Realization of meander permutations by boundary value problems

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Abstract

We consider Neumann boundary value problems of the form $u_{xx} + f(x, u, u_x) = 0$ on the unit interval $0 \leq x \leq 1$ for a certain class of dissipative nonlinearities f. Associated to these problems we have: (i) meanders in the phase space $(u, u_x) \in \mathbb{R}^2$, which are connected oriented simple curves on the plane intersecting a fixed oriented line (the u-axis) in n points corresponding to the solutions; and (ii) meander permutations $\pi_f \in S(n)$ obtained by ordering the intersection points first along the u-axis and then along the meander. The meander permutation π_f is the permutation defined by the braid of solutions in the space (x, u, u_x). It was recently shown by Fiedler and Rocha that π_f determines the global attractor of the dynamical system generated by the semilinear parabolic differential equation $u_t = u_{xx} + f(x, u, u_x)$, up to C^0 orbit equivalence. Therefore, these permutations are of considerable importance in the classification problem of the (Morse-Smale) attractors for these dynamical systems.

In this paper we present a purely combinatorial characterization of the set of meander permutations that are realizable by the above boundary value problems.

1 Introduction

The term *meander* was introduced by Arnold in [Arn88] to denote a connected oriented non-self-intersecting curve in the plane intersecting a fixed oriented base line in n points. The intersections are assumed to be strict crossings. The permutation defined by ordering the intersection points, first along the base line and then along the meander, is called a *meander permutation*.

Meanders arise in a natural way in the study of second-order boundary value problems with separated boundary conditions. To be specific, consider a Neumann boundary value problem

$$\begin{array}{ll} v_{xx} + f(x, v, v_x) = 0 & , \ 0 < x < 1 \\ v_x = 0 \ , & \quad x = 0 \text{ or } 1 \ , \end{array} \tag{1.1}$$

having exactly n solutions. Let $u = u(x, a)$ denote the solution of the associated initial value problem

$$\begin{aligned} u_{xx} + f(x, u, u_x) &= 0 \ , \\ u(0, a) &= a \ , \\ u_x(0, a) &= 0 \ . \end{aligned} \tag{1.2}$$

If the solution $u(\cdot, a)$ is defined for $0 \leq x \leq 1$ and all a, the set

$$\mathcal{S} := \{(u(1, a), u_x(1, a)) | a \in \mathrm{I\!R}\} \tag{1.3}$$

is a curve in the phase plane (u, u_x) of (1.2) intersecting the horizontal line $u_x = 0$ at exactly those n points which correspond to the solutions of (1.1). If the intersections are strict crossings, then $\mathcal{S}$ is a meander.

This shooting method (1.2) of solving boundary value problems (1.1) has far reaching consequences when applied to the determination of stationary solutions of scalar semilinear parabolic equations

$$\begin{array}{ll} u_t = u_{xx} + f(x, u, u_x) & , \ 0 < x < 1 \\ u_x = 0 \ , & \quad x = 0 \text{ or } 1 \ . \end{array} \tag{1.4}$$

In fact, although all the information encoded in the meander $\mathcal{S}$ is obtained from the ODE (1.2), Fiedler and Rocha [FR96] have recently shown that the meander permutation corresponding to $\mathcal{S}$ contains sufficient information to determine the global attractor of the PDE (1.4) up to global orbit equivalence. We next describe this result in some detail. It provides the key motivation for the main result of the present paper which deals with the realization of meander permutations by boundary value problems. Strangely enough, the quite elementary realization question of which permutations actually arise in second order two point boundary value problems seems to have escaped the attention of the quite extensive literature on the subject.

For smooth $f \in C^2$, the equation (1.4) generates a local semiflow in an appropriate Sobolev space X, for example the state space $X \subset H^2(0,1)$ of functions $u : [0,1] \to \mathbb{R}$ with Lebesgue square integrable second x-derivative u_{xx} and vanishing u_x at $x = 0, 1$; see [Hen81], [Paz83]. Under additional conditions on the nonlinearity f, the dynamical system is global and dissipative, that is, the solutions of (1.4) are defined for all $t \geq 0$ and there exists a large ball $B \subset X$ attracting all solutions. Sufficient conditions on f ensuring this are sign conditions of the form $f(x,u,0) \cdot u < 0$ for $|u|$ large enough, uniformly in x, and growth conditions of the form $|f(x,u,p)| < c_1(u) + c_2(u)|p|^\gamma$ for $\gamma < 2$ and some continuous functions c_1, c_2, (see [Ama85]). In the following, for brevity, a nonlinearity f satisfying these assumptions will be called *dissipative*. Finally, dissipative dynamical systems (1.4) possess global attractors $\mathcal{A}_f$, which are maximal compact invariant subsets of X and attract all bounded sets. See [Hal88], [Lad91], [BV89], for example.

The infinite dimensional dynamical system (1.4) has been widely studied and the characterization of its global attractor $\mathcal{A}_f$ has made significant progress. Generically in f, the flow defined by (1.4) is Morse-Smale ([Hen85], [Ang86]), and its global attractor possesses a Morse decomposition. In this case, the set

$\mathcal{A}_f$ is composed of a finite set of hyperbolic equilibria and a set of heteroclinic orbits connecting them. Let $v_j, j = 1, \ldots, n$ denote the solutions of (1.1), ordered by their values at $x = 0$, that is

$$v_1 < v_2 < \cdots < v_n \ , \text{ at } x = 0 \ . \tag{1.5}$$

Then, $\mathcal{E}_f := \{v_1, \ldots, v_n\}$ is the set of equilibria of (1.4). Reordering $\mathcal{E}_f$ according to the values of v_j at $x = 1$ defines a permutation $\pi \in S(n)$

$$v_{\pi(1)} < v_{\pi(2)} < \cdots < v_{\pi(n)} \ , \text{ at } x = 1 \ . \tag{1.6}$$

This permutation $\pi = \pi_f$ is the meander permutation corresponding to the meander (1.3).

The globally minded way of comparing flows for different nonlinearities is to compare the corresponding attractors through the notion of global orbit equivalence. Two attractors $\mathcal{A}_f$ and $\mathcal{A}_g$ are *globally orbit equivalent*, $\mathcal{A}_f \cong \mathcal{A}_g$, if there exists a homeomorphism

$$h : \mathcal{A}_f \to \mathcal{A}_g \tag{1.7}$$

mapping orbits of $\mathcal{A}_f$ onto orbits of $\mathcal{A}_g$ preserving the time direction.

Then, under the above conditions, Fiedler and Rocha have shown

Theorem 1.1 [FR98]. *The global attractors $\mathcal{A}_f$ and $\mathcal{A}_g$ are globally orbit equivalent if their meander permutations coincide. In short:*

$$\pi_f = \pi_g \implies \mathcal{A}_f \cong \mathcal{A}_g \ . \tag{1.8}$$

This result continues a long line of research on the set of heteroclinic orbit connections for (1.4). For reference we point out [Zel68], [CI74], [Mat78],

[CS80], [Mat82], [Hen85], [Ang86] [BF88], [BF89], [FR91], [Roc91] and [FR98]. The approach opens good perspectives for the classification of Morse-Smale attractors for the semilinear parabolic equations (1.4). However, one should also address the modeling question of determining all meander permutations actually realizable by (1.2). In this paper we present a purely combinatorial characterization of the set of these realizable meander permutations.

In the following, a permutation $\pi \in S(n)$ with n odd is called *dissipative* if it satisfies $\pi(1) = 1$ and $\pi(n) = n$. Furthermore, as already mentioned in the first paragraph, the permutation π is a *meander permutation* if it arises from a meander.

The shooting meander (1.3) determines the Morse indices of the equilibria $v_k \in \mathcal{E}_f$ of (1.4), $k = 1, \ldots, n$, that is, the dimensions of the corresponding unstable manifolds, $i(v_k) = \dim W^u(v_k)$, (see [Roc85] for a proof). In fact, these indices are determined explicitly by the meander permutation $\pi = \pi_f$ as

$$i(v_k) = \sum_{j=1}^{k-1} (-1)^{j+1} \text{sign } (\pi^{-1}(j+1) - \pi^{-1}(j)) \tag{1.9}$$

where an empty sum denotes zero, (see [FR96], Proposition 2.1). On the other hand, given any permutation $\pi \in S(n)$ one can always define an index vector $(i_k)_{1\leq k\leq n}$ using (1.9) or, more practically, using the recursion

$$\begin{aligned} i_1 &= 0 \,, \\ i_{k+1} &= i_k + (-1)^{k+1} \text{sign } (\pi^{-1}(k+1) - \pi^{-1}(k)) \end{aligned} \tag{1.10}$$

for $k = 1, \ldots, n-1$. Then, a permutation π will be called *Morse* if its index vector satisfies $i_k \geq 0$ for all $1 \leq k \leq n$.

We say that a permutation π is *realizable* by a boundary value problem (1.1) if there is a nonlinearity f such that

$$\pi = \pi_f \ . \tag{1.11}$$

We denote by MS the (generic) set of nonlinearities f for which the semiflow generated by (1.4) is Morse-Smale. Our main result in the present paper is the following characterization of the realizable permutations

Theorem 1.2 *A permutation $\pi \in S(n)$ is realizable with a dissipative nonlinearity f in MS if and only if n is odd and π is a dissipative Morse meander permutation.*

The "only if" part of this theorem follows from the characterization of the shooting meanders (1.3) gathered in [FR96], Proposition 2.1. In fact, it is just a restatement of earlier work by Fusco and Rocha; see [FR91], [Roc91] for example. We prove the "if" part in Section 4, using induction. We construct dissipative Morse meander permutations $\pi \in S(n)$ from realizable permutations $\pi' \in S(n-2)$ by a realizable deformation of the corresponding meanders. In Section 2 we present a characterization of the shooting surfaces generated by the initial value problem (1.2) in terms of the nonlinearity f. This characterization is necessary for the construction of the realizable deformations of the meanders (1.3). In Section 3 we introduce an index vector that is helpful in understanding the sequence of necessary meander deformations. For an alternative proof of theorem 1.2, with more emphasis on combinatorial rather than analytical aspects, see [Wol96]. Finally, in Sections 5 and 6, we illustrate our construction by an example, discuss some aspects of the nonuniqueness of this construction, and explore implications of the realization for the classification of attractors of Morse-Smale dynamical systems generated by semilinear parabolic equations and their space discretizations.

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2 Characterization of shooting surfaces

The proof of Theorem 1.2 is based on a geometric characterization of the manifold of solutions of equation (1.2) developed by Fusco and Rocha in [FR91]. This characterization allows the construction of nonlinearities f corresponding to a desired solution manifold for equation (1.2). This, in turn, allows the construction of families of nonlinearities performing realizable deformations of the meanders (1.3), to be used in the construction of the realizable permutations. For completeness, we recall the results of [FR91] and add some details necessary for our purposes.

To each nonlinearity f we associate a shooting surface

$$\mathcal{S}_f : (x, a) \mapsto (x, u(x, a), u_x(x, a)) \in [0, 1] \times \mathbb{R}^2 \tag{2.1}$$

defined by all trajectories of the initial value problem (1.2), $a \in \mathbb{R}$. For simplicity, here we assume that all trajectories of (1.2) are defined for $0 \leq x \leq 1$. This will always be the case when f satisfies, for example, a sublinear growth condition on u and v. We can clearly include such a condition in the construction of our nonlinearities. Then, let

$$\mathcal{S}_f^x : a \mapsto (u(x, a), u_x(x, a)) \in \mathbb{R}^2 \tag{2.2}$$

denote the section of $\mathcal{S}_f$ at x. The curve $\mathcal{S}_f^{x=1}$ corresponds to the meander $\mathcal{S}$ defined in (1.3) and, therefore, determines the meander permutation π_f. On the other hand, $\mathcal{S}_f^{x=0}$ always corresponds to the u-axis.

Let $K_f \subset [0, 1) \times \mathbb{R}^2$ denote the set of points (x, u, p) of the shooting surface $\mathcal{S}_f$ where the tangent to $\mathcal{S}_f^x$ in the (u, u_x) plane is vertical. Then, K_f is defined by the condition

$$u_a(x, a) = 0 \tag{2.3}$$

for the partial derivative u_a of the parametrization of $\mathcal{S}_f$. We call K_f the *critical set.* Note that K_f is indeed the set of critical points of the projection of the shooting surface $\mathcal{S}_f$ onto its (x, u) coordinates. Since on $\mathcal{S}_f$, u_a and p_a cannot vanish simultaneously we have that $u_{ax} = p_a \neq 0$ on K_f. The implicit function theorem therefore implies that the critical set K_f is a one dimensional manifold, locally parametrized over a. The tangent to the trajectory of (1.2) passing through the point $(x, u, p) \in \mathcal{S}_f$ has the form $(1, p, -f(x, u, p))$. By (2.3), the tangent space to the manifold $\mathcal{S}_f$ therefore also contains the vector $(1, p, 0)$, at every point $(x, u, p) \in K_f$. This provides a local constraint which a smooth two dimensional manifold $\mathcal{S}_*$ has to satisfy in order to be realizable, that is, such that $\mathcal{S}_* = \mathcal{S}_f$ for some nonlinearity f. Given $\mathcal{S}_*$, let K_* denote its critical set: the subset of points (x, u, p) where the tangent to $\mathcal{S}_*^x$ in the (u, p) plane is vertical. The critical set K_* is required to be a smooth one dimensional manifold and, at every point $(x, u, p) \in K_*$, the tangent space to the manifold $\mathcal{S}_*$ must also contain the vector $(1, p, 0)$.

This condition together with a nondegeneracy condition on $\mathcal{S}_*$ in a neighborhood of its critical set K_* is necessary and sufficient for $\mathcal{S}_*$ to be realizable (see [FR91] for details). The nondegeneracy condition is the following. Let $\xi : \mathcal{S}_* \to \mathbb{R}$ denote the vertical p-component of the normal vector to $\mathcal{S}_*$. On the critical set K_* this function ξ is zero. Then, the nondegeneracy condition is

$$\nabla \xi \neq 0 \quad \text{on} \quad K_* , \tag{2.4}$$

the gradient being with respect to the manifold parameters, $\mathcal{S}_* : (x, \alpha) \mapsto (x, u(x, \alpha), p(x, \alpha)) \in [0, 1] \times \mathbb{R}^2$.

To summarize, we have the following characterization of realizable manifolds.

Theorem 2.1 [FR91]. *For $k \geq 1$, consider a C^{k+2} smooth two dimensional manifold $\mathcal{S}_* \subset [0, 1] \times \mathbb{R}^2$. Assume the $\{x = 0\}$ section is the u-axis, and $\mathcal{S}_*$ is tangent to the field of planes*

$$\Sigma : (x, u, p) \mapsto (x, u, p) + \mathit{span}\,\{(0, 0, 1), (1, p, 0)\} \tag{2.5}$$

at its critical subset $K_* \subset \mathcal{S}_*$. Finally, assume the nondegeneracy condition (2.4). Then there is a C^k smooth nonlinearity f such that $\mathcal{S}_* = \mathcal{S}_f$.

The nonlinearity f is uniquely determined on $\mathcal{S}_*$ and extends smoothly (albeit nonuniquely) to $[0,1] \times \mathrm{IR}^2$.

This characterization puts most of the constraints for the realization of a manifold $\mathcal{S}_*$ on its critical subset K_*, allowing the rest of the manifold to vary freely. The critical set K_* must be an integral curve of the field of planes (2.5). This fact will be used later to compute curves K_* with a prescribed (u,p)-projection.

An immediate consequence of the constraint on the critical set K_f is an equivalent constraint on the section curves $\mathcal{S}_f^x$. As x varies, the curves $\mathcal{S}_f^x$ move such that its points $(u,p) \in \mathcal{S}_f^x$ with vertical tangent move with normal velocity equal to p. These curves $\mathcal{S}_f^x$ correspond to deformations of any given section curve $\mathcal{S}_f^{x=x_0}$, and are only constrained by the information on the critical set K_f. Therefore, the knowledge of the (u,p)-projection of the critical set K_* is sufficient for the construction of deformation curves $\mathcal{S}_*^x$ corresponding to a realizable manifold $\mathcal{S}_*$.

However, given an arbitrary curve K^0 satisfying the integrability condition it is not possible, in general, to assert the existence of a realizable global manifold $\mathcal{S}_* = \mathcal{S}_f$ such that its critical set $K_* = K_f \subset \mathcal{S}_f$ satisfies $K_* = K^0$. For example, there are closed curves K^0 satisfying the integrability condition, but no smooth manifold $\mathcal{S}_f$ can have a closed curve as part of its subset K_f. This is a consequence of the parametrization over a of the critical set K_f. Therefore, in our realization proof we proceed step by step avoiding all global problems with the construction of shooting surfaces. In each step we construct the desired manifold by deformation of a previously given shooting surface $\mathcal{S}_f$ through a realizable deformation of its section curves $\mathcal{S}_f^x$. This deformation is obtained by extending $\mathcal{S}_f^x$ to values $x > 1$. At the very end of this iterative extension process, the result is rescaled back to the interval $0 \leq x \leq 1$.

To set up a sequence of deformation steps leading to the final result we need to consider the detailed structure of the critical set K_f. We have seen that K_f is a one dimensional manifold, locally parametrized over a. Since the orbits of (1.2) correspond to the curves $a =$ const. on $\mathcal{S}_f$, the critical set K_f is everywhere transverse to the orbits of the flow on $\mathcal{S}_f$ defined by (1.2) ([FR91], Proposition 1.4).

The dissipativeness of f implies that on the shooting surface $\mathcal{S}_f$ the projection $(x, u, p) \to (x, u, 0)$ is a local diffeomorphism for large $|u|$. Therefore, the critical set K_f is a finite union of connected curves, $K_f = \bigcup_j K_j$, and each critical curve K_j starts and ends at points with $x = 1$, corresponding to points of $\mathcal{S}_f^{x=1}$ with vertical tangent.

Let $x = q_j(a)$ denote the parametrization of the x-coordinate of the critical curve K_j obtained from (2.3) by the implicit function theorem. Differentiating (2.3) with respect to a we obtain

$$u_{aa}(q_j(a), a) + p_a(q_j(a), a)q_j'(a) = 0 \ . \tag{2.6}$$

Therefore, on K_j the following relation holds

$$q_j'(a) = -\frac{u_{aa}}{p_a} \ . \tag{2.7}$$

Consider a point $(x, u(x, a), p(x, a))$ on K_j where $q_j' < 0$. Then, the second derivative u_{aa} has the same sign as p_a. Hence $\mathcal{S}_f^x$ folds clockwise, at that point, going up if $p_a > 0$, or going down if $p_a < 0$. At points where $q_j' > 0$, the section $\mathcal{S}_f^x$ folds anticlockwise in a similar way. See Fig. 2.1.

In the (x, a)-space the curves q_j are nonintersecting graphs over a. At $x = 1$, they therefore start in a point with $q_j' \leq 0$ and terminate, again at $x = 1$,

Figure 2.1: Stylized section curve $\mathcal{S}_f^{x=1}$ with the folding points (clockwise + and anticlockwise -) and corresponding stylized function graphs q_j, $j = 1, 2, 3$, in (x, a)-space.

in a point with $q_j' \geq 0$. Generically in f, the q_j are Morse functions: all critical points are nondegenerate, and all critical values are mutually distinct and different from $x = 1$. In this generic case we conclude that the critical curves K_j always start at points with $x = 1$ corresponding to points of $\mathcal{S}_f^{x=1}$ where this section curve folds clockwise, and terminate at points where $\mathcal{S}_f^{x=1}$ folds anticlockwise. See Fig. 2.2 for an example including section curves at different values of x.

In particular, the curves q_j define a unique one-to-one correspondence between clockwise and anticlockwise folding points in the (x, a)-space.

In the following section we introduce an index that, for certain curve sections (canonical meanders), allows the determination of the folding type of the vertical tangent points directly from the corresponding permutation π_f. This folding index will be explored in the construction of the realizable meander permutations.

Figure 2.2: Three section curves $\mathcal{S}_f^x$ corresponding to a Morse function q_j.

3 The folding index

Every smooth meander with only strict crossings with its base line can be isotopically transformed into a meander such that the arcs joining its intersection points with the base line are semicircles. Moreover, one can take the unbounded (first and last) arcs of the meander such that the meander possesses a vertical tangent at any intersection points with the base line. See Fig. 2.1 for an example of a meander in this stylized form. We will always number the intersection points $1, \ldots, n$ along the meander $\mathcal{S}$. This corresponds to the ordering (1.5). Therefore, their position along the baseline will be $\pi^{-1}(1), \ldots, \pi^{-1}(n)$, that is, the position at $x = 1$ of the kth equilibrium (numbered by the order at $x = 0$) is $\pi^{-1}(k)$.

Let $n \in \mathbb{N}$ be odd and let $\pi \in S(n)$ be a dissipative Morse meander permutation. To the permutation π we associate an index vector $\sigma_\pi = (\sigma_\pi(k))_{1 \leq k \leq n}$ in the following way:

$$\begin{array}{rcl} \sigma_\pi(1) & = & \sigma_\pi(n) \;=\; 0 \;, \\ \sigma_\pi(k) & = & \frac{1}{2}[i_{k+1} - i_{k-1}] \;,\; 1 < k < n \;. \end{array} \tag{3.1}$$

Here i_k are defined as in (1.9), (1.10).

Let $\mathcal{S}$ denote a meander in the above special form corresponding to the permutation π. The index $\sigma_\pi(k)$ compares the adjacent indices i_{k+1} and i_{k-1}, indicating the sign of their difference. Indeed, using (1.10) we have that

$$\begin{aligned}\sigma_\pi(k) \;=\; & \frac{(-1)^{k+1}}{2}\Big[\mathrm{sign}\left(\pi^{-1}(k+1)-\pi^{-1}(k)\right) \\ & -\mathrm{sign}\left(\pi^{-1}(k)-\pi^{-1}(k-1)\right)\Big]\ ,\end{aligned} \tag{3.2}$$

for $1 < k < n$. In particular $\sigma_\pi(k) \in \{-1, 0, 1\}$. By construction, $\mathcal{S}$ has a vertical tangent at each point k. Since $(i_k)_{1\le k\le n}$ count completed clockwise half-turns of the tangent to the meander $\mathcal{S}$ along its orientation, $\sigma_\pi(k)$ determines the folding type of the meander $\mathcal{S}$ at the point k. If $\sigma_\pi(k) = +1$, then $\mathcal{S}$ folds clockwise at the point k. If $\sigma_\pi(k) = -1$, then $\mathcal{S}$ folds anticlockwise. If $\sigma_\pi(k) = 0$, then $\mathcal{S}$ does not fold at k. In this last case, the vertical tangent can be eliminated by a simple perturbation of the curve $\mathcal{S}$ without introducing any further vertical tangents of $\mathcal{S}$. Let $\bar{\mathcal{S}}$ denote such a perturbation of $\mathcal{S}$. Then, the meanders $\mathcal{S}$ and $\bar{\mathcal{S}}$ share the same permutation π. This form $\bar{\mathcal{S}}$ will be called the *canonical form* of the meander. To prove Theorem 1.2 we will show the existence of nonlinearities f such that $\mathcal{S}_f^{x=1} = \bar{\mathcal{S}}$.

The purpose of working with $\bar{\mathcal{S}}$ rather than $\mathcal{S}$ is two-fold. On the one hand it avoids the consideration of the (nongeneric) degenerate situation where the extension of the critical set K_f to the section at $x = 1$ contains an isolated point. On the other hand $\bar{\mathcal{S}}$ also avoids the consideration of points k with index $i_k = 0$ where $\mathcal{S}$ has a vertical tangent. In fact, a meander with such a point would not be realizable by any f due to the following restriction. Let $\vartheta = \vartheta(x, a)$ denote the angle swept by the unit tangent vector to the section curve $a \mapsto \mathcal{S}_f^{\bar{x}}(a)$ as $\bar{x}$ varies from 0 to x. Then, $\vartheta \in (-\frac{\pi}{2}, +\infty)$ and the equilibrium v_k with initial value $u(x = 0, a) = a$ has index

$$i_k = 1 + \text{Integer part of } [\vartheta(1, a)/\pi]\ , \tag{3.3}$$

(see [Roc85]). This prohibits vertical tangents at equilibria with $i_k = 0$.

As it was already pointed out, the indices $(i_k)_{1\le k\le n}$ correspond to the complete clockwise half-windings of the tangent vector to the canonical meander $\bar{\mathcal{S}}$ along its orientation. Since π is dissipative we must also have $i_n = i_1 = 0$. Moreover, (1.10) implies that $i_{n-1} = i_2 = 1$. Therefore, the folding index defined in (3.1) satisfies

$$\sum_{k=1}^{n} \sigma_\pi(k) = 0 \ . \tag{3.4}$$

In particular, the meander $\bar{\mathcal{S}}$ must have as many clockwise as anticlockwise folding points. (A different proof of this observation was given at the end of Section 2.) Defining

$$\mu = \mu(\pi) := \#\{k : \sigma_\pi(k) = +1\} = \#\{k : \sigma_\pi(k) = -1\} \ , \tag{3.5}$$

our realization of π by a nonlinearity f with $\mathcal{S}_f^{x=1} = \bar{\mathcal{S}}$ must be such that the corresponding set $K_f = K_1 \cup \ldots \cup K_\mu$ is composed of μ curves pairwise connecting the points with $\sigma_\pi(k) = +1$ to the points with $\sigma_\pi(k) = -1$.

Since n is odd and $i_1 = i_n = 0$, we always have $\mu(\pi) \le \frac{n-3}{2}$. When $\mu(\pi) = 0$, our canonical meander is very simple and we will show through an example that the permutation π is realizable. When $\mu(\pi) > 0$, our proof of Theorem 1.2 proceeds by induction with respect to $\mu(\pi)$ and odd values of n. In fact, in Section 4 we prove the following

Lemma 3.1 *Given a dissipative Morse meander permutation $\pi \in S(n)$, $n \ge 3$, there is a dissipative Morse meander permutation $\pi' \in S(n-2)$ with $\mu(\pi') \le \mu(\pi)$ such that a canonical meander $\bar{\mathcal{S}}$ corresponding to π can be obtained by a realizable deformation from a canonical meander $\bar{\mathcal{S}}'$ corresponding to π'.*

After a finite number of steps one must have $\mu(\pi') < \mu(\pi)$ and eventually reach $\mu(\pi') = 0$ completing the induction.

To identify the desired permutation π', we look at the folding index vector $(\sigma_\pi(k))_{1\le k\le n}$. Along an oriented section curve $\mathcal{S}_f^{x=1}$, a point with $\sigma_\pi(k)=-1$ must have another point with $\sigma_\pi(k')=+1$, $k'<k$, preceding it. In fact, we have the following

Lemma 3.2 *If $\sigma_\pi(j)=0$ for $j=1,\ldots,k-1$, then $\sigma_\pi(k)\ge 0$.*

Proof: Indeed, if k is even then $i_\nu=0$ for all odd $\nu\le k-1$ by (3.1). Hence $\sigma_\pi(k)=\frac{1}{2}i_{k+1}\ge 0$. If k is odd, then $i_\nu=1$ for all even $\nu\le k-1$. Hence $\sigma_\pi(k)=\frac{1}{2}(i_{k+1}-1)$. Moreover, $i_\nu=0$ for all odd $\nu\le k-2$. Therefore, $i_k=i_{k-2}+2\sigma_\pi(k-1)=0$. Now (1.10) implies $0\le i_{k+1}=i_k\pm 1=1$. Consequently $\sigma_\pi(k)=0$. This proves lemma 3.2.

We now describe the explicit construction of the reduced permutation π' from π, which is at the heart of lemma 3.1. Our description also serves as an outline for the proof given in Section 4. Define κ to be the first point of anticlockwise intersection, $\sigma_\pi(\kappa)=-1$. Let ι denote the last clockwise intersection, $\sigma_\pi(\iota)=+1$, preceding it. In other words,

$$\kappa=\kappa_\pi:=\min\{j:\sigma_\pi(j)=-1\} \tag{3.6}$$

$$\iota=\iota_\pi:=\max\{j:j<\kappa,\sigma_\pi(j)=+1\}\ . \tag{3.7}$$

Note that the shooting surface $\mathcal{S}_f$ of our realization of π must have a critical curve K_j connecting the point ι to the point κ, if $\mu(\pi)>0$.

The immediate purpose of the meander deformation process leading to the permutation π' is the removal of the points ι and $\iota+1$. This process is best illustrated by sketching examples.

We consider the case where ι is even (at even points, the meanders $\mathcal{S}$ and $\bar{\mathcal{S}}$ cross the u-axis from the upper half-plane into the lower half-plane); ι odd can be treated similarly. Clockwise winding, $\sigma_\pi(\iota)=+1$, implies $\pi^{-1}(\iota)>$

Figure 3.1: Meander deformation in the simplest case, ι even and $\pi^{-1}(\iota) = \pi^{-1}(\iota+1)+1$.

$\pi^{-1}(\iota + 1)$. In the simplest case we have $\pi^{-1}(\iota) = \pi^{-1}(\iota + 1) + 1$. Then the arcs of the meander $\mathcal{S}$ between the points $\iota - 1$ and $\iota + 2$ look like the first illustration in Fig. 3.1. Recall that the points are numbered along the meander.

In this case there are no points of $\mathcal{S}$ on the u-axis between ι and $\iota + 1$. Therefore, the "nose" region bounded by the arc of $\mathcal{S}$ joining the points $\iota - 1$ to $\iota + 2$ and a half circle in the upper half-plane joining these same points does not contain any other points of the meander $\mathcal{S}$. Our deformation, then, consists simply in the retraction of the arc of $\mathcal{S}$ to the half circle. This *nose retraction* is also illustrated in Fig. 3.1. It is a simple task to adapt this deformation to the canonical meander $\bar{\mathcal{S}}$. The actual realization of this deformation using canonical meanders will be presented in the next Section. It leads to a meander $\bar{\mathcal{S}}'$ with permutation π' satisfying $\mu(\pi') \leq \mu(\pi)$. Strict inequality will hold only when $\kappa = \iota + 2$.

In the case $\pi^{-1}(\iota) > \pi^{-1}(\iota+1)+1$ there are points of $\mathcal{S}$ between ι and $\iota+1$ on the u-axis and the nose retraction operation must proceed very carefully. Still, there is a tubular neighborhood of the arc of $\mathcal{S}$ between $\iota+1$ and $\iota+2$ which has no other points of $\mathcal{S}$, see the shaded region in Fig. 3.2. Our meander deformation in this case consists in the *parallel transport* of points

Figure 3.2: Meander deformation by parallel transport of the points between $\pi^{-1}(\iota)$ and $\pi^{-1}(\iota+1)$, ι even.

of $\mathcal{S}$ along this tubular neighborhood. This leads to a meander $\tilde{\mathcal{S}}$ with a corresponding permutation $\tilde{\pi} \in S(n)$ satisfying $\tilde{\pi}^{-1}(\iota) = \tilde{\pi}^{-1}(\iota+1)+1$, where $\iota = \iota_\pi = \iota_{\tilde{\pi}}$. As the proof unravels, it will be clear that by suitable deformations the above tubular neighborhood can be made large enough to accommodate parallel transport. Then, an application of the previous nose retraction operation to the meander $\tilde{\mathcal{S}}$ leads to the desired permutation π' with $\mu(\pi') \leq \mu(\pi)$. Therefore, in this case the process of identification of the permutation π' and its subsequent deformation realization consists of two operations: from $\pi \in S(n)$ to $\tilde{\pi} \in S(n)$ (a *nose cleaning* deformation) and from $\tilde{\pi}$ to $\pi' \in S(n-2)$ (the nose retraction).

The realization of the nose cleaning parallel transport will also be presented in the next section.

4 Proof of the realization theorem

In this section we prove Lemma 3.1 and, therefore, by induction we obtain the realization Theorem 1.2. An outline of the relevant constructions was given at the end of the previous section.

As pointed out already, when $\mu(\pi) = 0$ our canonical meander is particularly simple. Indeed, $\pi = (1\ 2\ \dots\ n)$ with n odd due to dissipativeness. See Fig. 4.1 for an example with $n = 7$. In this case the permutation π is realized by the boundary value problem (1.1) with, for example, the following nonlinearity

$$f = f(v) = \lambda \prod_{k=1}^{n} (k - v) \ , \ 0 < \lambda < \frac{\boldsymbol{\pi}^2}{(n-2)!} \ . \tag{4.1}$$

In fact, one easily verifies that (1.1) with this f has exactly the n constant solutions $v_k = k$, with $k = 1, \dots, n$. These solutions are the equilibria of the corresponding problem (1.4). For the indicated range of λ, the equilibria are hyperbolic and have Morse index $i(v_k) = 0$, if k is odd, and $i(v_k) = 1$, if k is even. The critical set K_f in this case is empty.

When $\mu(\pi) > 0$ our induction involves two steps. In the first step we determine the permutation $\pi' \in S(n-2)$ described at the end of the previous section. This permutation sets up the process of nose cleaning and nose retraction. The permutation π' has either $\mu(\pi') = \mu(\pi) - 1$ or $\mu(\pi') = \mu(\pi)$. In either case we show that $\pi' \in S(n-2)$ is a dissipative Morse meander permutation. Therefore, π' is realizable by the induction hypothesis. There is a nonlinearity f' with a shooting surface $\mathcal{S}_{f'}$ such that $\bar{\mathcal{S}}' = \mathcal{S}_{f'}^{x=1}$ is a canonical meander with permutation π'. Let $\mathcal{S}_* = \mathcal{S}_{f'}$.

The second step consists in extending $\mathcal{S}_*$ to $x > 1$, introducing a family of section curves $\{\mathcal{S}_*^x : x > 1\}$ which corresponds to the meander deformations described before. As x increases from 1, these section curves evolve from the

Figure 4.1: Meander $\mathcal{S}$ with permutation $\pi = (1\ 2\ \dots\ n)$.

canonical meander $\bar{\mathcal{S}}'$ with permutation π' to a canonical meander $\bar{\mathcal{S}}$ with the desired permutation π. Therefore, if the extended manifold $\mathcal{S}_*$ satisfies the conditions of Theorem 2.1, there exists a dissipative nonlinearity f realizing it, that is, $\mathcal{S}_f = \mathcal{S}_*$. Rescaling the x-variable back to the interval $[0, 1]$, afterwards, completes the proof.

Before proving these two steps, we consider the extension process in more detail.

4.1 Realization of deformations and parallel transport

The canonical form of the meander $\mathcal{S}_*^{x=1} = \bar{\mathcal{S}}$ is very convenient for the application of Theorem 2.1. All points of $\bar{\mathcal{S}}$ with a vertical tangent occur at points k on the u-axis and are nondegenerate in the sense that they are folding points (with a folding index $\sigma_\pi(k) \neq 0$). These are the points of the section curve $\mathcal{S}_*^{x=1}$ belonging to the critical set K_*. To verify the conditions of Theorem 2.1 we need to consider only the evolution of these points as x increases from 1. Mainly, K_* must be an integral curve of the plane field (2.5). As x varies, these points have to move with normal component of the velocity equal to p. Hence, as long as these points remain on the u-axis, they will have $p = 0$ and the corresponding set K_* will be a straight line ($u =$ const., $p = 0$). Therefore, an extension $\mathcal{S}_*$ of the form $\mathcal{S}_*^x = \bar{\mathcal{S}}$ for $x > 1$ satisfies the conditions of Theorem 2.1. We use this fact to restrict our section curve deformations to an open set $(u, p) \in U \subset \mathbb{R}^2$, preserving

Figure 4.2: Integral curve of (2.5) with (u,p)-projection corresponding to a half circle.

its shape outside U. For example, in the meander deformation represented in Fig. 3.1 we take a neighborhood U containing the shaded region. During the deformation for $x > 1$, we continue the critical curves K_* with (u,p)-projection in U along integral curves of the plane field (2.5). In order to remain in the shaded region throughout the extension, we choose critical curves whose (u,p)-projection correspond to half circles. An integral curve of the plane field (2.5) with (u,p)-projection corresponding to the half circle $\{(u,p) : (u-u_0)^2 + p^2 = r^2, u > 0\}$ has the form $u = u_0 - r\cos(x-1)$, $p = r\sin(x-1)$ for $1 < x < 1+\boldsymbol{\pi}$. The curve in the (x,u,p)-space resulting from joining this arc to the two line segments corresponding to $u = u_0 \pm r$, $p = 0$ is continuous but not C^1. However, we can smooth out this curve near the points with $x = 1$ and $x = 1 + \boldsymbol{\pi}$ obtaining a C^∞ smooth integral curve of

Figure 4.3: Manifold $\mathcal{S}_*$ containing the curve K_* and section curves $\mathcal{S}_*^{x_i}$, $i = 1, 2, 3, 4$ corresponding to a nose retraction deformation.

(2.5). All it takes is a C^∞ smoothing of its (u, x)-projection, followed by a definition of its p-component as the derivative $p = \frac{du}{dx}$. We use this type of curves to continue the critical set K_* in the region of deformation. Therefore, in an x-interval of length $\boldsymbol{\pi}$ we can achieve a smooth deformation that moves one point of the set K_* from a position with coordinates $u = u_0 - r$, $p = 0$ to a position with coordinates $u = u_0 + r$, $p = 0$ (see Fig. 4.2). Once the critical curve K_* is defined, it is easy to define the shooting surface $\mathcal{S}_*$ through the family of its section curves $\mathcal{S}_*^x$. In fact, we preserve the fibration of $(x, u, p) \in \mathbb{R}^3$ by lines parallel to the p-axis: hence the name "parallel transport". $\mathcal{S}_*$ does not contain any further folds or other points of tangency to the field (2.5) except the points of the critical curve K_*. This is illustrated in Fig. 4.3.

During parallel transport, our meander deformation involves several components of the critical set K_*. To continue these components for $x > 1$ we use

Figure 4.4: Parallel transport introducing a gap between u_2 and u_3.

curves with (u, p)-projections corresponding to half circles, all with the same radius (see Fig. 4.4). This ensures that, as x increases from 1, all the folding points of $\mathcal{S}_*^x$ involved in the deformation have the same height p. Therefore, all these points move with the same speed $\frac{du}{dx}$, allowing for the realization of the parallel transport. The manifold $\mathcal{S}_*$ is, also in this case, easily defined by giving its section curves $\mathcal{S}_*^x$. See illustration in Fig. 4.5.

We now address the proof of Lemma 3.1.

4.2 Determination of π' from π

Let n be odd and $\pi \in S(n)$ be a dissipative Morse meander permutation such that $\mu(\pi) > 0$. The leading point to be removed in the operation is identified by the clockwise winding $\iota = \iota_\pi$; see (3.7). In this subsection, we explicitly

Figure 4.5: Manifold $\mathcal{S}_*$ and two section curves $\mathcal{S}_*^{x_i}$ for a parallel transport deformation.

describe the formal reduction process from $\pi \in S(n)$ to $\pi' \in S(n-2)$, see (4.2), (4.3) below. We also prove that π', thus defined, inherits from π the property of being a dissipative Morse meander permutation.

Following the outline on the previous section, we consider separately two cases regarding the relative positions along the base line of the points ι and $\iota + 1$. If these positions, $\pi^{-1}(\iota)$ and $\pi^{-1}(\iota + 1)$, are consecutive we use a nose retraction like the illustration in Fig. 3.1. Otherwise, we proceed in two steps: a parallel transport – nose cleaning – like the illustration in Fig. 3.2, followed by a nose retraction. Besides these, we also consider separately the cases of even or odd ι, corresponding to the two different directions in which the meanders can cross the base line.

Consider the case ι even, see Figs. 3.1, 3.2. The simplest case corresponds to $\pi^{-1}(\iota) = \pi^{-1}(\iota+1)+1$ – the nose retraction. The permutation $\pi' \in S(n-2)$ is then obtained from the permutation π simply by deleting the points with order ι and $\iota+1$ along the meander, and reordering all the points along the base line. More specifically, but less transparently,

$$(4.2)\quad \pi'(j) = \begin{cases} \pi(j) & \text{if } 1 \leq j < \pi^{-1}(\iota+1) \text{ and } \pi(j) < \iota \\ \pi(j)-2 & \text{if } 1 \leq j < \pi^{-1}(\iota+1) \text{ and } \pi(j) > \iota+1 \\ \pi(j+2) & \text{if } \pi^{-1}(\iota) < j \leq n-2 \text{ and } \pi(j+2) < \iota \\ \pi(j+2)-2 & \text{if } \pi^{-1}(\iota) < j \leq n-2 \text{ and } \pi(j+2) > \iota+1 \end{cases}$$

For example, the first line of (4.2) refers to those points j on the base line, which are to the left of $\pi^{-1}(\iota+1)$ and are traversed before the ι-th intersection along the meander: $\pi(j) < \iota$.

The permutation π' is dissipative because $1 < \iota < n$. Also, by construction (i.e. the nose retraction depicted in Fig. 3.1) π' is a meander permutation. Finally, since the retraction does not change the number of clockwise half-windings of the unit tangent to the meander at the points $1, 2, \ldots, \iota-1, \iota+2, \ldots, n$ we conclude that π' is Morse.

If ι is odd, the same result holds. The simplest case now corresponds to the condition $\pi^{-1}(\iota+1) = \pi^{-1}(\iota)+1$. To obtain π' it is then only necessary to interchange $\pi^{-1}(\iota+1)$ with $\pi^{-1}(\iota)$ in (4.2).

If ι is even, but $\pi^{-1}(\iota) > \pi^{-1}(\iota+1)+1$, we first consider the parallel transport deformation (alias nose cleaning) leading to the permutation $\tilde{\pi} \in S(n)$. See Fig. 3.2. This permutation is obtained by interchanging the δ_1 intersection points strictly between $\pi^{-1}(\iota)$ and $\pi^{-1}(\iota+1)$ in the base line order with the δ_2 points between $\pi^{-1}(\iota+1)$ and $\pi^{-1}(\iota+2)$ (including these end points). The base line orders within the two exchanged sets are preserved, respectively. Note that $\delta_1 = \pi^{-1}(\iota) - \pi^{-1}(\iota+1) - 1$ and $\delta_2 = \pi^{-1}(\iota+1) - \pi^{-1}(\iota+2) + 1$.

With interpretation similar to (4.2), we therefore obtain

$$\tilde{\pi}(j) = \begin{cases} \pi(j) & \text{if} \quad 1 \leq j < \pi^{-1}(\iota+2) \\ \pi(j+\delta_2) & \text{if} \quad \pi^{-1}(\iota+2) \leq j < \pi^{-1}(\iota+2)+\delta_1 \\ \pi(j-\delta_1) & \text{if} \quad \pi^{-1}(\iota+2)+\delta_1 \leq j < \pi^{-1}(\iota) \\ \pi(j) & \text{if} \quad \pi^{-1}(\iota) \leq j \leq n \end{cases} \tag{4.3}$$

Again, one verifies immediately that $\tilde{\pi}$ is dissipative. Furthermore, $\tilde{\pi}$ is a meander permutation by construction (the deformation by parallel transport depicted in Fig. 3.2). Finally, $\tilde{\pi}$ is also Morse since parallel transport does not change the number of clockwise half-windings of the unit tangent to the meander at the reference points.

As before, if ι is odd and $\pi^{-1}(\iota) > \pi^{-1}(\iota+1)+1$ a similar result holds. To obtain $\tilde{\pi}$ it is only necessary to interchange $\pi^{-1}(\iota)$ with $\pi^{-1}(\iota+2)$ in (4.3) and take $\delta_1 = \pi^{-1}(\iota+2) - \pi^{-1}(\iota+1) + 1$ and $\delta_2 = \pi^{-1}(\iota+1) - \pi^{-1}(\iota) - 1$.

To conclude, given a dissipative Morse meander permutation $\pi \in S(n)$, (4.3) leads to a dissipative, Morse, meander permutation $\tilde{\pi} \in S(n)$. Applying (4.2) to $\tilde{\pi}$ one obtains the desired dissipative Morse meander permutation $\pi' \in S(n-2)$. This proves the first part of Lemma 3.1.

4.3 Realization of π from π'

We complete the proof of Lemma 3.1 by showing that $\pi \in S(n)$ is realizable from a realization f' of the permutation $\pi' \in S(n-2)$. We need to show that the inverse nose cleaning deformations leading from π' to $\tilde{\pi} \in S(n-2)$ and $\pi \in S(n)$ are always realizable by extending $\mathcal{S}_* = \mathcal{S}_{f'}$ to some interval $1 \leq x \leq T$ with $T > 1$.

First, we note that the realization of π from $\tilde{\pi}$ by parallel transport is very simple. Turning again to Fig. 3.2 we see that in order to perform the

Figure 4.6: Realization of π from $\tilde{\pi}$ in two steps: a. Introduction of a gap between $\pi^{-1}(\iota)$ and $\pi^{-1}(\iota+1)$; b. Interchange of points by parallel transport.

deformation by parallel transport it is necessary to have a sufficiently large gap between the points $\pi^{-1}(\iota)$ and $\pi^{-1}(\iota+1)$ on the base line. This is easily achieved by an initial parallel transport (as one depicted in Fig. 4.4) of $\pi^{-1}(\iota+1)$ and all the points to the right of it. Extending $\mathcal{S}_*$ to an interval with $T = 1 + \boldsymbol{\pi}$ provides as large a gap as necessary. A subsequent parallel transport, taking the desired points to the enlarged gap (see Fig. 4.6), realizes the permutation π from $\tilde{\pi}$ by a further extension of $\mathcal{S}_*$ to $1 \leq x \leq T = 1+2\boldsymbol{\pi}$. The inverse nose retraction realization of π (or $\tilde{\pi}$, if necessary) from π' is most delicate in the case where $\kappa_\pi = \iota_\pi + 2$, see (3.6), (3.7) and Fig. 3.1. In this case, $\iota_\pi + 2$ is a folding point of the meander corresponding to π folding anticlockwise, $\sigma_\pi(\iota_\pi + 2) = -1$. Referring back to Fig. 3.1, the deformation leading from π' to π corresponds, pictorially, to a nose creation where a pair of folding points is introduced. Hence, this is the case where $\mu(\pi) = \mu(\pi')+1$. We start with an extension of $\mathcal{S}_*$ to some interval $1 \leq x \leq 1 + \varepsilon_1$, such that a *cusp* arises at $\mathcal{S}_*^{1+\varepsilon_1}$ close to the point corresponding to $\iota_{\pi'}$. Locally, such a cusp corresponds to a manifold of the form

$$u = (x - 1 - \varepsilon_1)p - \frac{\Omega}{3}(p - \varepsilon_2)^3 + \varepsilon_3 \tag{4.4}$$

Note that, indeed, the (x, u)-projection of the critical curve K_* in a neighborhood $V_{\pi'}$ of $(x = 1 + \varepsilon_1, u = \varepsilon_3, p = \varepsilon_2)$ is a cusp. As x increases beyond $1 + \varepsilon_1$, it corresponds to the appearance of a pair of points on $\mathcal{S}_*^x$ with vertical tangents and opposite folding indices σ. Therefore, for $x > 1 + \varepsilon_1$ a new connected component $K_{\mu(\pi)}$ is added to the critical set K_* which becomes $K_* = \bigcup_{1 \leq j \leq \mu} K_j$. To continue $\mathcal{S}_*$ for values of $x > 1 + \varepsilon$, $\varepsilon > \varepsilon_1$, we adjust the parameters Ω, $(\varepsilon_j)_{j=1,2,3}$, in such a way that one of the branches of the cusp continues as a straight line and the other as a sinusoidal curve as depicted in Fig. 4.7. As the (x, u)-projection of the critical set K_* traverses the sinusoidal curve, its (u, p)-projection describes a half circle with a prescribed radius. This continuation of the cusp only needs to be C^1 smooth, resulting in an integral curve to the plane field (2.5) which is only continuous. Smoothing out near $\partial V_{\pi'}$ as before, we obtain a critical curve $K_{\mu(\pi)}$ which is C^∞ smooth and leads to the desired deformation (a nose creation). If necessary, we use a preparatory parallel transport as above to introduce a sufficiently wide gap between the points where we need to insert "our" nose. This completes the realization of the permutation π from π' with an extension of $\mathcal{S}_*$ to an interval with $T = 1 + \varepsilon + 2\boldsymbol{\pi}$.

The realization of π (or $\tilde{\pi}$) from π' in the case where $\kappa_\pi > \iota_\pi + 2$ does

Figure 4.7: Introduction of a fold corresponding to the appearance of two folding points and creation of a “nose” deformation.

not require the introduction of a cusp. In fact, by (3.6) and (3.7), in this case we have $\sigma_\pi(\iota_\pi + 2) = \sigma_\pi(\iota_\pi + 1) = 0$ and $\sigma_\pi(\iota_\pi) = +1$. One can then show that $\sigma_{\pi'}(\iota_{\pi'}) = +1$ and the meander corresponding to π' has a clockwise folding point at the point $\iota_{\pi'} = \iota_\pi$. Therefore, the nose retraction deformation required to realize π from π' only involves the extension of an existing critical set K_* as illustrated in Fig. 4.3. This concludes the proof of Lemma 3.1.

5 Example

As a specific example, we consider the realization of the permutation

$$\pi = (1\ 10\ 7\ 4\ 3\ 8\ 9\ 2\ 5\ 6\ 11) \in S(11) \tag{5.1}$$

to illustrate our construction. Note that π is a dissipative Morse meander permutation. In fact, its Morse vector is given by

$$(i_k)_{1 \leq k \leq 11} = (0, 1, 2, 1, 0, 1, 2, 3, 2, 1, 0)\ , \tag{5.2}$$

and Fig. 5.1 shows a meander with permutation π. Then, the iteration procedure of nose cleaning and nose retraction used in our realization proof (see Lemma 3.1 and the end of Section 3), leads to the following sequence of dissipative Morse meander permutations:

$$\begin{array}{lllll}
\pi_1 & = & \pi & = & (1\ 10\ 7\ 4\ 3\ 8\ 9\ 2\ 5\ 6\ 11) \in S(11) \\
\pi_2 & = & \tilde{\pi}_1 & = & (1\ 10\ 7\ 8\ 9\ 4\ 3\ 2\ 5\ 6\ 11) \in S(11) \\
\pi_3 & = & \pi_2' & = & (1\ 8\ 5\ 6\ 7\ 2\ 3\ 4\ 9) \in S(9) \\
\pi_4 & = & \pi_3' & = & (1\ 6\ 5\ 2\ 3\ 4\ 7) \in S(7) \\
\pi_5 & = & \tilde{\pi}_4 & = & (1\ 2\ 3\ 6\ 5\ 4\ 7) \in S(7) \\
\pi_6 & = & \pi_5' & = & (1\ 2\ 3\ 4\ 5) \in S(5)
\end{array} \tag{5.3}$$

The corresponding canonical meanders $\mathcal{S}^i$, $1 \leq i \leq 6$, are shown in Fig. 5.1. Notice that we use twice the parallel transport operation of nose cleaning and three times the nose retraction.

Our realization of π leads to a manifold $\mathcal{S}_*$ with section curves $\mathcal{S}_*^{x_i} = \mathcal{S}^{7-i}$, for $1 \leq i \leq 6$ and $0 < x_1 < x_2 < \ldots < x_6 = 1$. The section curve $\mathcal{S}_*^{x_6}$ has three clockwise and three anticlockwise folding points corresponding to the folding index vector

$$(\sigma_\pi(k))_{1 \leq k \leq 11} = (0, +1, 0, -1, 0, +1, +1, 0, -1, -1, 0)\ . \tag{5.4}$$

Figure 5.1: Sequence of reduced meanders $\mathcal{S}^i$ with permutations π_i, $1 \leq i \leq 6$.

Therefore, the critical curve set K_* has three connected components, $K_* = K_1 \cup K_2 \cup K_3$, and the stylized form of the graphs of the corresponding functions q_j, $j = 1, 2, 3$, is shown in Fig. 5.2. The graphs of these functions bring forward the existence of a natural partial ordering $\succ$ on the set $\{q_j : j = 1, 2, 3\}$. In the present case we have $q_1, q_2 \succ q_3$. The example in Fig.

Figure 5.2: Stylized graphs q_j, $j = 1, 2, 3$, corresponding to the curves K_j of the manifold $\mathcal{S}_*$, with partial ordering $q_1, q_2 \succ q_3$.

2.1, in contrast, shows the ordering $q_1 \succ q_2 \succ q_3$. This partial ordering is, in any case, encoded in the folding index vector σ_π, and can be derived from it. This completes our construction of a shooting surface and of f such that $\pi_f = \pi$, for π given by (5.1).

It is clear that our construction of the realizable manifold $\mathcal{S}_*$ is non-unique. Many other sequences of section curves and corresponding meander permutations can lead to a realization of the same permutation π. However, the same partial ordering $\succ$ must be present in all the realizations of π and, is therefore a characteristic invariant of these realizations. The number of curves q_j determined by σ_π is equal to the minimum number of folds appearing in any manifold $\mathcal{S}_*$ realizing π. Then, the partial ordering $\succ$ determines the order of appearance of the minimal set of folding points in the section curves $\mathcal{S}_*^x$, as x increases. Therefore, we have just exhibited one particular way of constructing a realization of $\mathcal{S}_*^{x=1}$ with the minimum number of folds, alias minimal number of connected components of the critical set K_*.

6 Discussion

Theorem 1.2 settles the modeling question raised in [FR96] and referred to in the Introduction. All dissipative Morse meander permutations actually arise from (1.4). We recall in Table 5.1 the number $m(n)$ of different dissipative Morse meander permutations $\pi \in S(n)$, for odd $n \leq 17$.

n	1	3	5	7	9	11	13	15	17
$m(n)$	1	1	2	7	32	175	1083	7342	53372

Table 5.1: Numbers $m(n)$ of dissipative
Morse meander permutations in $S(n)$.

Theorem 1.1 asserts the global orbit equivalence of all Morse-Smale global attractors for (1.4) corresponding to the same dissipative Morse meander permutation π. Therefore, $m(n)$ is an upper bound for the number $c(n)$ of orbit equivalence classes of attractors with n hyperbolic equilibria. This number is, in general, smaller than $m(n)$. For example, the conjugating linear homeomorphisms

$$(6.1) \qquad (h(v))(x) = -v(x) \ , \ (h(v))(x) = v(-x) \ , \ (h(v))(x) = -v(-x)$$

applied to (1.4), lead respectively to transformed permutations, (see [FR96]),

$$(6.2) \qquad \tau\pi\tau^{-1} \ , \ \pi^{-1} \ , \ \tau\pi^{-1}\tau^{-1} \ .$$

Here τ denotes the reflection $\tau = (n, n-1, \ldots, 1) \in S(n)$. Therefore, the Morse-Smale attractors associated to π and the permutations in (6.2) are trivially globally orbit equivalent. Moreover, orbit equivalence fails to distinguish some Morse-Smale attractors for (1.4) not related by (6.1). This reduces the number of equivalence classes $c(n) \leq m(n)$ even further. See [Fie94] for specific examples with $n = 9$.

The realization problem which we have addressed here for general nonlinearities $f = f(x, u, u_x)$ also arises naturally for various subclasses. For example, the combinatorial characterization of those permutations $\pi = \pi_f$ associated to nonlinearities

$$f(u), f(u_x), f(x,u), f(x,u_x), f(u,u_x) \tag{6.3}$$

is, to our knowledge, open. It is of course related to modelling questions concerning spatial (in-)homogeneity and presence or absence of drift terms.

Elementary symmetry issues are also relevant in this context. For example consider $f = f(x, u, p)$ such that

$$f(x,u,p) = f(1-x, u, -p) \ . \tag{6.4}$$

Then $(h(v))(x) = v(-x)$ is an automorphism of the attractor, and $\pi_f = \pi_f^{-1}$ is an involution. It is unclear, at present, whether or not symmetric f, in the sense of (6.4), realize all involutive dissipative Morse meander permutations. Analogous questions can, and should, of course be asked, and answered, at least for the other automorphic symmetries $\pi = \tau\pi\tau^{-1}$, $\pi = \tau\pi^{-1}\tau^{-1}$ indicated in (6.2).

The problem of computing the numbers $m(n)$ is related to classical important problems in combinatorics which are largely unsolved (see [Ros84], [LZ92]). In addition to the asymptotic behavior for large n of the number of dissipative meander permutations in $S(n)$, the following estimates are known:

$$cat(n) \leq c(n) \leq m(n) \leq cat^2(n) \tag{6.5}$$

where $cat(n)$ denote the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. We refer to [LZ92], [LZ93] and [Wol96] for more details.

By space discretization of (1.4) one obtains dissipative Jacobi systems. These have the form

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}) \ , \ i = 0, \ldots, n \tag{6.6}$$

where each f_i has strictly positive partial derivatives with respect to the off-diagonal entries. Moreover, the Neumann boundary conditions take the form

$$u_{-1} := u_0 \ , \ u_{n+1} := u_n \ . \tag{6.7}$$

Apart from being finite dimensional, these systems have exactly the same properties as the semilinear parabolic systems (1.4). In particular, they possess attractors which are Morse-Smale when all the equilibria are hyperbolic, [FO88]. Let $\mathcal{A}^{disc}$ denote the set of all (equivalence classes of) global Morse-Smale attractors for (6.6) – the spatially *discrete* case. Similarly, let $\mathcal{A}^{cont}$ denote the set of all global Morse-Smale attractors for (1.4) – the spatially *continuous* case. Then, using Theorem 1.2 one can prove that

$$\mathcal{A}^{cont} = \mathcal{A}^{disc} \ , \tag{6.8}$$

([FR98], Theorem 8.2). Therefore, the class of Morse-Smale attractors for Jacobi systems (6.6) exactly matches the class for (1.4).

Let $\mathcal{A}^{MS}$ denote the above class of Morse-Smale attractors. Surprisingly enough, this class is also fairly independent of the separated boundary conditions used. It is argued in [Fie96] that the class of attractors for (1.4) with mixed type linear boundary conditions

$$\begin{aligned} \tau_0 u_x(.,0) - (1-\tau_0)u(.,0) = 0 \quad &, \ 0 \le \tau_0 \le 1 \\ \tau_1 u_x(.,1) + (1-\tau_1)u(.,1) = 0 \quad &, \ 0 \le \tau_1 \le 1 \end{aligned} \tag{6.9}$$

does not depend on the parameters (τ_0, τ_1). By this homotopy, $\mathcal{A}^{MS}$ includes also the case of Dirichelet boundary conditions. Moreover, it even includes mildly nonlinear boundary conditions of the form

$$u_x(.,0) = g_0(u(.,0)) \ , \ u_x(.,1) = g_1(u(.,1)) \ . \tag{6.10}$$

Therefore, with such separated boundary conditions one cannot leave the class $\mathcal{A}^{MS}$.

For the Jacobi systems (6.6), Morse-Smale attractors with the same permutation are Morse-Smale homotopic (see [FR98]). It is not known if the same statement holds for (1.4). We recall that along a Morse-Smale homotopy all the (transverse) intersections of stable and unstable manifolds, and their filtration of submanifolds corresponding to the different rates of approach to the equilibria (see [FR91]), are preserved. For this reason, the corresponding heteroclinic connections should be part of a detailed geometric description of the attractors for (1.4). We believe this information to be useful simultaneously for the study of the cell structure of the attractors, and for obtaining a stronger notion of attractor equivalence.

For periodic boundary conditions, however, the full picture is still far from understood. For most recent progress we refer to [MN97]. The classification and geometric characterization of the attractors for all such systems remains a challenging open problem.

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