# A Lyapunov function for tridiagonal competitive-cooperative systems

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**Abstract**

We construct a Lyapunov function for a tridiagonal competitive-cooperative systems. The same function is a Lyapunov function for Kolmogorov tridiagonal systems, which are defined on a closed positive orthant in $\mathbf{R}^n$. We show that all bounded orbits converge to the set of equilibria. Moreover, we show that there can be no heteroclinic cycles on the boundary of the first orthant, extending the result of H.I Freedman and H. Smith [F-S].

**Key words.** Kolmogorov tridiagonal systems, Lyapunov function, heteroclinic cycles.

**AMS subject classifications.** 34C37, 58F25

## 1 Results

We consider a system of differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_i &= f_i(x_{i-1}, x_i, x_{i+1}), \qquad i = 2, \dots, n-1 \\ \dot{x}_n &= f_n(x_{n-1}, x_n) \end{aligned} \tag{1}$$

where functions $f_i$ are defined on an non-empty open subset $A$ of $R^n$. We assume that the $f_i$ and their partial derivatives are continuous on $A$. We also assume that there are $\delta_i \in \{-1, +1\}$, such that

$$\delta_i \frac{\partial f_i}{\partial x_{i+1}} > 0, \quad \delta_i \frac{\partial f_{i+1}}{\partial x_i} > 0, \quad 1 \le i \le n-1.$$

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This assumption implies that the Jacobi matrix $\partial f/\partial x$, corresponding to (1), is tridiagonal and sign symmetric in the sense that $\partial f_i/\partial x_{i+1}$ and $\partial f_{i+1}/\partial x_i$ have the same sign $\delta_i$. If $\delta_i = -1$, for all $i$, then (1) is called *competitive.* If $\delta_i = 1$, for all $i$, then (1) is called *cooperative.* We introduce new variables, following Smith [S1]. We let $\bar{x}_i = \mu_i x_i$, $\mu_i \in \{\pm 1\}$, $1 \le i \le n$, with $\mu_1 = 1$, $\mu_i = \delta_{i-1}\mu_{i-1}$. Then the system (1) transforms into a new system of the same type with new

$$\bar{\delta}_i = \mu_i \mu_{i+1} \delta_i = \mu_i^2 \delta_i^2 = 1.$$

Therefore we can always assume, without loss of generality, that the competitive-cooperative system (1) is in fact cooperative and

$$\frac{\partial f_i}{\partial x_{i+1}} > 0, \quad \frac{\partial f_{i+1}}{\partial x_i} > 0, \quad 1 \le i \le n-1. \tag{\textbf{H1}}$$

Our goal in this paper is to construct a Lyapunov function for a class of equations (1). A Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$ for (1) is a real valued function which is non-increasing along trajectories of (1) and is strictly decreasing along all non-equilibrium trajectories. We assume that the system (2) is *dissipative.* We spell out the precise form of the assumption below. We remark that this assumption implies that all trajectories of (2) eventually enter a compact region of phase space. By LaSalle's invariance principle (see Hale and Koçak [HK]) the existence of a Lyapunov function then implies that each trajectory converges to the set of equilibria. Our dissipativeness condition takes the form

$$f_i(x_{i-1}, x_i, x_{i+1})x_i < 0, \quad \text{for} \quad |x_i| \ge C, |x_{i\pm 1}| \le |x_i| \tag{\textbf{H2}}$$

and some large constant $C$.

It is easy to see that this assumption forces any trajectory to enter the box $\{x \in \mathbf{R}^n \mid |x_i| \le C \text{ for all } i\}$, at some finite time, and then remain there.

Our final assumption on the functions $f_i$ is more technical. Observe that assumption (H1) implies that for any fixed $x_i$ the set $z(f_i, x_i)$ of points $(x_{i-1}, x_{i+1})$ satisfying $f_i(x_{i-1}, x_i, x_{i+1}) = 0$ is a curve in the $(x_{i-1}, x_{i+1})$-plane. Furthermore $z(f_i, x_i)$ is monotone with respect to both $x_{i-1}$ and $x_{i+1}$. The following assumption implies that for all $x_i$ the curve $z(f_i, x_i)$ is unbounded in both $x_{i-1}$ and $x_{i+1}$ directions. Assume

$$\lim_{x_k \to \infty} f_i(x_{i-1}, x_i, x_{i+1}) > 0, \quad \lim_{x_k \to -\infty} f_i(x_{i-1}, x_i, x_{i+1}) < 0 \tag{\textbf{H3}}$$

for both $k = i-1$ and $k = i+1$ and all $x_i, x_{i-1}$ and $x_{i+1}$.

We state the main Theorem of this paper.

**Theorem 1.1** *The system (1) with assumptions* **(H1), (H2), (H3)** *admits a Lyapunov function.*

The system (1) with assumption (H1) is a *monotone dynamical system.* There is an extensive literature on monotone dynamical systems, starting with the work of Hirsch [Hi1]-[Hi4] for monotone semiflows. The results of Hirsch and later improvements by Matano [M], Smith and Thieme [ST1, ST2], and Polàčik [P] established that most orbits of a strongly order-preserving semiflow converge to the set of equilibria. For references on the theory of monotone semiflows see the recent monograph by Smith [S2].

For the system (1) more is known: Smilie [Sm] has shown that all trajectories converge to the set of equilibria. He used an integer-valued Lyapunov function (nodal properties) to prove his result. As we can see, the main consequence of the existence of real valued Lyapunov function $V$ for the system (1), namely that all trajectories of (1) converge to the set of equilibria, is not new and was proved by Smilie [Sm].

The importance of the existence of the Lyapunov function $V$ comes from the fact that it can be used in a more general setting. We now consider the class of *Kolmogorov* systems, which model an interaction of populations, where every population interact only with "neighboring" populations. They have the form

$$\begin{aligned} \dot{x}_1 &= x_1 f_1(x_1, x_2) \\ \dot{x}_i &= x_i f_i(x_{i-1}, x_i, x_{i+1}) \\ \dot{x}_n &= x_n f_n(x_{n-1}, x_n) \end{aligned} \tag{2}$$

for $i = 2, \dots, n-1$. We assume that the functions $f_i$ satisfy (H1), (H2), and (H3). Since $x_i$'s represent population densities we restrict ourselves to the closed positive orthant

$$\mathcal{O} := \{x \in \mathbf{R}^n \mid x_i \geq 0 \text{ for } i = 1, \dots, n\}.$$

For specific biological systems modeled by (2), see Freedman and Smith [F-S] and the references herein.

The results of Smilie do not apply to the system (2) directly. The crucial assumption (H1) for the right hand side of (2) holds only in the interior of the positive orthant. We also notice that $x_j(0) = 0$ implies $x_j(t) = 0$, for all $t$. Therefore every boundary hyperplane of the first orthant is invariant under (2). Furthermore, if we restrict the set of equations to such a boundary hyperplane, we obtain a decoupled system of the same type as (2). Consequently, all faces and subfaces of the boundary of the positive orthant are invariant under (2) and the system (2), when restricted to such a set, is of the same type as the system (2) itself. The assumption (H1) is satisfied only in the interior of the positive orthant and, by the previous argument in the interior of every face and subface of the boundary of the positive orthant.

The question arises whether all trajectories still converge to the set of equilibria in this case. The result is obviously true, if the trajectory stays bounded away from the boundary of the region where is starts, which may be the positive orthant $\mathcal{O}$ itself, or a face, or a subface of the boundary. But in the context of population dynamics, the trajectories approaching the boundary of a given region are important, since they represent a situation when a certain population goes extinct.

This problem was studied by H.I. Freedman and H. Smith [F-S]. Under some nondegeneracy assumptions they were able to show that every bounded orbit converges either to an equilibrium, or to a cycle of equilibria on the boundary of $\mathcal{O}$. A *cycle of equilibria* is a nonempty finite set of equilibria $\{E_1, \dots, E_n\}$ such that

$$E_1 \to E_2 \to \dots \to E_n \to E_1.$$

Here we write $E_1 \to E_2$ if $E_1$ and $E_2$ are equilibria of (2), not necessarily distinct, such that there exists a solution $x \in \mathcal{O}$ with $\lim_{t \to -\infty} x(t) = E_1$ and $\lim_{t \to \infty} x(t) = E_2$. For illustration of such a boundary cycle of equilibria see Figure 1.

Figure 1: Heteroclinic cycle on the boundary of the positive orthant.

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In every boundary component the flow is convergent to the set of equilibria, but the boundary components are assembled together in such a way that they produce a cycle of equilibria.

The Lyapunov function $V$ constructed in Theorem 1.1 turns out to be a Lyapunov function for system (2). Furthermore, $V$ is defined, continuous on the boundary of $\mathcal{O}$, and restricts to the Lyapunov function on every face and subface of the boundary of $\mathcal{O}$.

**Theorem 1.2** *Consider system (2) on the closed positive orthant $\mathcal{O}$. Assume that the functions $f_i$ satisfy the assumptions (H1), (H2) and (H3). Then there is a Lyapunov function $V : \mathcal{O} \to \mathbf{R}$, which is strictly decreasing outside the set of equilibria.*

We can now rule out the existence of the cycles of equilibria on the boundary of $\mathcal{O}$.

**Corollary 1.3** *Consider system (2) on the closed positive orthant $\mathcal{O}$. Assume that the functions $f_i$ satisfy the assumptions (H1), (H2), and (H3). Then every bounded trajectory with initial data in the closed positive orthant $\mathcal{O}$ converges to the set of equilibria. If the set of equilibria is finite, every trajectory converges to a single equilibrium. Moreover, cycles of equilibria in $\mathcal{O}$ do not exist.*

The cycles of equilibria do not exist in the large class of Lotka-Voltera systems. We now briefly describe result of Fiedler and Gedeon [FG].

Consider the system

$$\dot{x}_i = x_i(c_i - \sum_{j=1}^{n} \beta_{ij} f_j(x_j)) \tag{3}$$

on the positive orthant $\mathcal{O}$ in $\mathbf{R}^n$. If $x_i$ describes the population size of a certain species, then the constants $\beta_{ij}$ describe the interaction between the species. Let $\Upsilon$ be the undirected graph with $n$ vertices, where the edge $j$ is connected to the vertex $i$ by the edge $e_{ij}$ if, and only if, $\beta_{ij} \neq 0$. We assume $\beta_{ij}\beta_{ji} > 0$ for every edge $e_{ij}$. Therefore the definition of $\Upsilon$ makes sense.

Consider the system (3) and assume that $\frac{df_j}{dx_j} > 0$ for all $j$, that the interaction graph $\Upsilon$ is a tree and that $\beta_{ij}\beta_{ji} > 0$ for every edge $e_{ij}$. Then, by the result of Fiedler and Gedeon [FG], every bounded trajectory of system (3) converges to the set of equilibria and boundary cycles of equilibria do not exist.

# 2 Proofs

**Proof of Theorem 1.1.** Motivated by a somewhat analogous result of Matano [M] concerning parabolic partial differential equations with gradient dependence, we seek a Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$ of the form $V = -\sum_{i=1}^{n-1} g_i(x_i, x_{i+1})$. By differentiating and collecting terms we get

$$\dot{V} = -\sum_{i=1}^{n} \dot{x}_i(\partial_i g_i + \partial_i g_{i-1})$$

where we used the notation $\partial_i := \frac{\partial}{\partial x_i}$. We also note that $\partial_1 g_0 = 0$ and $\partial_n g_n = 0$ in the expression for $\dot{V}$. Below, we will construct functions $a_i$ such that

**(a)** $a_i f_i = \partial_i g_i + \partial_i g_{i-1}$, and $a_i > 0$, for $i = 1, \ldots, n$.

**(b)** $\partial_{i-1}(a_i f_i) = \partial_i(a_{i-1} f_{i-1})$, for $i = 2, \ldots, n$.

The property (a) implies that

$$\begin{aligned} \dot{V} &= -\sum_i \dot{x}_i(\partial_i g_i + \partial_i g_{i-1}) \\ &= -\sum_i \dot{x}_i a_i f_i \\ &= -\sum_i a_i (\dot{x}_i)^2 \leq 0, \end{aligned}$$

since $a_i > 0$, for all $i$. Also $\dot{V}(x(t)) = 0$ if, and only if, $\dot{x}_i(t) = 0$ for all $i$. This implies that $\dot{V}(x(t)) = 0$ if and only if $x(t)$ is an equilibrium. Therefore $V$ is a Lyapunov function.

We see that we have to construct functions $a_i$, $i = 1, \ldots, n$, which satisfy property (a).

Condition (b) is a consequence of (a) and the form of the function $V$. Indeed, (a) implies that

$$\partial_i(a_{i-1} f_{i-1}) = \partial_i \partial_{i-1} g_{i-1} + \partial_i \partial_{i-1} g_{i-2} = \partial_i \partial_{i-1} g_{i-1}$$

since $g_{i-2}(x_{i-2}, x_{i-1})$ does not depend on $x_i$. A similar computation leads to

$$\partial_{i-1}(a_i f_i) = \partial_i \partial_{i-1} g_{i-1}.$$

Therefore, if (a) is satisfied and $g_i$ is a function of $x_i$ and $x_{i+1}$ only then condition (b) must hold. In the inductive argument below we shall first construct functions $a_i > 0$ which satisfy (b), and then use (b) to construct functions $g_i$ with property (a).

We construct the functions $a_i$, $i = 1, \ldots n$, by induction. The first step of the induction will be to construct functions $a_1, g_0$ and $g_1$ such that (a) and (b) are satisfied. We set $g_0 \equiv 0$ and $a_1 \equiv 1$. To determine $g_1(x_1, x_2)$ we set

$$\partial_1 g_1 := a_1 f_1 = f_1(x_1, x_2),$$

or, more explicitly,

$$g_1(x_1, x_2) = \int_0^{x_1} f_1(\zeta_1, x_2) d\zeta_1.$$

This choice of $a_1, g_0$ and $g_1$ satisfies conditions (a) above. Condition $(b)$ is vacuous for $i = 1$.

Having defined $a_1$, the condition (b) with $i = 2$ poses a restriction on $a_2$. The function $a_2$ must satisfy

$$\partial_1(a_2 f_2) = \partial_2(a_1 f_1).$$

Similarly, once the function $a_{i-1}$ has been defined, the function $a_i$ to be constructed in the next step of the induction must satisfy

$$\partial_{i-1}(a_i f_i) = \partial_i(a_{i-1} f_{i-1}).$$

With this in mind we define an auxiliary function

$$\gamma_1(x_1, x_2) := \partial_2(a_1 f_1).$$

Observe that $\partial_2(a_1 f_1) = \partial_2 f_1 > 0$, by assumption on function $f_1$, and so $\gamma_1 = \gamma_1(x_1, x_2) > 0$. So far, we have defined functions $a_1, g_0, g_1$ which satisfy (a), (b) for $i = 1$, and $\gamma_1$.

Now we proceed with the induction step. For technical reasons our induction hypothesis will not be statements (a) and (b) above but a slightly more complicated set of assumptions. We assume that we have constructed functions $a_k$, $g_k$ and $\gamma_k$, where $\gamma_k = \partial_{k+1}(a_k f_k)$, for $k = 1, \ldots, i-1$ with the following properties:

**(A1)** $a_k > 0$ are continuous; $\partial_{k-1}(a_k f_k)$ and $\partial_{k+1}(a_k f_k)$ exist and are continuous.

**(A2)** $\partial_{k-1}(a_k f_k) = \partial_k(a_{k-1} f_{k-1})$

**(B)** $a_k f_k = \partial_k g_k + \partial_k g_{k-1}$, and $g_k = g_k(x_k, x_{k+1})$

**(C)** $\gamma_{k-1}(x_{k-1}, x_{k-2}) > 0$.

Observe, that these conditions are satisfied for $k = 1$ by the above construction. Observe that (A2) is equivalent to (b), and (A1), (B) are equivalent to (a). Condition (C) is needed in the induction process.

Before we proceed with the induction step we introduce some notation. We denote $z(f_i) := \{x \mid f_i(x) = 0\}$ the zero set of the function $f_i$. In $\mathbf{R}^3$, spanned by coordinate axis $x_{i-1}, x_i, x_{i+1}$, the zero set $z(f_i)$ is a graph over the $(x_{i-1}, x_i)$ plane. Indeed, the equation $f_i(x_{i-1}, x_i, x_{i+1}) = 0$ can be solved for

$$x_{i+1} = y_{i+1}(x_{i-1}, x_i),$$

since $\partial_{i+1} f_i > 0$ Similarly, since $\partial_{i-1} f_i > 0$ there is a function $\eta_{i-1}$ with $x_{i-1} = \eta_{i-1}(x_i, x_{i+1})$ solving

$$f_i(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i-1}) = 0.$$

The assumption (H3) implies that both functions $\eta_{i-1}$ and $y_{i+1}$ are defined on the whole real line $\mathbf{R}$. We shall need this fact below in the construction of functions $a_i$. It is easy to see that $\partial_{i-1} y_{i+1}(x_i, x_{i-1}) < 0$ and

$$\partial_{i+1}\eta_{i-1}(x_i, x_{i+1}) < 0. \tag{4}$$

We now proceed with the induction step. We construct functions $a_i, g_i$ and $\gamma_i$ satisfying properties (A1-A2), (B) and (C) for $k := i$. The construction will achieved in three steps. In the first step we will define $a_i$ and verify properties (A1-A2). In the second step we will define $g_i$ and show that (B) holds. The last step will be to check that $\gamma_i = \partial_{i+1}(a_i f_i)$ satisfies (C).

**Step 1. Construction of $a_i$.**

The function $a_i$ must satisfy condition (A2) where the right hand side $\partial_i(a_{i-1} f_{i-1})$ is the function $\gamma_{i-1}$ already constructed in the previous step of the induction.

We first define $a_i$ on the zero set $z(f_i)$ of $f_i$. On $z(f_i)$, which can be written as $x_{i+1} = y_{i+1}(x_{i-1}, x_i)$, the condition (A2) takes the form

$$\gamma_{i-1} = \partial_{i-1}(a_i f_i) = a_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i)) \cdot \partial_{i-1} f_i(x_{i-1}, x_i, y_{i+1}(x_{i-1,} x_i)),$$

because $f_i = 0$. In order to satisfy (A2) we must therefore define the function $a_i$ on the set $z(f_i)$ by the identity

$$a_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i)) \cdot \partial_{i-1} f_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i)) = \gamma_{i-1}(x_{i-1}, x_i). \tag{5}$$

To simplify notation we denote $\alpha_i(x_{i-1}, x_i) := a_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i))$. Observe, that

$$\alpha_i > 0 \tag{6}$$

since $\partial_{i-1} f_i > 0$ by assumption (H1) and $\gamma_{i-1} > 0$ by induction hypothesis.

Now we want to define $a_i$ outside the zero set $z(f_i)$ of $f_i$. We set

$$a_i(x_{i-1}, x_i, x_{i+1}) = \frac{1}{f_i} \int_{\eta_{i-1}(x_i, x_{i+1})}^{x_{i-1}} \partial_{i-1} f_i(\zeta_{i-1}, x_i, y_{i+1}(\zeta_{i-1}, x_i)) \alpha_i(\zeta_{i-1}, x_i) d\zeta_{i-1}. \tag{7}$$

Observe that this definition makes sense only for those $(x_{i-1}, x_i, x_{i+1})$ for which $y_{i+1}(\zeta_{i-1}, x_i)$ and $\eta_{i-1}(x_i, x_{i+1})$ are defined. By assumption (H3), these functions are defined for all $(x_{i-1}, x_i, x_{i+1}) \in \mathbf{R}^3$. So the definition of $a_i$ does make sense, and $a_i$ is defined for all $(x_{i-1}, x_i, x_{i+1}) \in \mathbf{R}^3$.

We now check properties (A1) and (A2) for $a_i$. We show first that $a_i$ is continuous and that $\partial_{i-1}(a_i f_i)$ and $\partial_{i+1}(a_i f_i)$ exist and are continuous. These properties obviously hold at points $(x_{i-1}, x_i, x_{i+1})$ which do not belong to the zero set $z(f_i)$ of the function $f_i$. The calculation for the points in the set $z(f_i)$ is straightforward, but tedious. In order not to disrupt the argument we postpone the proof to the Appendix.

Now we show that $a_i$ is positive. By assumption $\partial_{i-1} f_i > 0$ and by (6) also $\alpha_i > 0$. Furthermore, $f_i = f_i(x_{i-1}, x_i, x_{i+1})$ is positive for $x_{i-1} > \eta_i(x_i, x_{i+1})$, and negative when $x_{i-1} < \eta_{i-1}(x_i, x_{i+1})$. In the first case the right hand side of (7) is positive since all the entries are positive; in the second case $f_i < 0$ but since the order of integration changes, the right hand side is still positive. For $x_{i-1} = \eta_i(x_i, x_{i+1})$ the function $a_i$ is positive by (6). Therefore $a_i > 0$ for all $(x_{i-1}, x_i, x_{i+1}) \in \mathbf{R}^3$. Thus $a_i$ satisfies (A1).

Now we check condition (A2) at an arbitrary point $(x_{i-1}, x_i, x_{i+1})$. For $(x_{i-1}, x_i, x_{i+1}) \notin z(f_i)$ we have

$$\begin{aligned} \partial_{i-1}(a_i f_i) &\overset{(7)}{=} \partial_{i-1} f_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i)) \alpha_i(x_{i-1}, x_i) \\ &= a_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i)) \partial_{i-1} f_i(x_{i-1}, x_i, y_{i+1}(x_{i-1}, x_i)) \\ &\overset{(5)}{=} \gamma_{i-1}(x_{i-1}, x_i) \\ &= \partial_i(a_{i-1} f_{i-1}), \end{aligned} \tag{8}$$

where the last equality is the definition of $\gamma_{i-1}$. Since $\partial_{i-1}(a_i f_i)$ and $\partial_{i+1}(a_i f_i)$ are continuous, (8) holds for all $(x_{i-1}, x_i, x_{i+1})$. This verifies (A2).

Therefore $a_i$, as defined in $(5, 7)$ indeed satisfies (A1-A2).

**Step 2. Construction of $g_i$.** The goal in this step is to define the function $g_i$ such that (B) holds. Condition (B) requires that $\partial_i g_i = a_i f_i - \partial_i g_{i-1}$. The function $g_{i-1}$ is already constructed, by the induction hypothesis, and $a_i f_i$ has been constructed in Step 1. Our expression for $\partial_i g_i$ does not depend on $x_{i-1}$ since

$$\begin{aligned} \partial_{i-1}(a_i f_i - \partial_i g_{i-1}) &= \partial_i(a_{i-1} f_{i-1}) - \partial_{i-1}\partial_i g_{i-1} \\ &= \partial_i(\partial_{i-1} g_{i-1} + \partial_{i-1} g_i) - \partial_{i-1}\partial_i g_{i-1} \\ &= \partial_{i-1}\partial_i g_i(x_i, x_{i+1}) = 0. \end{aligned}$$

In the first line, we used that $\partial_{i-1}(a_i f_i) = \partial_i(a_{i-1} f_{i-1})$, which is (A2) for $k = i$ and was verified in (8). In the second line, we used the induction hypothesis (B) for $k = i - 1$. Therefore we define

$$g_i = g_i(x_i, x_{i+1}) := \int_0^{x_i} ((a_i f_i)(x_{i-1}, \zeta_i, x_{i+1}) - \partial_i g_{i-1}(x_{i-1}, \zeta_i)) d\zeta_i.$$

Then the condition (B) holds for $k = i$.

**Step 3.** $\gamma_i > 0$

The remaining step in the induction is to show (C) for $\gamma_i := \partial_{i+1}(a_i f_i)$ i.e. that $\gamma_i > 0$.

We differentiate

$$\begin{aligned}
\gamma_i = \partial_{i+1}(a_i f_i) &= \partial_{i+1} \int_{\eta_{i-1}(x_i,x_{i+1})}^{x_{i-1}} \partial_{i-1} f_i(\zeta_{i-1}, x_i, y_{i+1}(\zeta_{i-1}, x_i))\alpha_i(\zeta_{i-1}, x_i) d\zeta_{i-1} \\
&= -\partial_{i+1}\eta_{i-1}(x_i, x_{i+1}) \cdot \partial_{i-1} f_i(\eta_{i-1}(x_i, x_{i+1}), x_i, y_{i+1}(\eta_{i-1}, x_i))\alpha_i(\eta_{i-1}(x_i, x_{i+1}), x_i) \\
&= -\partial_{i+1}\eta_{i-1}(x_i, x_{i+1}) \cdot \partial_{i-1} f_i(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1})\alpha_i(\eta_{i-1}(x_i, x_{i+1}), x_i)
\end{aligned}$$

since $x_{i+1} = y_{i+1}(\eta_{i-1}(x_i, x_{i+1}), x_i)$. We see that the function $\gamma_i$ is function of $x_i$ and $x_{i+1}$ only. Furthermore, since $-\partial_{i+1}\eta_{i-1} > 0$ by (4), $\partial_{i-1} f_i > 0$ by assumption (H1), and $\alpha_i > 0$ by (6), we see that

$$\gamma_i = \partial_{i+1}(a_i f_i) > 0.$$

This finishes the induction step and thus proves existence of the Lyapunov function $V$. □

**Proof of Theorem 1.2.** The Lyapunov function $V$ constructed in Theorem 1.1 is also a Lyapunov function for the system (2) on the closed positive orthant $\mathcal{O}$. Indeed,

$$\begin{aligned}
\dot{V} &= -\sum_i \dot{x}_i(\partial_i g_i + \partial_i g_{i-1}) \\
&= -\sum_i \dot{x}_i a_i f_i \\
&= -\sum_i a_i x_i (f_i)^2 \leq 0
\end{aligned}$$

since $a_i > 0$ and $x_i \geq 0$ in the positive orthant $\mathcal{O}$. The derivative $\dot{V} = 0$ if, and only if $x_i(f_i)^2 = 0$ for all $i$. This is equivalent to $x_i f_i = 0$ for all $i$. Since $\dot{x}_i = x_i f_i$, we have $\dot{V} = 0$ if, and only if, $\dot{x}_i = 0$ for all $i$. This implies that $\dot{V}(x) = 0$ if, and only if, $x$ is an equilibrium. Therefore $V$ is a Lyapunov function for (2). □

**Proof of Theorem 1.3.** Observe that the Lyapunov function $V = -\sum g_i(x_i, x_{i+1})$ is defined on the closed positive orthant $\mathcal{O}$. Since we assume (H2) for the functions $f_i$ in (2), we see immediately that $|x_i| > C$ for the maximal $|x_i|$ implies $\dot{x}_i < 0$. It follows that each trajectory enters the positively invariant box $Q := \{x \in \mathbf{R}^n \mid |x_i| \leq C\}$ in finite time. Since the positive orthant $\mathcal{O}$ is invariant under (2), each trajectory starting in $\mathcal{O}$ enters the set $\mathcal{O} \cap Q$ in finite time. By the LaSalle's invariance principle each trajectory in $\mathcal{O} \cap Q$ converges to the set where $\dot{V} = 0$, which is the set of equilibria.

So there is no chain recurrent set in $\mathcal{O}$, apart from the set of equilibria. In particular, there are no cycles of equilibria in $\mathcal{O}$. □

**Remark 2.1** We assumption (H3) in the definition of the function $a_i$; it guarantees that we can define $a_i$ for all points $(x_{i-1}, x_i, x_{i+1})$, since for each such point the values $\eta_{i-i}(x_i, x_{i+1})$ and $y_{i+1}(x_{i-1}, x_i)$ are defined. Suppose now that (H3) is violated and that for some fixed $x_i = x_i^*$ the zero set $z(f_i, x_i^*)$ of $f_i$ with $x_i = x_i^*$ fixed is a curve which satisfies

$$\lim_{x_{i-1} \to -\infty} z(f_i, x_i^*) = a_i^-(x_i^*) > -\infty, \quad \lim_{x_{i-1} \to \infty} z(f_i, x_i^*) = a_i^+(x_i^*) < \infty.$$

Then the function $a_i$ can only be defined in points $(x_{i-1}, x_i, x_{i+1})$ for which $a_i^-(x_i^*) < x_{i+1} < a_i^+(x_i^*)$. We also observe that the absolute value $|a_i|$ grows without bounds as $(x_{i-1}, x_i, x_{i+1})$ approaches the boundary of this region. Similar restrictions occur if we assume that

$$\lim_{x_{i+1} \to -\infty} z(f_i, x_i^*) = a_i^-(x_i^*) > -\infty, \quad \lim_{x_{i+1} \to \infty} z(f_i, x_i^*) = a_i^+(x_i^*) < \infty.$$

However, observe that we do not need to define the Lyapunov function on all of $\mathbf{R}^n$. It follows from the dissipativness assumption (H2) that each trajectory enters the box $Q := \{x \in \mathbf{R}^n \mid |x_i| \leq C\}$ in finite time. We need to define our Lyapunov function $V$ and thus functions $a_i$ only for $x \in Q$. Thus we can still construct the Lyapunov function $V$ provided that $|a_i^+(x_i^*)| > C$ and $|a_i^-(x_i^*)| > C$ for all $i$, with $C$ given in assumption (H2).

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# 3 Appendix

We show that the function $a_i$ is continuous and the partial derivatives $\partial_{i-1}(a_if_i)$ and $\partial_{i+1}(a_if_i)$ exist and are continuous.

For all $(x_{i-1}, x_i, x_{i+1}) \notin z(f_i)$ this follows from the definition of $a_i$. We hence consider $(x_{i-1}, x_i, x_{i+1}) \in z(f_i)$.

We first prove continuity of $a_i$. Write $x_{i-1} = \eta_{i-1}(x_i, x_{i+1}) + h$ and expand $f_i$ with respect to $h$ at the point $(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1}) \in z(f_i)$:

$$f_i(\eta_{i-1}(x_i, x_{i+1}) + h, x_i, x_{i+1}) = h(\partial_{i-1}f_i(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1}) + \tau(h, x_i, x_{i+1})). \tag{9}$$

Here the remainder $\tau(h, x_i, x_{i+1})$ is a continuous function with $\tau(0, x_i, x_{i+1}) = 0$.

The integral in definition (7) of $a_i$ becomes

$$\int_{\eta_{i-1}(x_i,x_{i+1})}^{\eta_{i-1}(x_i,x_{i+1})+h} \partial_{i-1}f_{i-1}(\zeta_{i-1}, x_i, y_{i+1}(\zeta_{i-1}, x_i))\alpha_i(\zeta_{i-1}, x_i)d\zeta_{i-1},$$

for $h \neq 0$ with a (uniformly) continuous integrand. Indeed, $\alpha_i$, which is the restriction of $a_i$ to $z(f_i)$, is continuous by definition (5). From the standard integration theory, we immediately obtain

$$\begin{aligned}
\lim_{h\to 0, h\neq 0} \quad & \frac{1}{h}\int_{\eta_{i-1}(x_i,x_{i+1})}^{\eta_{i-1}(x_i,x_{i+1})+h} \partial_{i-1}f_{i-1}(\zeta_{i-1}, x_i, y_{i+1}(\zeta_{i-1}, x_i))\alpha_i(\zeta_{i-1}, x_i)d\zeta_{i-1} \\
= \quad & (a_i\partial_{i-1}f_i)(\eta_{i-1}(x_i, x_{i+1}), x_i, y_{i+1}(\eta_{i-1}, x_i)) \\
= \quad & a_i\partial_{i-1}f_i)(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1}).
\end{aligned}$$

The limit is locally uniform with respect to $x_i, x_{i+1}$ by continuity of all functions involved. Inserting the expansion (9) of $f_i$, we obtain

$$\begin{aligned}
\lim_{h\to 0, h\neq 0} \quad & \frac{1}{f_i}\int_{\eta_{i-1}(x_i,x_{i+1})}^{\eta_{i-1}(x_i,x_{i+1})+h} \partial_{i-1}f_{i-1}(\zeta_{i-1}, x_i, x_{i+1})\alpha_i(\zeta_{i-1}, x_i)d\zeta_{i-1} \\
= \quad & \lim_{h\to 0, h\neq 0} \frac{h\cdot(a_i\partial_{i-1}f_i)(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1})}{h\cdot(\partial_{i-1}f_i(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1}) + \tau(h, x_i, x_{i+1}))} \\
= \quad & a_i(\eta_{i-1}(x_i, x_{i+1}), x_i, x_{i+1}),
\end{aligned}$$

locally uniformly with respect to $x_i, x_{i+1}$. This proves continuity of $a_i$, defined by (5) and (7).

Now we show that the partial derivatives $\partial_{i-1}(a_if_i)$ and $\partial_{i+1}(a_if_i)$ exist and are continuous.

This is immediate from the integral representation (7) of $a_if_i$, which holds for all $(x_{i-1}, x_i, x_{i+1})$; from continuity of the integrand, which does not contain $x_{i+1}$ and $x_{i-1}$; from differentiability of $\eta_{i-1}$, and from the fundamental theorem of calculus.

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