Orbit equivalence of global attractors of semilinear parabolic differential equations

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Abstract

We consider global attractors $\mathcal{A}_f$ of dissipative parabolic equations

$$u_t = u_{xx} + f(x, u, u_x)$$

on the unit interval $0 \leq x \leq 1$ with Neumann boundary conditions. A permutation π_f is defined by the two orderings of the set of (hyperbolic) equilibrium solutions $u_t \equiv 0$ according to their respective values at the two boundary points $x = 0$ and $x = 1$. We prove that two global attractors, $\mathcal{A}_f$ and $\mathcal{A}_g$, are globally C^0 orbit equivalent, if their equilibrium permutations π_f and π_g coincide. In other words, some discrete information on the ordinary differential equation boundary value problem $u_t \equiv 0$ characterizes the attractor of the above partial differential equation, globally, up to orbit preserving homeomorphisms.

1 Introduction and result

On the unit interval $0 \leq x \leq 1$, the interplay of linear diffusion with nonlinear, spatially heterogeneous reaction and drift terms can be modeled by the scalar parabolic partial differential equation

$$u_t = u_{xx} + f(x, u, u_x). \tag{1.1}$$

Equations of this form arise in many applied contexts. We just mention population dynamics in mathematical biology [CS80], reactor dynamics in chemical engineering [Ari75], viscosity limits to hyperbolic conservation laws [Smo83], clustering effects in astrophysics [Wol92a], [Wol92b], and phase transitions in materials sciences [CP88], [FH89].

The present paper aims at a global analysis of the long time behavior of the infinite dimensional dynamical system (1.1). For definiteness, we impose Neumann boundary conditions

$$u_x = 0, \qquad \text{for } x = 0, 1. \tag{1.2}$$

Let f be twice continuously differentiable: $f \in C^2$. By standard semigroup theory, the (local) solutions $u = u(t, x)$ of (1.1) with initial condition $u(0, x) = u_0(x)$ then define a dynamical system

$$0 \leq t \mapsto u(t, \cdot) \in X, \tag{1.3}$$

see [Hen81] or [Paz83]. The state space X can be picked here as the Sobolev space H^2 of x-profiles

$$x \mapsto u(t, x) \tag{1.4}$$

with Lebesgue square integrable second x-derivative u_{xx}, intersected with the appropriate Neumann conditions,

$$X = H^2([0, 1], \mathrm{IR}) \cap \{u_x = 0 \text{ at } x = 0, 1\}. \tag{1.5}$$

By Sobolev embedding, the x-profiles are at least once continuously differentiable, $X \subseteq C^1$. Moreover, the time-t maps

$$u_0 \mapsto u(t, \cdot; u_0) \in X \tag{1.6}$$

are compact for $t > 0$, by the smoothing action of diffusion.

We assume the nonlinearity f to be *dissipative*: there exists a fixed large ball $B \subseteq X$ which attracts each solution. In particular, solutions (1.3) are defined for all $t \geq 0$, and for any $u_0 \in X$ there exists $t_0 = t_0(u_0) \geq 0$ such that $u(t, \cdot) \in B$, for all $t \geq t_0$. Then our dynamical system possesses a *global attractor* $\mathcal{A}_f$. This set can be characterized as the maximal compact invariant subset of X. It is also the smallest subset of X which attracts all bounded sets. Equivalently, $\mathcal{A}_f$ consists of all those $u_0 \in X$ which, for $t = 0$, lie on an orbit $u(t, \cdot) \in X$ of (1.1) which is defined and uniformly bounded for all real times t, both positive and negative. See [Hal88], [Tem88], [Lad91], [BV89] for reference. We state explicit sufficient conditions for f to be dissipative, which we assume to hold in this paper. There exists a constant C_1 such that

$$f(x, u, 0) \cdot u < 0 \tag{1.7}$$

for $|u| \geq C_1$, and moreover there exist continuous functions a, b as well as an exponent $\gamma < 2$ such that

$$|f(x, u, p)| < a(u) + b(u)|p|^\gamma \tag{1.8}$$

for all x, u, p. For dissipativeness under conditions (1.7), (1.8), viz. for bounds on $\sup |u_x|$, see [Ama85], theorem 5.3. We defer a discussion of these rather restrictive conditions to section 8.

Let $\mathcal{E}_f \subset X$ denote the set of equilibria, that is, of solutions $u \in X$ of the Neumann boundary value problem for the second order ordinary differential equation

$$0 = u_{xx} + f(x, u, u_x). \tag{1.9}$$

We assume all equilibria to be *hyperbolic*, that is, the linearized equation

$$0 = \eta_{xx} + f_p \cdot \eta_x + f_u \cdot \eta \tag{1.10}$$

with Neumann boundary condition $\eta_x = 0$, for $x = 0, 1$, possesses only the trivial solution $\eta \equiv 0$. Here the partial derivatives f_u and f_p of $f = f(x, u, p)$ are evaluated at $u = u(x)$, $p = u_x(x)$, for some equilibrium $u \in \mathcal{E}_f$. Equivalently, the real part of all eigenvalues λ of the linearization

$$\lambda\eta = \eta_{xx} + f_p \cdot \eta_x + f_n \cdot \eta \tag{1.11}$$

is nonzero, for all $u \in \mathcal{E}_f$. In fact, all eigenvalues must be real, because the Sturm-Liouville problem (1.11) is selfadjoint, under an appropriately weighted L^2 scalar product. We note that hyperbolicity of $\mathcal{E}_f$ is a generic assumption on f, see [Smo83], [BP87].

By the implicit function theorem, hyperbolic equilibria are isolated in X. Because $\mathcal{E}_f \subseteq \mathcal{A}_f$ is compact, the set of equilibria is finite

$$\mathcal{E}_f = \{u^1, \ldots, u^N\}. \tag{1.12}$$

Here the numbering is chosen such that

$$u^1 < u^2 < \ldots < u^N, \text{ at } x = 0. \tag{1.13}$$

Indeed, the equilibria are strictly ordered by their values at $x = 0$, due to uniqueness of the initial value problem for (1.9) and because $u_x^k = 0$ for all k, at $x = 0$. For the same reason, the equilibria can be ordered by their other boundary value, at $x = 1$. This defines a permutation $\pi = \pi_f$ such that

$$u^{\pi(1)} < u^{\pi(2)} < \ldots < u^{\pi(N)}, \text{ at } x = 1. \tag{1.14}$$

Note that the permutation π_f only encodes ordering information on the boundary value problem for the *ordinary* differential equation (1.9). We call π_f the *shooting permutation* associated to the equilibria of f.

To compare the global attractors $\mathcal{A}_f, \mathcal{A}_g$ of (1.1) for different dissipative nonlinearities f, g, we recall the usual notion of C^0 *global orbit equivalence* of $\mathcal{A}_f$ and $\mathcal{A}_g$: there exists a homeomorphism

$$H: \quad \mathcal{A}_f \to \mathcal{A}_g \tag{1.15}$$

which maps f-orbits $\{u(t,\cdot) \in X | t \in \mathrm{IR}\} \subseteq \mathcal{A}_f$ onto g-orbits, preserving the time direction of the orbits. We use the notation

$$\mathcal{A}_f \cong \mathcal{A}_g \tag{1.16}$$

for global orbit equivalence. With this definition we can now state our main result.

Theorem 1.1 *Let $f, g \in C^2$ satisfy dissipation conditions (1.7), (1.8). Assume they possess only hyperbolic equilibria with associated shooting permutations π_f, π_g. Then the global attractors $\mathcal{A}_f$ and $\mathcal{A}_g$ are globally orbit equivalent, $\mathcal{A}_f \cong \mathcal{A}_g$, if*

$$\pi_f = \pi_g. \tag{1.17}$$

An outline of the proof will be given in section 2. The proof itself fills sections 3 to 7. We conclude with a detailed discussion, in section 8. We conclude this introduction indicating the two main features which make our theorem work: *Morse structure* and *nodal properties.* For a survey see [Fie89].

The *Morse, gradient,* or *variational structure* of (1.1) is given explicitly, for $f = f(x, u)$ independent of $p = u_x$, in terms of the *Lyapunov functional* V on X defined by

$$V(u) := \int_0^1 \left(\frac{1}{2}(u_x)^2 - F(x,u) \right) dx. \tag{1.18}$$

Here F is a primitive function of f with respect to u, that is $F_u = f$. Inserting solutions $u(t, x)$ of (1.1), we obtain

$$\frac{d}{dt} V(u(t,\cdot)) = - \int_0^1 (u_t)^2 dx. \tag{1.19}$$

In particular, V is bounded on the compact attractor $\mathcal{A}_f$, and is strictly decreasing with time, except of course on the equilibrium set $\mathcal{E}_f$. A similar, though less explicit, Lyapunov functional $V : \mathcal{A}_f \to \mathbb{R}$ with these properties also exists in the general case where $f = f(x, u, p)$ is allowed to depend on $p = u_x$, see [Zel68], [Mat88].

By its Morse structure, our global attractor $\mathcal{A}_f$ decomposes into equilibria $\mathcal{E}_f$ and their *heteroclinic* or *connecting* orbits. These orbits, by definition, limit onto (different) equilibria for $t \to +\infty$ and $t \to -\infty$, respectively. Indeed, $\frac{d}{dt}V = 0$ on α- and ω-limit sets, and therefore these sets are given by a single, isolated hyperbolic equilibrium, each. In short

$$\mathcal{A}_f = \mathcal{E}_f \cup \{\text{heteroclinic orbits}\}. \tag{1.20}$$

Investigation of the geometry of the heteroclinic set is an ongoing topic of research. We mention [Hen81], [Hen85], [Ang86], [BF88], [BF89], [AF88], [FR91], for partial results on equations of the form (1.1). Recently, [FR94] have settled the question for which $u^i, u^j \in \mathcal{E}_f$ there does exist a heteroclinic connection from u^i to u^j. The complete answer can be given by an explicit and constructive process, which only uses information on the equilibrium permutation π_f. The precise global geometry of the connections, however, remains unresolved.

The crucial importance of *nodal properties* for the qualitative understanding of equations of the form (1.1) was first noted by [Mat82]. Take any two solutions $u^1(t, \cdot), u^2(t, \cdot)$. Let $z(u^1 - u^2)$, for fixed t, denote the number of strict sign changes (nodes) of the x-profile $x \mapsto u^1(t, x) - u^2(t, x)$. Then

$$t \mapsto z(u^1(t, \cdot) - u^2(t, \cdot)) \tag{1.21}$$

is nonincreasing with time t. In fact, $z(u^1(t, \cdot) - u^2(t, \cdot))$ drops strictly whenever the x-profile $u^1 - u^2$ possesses a multiple zero $u^1 - u^2 = 0$, $u^1_x - u^2_x = 0$.

Indeed, the nodal property follows because the difference $\tilde{u} := u^1 - u^2$ satisfies a (time-dependent) linear parabolic equation. For comprehensive nodal properties in the linear case see the recent account by [Ang88]. The autonomous linear case is classical [Stu36], [Pol33].

Nodal properties have many important, and sometimes surprising consequences. For example, just by hyperbolicity of the equilibria $\mathcal{E}_f$ alone and without further genericity or nondegeneracy assumptions on f, our dynamical system (1.1) turns out to be *Morse-Smale*, see [Hen85], [Ang86]. In fact, unstable and stable manifolds W^u and W^s of any two equilibria necessarily intersect transversely, by the above nodal property. Together with the Morse structure this makes our dynamical Morse-Smale system structurally stable: small dissipative perturbations of the nonlinearity f to $\tilde{f}$ yield global attractors $\mathcal{A}_f, \mathcal{A}_{\tilde{f}}$ which are globally (C^0) orbit equivalent. In finite dimensions this result is due to Palis and Smale, see [Pal69], [PS70], [PdM82]. The infinite dimensional case is due to [Oli83]; see also [HMO84].

In view of Morse-Smale stability our strategy for a proof of our theorem seems simple. If we find a homotopy $f^\tau, 0 \leq \tau \leq 1$, from $f^0 = f$ to $f^1 = g$, such that the equilibrium set $\mathcal{E}_{f^\tau}$ remains hyperbolic throughout, then $\mathcal{A}_f$ and $\mathcal{A}_g$ are orbit equivalent by Oliva, Palis, and Smale. Indeed we can cover the compact parameter interval $0 \leq \tau \leq 1$ by finitely many open intervals of structural stability. In the next section we explain why matters are not all that simple, in fact, and how our proof takes a few more meanders before completion.

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2 Outline of proof

Throughout, let the assumptions of theorem 1.1 hold. In this section we explain our overall strategy of proof for our theorem on global orbit equivalence. Structural stability of Morse-Smale systems will be the main actor, though mostly somewhat behind the scene. We first explain the difficulty which a naive homotopy f^τ is facing, in the spirit of the end of the introduction section 1. We then indicate how discretization will help, in section 3. Section 4 is devoted to normalizations which involve dissipativeness. Our crucial trick, gaining additional freedom for homotopies, is the insertion of additional, artificial discretization points, in section 5, which essentially produce a high-dimensionally unstable suspension of the original global attractor $\mathcal{A}_f$ in an augmented discrete system. Sections 6 and 7 will highlight the additional freedom gained by augmented discretization. In fact, the additional freedom reduces the question of finding a suitable homotopy f^τ, on the augmented discrete level, to a purely topological result on the fundamental group of a certain space of diffeomorphisms. With those details at hand, the outline of proof given below will then have solidified into a complete proof.

Let us first try the most naive, standard homotopy

$$f^\tau = (1-\tau)f + \tau g, \quad 0 \le \tau \le 1, \tag{2.1}$$

between dissipative nonlinearities f, g with global attractors $\mathcal{A}_f, \mathcal{A}_g$. By dissipation condition (1.7), (1.8) global attractors respect bounds $|u| \le C_1, |u_x| \le C_2$, uniformly in t, x. Modifying f, g to become

$$f = g = -u, \tag{2.2}$$

for $|u| \ge 2C_1$ or $|u_x| \ge 2C_2$, as explained in section 4 below, ensures dissipativeness throughout the homotopy. The homotopy parameter τ induces a bifurcation diagram for the sets $\mathcal{E}_{f^\tau}$ of equilibria. If all equilibria remain hyperbolic, for all $0 \le \tau \le 1$, then the Morse-Smale property implies equivalence of $\mathcal{A}_f$ and $\mathcal{A}_g$. However, bifurcations of saddle-node type could occur, for some $0 < \tau < 1$. Saddle-nodes are lethal for our Morse-Smale proof. To understand the difficulty in avoiding saddle-nodes, we use a shooting, or "time" map approach to the equilibrium ODE (1.9), alias

$$\begin{aligned} u_x &= p \\ p_x &= -f(x,u,p) \end{aligned} \tag{2.3}$$

with Neumann boundary conditions $p = 0$, at $x = 0, 1$. At $x = 0$, consider the line of initial conditions

$$u = a, \quad p = 0, \tag{2.4}$$

parametrized by $a \in \mathrm{I\!R}$. Letting x evolve forward, according to (2.3), we obtain solutions $u(x;a), p(x;a)$ and the *shooting curves*

$$a \mapsto S_f^x(a) := (u(x;a), p(x;a)) \in \mathrm{I\!R}^2 \tag{2.5}$$

in (u,p) space. See for example [FR91], [Roc91], [FR94] and the references there. Each shooting curve is a planar C^2 Jordan curve. Their collection for $0 \le x \le 1$ defines the shooting surface

$$S_f: \quad (x,a) \mapsto (x, S_f^x(a)) \in [0,1] \times \mathrm{I\!R}^2 \tag{2.6}$$

in (x,u,p) space. Note that the equilibria $\mathcal{E}_f$ are described by the values a at intersections of $S_f^{x=1}$ with the u-axis:

$$S_f^{x=1}(a) = (u(x=1;a), 0)\,, \tag{2.7}$$

corresponding to the Neumann boundary condition

$$p(x=1;a) = 0. \tag{2.8}$$

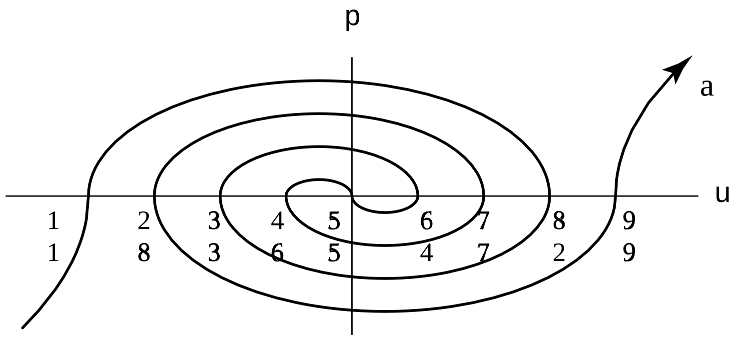

Figure 2.1: A stylized shooting curve $S_f^{x=1}$ for the Chafee-Infante problem, $(3\pi)^2 < \lambda < (4\pi)^2$.

It turns out that $\mathcal{E}_f$ is hyperbolic if, and only if, $S_f^{x=1}$ intersects the u-axis transversely. If $S_{f^\tau}^{x=1}$ touches, and crosses the u-axis during a homotopy f^τ, then a saddle-node bifurcation of equilibria occurs. Clearly, saddle-node bifurcations might occur during a naive standard homotopy $f^\tau = (1-\tau)f + \tau g$, with destructive consequences for the geometry of the global attractors $\mathcal{A}_{f^\tau}$.

There is a very special case though, where the naive, standard homotopy (2.1) works well. If the shooting surfaces of f and g coincide entirely,

$$S_f = S_g, \tag{2.9}$$

then S_{f^τ} is independent of τ, also at $x = 1$. Consequently, the global attractors $\mathcal{A}_f$ and $\mathcal{A}_g$ are then orbit equivalent. Therefore it will be sufficient to focus on just the shooting surfaces, when constructing a less naive homotopy f^τ.

As a simple specific example, we consider the *Chafee-Infante problem* [Cha74], [CI74]

$$f(x, u, p) = \lambda u(1 - u^2). \tag{2.10}$$

As in all dissipative systems, there exists a lowest and a highest equilibrium $\underline{u}, \bar{u}$; here $\underline{u} \equiv -1$, $\bar{u} \equiv +1$. In the (u,p)-plane the shooting curve points south-west, to the left of the lowest, and north-east, to the right of the highest equilibrium. Also observe the transverse intersections of $S_f^{x=1}$ with the u-axis corresponding to the equilibria $u^1, \dots, u^9$. Numbering the equilibria along the shooting curve $S_f^{x=1}$ provides the Chafee-Infante permutation

$$\pi_f = (2 \quad 8)\,(4 \quad 6) \tag{2.11}$$

in the case of Fig. 2.1. Parenthetically we note that the Morse indices, or unstable dimensions, of all equilibria are also determined by π_f; see again [Roc91], [FR94].

Consider a less naive homotopy f^τ from f to g next; this homotopy should preserve the transverse intersections of $S_{f^\tau}^{x=1}$ with the u-axis, enacting a Morse-Smale application. Since $\pi_f = \pi_g$, by assumption, there exists a diffeotopy

$$\varphi^\tau \tag{2.12}$$

of the (u,p) plane, $\varphi^\circ = id$, which provides a deformation from $S_f^{x=1}$ to $S_g^{x=1}$, such that transversality of the curves to the u-axis is preserved throughout the homotopy. Our proof would be complete, if we could now find a homotopy f^τ such that for $x = 1$

$$S_{f^\tau}^x \quad = \quad \varphi^\tau(S_f^x). \tag{2.13}$$

But we cannot. The obstruction lies in the second order system (2.3) which constrains the evolution of shooting curves S_f^x. Indeed, let $\mathbf{n} \in \mathbb{R}^2$ denote a unit normal to S_f^x at (u,p), and σ the normal speed of propagation of the curve S_f^x in the direction of $\mathbf{n}$, as x increases. Then σ is the normal projection of the vector field (2.3), that is,

$$\sigma = \mathbf{n} \cdot \begin{pmatrix} p \\ -f(x,u,p) \end{pmatrix}. \tag{2.14}$$

Varying f freely, this does not impose any restriction on the speed σ, unless $\mathbf{n} = (1,0)$ is horizontal. In that latter case, S_f^x possesses a vertical tangent (pictorically: a *nose*), and

$$\sigma = p. \tag{2.15}$$

Summarizing: *higher noses move faster to the right*, as x increases. It is this constraint on the shape of the shooting surfaces S_f, S_g which we could not overcome in our attempts to construct a homotopy f^τ which preserves the Morse-Smale type hyperbolicity condition (2.13) throughout $0 \leq \tau \leq 1$. Such a homotopy might in fact exist. But we did not secure a general algorithm for constructing one.

Therefore we discretize, in section 3. The discrete equations take the form

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}), \tag{2.16}$$

$i = 0, \ldots, n$, where each f_i has strictly positive partial derivatives with respect to its off-diagonal entries u_{i-1} and u_{i+1}. Following [FO88] we call (2.16) a *Jacobi system.* The Neumann boundary conditions become

$$u_{-1} := u_0, \quad u_{n+1} := u_n. \tag{2.17}$$

The global attractor of (2.16) is orbit equivalent to $\mathcal{A}_f$, for equidistant symmetric finite difference approximations f_i and large enough n (see section 3). Dissipativeness is also inherited (see section 4). We denote the global attractor of (2.16), (2.17) by $\mathcal{A}_f$, again.

We now describe the crucial trick, a $n(m-1)/2$-dimensionally unstable suspension of $\mathcal{A}_f$ in a singular perturbation manner, which will gain us enough additional freedom to realize the shooting curve diffeotopy φ^τ. For $i = 0, \ldots, n,\ k = 1, \ldots, m-1$ and for $0 < j := i + k/m < n$ we define the *augmented discrete system* as

$$\begin{aligned} \dot{u}_i &= f_i(u_{i-1/m}, u_i, u_{i+1/m}) \\ \varepsilon \dot{u}_j &= \quad u_{j-1/m} + u_{j+1/m}. \end{aligned} \tag{2.18}$$

Here $m \geq 5$ is chosen congruent 1 mod 4. We keep Neumann boundary conditions (2.17) in the adapted form

$$u_{-1/m} := u_0, \quad u_{n+1/m} := u_n, \tag{2.19}$$

and let $u = (u_0, u_{1/m}, \dots, u_{n-1/m}, u_n) \in \mathbb{R}^{nm+1}$. The linear $(n+1)$-dimensional subspace

$$M := \{u_{j-1/m} + u_{j+1/m} = 0, \text{ for all noninteger } 0 < j < n\} \tag{2.20}$$

is invariant under (2.18) because $m \equiv 1 \pmod 4$. Note that M contains the original global attractor $\mathcal{A}_f$ since

$$u_{i\pm 1/m} = u_{i\pm 1} \tag{2.21}$$

on M. For small $\varepsilon > 0$, the invariant manifold M is normally hyperbolic with $n(m-1)/2$-dimensional strong unstable and strong stable fibers. The maximal compact invariant set $\tilde{\mathcal{A}}_f$ of (2.17), (2.18) therefore coincides with $\mathcal{A}_f$. These facts will be proved in section 5, see lemma 5.1.

In section 6 we study the analogue of the shooting curves S_f^x for the augmented discrete system (2.18), rewritten as a Jacobi system

$$\dot{u}_j = f_j(u_{j-1/m}, u_j, u_{j+1/m}), \tag{2.22}$$

$j = 0, 1/m, \dots, n$. For equilibria, $\dot{u} = 0$, we obtain

$$0 = f_j(u_{j-1/m}, u_j, u_{j+1/m}). \tag{2.23}$$

Because the off-diagonal derivatives are strictly positive, these equations can be solved for $u_{j+1/m}$, defining the *shooting recursion*

$$u_{j+1/m} = \varphi_j^f(u_{j-1/m}, u_j). \tag{2.24}$$

Note that the partial derivative of φ_j^f with respect to $u_{j-1/m}$ is strictly negative. Given such a φ_j, we may also conversely define f_j by $f_j := u_{j+1/m} - \varphi_j$, and obtain a system (2.22) with Jacobi structure.

Rewriting (2.24) as a system

$$(2.25) \qquad \psi_j^f : \quad \begin{array}{rcl} u_j &=& v_{j-1/m} \\ v_j &=& \varphi_j^f(u_{j-1/m}, v_{j-1/m}) \end{array}$$

we see that the diffeomorphisms ψ_j^f replace f-evolution maps from $x = (j-1)/n$ to $x = j/n$, in the continuous case. Also, for non-integer j, the map ψ_j^f is a pure rotation R by 90 degrees,

$$(2.26) \quad \psi_j^f(u_{j-1/m}, v_{j-1/m}) = R(u_{j-1/m}, v_{j-1/m}) := (v_{j-1/m}, -u_{j-1/m}).$$

In particular

$$(2.27) \qquad \varphi_j^f(u_{j-1/m}, v_{j-1/m}) = -u_{j-1/m}$$

for non-integer j. If $j = i$ is integer, then ψ_i^f was already defined by the original discretization f_i. Let the *shooting quadruple*

$$(2.28) \qquad \tilde{\psi}_j^f := \psi_{j+3/m}^f \circ \ldots \circ \psi_j^f$$

be the composition of any four consecutive ψ-maps with non-integer $j, \ldots, j+3/m$. Clearly $\tilde{\psi}_j^f = id$ in this case of four rotations by 90 degrees. In Lemma 6.1 we prove that, by small perturbations of $\varphi_{j+3/m}^f, \varphi_{j+2/m}^f$, the shooting quadruple $\tilde{\psi}_j^f$ can be forced to realize *any* diffeomorphism near identity. This is the crucial additional freedom which will enable us to complete a non-naive structurally stable homotopy from $\mathcal{A}_f$ to $\mathcal{A}_g$. Indeed, between any two maps ψ_i with integer i we have now inserted $(m-1)/4 \in \mathbb{N}$ shooting quadruples $\tilde{\psi}_j$, each of which can realize any near identity diffeomorphism.

Our non-naive homotopy is constructed, in section 7, as follows. Consider the diffeomorphisms

$$(2.29) \qquad \Psi_i^f := \psi_i^f \circ \ldots \circ \psi_0^f,$$

$i = 1, \ldots, n$, which accumulate diffeomorphisms ψ_i^f, i integer, defined by (2.25). For g, we define Ψ_i^g analogously. Comparing with (2.13) we observe that Ψ_n^f induces the permutation π_f, by the discrete shooting curve intersecting the diagonal transversely in the original attractor region $\{|u| \leq, |v| \leq 1\}$; see section 6. Here we assume n is chosen large enough, of course. Similarly, Ψ_n^g induces $\pi_g = \pi_f$, for large n. In a neighborhood Ω of the attractor region, we now prescribe homotopy paths

$$\Psi_i^\tau, \quad 0 \leq \tau \leq 1 \tag{2.30}$$

from $\Psi_i^f = \Psi_i^0$ to $\Psi_i^g = \Psi_i^1$ in the space $\mathit{Diff}(\Omega)$ of diffeomorphism of Ω. We will assume $\Psi_i^f \equiv \Psi_i^g \equiv \Psi_i^\tau$, outside Ω, without loss of generality. At the end point $i = n$, corresponding to $x = 1$, we specifically prescribe Ψ_n^τ to provide a diffeotopy φ^τ of the discrete shooting curves from f to g, analogously to the construction of $\varphi_{x=1}^\tau$ in the continuous case; see (2.13), (7.9). Moreover we choose a standard homotopy

$$\psi_i^\tau := (1-\tau)\psi_i^f + \tau\psi_i^g \tag{2.31}$$

of diffeomorphisms (2.25) at integer values $j = i = 0, \ldots, n-1$. The maps ψ_j at non-integer values of j between $n-1$ and n will compensate for this mortal sin against hyperbolicity and the Morse-Smale property during the homotopy $0 \leq \tau \leq 1$.

In section 7, lemma 7.2, we will prove that there exists a continuous two-parameter family $\Psi(\tau, \xi) \in \mathit{Diff}(\Omega)$, $0 \leq \tau, \xi \leq 1$ of interpolating diffeomorphisms, which compensates for the mortal sin (2.31). More precisely, as indicated in Fig. 2.2

$$\begin{aligned} \Psi(\tau, \xi = 0) &= \Psi_{n-1}^\tau \\ \Psi(\tau, \xi = 1) &= (\psi_n^\tau)^{-1}\Psi_n^\tau \\ \Psi(\tau = 0, \xi) &= \Psi_{n-1}^f \\ \Psi(\tau = 1, \xi) &= \Psi_{n-1}^g \end{aligned} \tag{2.32}$$

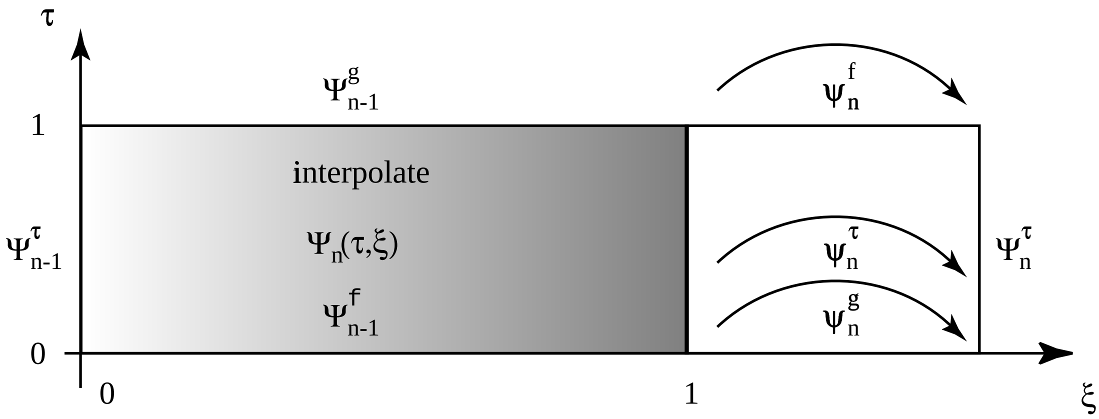

Figure 2.2: Constructing interpolating diffeomorphisms.

This fact uses essentially triviality of the fundamental group π_1 of the space $Diff_*(\Omega)$ of diffeomorphisms of Ω which fix $\partial\Omega$:

$$\pi_1(Diff_*(\Omega)) = 0, \tag{2.33}$$

by Smale's theorem. See lemma 7.1 below.

In the final step of our proof we fix m large enough to sufficiently discretize the ξ-variable in the interpolating family of diffeomorphisms $\Psi(\tau, \xi)$. In fact, all diffeomorphisms

$$\Psi^\tau_{n,k} := \Psi(\tau, \xi_{k+1}) \circ (\Psi(\tau, \xi_k))^{-1} \tag{2.34}$$

with $\xi_k := 4k/(m-1),\ k = 0, \ldots, (m-5)/4$, are uniformly close to identity in $Diff(\Omega)$. They can therefore be realized, at last, by quadruple shootings $\tilde{\psi}_j$ as in (2.28). This is the pay-off out of the freedom gained by $n(m-1)/2$-dimensionally unstable suspension of the global attractors $\mathcal{A}_f$ and $\mathcal{A}_g$.

Returning to the cumulative shooting maps Ψ^τ_n, a Morse-Smale homotopy is realized between the suspended attractors $\tilde{\mathcal{A}}_f$ and $\tilde{\mathcal{A}}_g$ and, consequently, between the original attractors $\mathcal{A}_f$ and $\mathcal{A}_g$. Therefore, $\mathcal{A}_f$ and $\mathcal{A}_g$ are indeed globally C^0 orbit equivalent, and our main theorem will be proved.

3 Discretization

As outlined in section 2, we discretize the original semilinear parabolic equation

$$u_t = u_{xx} + f(x, u, u_x). \tag{3.1}$$

In fact, equidistant semidiscretization in x of step size $1/n$ and the method of lines in t-direction yield

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}), \tag{3.2}$$

$i = 0, \ldots, n$, with f_i given explicitly by

$$f_i(u_{i-1}, u_i, u_{i+1}) = n^2(u_{i-1} - 2u_i + u_{i+1}) + f(i/n, u_i, n(u_i - u_{i-1})). \tag{3.3}$$

Clearly, u_i represents u at $x = i/n$. As noted before, Neumann boundary conditions become

$$u_{-1} := u_0, \quad u_{n+1} := u_n. \tag{3.4}$$

We do not advocate (3.2)–(3.4) as a particularly efficient discretization from a numerical point of view. Rather, we prefer this discretization because, for large enough n, it preserves

(i) nodal properties of solutions,

(ii) the Morse-Smale structure,

(iii) the global attractor $\mathcal{A}_f$, up to orbit equivalence, and

(iv) the shooting permutation π_f.

Properties (i)–(iii) are essentially known and are surveyed below. Property (iv) will be addressed in section 6.

Recall that $u := (u_0, \ldots, u_n) \in \mathrm{IR}^{n+1}$ will also denote solutions of (3.2), in addition to denoting solutions of (3.1); context will avoid confusion. Properties (i),(ii) hold for general systems (3.2), (3.4), if only f_i has positive off-diagonal partial derivatives, alias Jacobi structure, as in (2.16). This is indeed the case for the discretization (3.3) and large n, by our dissipation normalization (2.2).

Let the zero number $z(u)$ denote the number of strict sign changes of the sequence of coordinates $u_0, \ldots, u_n$. Then

$$t \mapsto z(u^1(t) - u^2(t)) \tag{3.5}$$

is noncreasing with time t, just as in the continuous case (1.21). In fact, $z(u^1(t) - u^2(t))$ drops strictly at $t = t_0$ whenever a component i of the nontrivial difference $\eta := u^1 - u^2$ vanishes and the difference components $i \pm 1$ are of equal sign ("multiple zero"),

$$\eta_i(t_0) = 0, \text{ and } \eta_{i-1}(t_0) \cdot \eta_{i+1}(t_0) \geq 0, \tag{3.6}$$

for some $i = 0, \ldots, n$. In case $\eta_{i-1} \cdot \eta_{i+1} > 0$ this follows easily from the monotonicity properties of f_i, which imply

$$\text{sign } \dot{\eta}_i = \text{sign } \eta_{i-1} = \text{sign } \eta_{i+1} \tag{3.7}$$

at $t = t_0$. For the general case see [MPS90], [FO88]. This settles claim (i).

The Morse-Smale structure (ii) of the finite dimensional system (3.2), (3.4) follows from the dropping properties (3.5), (3.6) of the zero number z; see [FO88], [Ang86]. As in the PDE case outlined in the introduction, hyperbolicity of all equilibria automatically entails transversality of their associated stable and unstable manifolds.

The properties (iii) and (iv) consider the particular nonlinearities f_i given by semidiscretization (3.3). First note that (3.2), (3.4) is a dissipative system, in that case. Indeed $|u_i| \geq 2C_1$ or $|n(u_{i+1} - u_i)| \geq 2C_2$ imply

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}) = n^2(u_{i-1} - 2u_i + u_{i+1}) - u_i. \tag{3.8}$$

In particular $\|u\| := \max|u_i(t)|$ decreases with time, as long as $\|u\| \geq 2C_1$.

By dissipativeness, (3.2)–(3.4) possesses a compact global attractor $\mathcal{A}_f^n$. Orbit equivalence of $\mathcal{A}_f^n$ to the global PDE attractor $\mathcal{A}_f$ was announced in [Hal94], see p. 24. A proof is in preparation [Rau95]. We indicate our own viewpoint, for convenience. Let $d = \dim \mathcal{A}_f = \max\{i(u) \,|\, u \in \mathcal{E}_f\}$ be the maximal Morse index of any equilibrium $u = u(x) \in \mathcal{E}_f$. Define a projection

$$P: \quad \mathcal{A}_f \to \mathbb{R}^{n+1} \tag{3.9}$$

letting $(Pu)_i := u(i/n)$, $i = 0, \ldots, n$, denote actual values of the x-profile u. By nodal properties on $\mathcal{A}_f$, the projection P is injective if $n+1 \geq d$. In fact, P turns out to be a diffeomorphic embedding of $\mathcal{A}_f$ into $\mathbb{R}^{n+1}$ [Mat95]. The main step here is to show that $\mathcal{A}_f$ possesses a unique d-dimensional tangent plane everywhere; not only on the d-dimensional unstable manifolds of its most unstable equilibria, but also, by a limiting procedure, on their closure – which constitutes all of $\mathcal{A}_f$. Along the lines of [FR94], section 3, these claims are consequences of the nodal properties (3.5), (3.6) for the flow induced, by linearization, on the Grassmannian of d-planes. Along any heteroclinic orbit from u to $\tilde{u} \in \mathcal{E}_f$, for example, these d-planes limit onto the span of the first d eigenfunctions of the target equilibrium $\tilde{u}$. They are then propagated further down in $\mathcal{A}_f$, along $W^u(\tilde{u})$, as unique tangent planes of a non-unique generalized d-dimensional "center"-unstable manifold. At $\tilde{u}$, of course, this generalized center-unstable manifold is tangent to the eigenfunctions of the first d eigenvalues. Propagating along $W^u(\tilde{u})$, its tangent spaces remain unique even though the generalized center-manifold is non-unique.

With the embedding P at hand, it is then relatively easy to compare the flow on $\mathcal{A}_f$ with the flows on the d-dimensional global attractors $\mathcal{A}_f^n$, for large n.

Indeed, for $n \to \infty$, all equilibria become hyperbolic, converging to $\mathcal{E}_f$ with eventually constant Morse indices. Their unstable manifolds converge, uniformly in C^1, and so do the unique d-dimensional tangent planes, on the attractors $\mathcal{A}_f^n$, to the non-unique center-unstable manifolds constructed above. The final observation is C^0 structural stability of Morse-Smale flows under C^1-small perturbations. Therefore $\mathcal{A}_f$ and $\mathcal{A}_f^n$ are globally orbit equivalent, for large n.

Henceforth, we will fix such a large n and denote $\mathcal{A}_f^n$ by $\mathcal{A}_f$, again. Also the ambiguity of notation in our u seems thoroughly justified, by now.

4 Dissipation

In this section we discuss conditions for and elementary consequences of dissipativeness, as defined in the introduction, for the continuous PDE case and the (augmented) discrete system.

Recall the dissipation conditions (1.7), (1.8) for the original PDE (1.1), (1.2). Dissipation, together with the parabolic comparison principle alias strong monotonicity (e.g. [Smo83], [Hir88], [Mat86]) implies that (1.1) possesses a maximal and a minimal equilibrium

$$\bar{u}, \underline{u} \in \mathcal{E}_f \subseteq \mathcal{A}_f \tag{4.1}$$

in the global attractor $\mathcal{A}_f$. Indeed $\bar{u}, \underline{u}$ arise, respectively, as ω-limit sets of large positive or negative initial conditions (taken for example x-independent). After a linear transformation $u \mapsto \tilde{u}$,

$$u := a(x)\tilde{u} + b(x) \tag{4.2}$$

we may assume

$$\bar{u}(x) \equiv +1, \quad \underline{u}(x) \equiv -1, \tag{4.3}$$

without loss of generality. Indeed the transformation (4.2) preserves the form of the PDE (1.1). In terms of the original x-dependent equilibria $\bar{u}, \underline{u}$ in (4.1), the coefficients a, b in (4.2) read

$$a = (\bar{u} - \underline{u})/2, \quad b = (\bar{u} + \underline{u})/2. \tag{4.4}$$

Solutions $\tilde{u}$, like u itself, therefore satisfy Neumann boundary conditions because $a_x, b_x = 0$ at $x = 0, 1$. We can now assume (4.3) to hold without loss of generality.

Lemma 4.1 *Let (4.3) hold and consider $u \in \mathcal{A}_f$. Then $|u(x)| \leq 1$ holds for all x.*

Proof: We argue indirectly, assuming $u(x_0) > 1$ without loss of generality. Since $u \in \mathcal{A}_f$, the α-limit set of u is an equilibrium $u^- \in \mathcal{E}_f$. Since $z(u-1)$ is nondecreasing, in backwards time, and by maximality of $\bar{u} \equiv 1$ in $\mathcal{E}_f$, the zero number z of $u^- - 1$ is nontrivial:

$$z(u^- - 1) \geq 1. \tag{4.5}$$

Now let $u_0(x) > u^-(x), 1$ be another initial condition. Necessarily $\omega(u_0) \equiv 1$. But because z is nonincreasing, in forward time,

$$z(1 - u^-) \leq z(u_0 - u^-) = 0. \tag{4.6}$$

The contradiction (4.5), (4.6) proves the lemma. □

As mentioned in the outline, (2.2), we now replace the dissipation conditions (1.7), (1.8) by the simpler condition

$$f(x, u, p) = -u, \tag{4.7}$$

for $|u| \geq C_1$ or $|p| \geq 2C$. Here C is a constant specified below.

Lemma 4.2 *Let f satisfy the dissipation conditions (1.7), (1.8) for $|u| \geq C_1 > 1$, with the normalization (4.3).*

Then $|u| \leq 1 < C_1$ and *$|u_x| \leq C$ on the global attractor $\mathcal{A}_f$, for some constant C. Moreover, there exists a nonlinearity $\tilde{f} = \tilde{f}(x, u, p) \in C^2$ which*

- *(i) coincides with f on $\mathcal{A}_f$ and in fact on the larger set where $|u| \leq (1 + C_1)/2$ and $|p| \leq 1.5C$,*
- *(ii) satisfies (4.7), and*
- *(iii) possesses the same global attractor*

(4.8) $$\mathcal{A}_{\tilde{f}} = \mathcal{A}_f,$$

of course with the same dynamics on it.

Proof: The bound

(4.9) $$|u| \leq 1$$

on $\mathcal{A}_f$ was proved in lemma 4.1. The bound

(4.10) $$|u_x| \leq C$$

follows from compactness of the global attractor $\mathcal{A}_f \subset X \subset C^1$ under dissipation conditions (1.7), (1.8) on f. The gradient bound (4.10) depends on C_1, a, b, γ, but not on f. Without loss of generality, we assume $a(u) \geq |u|$ in our construction of C.

We construct $\tilde{f}$ as follows. Let $\mathcal{V} \subset \mathbb{R}^2$ be a small open neighborhood of the set where $|u| \geq C_1$ or $|p| \geq 2C$, such that the open dissipation conditions (1.7), (1.8) still hold on $\mathcal{V}$. Let $\tilde{f} := (1-\tau)f - \tau u$, choosing $0 \leq \tau = \tau(u,p) \leq 1$ with $\tau \equiv 0$ outside $\mathcal{V}$, and $\tau(u,p) = 1$ for $|u| \geq C_1$ or $|p| \geq 2C$. Then

(i) $\tilde{f}(x,u,p) = f(x,u,p)$, for $(u,p) \notin \mathcal{V}$,

(ii) $\tilde{f}(x,u,p) = -u$, for $|u| \geq C_1$ or $|p| \geq 2C$, and

(iii) $\tilde{f}$ satisfies (1.7), (1.8) on $\mathcal{V}$.

Note that (iii) holds, with the same C_1, a, b, γ, by convexity of the dissipation conditions (1.7), (1.8) with respect to f. Clearly (i), (ii) above imply (i), (ii) in the lemma.

To see that (iii) implies (iii), we first note that $\mathcal{E}_f = \mathcal{E}_{\tilde{f}}$. Indeed, in view of the bounds (4.9), (4.10), the nonlinearities $\tilde{f}$ cannot possess an equilibrium u such that $(u(x), u_x(x)) \in \mathcal{V}$, for any fixed x or τ. To see that $\mathcal{A}_f = \mathcal{A}_{\tilde{f}}$, just recall from (1.20) that $\mathcal{A}_f$ is the union of unstable manifolds $W^u_f(u)$ of $u \in \mathcal{E}_f$. Because $|u| \leq 1 < C_1, |u_x| \leq C$ on $\mathcal{A}_f$, we conclude $W^u_{\tilde{f}}(u) = W^u_f(u)$ for all $u \in \mathcal{E}_{\tilde{f}} = \mathcal{E}_f$. Taking the union over $u \in \mathcal{E}_{\tilde{f}}$, this implies (iii), $\mathcal{A}_{\tilde{f}} = \mathcal{A}_f$, and the lemma is proved. □

We note that $\pi_{\tilde{f}} = \pi_f$, for the associated equilibrium permutations, because $\mathcal{A}_{\tilde{f}} = \mathcal{A}_f$ implies $\mathcal{E}_{\tilde{f}} = \mathcal{E}_f$. Therefore we can assume (4.7) to hold for f and g in our proof of theorem 1.1, without loss of generality.

We now consider discrete equations with positive off-diagonal partial derivatives,

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}), \tag{4.11}$$

for $i = 0, \ldots, n$, as in (2.16), (2.17). An obviously sufficient condition for dissipativeness of such Jacobi systems is, in general,

$$u_i f_i(u_{i-1}, u_i, u_{i+1}) < 0 \tag{4.12}$$

for $|u_i| \geq \max\{2C_1, |u_{i-1}|, |u_{i+1}|\}$, $i = 0, \ldots, n$. Indeed, $\|u\| = \max_i\{|u_i|\}$ then decreases monotonically down to level $2C_1$. In our particular case, where f_i arises by discretization (3.3), condition (4.7) is obviously satisfied. The global attractor of (4.11) is again called $\mathcal{A}_f$, in view of section 3. Our normalization (4.3) still holds, because all $f_i(\pm 1, \pm 1, \pm 1) = 0$. Therefore lemma 4.1 carries over to the discrete case.

5 Unstable suspension

In this section we discuss augmented discrete systems

$$
\begin{aligned}
\dot{u}_i &= f_i(u_{i-1/m}, u_i, u_{i+1/m}) \\
\varepsilon \dot{u}_j &= u_{j-1/m} + u_{j+1/m}
\end{aligned}
\tag{5.1}
$$

introduced in (2.18). Here $0 \le i \le n$ and $\varepsilon > 0$. For integer $0 < k < m$, the noninteger values $0 < j := i + k/m < n$ indicate the artificially inserted values u_j which augment the original system. We impose Neumann boundary conditions (2.19) on the flow of (5.1). Our aim is to recover the global attractor $\mathcal{A}_f$ of the original discretized flow

$$
\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1})
\tag{5.2}
$$

with Neumann boundary conditions (2.17). We will find $\mathcal{A}_f$ unstably suspended, in the linear subspace

$$
M := \left\{ u_{j-1/m} + u_{j+1/m} = 0, \text{ for all noninteger } 0 < j < n \right\}.
\tag{5.3}
$$

Lemma 5.1 *Assume $m \equiv 1$ (mod 4), $m \ge 5$. Then the subspace M of dimension $n+1$ is invariant under the flow (5.1), (2.19).*

The discretized flow (5.2), (2.17) is equivalent to the flow on M by the flow preserving linear isomorphism

$$
\begin{aligned}
I: \quad \mathbb{R}^{n+1} &\to M \subseteq \mathbb{R}^{nm+1} \\
(u_0, u_1, \ldots, u_n) &\mapsto (u_0, u_{1/m}, \ldots, u_n)
\end{aligned}
\tag{5.4}
$$

defined by the sign alternation

$$
\begin{aligned}
u_{i+k/m} &= (-1)^{k/2} u_i, \\
u_{i+1-k/m} &= (-1)^{k/2} u_{i+1},
\end{aligned}
\tag{5.5}
$$

for all $i = 0, \dots, n-1$, and all even $k = 2, \dots, m-1$. In particular

$$u_{i\pm 1} = u_{i\pm 1/m} \tag{5.6}$$

holds for this embedding, $i = 0, \dots, n$.

For small $\varepsilon > 0$, the subspace M is uniformly normally hyperbolic in $\mathbb{R}^{nm+1}$. The dimensions of the strong stable and the strong unstable fibers are equal, each given by

$$n(m-1)/2. \tag{5.7}$$

In particular,

$$\tilde{\mathcal{A}}_f := I(\mathcal{A}_f) \subseteq M \tag{5.8}$$

is the maximal compact invariant set, alias the set of bounded solutions, of the augmented discrete system (5.1), (2.19).

Proof: Because $m \equiv 1 \pmod 4$, the map I is indeed a well-defined linear isomorphism between $\mathbb{R}^{n+1}$ and M. In particular $\dim M = n+1$. Also (5.5) implies $\dot{u}_j = 0$, for noninteger j. This proves flow invariance of M. Because $m \equiv 1 \pmod 4$ we may choose $k := m-1$ even, in (5.5). Because $k/2$ is still even, this implies (5.6) for $i = 1, \dots, n-1$. For $i = 0, n$ equality (5.6) follows from the respective Neumann boundary conditions (2.17), (2.19). Inserting (5.6) into the first equation of the augmented discrete system (5.1) immediately recovers the original discretized flow (5.2), proving flow equivalence.

To prove the normal hyperbolicity of M, for $\varepsilon \searrow 0$, we follow standard singular perturbation reasoning. For small $\varepsilon > 0$ and rescaled time εt, the fast time transition layers are given approximately by the equations

$$\dot{u}_j = u_{j-1/m} + u_{j+1/m}, \tag{5.9}$$

for all noninteger $0 < j < n$ as above. The approximation is of class C^κ, for any κ. Since $\dot{u}_i = 0$ for integer i, in fast time, the values of $u_{j\pm 1/m}$ for integer $i = j \pm 1/m$ in (5.9) are to be considered as constants. Therefore the linearization of (5.9) at any point of M, coordinatized by $u_0, u_1, \ldots, u_n$ according to relations (5.5), decouples into n identical copies of equation (5.9). In the first copy, j ranges from $1/m$ to $(m-1)/m$, only. The constants u_i, when linearized, provide Dirichlet boundary conditions $u_0 = u_1 = 0$, and similarly for the other j-blocks. Eigenvectors of the right hand side of (5.9) are therefore given by $u_j = \sin(\pi j \ell)$ with associated real, distinct, nonzero eigenvalues

$$\lambda_\ell = 2\cos(\pi\ell/m), \tag{5.10}$$

$\ell = 1, \ldots, m-1$. Counting $\lambda_\ell > 0$ and $\lambda_\ell < 0$ then proves (5.7). Since the vector field inside M possesses globally bounded derivatives, normal hyperbolicity follows for small $\varepsilon > 0$.

Normal hyperbolicity of M holds uniformly in $\mathbb{R}^{nm+1}$. Indeed the flow (5.1) is already linear in the $\dot{u}_j$ equation. Moreover, the $\dot{u}_i$ equation becomes linear for large $\|u\|$, due to definition (3.3) of f_i and normalization (4.7) of f. Therefore M is uniformly normally hyperbolic.

By uniform normal hyperbolicity of M, the maximal compact invariant set $\tilde{\mathcal{A}}_f$ is contained in M. Because the isomorphism $I : \mathbb{R}^{n+1} \to M$ is a flow equivalence and $\mathcal{A}_f$ is the maximal compact invariant set in $\mathbb{R}^{n+1}$, this implies (5.8), $\tilde{\mathcal{A}}_f = I(\mathcal{A}_f)$, and the lemma is proved. □

6 Discrete shooting

In this section we study the analogue of the shooting curves S_f^x, introduced in (2.5), for the augmented discrete system (2.17). We make free use of the notation (2.21)–(2.28) for the two-term shooting recursion φ_j^f, $j = 0, 1/m, \ldots, n$, the incremental shooting diffeomorphism ψ_j^f in (u_j, v_j)-coordinates, the shooting quadruples $\tilde{\psi}_j^f$, and the cumulative shooting diffeomorphisms Ψ_i^f. We define the discrete analogue π_f^n for the equilibrium shooting permutation π_f. In lemma 6.1 we show that π_f does not change under discretization or augmentation. In lemma 6.2 we highlight the additional freedom gained by augmentation: realization of arbitrary shooting diffeomorphisms near identity by shooting quadruples $\tilde{\psi}_j^f$.

The shooting permutation π_f^n can be defined for general discrete Jacobi systems

$$\dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}), \tag{6.1}$$

$i = 0, \ldots, n$, for example with Neumann boundary conditions $u_{-1} = u_0$, $u_{n+1} = u_n$. Assume the equilibrium set $\{u^1, \ldots, u^N\} \subseteq \mathbb{R}^{n+1}$ is finite. Just as in the continuous case, the ordering at $i = 0$,

$$u_0^1 < \ldots < u_0^N, \tag{6.2}$$

defines a permutation $\pi = \pi_f^n$ by the ordering of the values

$$u_n^{\pi(1)} < \ldots < u_n^{\pi(N)} \tag{6.3}$$

at the other Neumann boundary $i = n$.

The discrete analogue of the shooting curves S_f^x are the C^1 Jordan curves

$$S_i^f := \Psi_i^f(\text{diag}) \;\subseteq \mathbb{R}^2. \tag{6.4}$$

Here Ψ_i^f is the cumulative shooting diffeomorphism, and diag $= \{u_{-1} = v_{-1}\}$ denotes the diagonal in $\mathbb{R}^2$. As in the continuous case, the permutation π_f^n

can also be viewed as relating the two possible orderings of the intersection points

$$S_n^f \cap \text{diag}\,, \tag{6.5}$$

along the diagonal diag $= \{u_n = v_n\}$, and along S_n^f, respectively. We note that transverse crossings, in (6.5), indicate hyperbolic equilibria of system (6.1), similarly to the continuous case.

Let π_f denote the original permutation associated to the equilibria $\mathcal{E}_f$, let π_f^n denote the permutation associated to the discretization (3.2)–(3.4), and finally $\tilde{\pi}_f^{mn}$ the permutation associated to the augmented system (2.17)–(2.18).

Lemma 6.1 *Let n be large enough and $m \equiv 1$ (mod 4), $m \geq 1$. Then*

$$\pi_f = \pi_f^n = \tilde{\pi}_f^{mn} \tag{6.6}$$

Proof: The convergence of discretized equilibria is uniform in x, for $n \to \infty$, if we interpolate linearly between the values u_i at $x = i/n$. More precisely, uniform convergence holds on bounded subsets of (u_i, v_i). The normalization $f = -u$ for large $|u|$, $|p|$ given in (2.2) and section 5 avoids the appearance of additional "spurious" equilibria in the discretized equations which otherwise might escape to infinity for $n \to \infty$. This proves $\pi_f = \pi_f^n$, by definitions (1.13)–(1.14) and (6.2)–(6.3).

Since the augmentation (2.17), (2.18) does not change the cumulative shooting diffeomorphism Ψ_f^n, at all, the claim $\pi_f^n = \tilde{\pi}_f^{mn}$ is obvious. This proves the lemma. □

Encouraged by lemma 6.1, we simply denote all three permutations in (6.6) by π_f, from now on.

We turn to realization of near identity diffeomorphisms by shooting quadruples

$$\tilde{\psi}_j = \psi_{j+3/m} \circ \ldots \circ \psi_j, \tag{6.7}$$

for noninteger $j, \ldots, j+3/m$. The unperturbed incremental shooting diffeomorphisms

$$\psi_{j+k/m} = R, \tag{6.8}$$

$k = 0, \ldots, 3$, are pure rotations by 90 degrees, independently of the original nonlinearity f. In particular, $\tilde{\psi}_j = id$ in the unperturbed case. To preserve the Jacobi structure of the augmented system (2.17), under perturbation, we require the incremental shooting diffeomorphism

$$\psi_{j'+1/m}(u_{j'}, v_{j'}) \; = \; (v_{j'}, \varphi_{j'+1/m}(u_{j'}, v_{j'})), \tag{6.9}$$

$j' = j + (k-1)/m$, to exhibit uniformly negative partial derivatives $\partial_{u_{j'}} \varphi_{j'+1/m}$. Note that the unperturbed case is

$$\varphi_{j'+1/m}(u_{j'}, v_{j'}) = -u_{j'}. \tag{6.10}$$

Lemma 6.2 *Let $\tilde{\psi}_j$, for noninteger j, be any diffeomorphism of $\mathbb{R}^2$ with Jacobian uniformly δ-close to identity, for some $0 < \delta < 1/2$.*

Then there exist four mappings $\varphi_j, \ldots, \varphi_{j+3/m}$ as in (6.9), such that their composition quadruple (6.7) realizes the prescribed diffeomorphism $\tilde{\psi}_j$.

More explicitly, the partial derivatives of $\varphi_j, \ldots, \varphi_{j+3/m}$ can be chosen $\frac{\delta}{1-\delta}$-close to those of the unperturbed case (6.10).

Proof: Without loss of generality, let $j' = 0, \ldots, 3/m$. Simplifying notation, in this proof we write $\tilde{\psi}$, for $\tilde{\psi}_j$, replacing j' by $0, 1, 2, 3$, and $j = j' + 1$ by $1, 2, 3, 4$. We then have to realize

$$\tilde{\psi} = \psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1 \tag{6.11}$$

near identity. In fact, we do not perturb

$$\psi_2 = \psi_1 = R. \tag{6.12}$$

Therefore it suffices to prove that $\psi_4 \circ \psi_3$ can realize any diffeomorphism $-\tilde{\psi}$ which is δ-close to $-id$ in C^1.

In terms of φ_4, φ_3, we can rewrite $\psi_4 \circ \psi_3$ as

$$(\psi_4 \circ \psi_3)(u, v) = (\varphi_3(u, v), \varphi_4(v, \varphi_3(u, v))). \tag{6.13}$$

This defines φ_3, ψ_3 in terms of the given diffeomorphism $-\tilde{\psi}$, trivially. We note two obvious estimates for the partial derivatives $\partial_1\varphi_3, \partial_2\varphi_3$:

$$\begin{aligned} |1 + \partial_1\varphi_3| &\leq \delta \\ |\partial_2\varphi_3| &\leq \delta \end{aligned} \tag{6.14}$$

In particular $\partial_1\varphi_3 < 0$ holds uniformly, if we choose $\delta < 1$. Moreover, φ_3 is δ-close to the unperturbed case.

Let $\varphi(u, v)$ denote the second component of the prescribed diffeomorphism $-\tilde{\psi}$,

$$\begin{aligned} |\partial_1\varphi| &\leq \delta \\ |1 + \partial_2\varphi| &\leq \delta \end{aligned} \tag{6.15}$$

To realize $\psi_4 \circ \psi_3 = -\tilde{\psi}$, we simply have to define

$$\varphi_4 := \varphi \circ \psi_3^{-1}. \tag{6.16}$$

With this definition, we now partially differentiate

$$\varphi_4(v, \varphi_3(u, v)) = \varphi(u, v), \tag{6.17}$$

with respect to u and v, to obtain

$$\begin{aligned} \partial_2\varphi_4 \cdot \partial_1\varphi_3 &= \partial_1\varphi \\ \partial_1\varphi_4 + \partial_2\varphi_4 \cdot \partial_2\varphi_3 &= \partial_2\varphi \end{aligned} \tag{6.18}$$

We have to show $\partial_1\varphi_4 < 0$, uniformly. Inserting estimates (6.14), (6.15) into the first equation of (6.18), we immediately see

$$|\partial_2\varphi_4| \;=\; |\partial_1\varphi| \;/\; |\partial_1\varphi_3| \;\leq \delta/(1-\delta). \tag{6.19}$$

Reinserting (6.19) into the second equation of (6.18) we find

$$|1+\partial_1\varphi_4| \leq |1+\partial_2\varphi| + |\partial_2\varphi_4\cdot\partial_2\varphi_3| \leq \delta + \frac{\delta}{1-\delta}\cdot\delta = \delta/(1-\delta). \tag{6.20}$$

Hence $\partial_1\varphi_4 < 0$, uniformly, provided we fix $\delta < 1/2$. Moreover all $(\partial_1\varphi_j, \partial_2\varphi_j)$ are $\frac{\delta}{1-\delta}$-close to the unperturbed case. □

7 Paths and loops of diffeomorphisms

This section contains details of the construction of our final, non-naive Morse-Smale homotopy in our class of augmented discretized systems (2.18). For an outline and notation see (2.29)–(2.34). We first define the planar region Ω, outside of which the nonlinearities f and g produce identical incremental and cumulative shooting diffeomorphisms. We then review the explicit construction of the prescribed homotopies Ψ_i^τ, $i = 1, \ldots, n$. Facts concerning their relation to Ω are collected in lemma 7.1. In proposition 7.2 we state the property $\pi_1(\mathit{Diff}(\Omega)) = 0$, which is used for the construction of the interpolating cumulative shootings $\Psi(\tau, \xi)$, in lemma 7.3. We conclude with a final discretization of ξ, which realizes the homotopy of cumulative shootings in our augmented discretized system (2.18).

We construct the disc $\Omega \subset \mathrm{IR}^2$ as follows. Consider the original iterations (u_i, v_i) for equilibria of the discretized system (2.16) or, equivalently, for the unperturbed, augmented discretization (2.18); see (2.25). For $i = 0, \ldots, n-1$ let

(7.1) $\quad \Omega_i^f := \{(u_{-1}, v_{-1}) \in \mathrm{IR}^2 \mid |u_i| \leq 2C_1 \text{ and } n|u_i - v_i| \leq 2C_2\}.$

The constants C_1, C_2 were used for normalization of f, g, in (2.2), and the superscript f emphasizes f-dependence of the iteration. Denote

$$\Omega^f := \bigcup_{i=0}^{n-1} \Omega_i^f \tag{7.2}$$

and let Ω be any closed planar disc such that

$$\Omega \supseteq \Omega^f \cup \Omega^g. \tag{7.3}$$

Consider any initial condition $(u_{-1}, v_{-1}) \notin \Omega$. Then the f- and g-trajectories of (u_{-1}, v_{-1}) coincide, by normalization. In particular

$$
\begin{aligned}
\varphi_i^f &= \varphi_i^g \\
\psi_i^f &= \psi_i^g \\
\Psi_i^f &= \Psi_i^g \\
S_i^f &= S_i^g
\end{aligned}
\tag{7.4}
$$

holds outside Ω, and on $\partial\Omega$, for all $i = 0, \ldots, n$; see (2.24), (2.25), (2.29), (6.4) for notation. The equality of the shooting curves "outside Ω" is meant to hold for corresponding initial conditions $u_{-1} = v_{-1}$ outside Ω, of course.

For $i = 0, \ldots, n-1$ we will now fall back on our original, naive idea of a standard homotopy for φ_i, ψ_i. Only for the iteration from $i = n-1$ to $i = n$ we will make use of the additional flexibility procured by augmentation. So, for $i = 0, \ldots, n-1$ and $0 \leq \tau \leq 1$ define

$$
\begin{aligned}
\varphi_i^\tau &:= (1-\tau)\varphi_i^f + \tau\varphi_i^g \\
\psi_i^\tau &:= (1-\tau)\psi_i^f + \tau\psi_i^g
\end{aligned}
\tag{7.5}
$$

and let the cumulative shootings Ψ_i^τ be defined via composition of ψ_i^τ, as before. Similarly, we obtain shooting curves S_i^τ, ψ_i^τ, as before. By definition of Ω, equations (7.4) expand to

$$
\begin{aligned}
\varphi_i^f &= \varphi_i^\tau = \varphi_i^g \\
\psi_i^f &= \psi_i^\tau = \psi_i^g \\
\Psi_i^f &= \Psi_i^\tau = \Psi_i^g \\
S_i^f &= S_i^\tau = S_i^g
\end{aligned}
\tag{7.6}
$$

outside Ω.

For the final step, $i = n$, such a naive homotopy is of course prohibitive because it does not preserve hyperbolicity. Instead we *prescribe* some planar diffeotopy Φ^τ, in the spirit of (2.12), from $\Phi^0 = \Psi_n^f$ to $\Phi^1 = \Psi_n^g$, such that transversality of the shooting curves to the diagonal is preserved throughout the diffeotopy:

$$\Phi^\tau(\text{diag}) \pitchfork \text{diag} \tag{7.7}$$

for all $0 \le \tau \le 1$. Such a planar diffeotopy exists because $\pi_f = \pi_g$ for the shooting permutations of $S_n^f = \Phi^0(\text{diag})$ and of $S_n^g = \Phi^1(\text{diag})$. By normalization of f, g outside Ω, we may assume

$$\Psi_n^f = \Phi^\tau = \Psi_n^g \tag{7.8}$$

there. The crucial task of this section is, now, to realize the prescribed homotopy Φ^τ by appropriate perturbations of the $m-1$ incremental shootings ψ_j, $j = n-1+1/m, \ldots, n-1/m$, such that

$$S_n^\tau = \Psi_n^\tau(\text{diag}) = \Phi^\tau(\text{diag}) \tag{7.9}$$

holds for the cumulative shootings Ψ_n^τ, in analogy to (2.13). The construction of Ψ_n^τ is prepared in the following two lemmas.

Recall that $\mathit{Diff}_*(\Omega)$ denotes the space of all diffeomorphisms of the planar disc Ω which fix the boundary $\partial\Omega$ pointwise. The topology on $\mathit{Diff}_*(\Omega)$ is the C^1-topology. As before, π_1 denotes the fundamental group.

Lemma 7.1

$$\pi_1(\mathit{Diff}_*(\Omega)) = 0. \tag{7.10}$$

Proof: This is Smale's theorem, see [Cer68], appendix. □

We can now construct the two-parameter interpolation $\Psi(\tau, \xi)$ mentioned in (2.32), for $i = n$. We have slightly changed notation: for clarity, the *prescribed* path Ψ_n^τ of diffeomorphisms is distinguished as Φ^τ, here, whereas the notation Ψ_n^τ is strictly reserved for cumulative shootings which are *realized* by perturbations of augmented discrete systems in the sense of (2.18), (2.19).

Lemma 7.2 *In the above setting, there exists a continuous two-parameter family* $\Psi = \Psi(\tau, \xi)$, $0 \le \tau, \xi \le 1$, *of planar diffeomorphisms with the following properties (see Fig. 2.2 and (2.32). Let* $\psi_n^\tau := (1-\tau)\psi_n^f + \tau\psi_n^g$. *Then, for all* $0 \le \tau, \xi \le 1$,

$$\begin{aligned} \Psi(\tau, \xi = 0) &= \Psi_{n-1}^\tau \\ \Psi(\tau, \xi = 1) &= (\psi_n^\tau)^{-1}\Phi^\tau \\ \Psi(\tau = 0, \xi) &= \Psi_{n-1}^{\tau=0} \\ \Psi(\tau = 1, \xi) &= \Psi_{n-1}^{\tau=1} \end{aligned} \tag{7.11}$$

Specifically outside Ω, *we have*

$$\Psi(\tau, \xi) = \Psi_{n-1}^\tau. \tag{7.12}$$

Proof: Outside Ω, the definition (7.12) of $\Psi(\tau, \xi)$ fulfills all the requirements of (7.11). Indeed, the second equality follows from (7.8) and the definitions of ψ_n, Ψ_n. The remaining equalities hold trivially.

We now construct $\Psi(\tau, \xi)$ inside Ω. Let $\tilde{\Psi}(\tau, \xi)$ be a contraction, in $\mathit{Diff}_*(\Omega)$, of the loop

$$\tau \mapsto \tilde{\Psi}(\tau, \xi = 1) := (\psi_n^\tau \circ \Psi_{n-1}^\tau)^{-1}\Phi^\tau, \tag{7.13}$$

$0 \le \tau \le 1$. In fact, $\tilde{\Psi}(\tau, 1) = id$ for $\tau = 0, 1$, and, outside Ω, for $0 \le \tau \le 1$; see (2.29), (7.5), (7.6), (7.8). Therefore $\tilde{\Psi}(\tau, \xi) \in \mathit{Diff}_*(\Omega)$ indeed exists, continuously in (τ, ξ), by triviality of the fundamental group as observed in lemma 7.1. Note that

$$\tilde{\Psi}(0, \xi) = \tilde{\Psi}(1, \xi) = \tilde{\Psi}(\tau, 0) = id, \tag{7.14}$$

throughout $0 \le \tau, \xi \le 1$. With the help of $\tilde{\Psi}$ we can now easily construct Ψ as

$$\Psi(\tau, \xi) := \Psi_{n-1}^\tau \circ \tilde{\Psi}(\tau, \xi). \tag{7.15}$$

Then the second equation of (7.11) follows from (7.15) and (7.13). The remaining three equations of (7.11) follow from (7.15) and (7.14). This completes the construction inside Ω.

A remaining issue is the C^1-matching of the diffeomorphisms $\Psi(\tau,\xi)$ along $\partial\Omega$. Pulling this question back to $\tilde{\Psi}(\tau,\xi)$, via (7.15), it is sufficient to provide diffeomorphisms which are identity in a small neighborhood of $\partial\Omega$ in Ω. This is easy to achieve, by first performing the above construction in a slightly smaller disc $\Omega' \subset \Omega$, and subsequently smoothing out the transition in a small annulus around $\partial\Omega'$ in Ω. This completes the proof of lemma 7.2.

In the remaining section we conclude our proof of theorem 1.1, specifying the intermediate incremental discretization steps ψ_j with noninteger $0 < j < n$, and summarizing the Morse-Smale homotopy from f to g.

For noninteger $0 < j < n-1$, we do not perturb ψ_j away from rotation by 90 degrees. In fact, we could do without any augmentations between $i = 0$ and $i = n-1$. This follows because the homotopies ψ_i^τ, $i = 0, \cdots, n-1$, are all realized within the class of maps $\varphi_i^\tau(u,v)$ with uniformly negative partial derivatives $\partial_u \varphi_i^\tau$. Standard homotopy $(1-\tau)\varphi_i^f + \tau\varphi_i^g$ does the job. We have nevertheless inserted all these artificial discretization points, just for notational convenience.

It is for noninteger $n-1 < j < n$, where augmentation seems really indispensable. Following (2.34), we choose $m \equiv 1 \pmod 4$ large enough such that each of the $(m-1)/4$ diffeomorphisms

$$\Psi_{n,k}^\tau := \Psi(\tau,\xi_{k+1}) \circ (\Psi(\tau,\xi_k))^{-1} \tag{7.16}$$

is δ-close to identity in C^1-norm. Here $\xi_k = 4k/(m-1)$, $k = 0, \ldots, (\text{m-5})/4$, $0 \le \tau \le 1$, $\delta < 1$, and $\Psi(\tau,\xi)$ is choosen as in lemma 7.2. By lemma 6.2, each of the diffeomorphisms can be realized as a shooting quadruple

$$\Psi^{\tau}_{n,k} = \psi^{\tau}_{n-1+(4k+4)/m} \circ \cdots \circ \psi^{\tau}_{n-1+(4k+1)/m} \tag{7.17}$$

with incremental shooting maps ψ_j, φ_j respecting the partial derivative condition of Jacobi systems. Only for $\tau = 0, 1$, the incremental shootings are rotations by 90 degrees; for in between τ they are perturbed. With this choice of homotopy, the cumulative shooting Ψ^{τ}_n realizes the prescribed diffeotopy φ^{τ} of discrete shooting curves,

$$\Psi^{\tau}_n = \Phi^{\tau} \tag{7.18}$$

for $0 \le \tau \le 1$, as required essentially in (2.13). Indeed, (7.11), (7.16), and (7.17) imply

$$\begin{aligned} \Psi^{\tau}_n &= \psi^{\tau}_n \circ \psi^{\tau}_{n-1+(m-1)/m} \circ \cdots \circ \psi^{\tau}_{n-1+1/m} \circ \Psi^{\tau}_{n-1} \\ &= \psi^{\tau}_n \circ \Psi^{\tau}_{n,(m-5)/4} \circ \cdots \circ \Psi^{\tau}_{n,0} \circ \Psi^{\tau}_{n-1} \\ &= \psi^{\tau}_n \circ \Psi(\tau, \xi_{(m-1)/4}) \circ \Psi(\tau, 0)^{-1} \circ \Psi^{\tau}_{n-1} \\ &= \psi^{\tau}_n \circ \Psi(\tau, 1) \\ &= \Phi^{\tau} \end{aligned} \tag{7.19}$$

which proves (7.18). This completes our construction of a hyperbolicity preserving homotopy of shooting curves from S^f_n to S^g_n, in an augmented discrete context.

However, there still remains a last homotopy step. By (7.19), we have constructed a shooting homotopy, for all *solutions of iterations*

$$0 = f_j(u_{j-1/m}, u_j, u_{j+1/m}), \tag{7.20}$$

with given $u_0, u_{-1/m}$. We have not yet constructed a Morse-Smale homotopy for the augmented discrete systems

$$\dot{u}_j = f_j(u_{j-1/m}, u_j, u_{j+1/m}), \tag{7.21}$$

under Neumann boundary conditions.

To complete this final step, consider nonlinearities $f_j, \tilde{f}_j$ which are both dissipative, as in (4.12), and in addition induce the same iteration via (7.20), that is,

$$\varphi_j = \tilde{\varphi}_j \tag{7.22}$$

for $\tilde{\varphi}_j := \varphi_j^{\tilde{f}}$ and all j. Here we think of

$$\tilde{f}_j(u_{j-1/m}, u_j, u_{j+1/m}) := u_{j+1/m} - \tilde{\varphi}_j(u_{j-1/m}, u_j). \tag{7.23}$$

Note that sign $\tilde{f}_j$ = sign f_j, and hence $\tilde{f}_j$ satisfies the same dissipation condition (4.12) as does f_j itself. We show that

$$\mathcal{A}_f \cong \mathcal{A}_{\tilde{f}} \tag{7.24}$$

holds for the corresponding maximal compact invariant sets (due to hyperbolic suspension, they are not attractors any more). Indeed, a standard homotopy $(1-\tau)f_j + \tau\tilde{f}_j$ works: none of the iterations $\varphi_j, \psi_j, \Psi_j$, or finally the shooting curves S_n^f changes at all during this homotopy. In particular, the homotopy is automatically Morse-Smale. This implies (7.24). Using the same argument on $g, \tilde{g}$, we summarize

$$\mathcal{A}_f \cong \mathcal{A}_{\tilde{f}} \cong \mathcal{A}_{\tilde{g}} \cong \mathcal{A}_g. \tag{7.25}$$

The first and last orbit equivalence are stated in (7.24). The central orbit equivalence follows from (7.18), (7.19), because via construction (7.23) our hyperbolicity preserving homotopy φ_j^τ induces a Morse-Smale homotopy $\tilde{f}_j^\tau$ from $\tilde{f}$ to $\tilde{g}$. This proves (7.25) and theorem 1.1. □

8 Discussion

Equipped with the proof of our main result, theorem 1.1, we collect some criticism, generalizations, and open questions. We begin with a discussion of dissipation conditions, narrowing our view to $f = f(x, u)$ and also widening it to more general semilinear and quasilinear PDEs. Keeping the space variable x one-dimensional, we address other separated as well as periodic boundary conditions. The latter case is basically as open as the case of monotone feedback delay equations. For dissipative Jacobi systems, not necessarily arising by space semidiscretization, we announce a result on global orbit equivalence of global attractors in analogy to theorem 1.1, see theorem 8.1. It turns out that, from our global attractor point of view, the class of all dissipative Jacobi systems is not richer than the class of all dissipative PDEs (1.1). The question of the mininal dimension $n+1$ of a Jacobi system which realizes a given global PDE attractor of dimension d remains open, however. We conclude with a discussion of the relation between connection equivalence and orbit equivalence of global attractors, and with the problem of a geometric, simplicial description of global attractors.

We recall our dissipation conditions (1.7), (1.8) which state that

$$\begin{aligned} f(x,u,0)\cdot u &< 0 \\ |f(x,u,p)| &< a(u) + b(u)|p|^{\gamma} \end{aligned} \tag{8.1}$$

for large $|u| + |p|$. These conditions are by no means necessary. For example, the second condition can be replaced with the crude unilateral constraint $(f_x + f_u p)p < 0$. We have used (8.1) in lemma 5.2 in order to normalize

$$f(x,u,p) = -u \tag{8.2}$$

for large $|u| + |p|$. Actually, only convexity of (8.1) with respect to f was used. More generally, any cut-off which replaces a general, dissipative f by an f satisfying (8.2), while preserving the global attractor, would enable us to assume (8.2) without loss of generality. Such a cut-off replacement can be achieved in the class of abstract nonlinearities $f : X^\alpha \to X$, where X^α denotes a fractional power space related to diffusion. See for example [MPS88]. But our present proof requires a cut-off $f = f(x, u, p)$ which preserves, for example, the shooting description of equilibria and still does not enlarge the global attractor $\mathcal{A}_f$, the equilibrium set $\mathcal{E}_f$, or the shooting permutation π_f.

Generalizations to space dependent diffusion terms

$$u_t = c(x)u_{xx} + f(x, u, u_x) \tag{8.3}$$

with uniformly positive $c(x)$ can be achieved easily, for example, by a reparametrization of x. The fully nonlinear, but still uniformly parabolic and dissipative case

$$u_t = f(x, u, u_x, u_{xx}) \tag{8.4}$$

is not sufficiently investigated on a technical level, at present, to apply our ideas directly. We still expect global attractors to be characterized by shooting permutations, up to orbit equivalence. If this is the case, we argue below that these "more nonlinear" equations will not generate additional Morse-Smale attractor types beyond the semilinear case.

The maximal compact invariant set $\mathcal{A}_f$ can also be associated to non-dissipative nonlinearities f. This observation applies to equations with finite time blow up and, *mutatis mutandis*, to degenerate diffusion problems of porous media type. For a recent survey see []. In principle, our approach via shooting permutations should be applicable to the structure of $\mathcal{A}_f$ and might, in conjunction with nodal properties, even lead to qualitative information on some blow up trajectories. To our knowledge, this direction has not been followed in the existing literature.

Natural separated boundary conditions of mixed type can be written as

$$(8.5) \qquad \begin{aligned} (1-\tau)u - \tau u_x &= 0 \quad \text{at} \quad x=0 \\ (1-\tau)u + \tau u_x &= 0 \quad \text{at} \quad x=1 \end{aligned}$$

with $0 \leq \tau \leq 1$. We have treated Neumann boundary conditions, $\tau = 1$, so far. The Dirichlet case, $\tau = 0$, as well as the remaining mixed cases $0 < \tau < 1$ can be treated along the same lines. We just redefine the (oriented) shooting curves S_f^x as $x = 1$ images of the boundary condition (8.5) at $x = 0$, under x-shooting. Similarly, we adapt the shooting permutation π_f to encode the discrepancy of orderings of equilibria along the (oriented) boundary condition line (8.5), at $x = 1$, and along the shooting curve $S_f^{x=1}$, respectively. Imposing suitable dissipation conditions for f, theorem 1.1 remains valid:

$$(8.6) \qquad \pi_f = \pi_g \quad \Rightarrow \quad \mathcal{A}_f \cong \mathcal{A}_g.$$

Perhaps more surprisingly, all attractors already arise in the Neumann case treated here, up to orbit equivalence. New structurally stable attractor types to not arise, for any $0 \leq \tau < 1$. For more details see [Fie96]. Adaptations to mildly nonlinear boundary conditions seem feasible.

The case of periodic boundary conditions, $x \in S^1$, is inherently more difficult. Although a Morse-Smale structure persists, the variational structure is lost. In fact, nontrivially time periodic solutions appear, for $f = f(u,p)$ [AF88]. Even in the gradient case $f = f(x,u)$, where the Lyapunov functional (1.17), (1.18) works, we do not possess an analogue of the shooting permutation π_f. Of course, the braid type of the equilibrium profiles in (x,u,p)-space comes to mind.

Theorem 1.1 was proved via discretization

$$(8.7) \qquad \dot{u}_i = f_i(u_{i-1}, u_i, u_{i+1}),$$

$i = 0, \ldots, n$, subsequent augmentation, and a Morse-Smale homotopy. Skipping just the initial discretization step, we immediately obtain the analogous result for Jacobi systems (8.7) provided f_i is also dissipative, as stated in (4.12). We state this result next. Again $\mathcal{A}_f$ denotes the global attractor of (8.7) for $f = (f_0, \ldots, f_n)$ and under Neumann boundary conditions $u_{-1} = u_0, \ u_{n+1} = u_n$. The permutation π_f was defined in (6.2), (6.2) for hyperbolic sets $\mathcal{E}_f$ of equilibria.

Theorem 8.1 *Consider Jacobi systems (8.7). Let $f, g \in C^1$ satisfy dissipation condition (4.12). Assume $\mathcal{E}_f, \mathcal{E}_g$ are hyperbolic. Then $\pi_f = \pi_g$ implies $\mathcal{A}_f \cong \mathcal{A}_g$.*

Proof: See sections 2, 4–7. □

Let $\mathcal{A}^{dis}$ denote the set of all global attractors of *spatially discrete* dissipative Jacobi systems (8.7) as in theorem 8.1, up to orbit equivalence. We do not require (8.7) to arise by discretization. Similarly, let $\mathcal{A}^{con}$ denote the same set for *spatially continuous* PDEs (1.1) as in theorem 1.1. By discretization, section 3, we know $\mathcal{A}^{con} \subseteq \mathcal{A}^{dis}$.

Theorem 8.2 *In the above setting,*

$$\mathcal{A}^{con} = \mathcal{A}^{dis}. \tag{8.8}$$

Proof: We show how (8.8) follows from further results in three related papers.

It remains to show $\mathcal{A}^{con} \supseteq \mathcal{A}^{dis}$: the class (8.7) of spatially discrete Jacobi systems is not richer than the class (3.2), (3.3) of spatially discretized systems. Consider any permutation π_f, for (8.7). From the permutation π_f, the Morse indices $i(u^k) \geq 0$ of equilibria $u^k \in \mathcal{E}_f$, $k = 1, \ldots, N$, can be determined explicitly:

$$i(u^k) = \sum_{l=1}^{k-1} (-1)^{l+1} \mathrm{sign}(\pi_f^{-1}(l+1) - \pi_f^{-1}(l)). \tag{8.9}$$

Here $i(u^1) = i(u^N) = 0$, by dissipation. See [FR96a] for a proof. Perhaps not surprisingly, in view of discretization, the same formula is valid in the spatially continuous case, see [FR94]. Moreover it turns out that any such permutation π_f also *arises* as

$$\pi_{\tilde{f}} = \pi_f \tag{8.10}$$

for a suitable dissipative nonlinearity $\tilde{f}$, in the spatially continuous case, see [FR96b]. As in section 3, let $\mathcal{A}^n_{\tilde{f}}$ denote a suitable discretization of $\mathcal{A}_{\tilde{f}}$ such that

$$\mathcal{A}_{\tilde{f}} \cong \mathcal{A}^n_{\tilde{f}}. \tag{8.11}$$

For n large enough we also conclude $\pi_{\tilde{f}} = \pi^n_{\tilde{f}}$; see lemma 6.1. Combined with (8.10) this implies $\pi^n_{\tilde{f}} = \pi_{\tilde{f}} = \pi_f$. By theorem 8.1 we obtain

$$\mathcal{A}^n_{\tilde{f}} \cong \mathcal{A}_f, \tag{8.12}$$

for these spatially discrete attractors. Together, (8.11) and (8.12) prove $\mathcal{A}^{con} \supseteq \mathcal{A}^{dis}$, (8.8), and the theorem. □

By theorem 8.2, or section 3, any global attractor $\mathcal{A}_{\tilde{f}}$ in $\mathcal{A}^{con}$ can be realized as some attractor $\mathcal{A}_f$ in $\mathcal{A}^{dis}$ – for example by discretization. This raises the following open question: What is the minimal dimension $n+1$, for given $\mathcal{A}_{\tilde{f}}$, such that

$$\mathcal{A}_f \cong \mathcal{A}_{\tilde{f}} \tag{8.13}$$

for some dissipative Jacobi system $f = (f_0, f_1, \ldots, f_n)$ of form (8.7)? We recall from (3.9) that the canonical projection

$$P: \quad \mathcal{A}_{\tilde{f}} \to \mathrm{IR}^{n+1}, \tag{8.14}$$

$(Pu)_i := u(x_i),\ \ 0 \le x_1 < \ldots < x_n \le 1$, is injective and differentiable, for

$$n + 1 = \dim\ \mathcal{A}_{\tilde f}. \tag{8.15}$$

However, the flow induced by P on $P\mathcal{A}_{\tilde f}$ need not be a tri-diagonal Jacobi system. Still the optimal bound (8.15) might be the answer to our question. In view of our proof of theorem 8.2, the question reduces in principle to an investigation of realization of shooting permutations $\pi_{\tilde f}$ by discrete cumulative shootings.

In [FR94], we have introduced the weaker notion of connection equivalence. Here global attractors $\mathcal{A}_f, \mathcal{A}_g$ are called *connection equivalent*, $\mathcal{A}_f \sim \mathcal{A}_g$, if there exists a bijection $h : \mathcal{E}_f \to \mathcal{E}_g$ between the equilibria which preserves Morse index and the existence of heteroclinic connections. More precisely, there exists a heteroclinic connection from u to $\tilde u$ in $\mathcal{A}_f$ if, and only if, there exists a connection from $h(u)$ to $h(\tilde u)$ in $\mathcal{A}_g$. By definition,

$$\mathcal{A}_f \cong \mathcal{A}_g \quad \Rightarrow \quad \mathcal{A}_f \sim \mathcal{A}_g. \tag{8.16}$$

In [FR94], corollary 6.1 we have noted that $\pi_f = \pi_g$ implies $\mathcal{A}_f \sim \mathcal{A}_g$. In [FR94], (6.11) we have conjectured that $\pi_f = \pi_g$ implies $\mathcal{A}_f \cong \mathcal{A}_g$; this is now proved in theorems 1.1 and 8.1. We ask, this time, whether the converse of (8.16) also holds:

$$\mathcal{A}_f \sim \mathcal{A}_g \quad \Rightarrow \quad \mathcal{A}_f \cong \mathcal{A}_g. \tag{8.17}$$

This is nontrivial because, in [Fie94], [FR94] we have found shooting permutations π_f, π_g with surprising connection equivalence $\mathcal{A}_f \sim \mathcal{A}_g$, in spite of π_f, π_g not being even conjugate. The simplest examples require $N = 9$ equilibria.

Although we have now examined orbit equivalence of global attractors in much detail, we are still lacking a good geometric description. Given π_f, we would like to be able to draw a "picture" of $\mathcal{A}_f$, on a reasonably descriptive level of abstraction. One approach would attempt to obtain a simplicial complex from the decomposition of $\mathcal{A}_f$ into equilibria and their manifolds of (transverse) heteroclinic connections. The combinatorics of such complexes, tied in with the shooting permutations π_f, might also hold the key to the orbit equivalence question (8.17). At the same time, it might shed some light on the surprising connection equivalences mentioned above. Providing good geometric attractor models which tie in with their dynamics remains a challenge!

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