index; see also [FM88]. Let $\Sigma \subseteq X$ be an invariant set of an admissible nonlinear semiflow $u(t) = T(t)u_0$ on a Banach space $X$. (Here, admissibility essentially involves a compactness condition which is satisfied for the parabolic equation (1.1).) Invariance is understood in positive and, where defined, negative time direction. We call a closed neighborhood $\mathcal{N}_1$ of $\Sigma$ an *isolating neighborhood* of $\Sigma$, if $\Sigma$ is contained in the interior of $\mathcal{N}_1$ and if, in addition, $\Sigma$ is the maximal invariant subset of $\mathcal{N}_1$. The set $\Sigma$ is an *isolated invariant set*, if such an isolating neighborhood exists.

Assume $\Sigma$ is an isolated invariant set. In fact, a particular isolating neighborhood $\mathcal{N}_1$ can then be chosen such that any point $u_0$ in the (topological) boundary $\partial\mathcal{N}_1$ leaves $\partial\mathcal{N}_1$ for sufficiently small positive and (where defined) negative time direction. Moreover, boundary points which come from the interior of $\mathcal{N}_1$, hit $\partial\mathcal{N}_1$, and then reenter the interior of $\mathcal{N}_1$ can be avoided. In other words, $\partial\mathcal{N}_1$ consists of

- strict ingress,
- bounce off, and
- strict egress

points, with respect to the set $\mathcal{N}_1$. Denote by $\mathcal{N}_2 \subseteq \partial\mathcal{N}_1$ the *exit set* of $\mathcal{N}_1$, which consists of all boundary points which are not strict ingress points.

The *Conley index* $C(\Sigma)$ of $\Sigma$ is the homotopy type $[\mathcal{N}_1, \mathcal{N}_2]$ of the isolating neighborhood $\mathcal{N}_1$ relative to its exit set $\mathcal{N}_2$. Denoting by $[\mathcal{N}_1/\mathcal{N}_2]$ the homotopy type of $\mathcal{N}_1$ with $\mathcal{N}_2$ collapsed to a (distinguished) point, we have

$$C(\Sigma) := [\mathcal{N}_1, \mathcal{N}_2] = [\mathcal{N}_1/\mathcal{N}_2]. \tag{4.2}$$

The main point is now that the Conley index $C(\Sigma)$ does not depend on the particular choice of the isolating neighborhood $\mathcal{N}_1$ of $\Sigma$. Moreover, $C(\Sigma)$ is a homotopy invariant. In fact, $C(\Sigma)$ does not change under continuous deformations of the semiflow as long as $\mathcal{N}_1$ remains an isolating neighborhood. Note that $\Sigma$ itself may, and typically will, change. Also the exit set will change.

The Conley index can be used to detect connecting orbits, as follows. Suppose, the semiflow is gradient-like and $v, w$ are the only equilibria in $\Sigma$, hyperbolic of unstable dimensions $i(v)$ and $i(w)$. Suppose $\Sigma = \{v, w\}$, that is, there are no orbits connecting $v$ and $w$. Then

$$C(\Sigma) = C(v) \vee C(w) = \Sigma^{i(v)} \vee \Sigma^{i(w)} \tag{4.3}$$

is the wedge product of the (pointed) spheres $\Sigma^{i(v)}$ and $\Sigma^{i(w)}$ of the respective dimensions. Conversely, suppose we can prove

$$C(\Sigma) = \overline{0} \tag{4.4}$$

is (the homotopy type of) just a single (distinguished) point. Then (4.4) is incompatible with (4.3). In particular, $v$ and $w$ must be connected. Specializing to our partial differential equation (1.1), we can then conclude that $v$ connects to $w$; because the blocking definition 1.6 excludes connections from $w$ to $v$.

We can now give an outline of our proof of liberalism lemma 1.7. Given equilibria $v, w$, as in the beginning of this section, we define the closed cones

$$K_v := \{v\} \cup \{u \in X | z(u-v) = n, \ u \leq v \text{ at } x = 0\}, \tag{4.5}$$

$$K_w := \{w\} \cup \{u \in X | z(u-w) = n, \ u \geq w \text{ at } x = 0\}. \tag{4.6}$$

We also choose closed $\varepsilon$-neighborhoods $\mathcal{N}_\varepsilon(v)$, $\mathcal{N}_\varepsilon(w)$, for some small enough $\varepsilon > 0$. Define the closed set

$$\mathcal{N}_1^\varepsilon := \mathcal{N}_\varepsilon(v) \cup \mathcal{N}_\varepsilon(w) \cup (K_v \cap K_w). \tag{4.7}$$

For small enough $\varepsilon$, we may assume that $\mathcal{N}_\varepsilon(v)$, $\mathcal{N}_\varepsilon(w)$ do not contain equilibria besides $v, w$, respectively. By blocking definition 1.6, this implies that $\mathcal{N}_1^\varepsilon$ does not contain further equilibria:

$$\mathcal{N}_1^\varepsilon \cap E = \{v, w\}. \tag{4.8}$$

Now let $\Sigma$ denote the maximal invariant subset of $\mathcal{N}_1^\varepsilon$. Then

$$\Sigma = \{v, w\} \cup C(v, w) = \text{ clos } C(v, w). \tag{4.9}$$

Indeed, $\Sigma$ is contained in clos $C(v, w)$, by (4.8). Conversely, $C(v, w)$ is contained in the closed sets $K_v$ and $K_w$ by the zero number proposition 1.3 and (4.1). The same holds true for the invariant set clos $C(v, w) \subseteq \mathcal{N}_1^\varepsilon$. By maximality of $\Sigma$, this proves (4.9). In particular, $\Sigma$ is an isolated invariant set with isolating neighborhood $\mathcal{N}_1^\varepsilon$.

Following the idea (4.3), (4.4), we claim that

$$C(\Sigma) = \overline{0} \tag{4.10}$$

holds for the isolated invariant set $\Sigma = \text{ clos } C(v,w)$. Then $C(v,w)$ must be nonempty, and lemma 1.7 on "liberalism" will be proved.

We prove claim (4.10) in three steps. For comparison, we first construct a model equation of the form (1.1), with diffusion coefficient $a(x) \equiv 1$ and nonlinearity $g$ replacing $f$; see lemma 4.1 below. The model equation will contain a parameter $\mu$. Increasing $\mu$ through $\mu = 0$, a saddle node bifurcation will occur. For small $\mu > 0$, the two generated equilibria, $v_+$ and $v_-$, together with their connecting orbits $C(v_+, v_-)$, form an isolated invariant set

$$\Sigma_\mu := \{v_+, v_-\} \cup C(v_+, v_-). \tag{4.11}$$

By homotopy invariance of the Conley index

$$C(\Sigma_\mu) = \overline{0}. \tag{4.12}$$

In particular, $\Sigma_\mu$ will serve as a model for $\Sigma$ in (4.10).

In a second step, we transform $v_+$ into $v$ and $v_-$ into $w$; see lemma 4.2. This transformation is not a homotopy. It consists of a $C^4$-diffeomorphism

$$\begin{aligned} \xi : \ [0,1] &\to [0,1] \\ x &\mapsto \xi(x) \end{aligned} \tag{4.13}$$

of $x$-space; and a $u$-affine $C^4$-transformation

$$\begin{aligned} \Theta : [0,1] \times \mathrm{IR} &\to \mathrm{IR} \\ (x,u) &\mapsto \alpha(x)u + \beta(x) \end{aligned} \tag{4.14}$$

with uniformly nonzero (positive or negative) $\alpha$. Either transformation does not change the zero number of differences of points in $X$. Moreover, the cones

$K_{v_\pm}$ can be mapped onto $K_v, K_w$. The model equation is transformed into (1.1) with diffusion coefficient $\tilde{a}$ and nonlinearity $\tilde{f}$ replacing $a, f$. Similarly, the set $\Sigma_\mu$ is transformed into an isolated invariant set $\tilde{\Sigma}$ such that

$$C(\tilde{\Sigma}) = C(\Sigma_\mu). \tag{4.15}$$

In a third step, finally, we perform a homotopy from $\tilde{f}$ to $f$ such that the equilibria $v, w$ remain fixed and uniformly hyperbolic during the homotopy. This can be achieved by a slight modification of the standard homotopy

$$\tau f + (1-\tau)\tilde{f}.$$

The modification concerns addition of terms

$$\begin{aligned} &\chi_v \mu_v(\tau)(u - v(x)), \\ &\chi_w \mu_w(\tau)(u - w(x)) \end{aligned}$$

for $(x, u, u_x)$ near $(x, v(x), v_x(x))$, $(x, w(x), w_x(x))$, respectively. The coefficients $\chi_v, \chi_w$ denote standard cut-offs. The coefficients $\mu_v(\tau)$, $\mu_w(\tau)$, zero near $\tau = 0$ and 1, shift the spectra of the linearizations at $v, w$, such that uniform hyperbolicity of these equilibria is guaranteed during the homotopy. Consider $v, w$ and their connecting orbits $C_\tau(v, w)$ during this homotopy;

$$\Sigma(\tau) := \{v, w\} \cup C_\tau(v, w). \tag{4.16}$$

Recalling definitions (4.5)–(4.7), we observe

$$\Sigma(\tau) \subseteq K_v \cup K_w, \tag{4.17}$$

since zero dropping proposition 1.3 applies throughout the homotopy. Varying $\tau$, the equilibria $v, w$ do not bifurcate, due to uniform hyperbolicity. Therefore, choosing $\varepsilon > 0$ small enough again, the neighborhood $\mathcal{N}_1^\varepsilon$ defined in (4.7) is an isolating neighborhood, throughout the homotopy. Indeed,

$\Sigma(\tau)$ can never touch the boundary of $K_v$ or $K_w$, except at the points $v, w$, by zero dropping proposition 1.3. In particular, by homotopy invariance of Conley index, $C(\Sigma(\tau))$ does not depend on $\tau$. Hence

$$C(\Sigma) = C(\Sigma(0)) = C(\Sigma(1)) = C(\tilde{\Sigma}). \tag{4.18}$$

Combining (4.12), (4.15) and (4.18), we conclude

$$C(\Sigma) = C(\tilde{\Sigma}) = C(\Sigma_\mu) = \bar{0}. \tag{4.19}$$

This proves claim (4.10), and therefore liberalism lemma 1.7.

It only remains to formulate, and prove, the two lemmas promised above. We first provide the model equation.

**Lemma 4.1** *Let $n \geq 0$ be given. Consider the equation*

$$0 = u_{xx} + g(\mu, x, u, u_x), \quad 0 < x < 1, \tag{4.20}$$

*with Neumann boundary conditions. Define*

$$g(\mu, x, u, p) := n^2\pi^2 u + (u^2 + \frac{1}{n^2\pi^2}p^2 - \mu)\gamma(x), \tag{4.21}$$

*where $\gamma(x) := \cos(n\pi x)$.*

*Then (4.20) undergoes a saddle node bifurcation at $\mu = 0, u = 0$. The bifurcating equilibria are given by*

$$v_\pm(x) = \pm\sqrt{\mu}\gamma(x) \tag{4.22}$$

*for $\mu > 0$. Note that obviously*

$$z(v_+ - v_-) = n. \tag{4.23}$$

*For small $\mu > 0$ these equilibria are hyperbolic of unstable dimensions*

$$i(v_+) = n + 1, \; i(v_-) = n. \tag{4.24}$$

**Proof:**

The equilibria $v_\pm(x)$ can be identified by direct calculation. Now parametrize the stationary branch by real $s$ such that $\mu = s^2,\ v(s,x) := s\gamma(x)$. Positive $s$ correspond to $v_+$, negative $s$ to $v_-$. Again by explicit calculation, we observe that

$$\lambda_n(s) = 2s$$

is an eigenvalue of the linearization of (4.20) at the equilibrium $v(s,\cdot)$; the corresponding ($s$-independent) eigenfunction is given by

$$\varphi_n(x) = \gamma(x).$$

Here we use the notation of (3.13) for the eigenvalues; $z(\varphi_n) = n$. By standard perturbation theory, this observation implies that

$$i(v_+(x)) = n+1, \qquad i(v_-(x)) = n,$$

for small positive parameter values $\mu$. This proves (4.24) and the lemma. $\square$

For the entertainment of our readers, we remark that a saddle node bifurcation with properties (4.23), (4.24) can also be constructed with nonlinearities $g = \mu^2 \hat{g}(u)$, independent of $x, u_x$. An explicit example (for $n \geq 1$) is given by

$$\hat{g}(u) = u(u^2-1)(2-u^2). \tag{4.25}$$

We address the transformations $\xi, \Theta$ introduced in (4.13), (4.14) next.

**Lemma 4.2** *The $C^4$-transformations $\xi, \Theta$ given by (4.13), (4.14) can be viewed as bounded (affine) linear isomorphisms in phase space $X$. Either transformation does not change*

$$z(u_1 - u_2),$$

*when applied to $u_1$ and $u_2$ in $X$. Equations of the form (1.1) are transformed into equations of the same form, under $\xi$ and $\Theta$. Neumann boundary conditions are preserved.*

*Specific $C^4$-transformations can be chosen such that their composition $\Theta \circ \xi$ maps $v_+$ to $v$ and $v_-$ to $w$.*

**Proof:**

Explicitly, the induced transformations on $X$ are

$$\begin{aligned} \xi: \quad & u \mapsto \tilde{u}(x) && := u(\xi^{-1}(x)), \\ \Theta: \quad & u \mapsto \tilde{u} && := \alpha u + \beta, \end{aligned}$$

with uniformly nonzero $\alpha, \xi_x$ and with $\alpha, \beta, \xi \in C^4$. Clearly these transformations are bounded affine linear in the $W^{2,2}-$topology of $X$. By definition,

$$z(\xi u_1 - \xi u_2) = z(u_1 - u_2),$$

for all $u_1, u_2 \in X$, and likewise for $\Theta$ replacing $\xi$. The induced transformations on differential equation (1.1) are easy to compute. For example $\xi$ transforms the diffusion coefficient $a(x)$ into

$$\tilde{a}(x) = (\xi_x)^2 a(x) > 0.$$

Neumann conditions are also preserved. (To preserve other boundary conditions, e.g. of mixed type, it is sufficient to assume $\xi_x = 1$ at $x = 0, 1$.)

It remains to construct $\Theta, \xi$ such that $\Theta \circ \xi$ maps $v_+$ to $v$ and $v_-$ to $w$. Note that, for any $\Theta, \xi$, we have

$$\Theta \circ \xi = \xi \circ \tilde{\Theta}$$

for $\tilde{\Theta}(u) = \tilde{\alpha} u + \tilde{\beta}$, $\tilde{\alpha}(x) = \alpha(\xi(x))$, $\tilde{\beta}(x) = \beta(\xi(x))$. Therefore, we may reduce our construction to the special case

$$\begin{aligned} v_- \equiv w \equiv 0, \\ z(v_+) = n = z(v). \end{aligned}$$

Picking $\beta \equiv 0$ will get $v_- \equiv 0$ mapped to $w \equiv 0$ under $\Theta \circ \xi$. Note that $v_+, v$ each possess precisely $n$ zeros, all of which are simple. For $\xi$ we choose a smooth diffeomorphism of $[0,1]$ which maps the zeros of $v_+$ to zeros of $v$. After this transformation, we may assume that the zero sets of $v_+$ and $v$ coincide. Finally define

$$\alpha(x) := \begin{cases} v(x)/v_+(x), & \text{for } v_+(x) \neq 0 \\ v_x(x)/v_{+,x}(x), & \text{for } v_+(x) = 0. \end{cases} \tag{4.26}$$

By the usual l'Hospital procedure, $\alpha \in C^4$. Indeed $v, w \in C^5$, since we have assumed $a, f \in C^3$, without loss of generality, at the beginning of this section. Moreover $v_\pm$ are analytic to begin with. This completes the proof of lemma 4.2, and of liberalism lemma 1.7. □

# 5 Three examples

In this section, we illustrate our results with three specific permutations $\pi \in S_{11}$ involving 11 equilibria. We include a description of what we believe the respective attractors look like.

**Example 5.1** $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 8 & 7 & 2 & 3 & 6 & 9 & 10 & 5 & 4 & 11 \end{pmatrix}$

This example which corresponds to an explicit choice of a dissipative nonlinearity $f = f(x, u)$ with diffusion $a \equiv 1$ was investigated in [Roc92]. It is known that the corresponding attractor does not arise from a single equilibrium by a sequence of pitchfork bifurcations, in contrast to the Chafee-Infante problem. Proposition 2.1 provides us with the vector of Morse indices

$$(i(v_n))_n = (0, 1, 0, 1, 2, 1, 2, 1, 0, 1, 0). \tag{5.1}$$

Just tabulate $\pi^{-1}(n)$ and use (2.3), (2.4). Do it, it's fun! Using recursions (2.5), (2.6) we arrive at the (symmetric) matrix of $z(v_n - v_m)$. For convenience, we insert $i(v_n)$ along the diagonal.

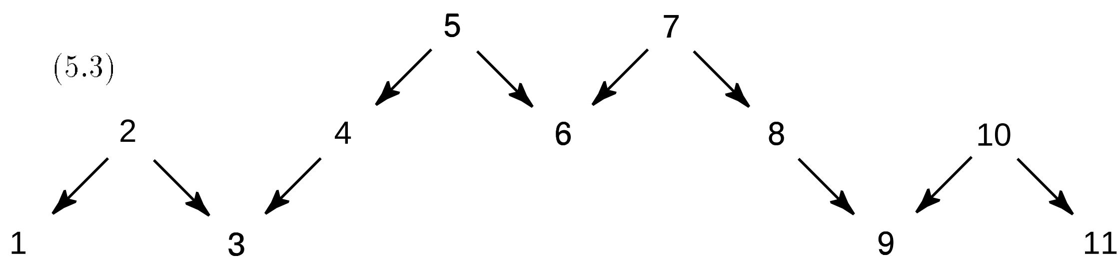
$$(z(v_n - v_m))_{n,m} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

See also figure 1.1 of the equilibrium profiles. The matrix (5.2) contains all the information which is necessary to determine the connecting orbits. By cascading lemma 1.5, it is sufficient to determine those connections $v \searrow w$ for which

$$i(v) = i(w) + 1.$$

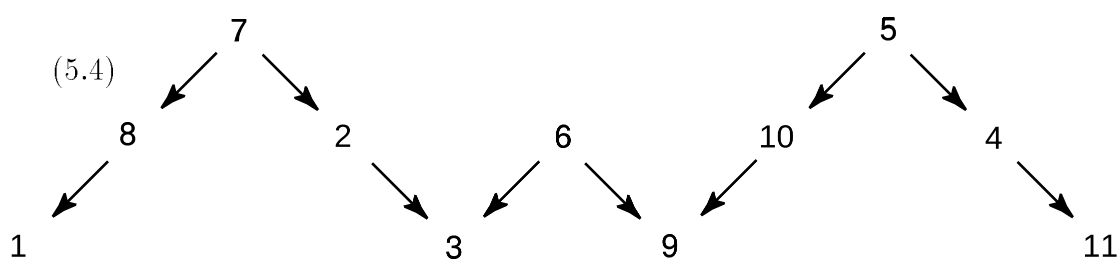
We invoke corollary 2.2 and lemma 1.7 to determine some of these connections. By corollary 2.2 (i) and the Morse vector (5.1) we obtain the connections

(5.3)




Here rows correspond to equal Morse indices. Columns correspond to the order of the values $v_n(x)$ at $x = 1$, i.e. to increasing values of $n$. Only the numbers $n$ are shown. Similarly, corollary 2.2 (ii) and the Morse vector (5.1) yield the connections

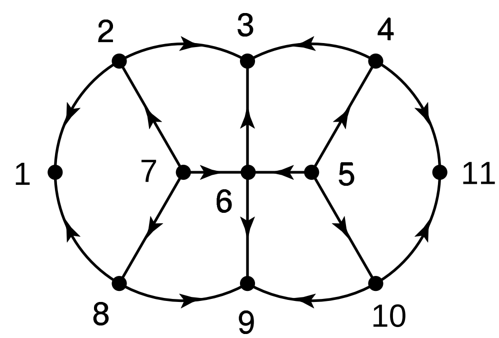
(5.4)




This time, columns correspond to the order of the values $v_n(x)$ at $x = 1$, i.e. to increasing values of $\pi(n)$. It turns out, in this example, that all other one-dimensional connections are blocked. This must and can be checked, e.g.

via the matrix (5.2) of zero numbers. Thus (5.3), (5.4) represent the directed graph of all one-dimensional connections. This graph is planar:

(5.5)




By cascading lemma 1.5, this graph also determines the higher-dimensional nonempty connection manifolds $C(v, w)$.

We can view (5.5) as a planar flow with arrows indicating separatrices. Then (5.5) decomposes into six quadrangles, e.g. $\{1, 2, 7, 8\}$, which correspond to two-dimensional cells $C(v, w)$ of connecting orbits, e.g. $C(v_7, v_1)$. We believe, but do not prove here, that this planar flow describes the global attractor of any dissipative $f$ with the same permutation $\pi$, up to $C^0$ flow equivalence.

**Example 5.2** $\quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 10 & 3 & 8 & 5 & 6 & 7 & 4 & 9 & 2 & 11 \end{pmatrix}$

This permutation corresponds to the well-known Chafee-Infante problem

(5.6) $$f = \lambda^2 u(1 - u^2), \quad a \equiv 1,$$

with $4 < \lambda/\underline{\pi} < 5, \quad \underline{\pi} = 3.14\ldots$. Proposition 2.1 implies

(5.7) $$i(v_n)_n = (0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0)$$

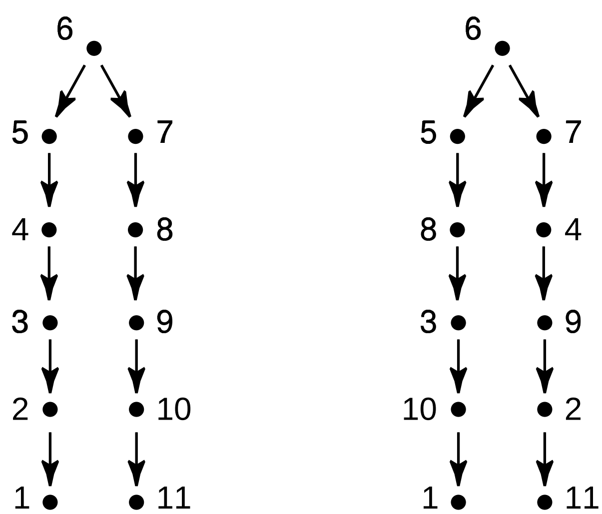
for the Morse vector. As before, we arrive at the matrix of zero numbers

$$(5.8) \qquad (z(v_n - v_m))_{n,m} \quad = \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 4 & 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that is, $z(v_n - v_m) = 5 - \max\{|6-m|, |6-n|\}$.

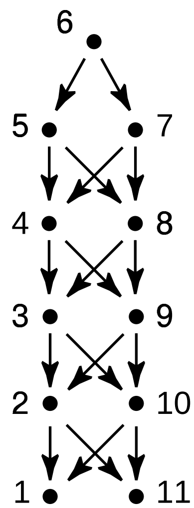
Again, we determine the connections $v \searrow w$ for which $i(v) = i(w) + 1$. Invoking corollary 2.2 (i), (ii) and lemma 1.7, we obtain the connections

(5.9)




respectively, from the Morse vector (5.7) and the permutation $\pi$. In particular, $v$ connects to $w$ if $i(v) = i(w) + 1$. Thus we need not worry about blocking, this time. The graph of one-dimensional connections is given by

(5.10)




All other $\lambda$-values with noninteger $\lambda/\underline{\pi}$ lead to similar graphs. A model for the corresponding Chafee-Infante flows on the attractor seems to be given in polar coordinates on the closed unit ball in $\mathbb{R}^N$ by

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\varphi} &= h(\varphi). \end{aligned} \tag{5.11}$$

Here the flow of $\varphi$ on the unit sphere $S^{N-1}$ is induced by the linear differential equation

$$\dot{x}_j = -jx_j, \quad 1 \le j \le N. \tag{5.12}$$

Note that (5.12) defines a flow on (half-) lines through the origin and, thereby, a flow on $S^{N-1}$, similarly to our Grassmann construction in section 3. We believe, but do not prove here, that the flow (5.11) is $C^0$-equivalent to the Chafee-Infante flow on the global attractor, for $N-1 < \lambda/\underline{\pi} < N$. Similarly, for $N = 5$, it should serve as a model for any nonlinearity $f$ with permutation $\pi$.

**Example 5.3** $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 10 & 9 & 2 & 3 & 8 & 5 & 6 & 7 & 4 & 11 \end{pmatrix}$

The previous two examples were symmetric under the flip $n \mapsto 12 - n$, alias $v \mapsto -v$. In addition, $\pi = \pi^{-1}$ in the Chafee-Infante example 5.2. The present permutation does not exhibit any such symmetries. It can be realized by an appropriate choice of $f$; cf. section 6. By proposition 2.1 the Morse vector is given by

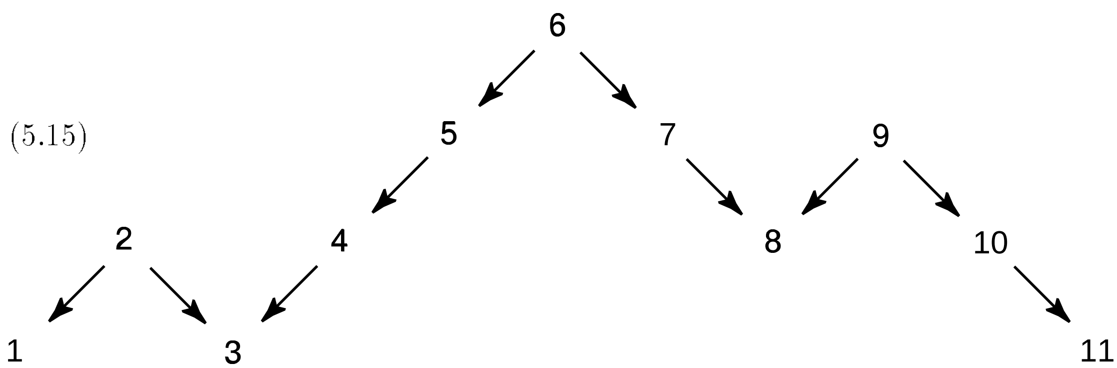
$$(i(v_n))_n = (0,1,0,1,2,3,2,1,2,1,0). \tag{5.13}$$

As before, we obtain the matrix of zero numbers

$$(z(v_n - v_m))_{n,m} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.14}$$

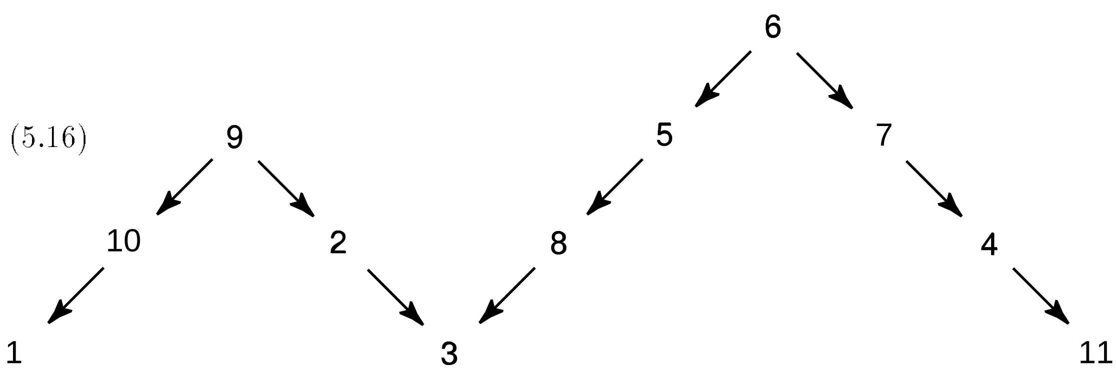
Similarly to example 5.1, corollary 2.2 (i) and lemma 1.7 yield the following connections

(5.15)




The same proceedure, but using corollary 2.2 (ii) instead, yields

(5.16)

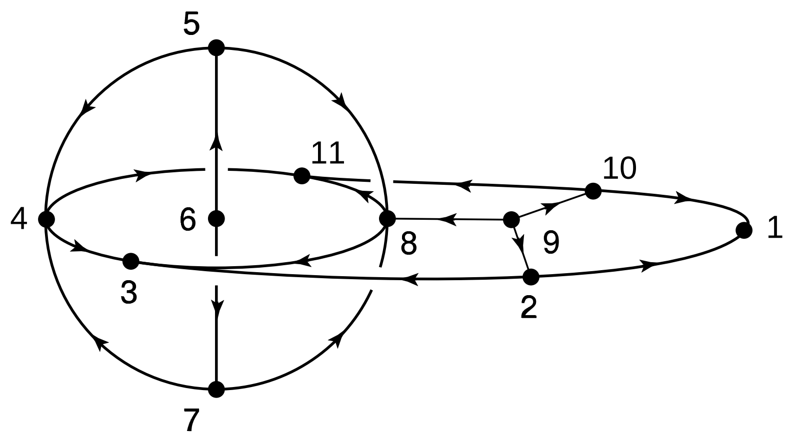



A careful (!) analysis of blocking of one-dimensional connections reveals precisely one additional connection besides those already found in (5.15), (5.16):

(5.17) $$8 \searrow 11.$$

This connection exists by liberalism lemma 1.7. The completed connection graph is sketched as follows.

(5.18)




Again it is tempting to view this as a three-dimensional flow: a Chafee-Infante flow on the 3-ball described by the equilibria $\{3, 4, 5, 6, 7, 8, 11\}$ and a 2-cell, given by $\{1, 2, 3, 8, 9, 10, 11\}$, which is attached on the half-equator $\{3, 8, 11\}$. Strictly speaking, however, we do not prove such a statement here.

# 6 Discussion

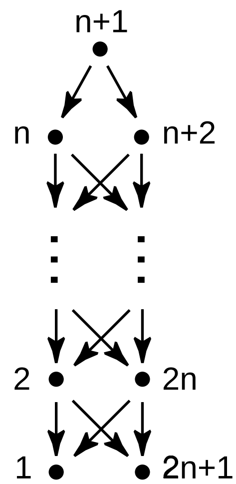
This section addresses the background, context and significance of our connecting orbits problem. We first give a brief account of earlier results. We then discuss the present result, focussing on the following question: can we "count" attractors with a prescribed number $\kappa$ of equilibria? We conclude with a list of some related open problems.

Historically, the Chafee-Infante problem

$$f = f(u) = \lambda^2 u(1 - u^2), \quad a \equiv 1, \tag{6.1}$$

with cubic nonlinearity $f$ was a long-standing source of inspiration; see also example 5.2. Originally, Chaffee and Infante investigated (ODE) steady states $v$ and their (PDE) Morse indices $i(v)$; see [CI74]. For $\lambda < 4\pi$ and Dirichlet boundary conditions, the corresponding attractors are sketched in Henry's classic, [Hen81]. The dimension of these attractors does not exceed 3. Conley and Smoller were the first to examine connecting orbits in a more global context. Using Conley index, they found connecting orbits from the trivial solution to the two bifurcated solutions with Morse index $i = n$, for $(n+1)\pi < \lambda < (n+2)\pi$, again under Dirichlet boundary conditions. See [Smo83, ch. 24.D]. Only after Matano had pointed out the relevance of zero numbers (or lap numbers) to these equations, [Mat82], further progress was made. We have already mentioned the transversality results, $W(v) \pitchfork W^s(w)$ proved by Angenent and Henry, [Ang86], [Hen85]. Using a refined version, which included center manifolds, Henry was able to track the changes in connecting orbits which occur at the pitchfork bifurcations at $\lambda = n$ in the Chafee-Infante problem. This way, Henry found all connecting orbits in the Chafee-Infante problem. Numbering equilibria for $n - 1 < \lambda < n$ by $v_1, ..., v_{2n+1}$ as in (5.16) we obtain the structure

(6.2)




The case of general dissipative

$$f = f(u) \tag{6.3}$$

was treated by Brunovský and Fiedler; see [BF88], [BF89]. Their approach was based on zero numbers and the "$y$-map". This map, defined on the unstable manifold of any equilibrium $v$, encodes information on

$$z(u(t) - v),$$

for all $t$ and all trajectories. In particular, the dropping times of $z$ can be prescribed arbitrarily. This provides connecting orbits from $v$ to $w$ with prescribed $z(w - v)$. A special case of cascading lemma 1.5, together with a Conley index type argument then detected all connecting orbits – after a rather involved induction process. From an ODE point of view, the case $f = f(u)$ is the case of an autonomous Hamiltonian steady state equation. For a detailed study of the corresponding boundary value problem, in particular for parameter dependence

$$f = \lambda^2 f(u),$$

see [Sch90]. It seems promising to recast these results in our permutation setting.

The permutation setting was introduced by Fusco and Rocha, [FR91]. It was used to determine the connecting orbits for all those attractors of equations (1.1) which arise from a unique, globally attracting equilibrium by a sequence of (nondegenerate) pitchfork bifurcations. In terms of the permutation $\pi$, it is then necessary (but not sufficient) that the sequence

$$\pi(1), \pi(2), \ldots, \pi(n)$$

contains three consecutive integers: $m, m+1, m+2$ or $m+2, m+1, m$.

For example, any attractor with permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 5 & 2 & 3 & 4 & 7 \end{pmatrix}$$

will be "pitchforkable". For a peaceful connection graph see figure 6.1.

Unfortunately, not all attractors are "pitchforkable" in that sense. Clearly, example 5.1 with

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 8 & 7 & 2 & 3 & 6 & 9 & 10 & 5 & 4 & 11 \end{pmatrix}$$

is a counterexample. The present paper can therefore be viewed as a synthesis of the two different approaches taken by Fusco, Rocha and Brunovský, Fiedler.

Our approach suggests a classification of attractors. Suppose we fix a number $\kappa$ of equilibria. How many attractors with $\kappa$ equilibria can there be? And what do they look like? By dissipativeness, $\kappa$ must be odd. Indeed the attractor arises, generically, out of a unique globally attracting equilibrium by a sequence of saddle node bifurcations. But how do we "count" attractors? We summarize some results on this question here; for more details see [Fie94].

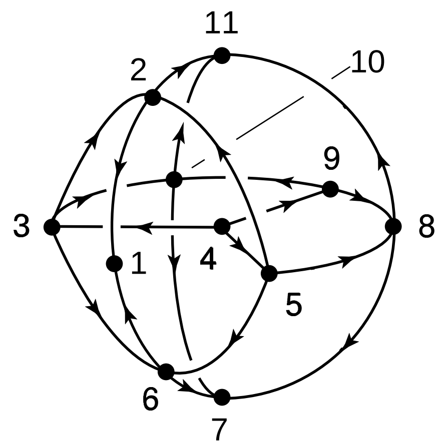



Figure 6.1: A non-pitchforkable attractor with eleven equilibria

Of course, we need a notion of equivalence before we can start counting. There are several possibilities. Consider $C^0$ flow equivalence first. Given two attractors $\mathcal{A}_0, \mathcal{A}_1$ we call them $C^0$ *flow equivalent* if there exists a ($C^0$) homeomorphism

$$h : \mathcal{A}_0 \to \mathcal{A}_1 \tag{6.4}$$

which conjugates the respective flows on $\mathcal{A}_0$ and $\mathcal{A}_1$. This notion "globalizes" the Grobman-Hartman result which shows $C^0$ flow equivalence, locally and in finite dimensions, near hyperbolic equilibria of the same unstable dimension. Assume all equilibria in $\mathcal{A}_0$ are hyperbolic. Then the flow on $\mathcal{A}_0$ is Morse-Smale, by the transversality results of [Ang86], [Hen85]. Therefore, by a result due to [Oli92], see also [PS70], the flow on $\mathcal{A}_0$ is structurally stable. In particular, the above flow equivalence (6.4) holds as long as the diffusion coefficient and the nonlinearity of $\mathcal{A}_1$ remain close enough to those of $\mathcal{A}_0$.

Another notion of equivalence, closer to the setting of the present paper, is connection equivalence. In our setting, consider two attractors $\mathcal{A}_0$, $\mathcal{A}_1$ with hyperbolic equilibria; the respective equilibrium sets are denoted by $E_0, E_1$.

We call $\mathcal{A}_0, \mathcal{A}_1$ *connection equivalent* if there exists a bijection.

$$\sigma : E_0 \longrightarrow E_1 \tag{6.5}$$

such that

(i) $i(\sigma(v)) = i(v)$

(ii) $v \searrow w \quad \Leftrightarrow \quad \sigma(v) \searrow \sigma(w)$

both hold, for all $v$, $w \in E_0$. Consider the oriented *connection graph* which consists of all equilibria, as vertices, and all one-dimensional heteroclinic connections, as directed edges. Then connection equivalence expresses that the respective connection graphs are isomorphic.

Clearly, flow equivalence implies connection equivalence. The converse need not be true in general. For example, connection equivalence distinguishes whether or not there exists a connecting orbit from $v$ to $w$. However, it does not distinguish how many connecting orbits there are (they are unique in case $i(v) = i(w) + 1$) or what the geometry of the set $C(v, w)$ of connecting orbits is. Thus connection equivalence is the weaker notion. Consider pairs $(a_0, f_0)$ and $(a_1, f_1)$ in (1.1) giving rise to permutations $\pi_0, \pi_1$. Then our main theorem 1.2 immediately implies

$$\pi_0 = \pi_1 \quad \Longrightarrow \quad \mathcal{A}_0 \text{ and } \mathcal{A}_1 \text{ are connection equivalent.} \tag{6.6}$$

Note that the converse of (6.6) is not true, in general. Indeed, different permutations may give rise to the "same" attractor. For example, transform $v$ into $-v$ in equation (1.1). Clearly the corresponding attractors are flow equivalent (by the linear map $h(v) = -v$) and connection equivalent. But the corresponding permutations $\pi_0, \pi_1$ may differ:

$$\pi_1 = \tau \pi_0 \tau^{-1} = \pi_0^\tau \tag{6.7}$$

The permutations are, by definition (1.10), conjugate by the involution

$$\tau = \begin{pmatrix} 1 & 2 & \dots & \kappa - 1 & \kappa \\ \kappa & \kappa - 1 & \dots & 2 & 1 \end{pmatrix} \tag{6.8}$$

Similarly, suppose we transform $x$ into $1 - x$ instead. Then

$$\pi_1 = \pi_0^{-1} \tag{6.9}$$

is the corresponding new permutation.

In example 5.3, the four permutations $\pi, \pi^\tau, \pi^{-1}, (\pi^\tau)^{-1}$ are all distinct; but they give rise to the same attractor. The Chafee-Infante problem, e.g. as in 5.2, illustrates the other extreme: all four permutations coincide. In the Rocha example 5.1 we observe $\pi^\tau = \pi \neq \pi^{-1}$.

Let us call the equivalences (6.7, 6.9) "trivial", for a moment. Besides these, nontrivial equivalences also occur. The simplest example involves $\kappa = 9$ equilibria. In cycle notation, it is given by the following permutations

$$\begin{aligned} \pi_0 &= (\ 2\ 8\ 4\ )(\ 3\ 7\ 5\ ), \\ \pi_1 &= (\ 2\ 8\ 4\ 6\ )(\ 3\ 7\ ). \end{aligned} \tag{6.10}$$

Since cycle lengths differ, these two permutations are not related by (6.7), (6.9), nor by any other conjugating permutation. We have examined the connecting orbits of $\pi_0$ by the methods of section 5. The result is shown in figure 6.2. With the renumbering

$$\sigma = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9 \\ 7\ 8\ 5\ 6\ 1\ 4\ 3\ 2\ 9 \end{pmatrix}$$

of the equilibria, the connecting orbits for $\pi_1$ are obtained. We repeat that $\pi_0, \pi_1$ are not *not* conjugated by $\sigma$. Still the corresponding attractors are connection equivalent, by $\sigma$.

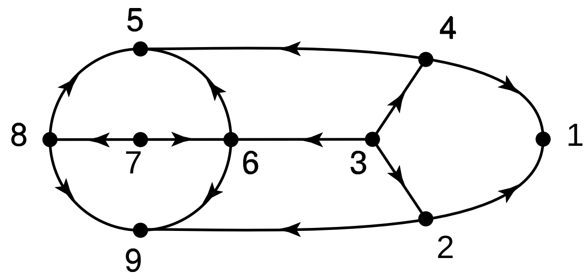



Figure 6.2: Connecting orbits for $\pi = (\ 2\ 8\ 4\ )\ (\ 3\ 7\ 5\ ), \kappa = 1$.

At this stage we can count global attractors with $\kappa$ hyperbolic equilibria, up to connection equivalence. Let c$(\kappa)$ denote their number. For comparison, we also give the number $j(\kappa)$ of permutations $\pi \in S_\kappa$ with $\pi(1) = 1, \pi(\kappa) = \kappa$, which give rise to a Jordan curve $S$; see table 6.1.
Note that

$$j(\kappa) = M_{\kappa-2},$$

where $M_n$ denote the *meandering numbers* mentioned by Arnol'd; see [Arn88], and also [LZ92]. Moreover we include the Morse number $m(\kappa)$, of those permutations for which all Morse indices $i(v_n)$ turn out to be nonnegative, according to proposition 2.1. Clearly permutations which arise from shooting curves, that is, from our differential equation (1.1), must belong to that class. There is a modelling question here, which we postpone for a moment: do all the $m(\kappa)$ permutations actually arise from (1.1)? By successive inspection of the non-pitchforkable cases, we believe the answer to be yes, at least for $\kappa \leq 15$. After a little programming and a case by case inspection in the spirit of section 5 we arrive at the results summarized in table 6.1. The 16 diagams of connection graphs involving nine equilibria are given in [Fie94]. We intend to provoke our readers to understand the numbers in table 6.1: how to generate them systematically, asymptotic behavior, interrelations. We do not know....

| $\kappa$ | $j(\kappa)$ | $m(\kappa)$ | $c(\kappa)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 |
| 5 | 2 | 2 | 2 |
| 7 | 8 | 7 | 5 |
| 9 | 42 | 32 | 16 |
| 11 | 262 | 175 | ? |
| 13 | 1828 | 1083 | ? |
| 15 | 13820 | 7342 | ? |
| 17 | 110954 | 53372 | ? |

Table 6.1: Counting attractors with $\kappa$ equilibria.

We have noted above that connection equivalence is, in general, a weaker notion than $C^0$ flow equivalence. In the present context, however, we may ask whether connection equivalence *implies* $C^0$ flow equivalence. More modestly and specifically, we ask whether

(6.11) $$\pi_0 = \pi_1 \quad \Rightarrow \quad \mathcal{A}_0 \text{ and } \mathcal{A}_1 \text{ are } C^0 \text{ flow equivalent?}$$

This is the strict analogue of (6.5) for flow equivalence. We conjecture that the answer to both questions is positive. The traditional approach to equivalence in calculus of variations goes back to Morse [Mor34] and is somewhat more cautious. Let $V$ be the gradient functional, as in (1.4). Then the Morse complex is constructed by successively attaching cells of the appropriate dimension $i(v)$, whenever $V$ is increased through a critical value $V(v)$ of an equilibrium $v$. This construction determines the homotopy type of the sublevel sets

$$V^\eta = \{u \in X \mid V(u) \leq \eta\},$$

for non-critical $\eta$. In particular, this procedure also works on the global attractor $\mathcal{A}$, replacing $X$.

Suppose that we knew the critical $V$-values $V(v)$, in our case, or that we could adjust them in some convenient way. The one-dimensional connections $v \searrow w$, for $i(v) = i(w) + 1$, are unique in our case; see [BF89], lemma 3.5. Therefore the Conley index connection matrix precisely provides the one-dimensional connections; see [Fra89], [FM92], [McC88] and the references there. In particular, the *homology* of the sublevel sets is then determined by the permutation $\pi$. This, of course, is saying less than homotopy equivalence of the sublevel sets. Moreover, due to effects like Reidemeister torsion, non-homeomorphic sets can even be homotopy equivalent and can, a forteriori, carry the same homology.

The answer to question (6.11) may be related to the modelling question which we have postponed above. Consider any permutation $\pi$ which is meandering. That is, assume $\pi$ is induced by a Jordan curve $S$ in the $(v, v_x)$-plane, transverse to the $v$-axis, intersecting at $v = 1$, ..., $\kappa$ say, beginning at $v = 1$ towards $v_x > O$ and terminating at $v = \kappa$ from $v_x < O$. Define the "Morse vector"

$$(i(m))_{1 \leq m \leq \kappa} \tag{6.12}$$

according to (2.1). Assume $i(m) \geq 0$, for all $m$. Does there exist a nonlinearity $f$ with corresponding permutation $\pi$? (We may choose $a \equiv 1$ without restriction; see (1.7)). By inspection, case by case, the answer to this question is affirmative at least for $\kappa \leq 11$.

A related modelling question is the following. Given nonlinearities $f_0, f_1$ inducing (for $a \equiv 1$) the same permutation $\pi_0 = \pi_1 = \pi$. Are $f_0, f_1$ then homotopic in that same class, that is, without ever adding or removing equilibria? If the answer to this question is positive, then (6.11) holds: the same permutation induces the same flow on the attractor, up to $C^0$ flow equivalence. This would follow from Oliva's result, [Oli92], applied along the $f$-homotopy. We repeat that there is also a more embracing question: are (possibly nontrivially) connection equivalent attractors of (possibly different) permutations necessarily $C^0$ flow equivalent? This question, of course, is out of reach of homotopies which preserve the number of equilibria.

From a modelling point of view it would also be useful to understand which permutations arise from more restrictive classes of nonlinearities. For example consider nonlinearities

$$\begin{aligned} f &= f(u), \text{ or} \\ f &= f(x,u), \end{aligned} \tag{6.13}$$

which do not depend on the drift term $u_x$. We have mentioned the Hamiltonian structure of the associated equilibrium ODE (1.7) above. Which attractors, alias permutations arise here? And are there any attractors which cannot be modelled already in this more restrictive class? For example, consider $f = f(u), a \equiv 1$. Since $f$ does not depend on $x$, we may replace $x$ by $1-x$. But $f$ remains unaffected by this transformation. Therefore

$$\pi = \pi^{-1}, \tag{6.14}$$

In particular, $\pi$ must decompose into cycles of length at most two. Specifically, examples 5.1, 5.3 or the non-pitchforkable example of fig. 6.2 do not belong to this class. Therefore, the corresponding attractors cannot be modelled by $x$-independent nonlinearities. The same argument applies to "reversible" nonlinearities $f(x,u,u_x)$ for which

$$f(x,\ u,\ p) \ = \ f(1-x,\ u,-p). \tag{6.15}$$

We do not give very specific applications, in this paper. Our main objective is classification. Still, we comment on some more applied aspects like basins of attraction, singular perturbations, and viscosity limits next.

Some people think they do not want to hear about unstable objects. They are still interested in stability regions, that is, domains (or basins) of attraction. They should be interested in unstable equilibria and connecting orbits as well. For example, suppose $w$ is some stable equilibrium of (1.1). What is the boundary $\partial B(w)$ of its basin of attraction $B(w)$? Clearly $\partial B(w)$ contains any equilibrium $e$ which connects to $w$, since $B(w) = W^s(w)$ contains all connecting orbits which end at $w$. By the $\lambda$–Lemma, $\partial B(w)$ contains the union of all stable manifolds $W^s(e)$ of such equilibria $e$. By arguments similar to the proof of lemma 3.2, we obtain equality

$$\partial B(w) = \partial W^s(w) = \bigcup_{e:\ e \searrow w} W^s(v). \tag{6.16}$$

If we are interested in the basin (boundary) inside the global attractor, we find similarly

$$\partial B(w) \cap \mathcal{A} = \bigcup_{v \searrow e \searrow w} C(v, e) \tag{6.17}$$

By (6.16), (6.17) it is clearly important to know all connecting orbits $C(v,\ e)$ it one is interested in a description of domains of attraction. For a specific example consider the global attractors $\mathcal{A}_\varepsilon$ of the family of singularly perturbed equations

$$u_t = \varepsilon^2 u_{xx} + f(x, u, u_x), \tag{6.18}$$

for $\varepsilon \searrow 0$. This is also known as the viscosity limit for the nonlinear hyperbolic equation

$$u_t - f(x, u, u_x) = 0. \tag{6.19}$$

The setting includes scalar one-dimensional conservation laws for which

$f = -(g(u))_x$; see [Smo83]. It also allows for reaction terms

$$u_t + g(u)_x = r(u)$$

An interesting example is the model

$$u_t = \varepsilon^2 u_{xx} + u u_x - a(x) u_x \tag{6.20}$$

for the density of self-gravitating clusters, where $a(x)$ is given; see [Wol92a], [Wol92b]. A bistable model for phase transition is the $x$-dependent cubic

$$u_t = \varepsilon^2 u_{xx} + (u^2 - 1)(a(x) - u), \tag{6.21}$$

again with given $a(x)$ meandering around the value zero; see for example [AMPP87], [FH89], [HS88] and the references there. Very slow evolution of steep fronts has been observed for (6.21). Thus the transient dynamics on the global attractor becomes relevant, with unstable equilibria indicating metastable states.

Our results indicate that complete ODE steady state information determines the global attractor to some extent. Obtaining that ODE information however, may be quite nontrivial. For example, the steady state equation of (6.18) includes the forced and singularly perturbed van der Pol oscillator. For an account of the dynamic complexity of this equation see for example [Lev81]. Even for $f$ periodic in $x$, the arising transverse homoclinic points, horseshoe structures etc. will have their counterparts in the associated permutation $\pi_\varepsilon$ and the attractor $\mathcal{A}_\varepsilon$. We are still far from understanding these relations.

Whenever anything is proved for Neumann boundary conditions people tend to ask for Dirichlet. And conversely. It is fairly easy to adapt our results to other boundary conditions, in principle. They should be reasonably linear and separated at $x = 0$ and 1. In the Dirichlet case, for example, we can define the permutation $\pi$ by ordering $v_x$ at $x = 0, 1$, respectively. For mixed type boundary conditions, we can use either ordering. We could treat different boundary conditions at $x = 0$ and 1. We may lose certain invariances like the transformations joining $\pi$ with its conjugates $\pi^\tau, \pi^{-1}$ in (6.7), (6.9). The detailed form of proposition 2.1 may also change. But in principle, theorem 1.2 and its method of proof will survive these modifications. Passing to periodic boundary conditions

$$\begin{aligned} u(x=0) &= u(x=1) \\ u_x(x=0) &= u_x(x=1) \end{aligned} \tag{6.22}$$

drastically complicates the situation. Even for equilibria, we do not know how to even define the permutation $\pi$. Moreover, the gradient structure disappears and time periodic solutions $u(t, \cdot)$ emerge; see [FMP89]. For the SO(2)-equivariant case $f = f(u, u_x)$ these periodic solutions are rotating waves, i.e. they are of the special form

$$u = u(x + ct),$$

for some wave speed $c$. Some connecting orbits between rotating waves have been found by a $y$-map method, see [AF88]. But a complete description has not been attempted, so far. For $f = f(x, u)$ independent of $u_x$, the system is still gradient-like with respect to the functional

$$V(u) = \int_0^1 \left( \frac{1}{2}(u_x)^2 - F(x, u) \right) dx,$$

where $F_u := f$. In particular, convergence to equilibrium is retained. In the SO(2)-equivariant case $f = f(u)$ we can shift equilibria $v$ (and, in fact, any solution $u$) to obtain another equilibrium (or another solution). In particular, the Neumann problem embeds into the complete semiflow, and our present methods describe part of the set of all connecting orbits. For the general case $f = f(x,u)$, we suspect that the connecting orbits are determined by the braid type of the braid of equilibria, under periodic boundary conditions. The (positive) braid is given by the equilibrium profiles

$$(x, v(x), v_x(x)) \in S^1 \times \mathbb{R}^2 \tag{6.23}$$

The linking number $\ell$ of two equilibria is related to the (even) zero number:

$$\ell(v_n - v_m) = z(v_n - v_m)/2.$$

In the case of Neumann boundary conditions, the profiles (6.23) of course also define a braid, this time in $[0,1] \times \mathbb{R}^2$. Since all profiles lie on the shooting surface $M$ defined in (2.10), in this particular case, the permutation $\pi$ determines the braid (up to diffeotopy). Like so many other questions in this paper, however, the problem of periodic boundary conditions must remain open. Open for *you* to join!

## References


[AF88] S. Angenent and B. Fiedler. The dynamics of rotating waves in scalar reaction diffusion equations. *Trans. Amer. Math. Soc.*, **307**, 545–568, (1988).

[AMPP87] S. Angenent, J. Mallet-Paret, and L.A. Peletier. Stable transition layers in a semilinear boundary value problem. *J. Diff. Eq.*, **67**, 212–242, (1987).

[Ang86] S. Angenent. The Morse-Smale property for a semi-linear parabolic equation. *J. Diff. Eq.*, **62**, 427–442, (1986).

[Ang88] S. Angenent. The zero set of a solution of a parabolic equation. *J. reine angew. Math.*, **390**, 79–96, (1988).

[Arn88] V. I. Arnol'd. The branched covering of $\mathbb{C}P^2 \to S^4$, hyperbolicity and projective topology. *Siberian Math. J.*, **29**, 717–726, (1988).

[BF86] P. Brunovský and B. Fiedler. Numbers of zeros on invariant manifolds in reaction-diffusion equations. *Nonlin. Analysis*, **10**, 179–193, (1986).

[BF88] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics Reported*, **1**, 57–89, (1988).

[BF89] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: The complete solution. *J. Diff. Eq.*, **81**, 106–135, (1989).

[BV89] A.V. Babin and M.I. Vishik. *Attractors in Evolutionary Equations.* Nauka, 1989. (Russian).

[CI74] N. Chafee and E. Infante. A bifurcation problem for a nonlinear parabolic equation. *J. Applicable Analysis*, **4**, 17–37, (1974).

[Con78] C. C. Conley. *Isolated Invariant Sets and the Morse Index.* CBMS Notes **38**. AMS, Providence, 1978.

[FH89] G. Fusco and J.K. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *J. of Dyn. and Diff. Eq.*, **1**, 111–137, (1989).

[Fie89] B. Fiedler. Discrete Ljapunov functionals and $\omega$-limit sets. *Math. Mod. Num. Anal.*, **23**, 415–431, (1989).

[Fie94] B. Fiedler. Global attractors of one-dimensional parabolic equations: sixteen examples. Preprint, (1994).

[FM88] R. Franzosa and K. Mischaikow. The connection matrix theory for semiflows on (not necessarily locally compact) metric spaces. *J. Diff. Eq.*, **71**, 270–287, (1988).

[FM92] B. Fiedler and K. Mischaikow. Dynamics of bifurcations for variational problems with $O(3)$ equivariance: a Conley index approach. *Arch. Rational Mech. Analysis*, **119**, 145–196, (1992).

[FMP89] B. Fiedler and J. Mallet-Paret. A Poincaré-Bendixson theorem for scalar reaction diffusion equations. *Arch. Rational Mech. Analysis*, **107**, 325–345, (1989).

[FR91] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *J. Diff. Eq.*, **91**, 75–94, (1991).

[Fra89] R. Franzosa. The connection matrix theory for Morse decompositions. *Trans. AMS*, **311**, 781–803, (1989).

[Hal88] J.K. Hale. *Asymptotic Behavior of Dissipative Systems.* Math. Surv. **25**. AMS, Providence, 1988.

[Hen81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lect. Notes Math. **840**. Springer-Verlag, New York, 1981.

[Hen85] D. Henry. Some infinite dimensional Morse-Smale systems defined by parabolic differential equations. *J. Diff. Eq.*, **59**, 165–205, (1985).

[HS88] J. Hale and K. Sakamoto. Existence and stability of transition layers. *Japan J. Appl. Math.*, **5**, 367–405, (1988).

[Lad91] O.A. Ladyzhenskaya. *Attractors for Semi-groups and Evolution Equations.* Cambridge University Press, 1991.

[Lev81] M. Levi. Qualitative analysis of the periodically forced relaxation oscillations. *AMS Memoirs*, **32**, (1981).

[LZ92] S. K. Lando and A. K. Zvonkin. Meanders. *Selecta Math. Sov.*, **11**, 117–144, (1992).

[Mat78] H. Matano. Convergence of solutions of one-dimensional semilinear parabolic equations. *J. Math. Kyoto Univ.*, **18**, 221–227, (1978).

[Mat82] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semi-linear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sec. IA*, **29**, 401–441, (1982).

[McC88] C. McCord. The connection map for attractor-repeller pairs. *Trans. Amer. Math. Soc.*, **301**, 195–203, (1988).

[Mor34] M. Morse. *The Calculus of Variations in the Large*. AMS, Providence, 1934.

[Oli92] W.M. Oliva. Stability of Morse-Smale maps. Preprint, (1992).

[PS70] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis*. Proc. Symp. in Pure Math. vol. XIV. AMS, Providence, 1970.

[Roc92] C. Rocha. Properties of the attractor of a scalar parabolic PDE. Preprint, (1992).

[Ryb82] K. Rybakowski. On the homotopy index for infinite dimensional semi-flows. *Trans. Amer. Math. Soc.*, **269**, 351–382, (1982).

[Sch90] R. Schaaf. *Global Solution Branches of Two Point Boundary Value Problems*. Lect. Notes Math. **1458**. Springer-Verlag, New York, 1990.

[Smo83] J. Smoller. *Shock Waves and Reaction Diffusion Equations*. Springer-Verlag, New York, 1983.

[Wol92a] G. Wolansky. Stationary and quasi-stationary shock waves for non-spatially homogeneous Burger's equation in the limit of small dissipation. I. *Indiana Univ. Math. J.*, **41**, 43–69, (1992).

[Wol92b] G. Wolansky. Stationary and quasi-stationary shock waves for non-spatially homogeneous Burger's equation in the limit of small dissipation. II. Preprint, (1992).

[Zel68] T.I. Zelenyak. Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable. *Differential Equations*, **4**, 17–22, (1968).