A class of convergent neural network dynamics

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Abstract

We consider a class of systems of differential equations in $\mathbf{R}^n$ which exhibits convergent dynamics. We find a Lyapunov function and show that every bounded trajectory converges to the set of equilibria. Our result generalizes the results of Cohen and Grossberg [1] for convergent neural networks. It replaces the symmetry assumption on the matrix of weights by the assumption on the structure of the connections in the neural network.

We prove the convergence result also for a large class of Lotka-Voltera systems. These are naturally defined on the closed positive orthant. We show that there are no heteroclinic cycles on the boundary of the positive orthant for the systems in this class.

Key words. Neural networks, Lotka-Voltera systems, Lyapunov function, heteroclinic cycles, convergence.

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1 Results

In this paper we shall prove a convergence theorem for a class of n -dimensional systems of differential equations of the form

$$\dot{x}_i = a_i(\mathbf{x})(\gamma_i(x_i) - \sum_{j=1}^{n} \beta_{ij} f_j(x_j)) \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$. Without loss of generality we put $\beta_{ii} = 0$. Before we spell out assumptions which we put on the system, we mention several applications.

Systems of type (1) can be considered as a model neural network consisting of n neurons. Each x_i represents an activation level of the corresponding neuron. The paradigm, put forward by Hopfield [2, 3], states that the stable equilibria of the system (1) represent stored memory. The trajectories, starting sufficiently close to an asymptotically stable equilibrium of (1), will converge to that equilibrium. From the point of view of a neural networks this models retrieval of a stored memory pattern upon presentation of a partial, corrupted pattern. This interpretation then leads to applications of systems (1) to parallel memory storage, content-addressable memory and global pattern formation, see Cohen and Grossberg [1].

When trying to design a system which could serve as a memory storage, the fundamental question is whether the dynamics of the system is convergent or at least quasi-convergent (Hirsch [4]). The system is *convergent*, if every trajectory of the system converges to an equilibrium. The system is *quasi-convergent*, if every trajectory of the system converges at least to the set of equilibria. Only after this question has been resolved one can actually try to design the system so that the equilibria are at the desired places, or, using again neural networks terminology, to design a system, which stores desired memory patterns.

In the context of neural networks a system analogous to system (1) was considered by Cohen and Grossberg [1]. Since we want to compare their assumptions with ours we list their assumptions below. They have assumed that

- the function $a_i(\mathbf{x})$ is a function of x_i only, and is positive
- the matrix $B = [\beta_{ij}]$ is symmetric and nonnegative
- $f_j \geq 0$

In addition, monotonicity of the f_j is required

$\frac{\partial f_j}{\partial x_j} > 0$ for all j. **(H1)**

Under these assumptions they constructed a Lyapunov function for the system (1) and proved quasi-convergence; each trajectory converges to the set of equilibria.

Systems of type (1) also arise in the context of mathematical biology. There x_i represents population size of a certain species. In this context, we say that the system (1) is *cooperative*

if all $\beta_{ij} \geq 0$ and *competitive* if all $\beta_{ij} \leq 0$. We note that the generalized Lotka-Volterra equations,

$$\dot{x}_i = x_i(c_i - \sum_{j=1}^{n} \beta_{ij} f_j(x_j)) \tag{2}$$

where c_i and β_{ij} are constants and f_j satisfy (H1), fall within the class of systems (1).

We now state the assumptions and our main result. We assume that the functions a_i are nonnegative,

$$a_i(\mathbf{x}) \geq 0 \qquad \text{for all } \mathbf{x} \in \mathbf{R}^n \text{ and all } i. \tag{H2}$$

The second assumption will address the structure of the neural network described by (1). Observe that if $\beta_{ij} = 0$ then the neuron j does not effect the activity of the neuron i. Thus there is no connection from the neuron j to the neuron i. Let Υ be the undirected graph with n vertices, where the vertex j is joined to the vertex i by the edge e_{ij} if, and only if, $\beta_{ij} \neq 0$. We assume that

$$\beta_{ij}\beta_{ji} > 0 \tag{H3}$$

for every edge e_{ij}. Therefore the definition of the graph Υ makes sense. Note that to every edge e_{ij} of the graph Υ there are two associated numbers, β_{ij} and β_{ji}. A *cycle* C in the graph Υ is a collection of edges $e_{i_1 i_2}, e_{i_2 i_3}, \ldots, e_{i_k i_1}$ with $k \geq 3$. Observe, that a pair $e_{i_2 i_1}, e_{i_1 i_2}$ does not form a cycle. We assume that along every cycle C

$$\Pi_C \beta_{ij} = \Pi_C \beta_{ji}, \tag{H4}$$

where Π denotes the product.

The main Theorem of this paper is

Theorem 1.1 *Consider system (1) with assumptions (H1-H4). Then there exists a Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$, non-increasing along the trajectories and strictly decreasing along all non-equilibrium trajectories of (1).*

Observe that our assumption (H2) relaxes the first assumption of Cohen and Grossberg. We do not need to assume that the functions f_j are positive and we do not need the competitiveness assumption $\beta_{ij} \geq 0$. We keep the assumption (H1) which is natural in the context of neural dynamics. The crucial difference in our approaches is the replacement of the symmetry hypothesis on the matrix of weights B. This hypothesis has its roots in the papers of Hopfield [2, 3], and considerable effort was spent trying to remove or weaken this hypothesis. That this hypothesis cannot be removed completely is known for some time. Consider the May and Leonard [5] model of the voting paradox

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1 - ax_2 - bx_3) \\ \dot{x}_2 &= x_2(1 - bx_2 - x_2 - ax_3) \\ \dot{x}_3 &= x_3(1 - ax_1 - bx_2 - x_3). \end{aligned} \tag{3}$$

Grossberg [6] and Schuster *et al* [7] considered the case $b > 1 > a$ and $a + b > 0$. They proved that all positive trajectories, except the uniform trajectories $x_1(0) = x_2(0) = x_3(0)$, persistently oscillate as $t \to \infty$. Observe that the matrix of coefficients

$$\begin{pmatrix} 1 & a & b \\ b & 1 & a \\ a & b & 1 \end{pmatrix}$$

can be chosen arbitrarily close to a symmetric matrix and still exhibits persistent oscillations. Therefore a Lyapunov function in the sense of Theorem 1.1 cannot exist for this system.

We replace the symmetry assumption by assumptions (H3) and (H4). These two are obviously satisfied if the matrix B is symmetric. Observe that in the system (3) the condition (H4) is not satisfied; along the cycle e_{12}, e_{23}, e_{31} we have

$$\beta_{12}\beta_{23}\beta_{31} = a^3 \neq b^3 = \beta_{21}\beta_{32}\beta_{13}.$$

Observe that the assumption (H4) is always satisfied in the absence of cycles, i.e. when the graph Υ is a tree.

Corollary 1.2 *Consider system (1) where the graph of connections Υ is a tree. Assume (H1), (H2) and (H3). Then the conclusion of Theorem 1.1 holds.*

In order to show that the system is quasi-convergent we need to show that forward trajectories are bounded. To this end, a natural assumption is the *dissipativeness* assumption

all trajectories eventually enter a positively invariant bounded set B. **(D)**

Remark 1.3 Dissipativness is guaranteed, for example, by the following set of assumptions:

Assume that all f_i are bounded, $a_i(\mathbf{x}) > 0$ for sufficiently large $|\mathbf{x}|$ and $\gamma_i(x_i)x_i \to -\infty$ for $|x_i| \to \infty$. It is easy to see that under these assumptions $\dot{x}_i x_i < 0$, for every i, if $|x_i|$ is sufficiently large.

We also remark that in the context of neural networks it is customary to assume that the functions f_j are bounded.

Corollary 1.4 *Consider system (1) with (H1-H4) and (D). Then the system (1) is quasi-convergent: every trajectory converges to the set of equilibria. If the set of equilibria is finite, then the system is convergent.*

This result has an interesting consequence for Lotka - Volterra systems. Since in this case the physically meaningful region of the phase space is the nonnegative orthant $\mathcal{O} := \{x \in \mathbf{R}^n \mid x_i \geq 0\}$ the assumption (H2) is automatically satisfied in $\mathcal{O}$.

Corollary 1.5 *Consider Lotka-Volterra system (2) on the closed positive orthant $\mathcal{O}$ with the graph Υ of interactions being a tree. Assume (H1) and (H3). Then all bounded trajectories converge to the set of equilibria. Moreover, heteroclinic cycles are excluded, even on the boundary of $\mathcal{O}$.*

Remark 1.6 Observe, that every boundary face of the positive orthant $\mathcal{O}$ is invariant under the flow of (2), and the system, restricted to this face, splits into subsystems of type (2). This idea can be used to show that if there are no nontrivial reccurent sets in the interior of the first orthant, then there are no nontrivial recurrent sets in the interior of any face and subface of the boundary $\partial\mathcal{O}$. This, however, does not preclude the existence of heteroclinic cycles on the boundary. Freedman and Smith [8] studied this problem in a more general setting of tridiagonal systems of differential equations. Using a discrete Lyapunov function there were able to show that there is no nontrivial reccurent set in the interior of the positive orthant and, consequently, in the interior of any face and subface of its boundary. However, they could not exclude heteroclinic cycles on the boundary. Although Corollary 1.5 does not cover all tridiagonal systems, we can prove (see Fiedler and Gedeon [9]), using a different Lyapunov function, that heteroclinic cycles on the boundary cannot exist in tridiagonal systems.

To see that assumption (H3) is sharp, consider a two dimensional predator-prey Lotka-Volterra model

$$\begin{aligned} \dot{x} &= x_1(1 - x_2) \\ \dot{y} &= x_2(-1 + x_1). \end{aligned}$$

This system satisfies (H1) and the graph of interactions is a tree. The condition (H3) is not satisfied and, as is well known (see for instance Hale and Koçak [10]), all solutions starting in the interior of the positive orthant are periodic. So, we must assume that the pairwise interactions are not of the predator-prey type, if oscillations in Lotka-Volterra are to be excluded.

2 Proofs

Proof of Theorem 1.1 We start with a preliminary construction.

Since $\beta_{ij}\beta_{ji} > 0$, by assumption (H3), the ratio β_{ij}/β_{ji} can be written as

$$\beta_{ij}/\beta_{ji} = \exp(v_i - v_j) \tag{4}$$

for some v_i and v_j. In fact, we can choose v_i arbitrarily and then use the equation (4) to compute v_j. For every pair (i, j) with $e_{ij} \in \Upsilon$ we have one equation of the type (4). In order to be able to solve the system of all such equations consistently we need a consistency condition. It is easy to see that the only obstruction comes from compatibility along the cycles in the graph Υ. If C is a cycle with vertices $i_1, \ldots, i_k$ then

$$\exp(v_{i_1} - v_{i_2}) \exp(v_{i_2} - v_{i_3}) \ldots \exp(v_{i_k} - v_{i_1}) = 1.$$

In terms of β_{ij}, this condition reads

$$\Pi_C \beta_{ij}/\beta_{ji} = 1,$$

which is assumption (H4). Thus, if (H4) is satisfied, then the system of equations (4) can be solved for v_i, $i = 1, \ldots, n$, with one of the v_i being chosen arbitrarily.

Having solved for v_i, we define

$$\begin{aligned} s_i &:= exp(-v_i) && (5)\\ b_i &:= s_i \int^{x_i} (\frac{df_i}{d\zeta_i}\gamma_i)(\zeta_i)d\zeta_i && (6)\\ q_{ij} &:= \frac{1}{2}s_i\beta_{ij}, \qquad q_{ji} := \frac{1}{2}s_j\beta_{ji} && (7) \end{aligned}$$

for every pair (i, j). Observe, that

$$\begin{aligned} q_{ij}/q_{ji} &= \frac{\beta_{ij}}{\beta_{ji}}\frac{s_i}{s_j} \\ &= exp(v_i - v_j)exp(-v_i)exp(v_j) = 1, \end{aligned}$$

for $i \neq j$, and so $q_{ij} = q_{ji}$ for all edges e_{ij}. Therefore the matrix $Q := [q_{ij}]$ is symmetric. However, there is no change of variables known to us which would change the system (1) with weight matrix B into the system with the weight matrix Q - thus in effect symmetrizing the matrix B.

Finally, we define the function $V : \mathbf{R}^n \to \mathbf{R}$

$$V(\mathbf{x}) := -\sum_{i=1}^{n}(b_i(x_i) - \sum_{j=1}^{n} q_{ij}f_i(x_i)f_j(x_j)). \tag{8}$$

In order to prove that V is a Lyapunov function we need to show that it is non-increasing along all trajectories, and strictly decreasing along all non-equilibrium trajectories of system (1).

We calculate the derivative of V, abbreviating $f_i' = \frac{df_i}{dx_i}$ to simplify notation.

$$\begin{aligned} \dot{V} &= -\sum_{i=1}^{n}(\frac{db_i}{dx_i}\dot{x}_i - \sum_{j=1}^{n} q_{ij}(f_i'f_j(x_j)\dot{x}_i + f_j'f_i(x_i)\dot{x}_j)) \\ &= -\sum_{i=1}^{n}(s_if_i'\gamma_i\dot{x}_i - \sum_{j=1}^{n}(q_{ij}f_i'f_j(x_j)\dot{x}_i + q_{ji}f_i'f_j(x_j)\dot{x}_i)) = (*) \end{aligned}$$

since, by rearranging the summations,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}f_i'f_j(x_j)\dot{x}_i + q_{ij}f_j'f_i(x_i)\dot{x}_j = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}f_i'f_j(x_j)\dot{x}_i + q_{ji}f_i'f_j(x_j)\dot{x}_i.$$

We continue our computation

$$\begin{aligned} (*) &= -\sum_{i=1}^{n}(s_if_i'\dot{x}_i(\gamma_i - \sum_{j=1}^{n}\frac{q_{ij}+q_{ji}}{s_i}f_j(x_j)) \\ &= -\sum_{i=1}^{n}(s_if_i'\dot{x}_i(\gamma_i - \sum_{j=1}^{n}\frac{2q_{ij}}{s_i}f_j(x_j)) \\ &= -\sum_{i=1}^{n} s_if_i'a_i(\mathbf{x})[\gamma_i - \sum_{j=1}^{n}\beta_{ij}f_j(x_j)]^2 \end{aligned}$$

where we used equations (1) and that $\beta_{ij} = 2q_{ij}/s_i$. To finish the computation we observe that s_i are positive constants, $f_i' > 0$ by the assumption (H1), and $a_i(\mathbf{x}) \geq 0$ by assumption (H2). Thus

$$\dot{V} \leq 0.$$

Since $s_i f_i'$ is strictly positive, $\dot{V} = 0$ if, and only if,

$$a_i(\mathbf{x})[\gamma_i - \sum_{j \in \mathcal{N}_i} \beta_{ij} f_j(x_j)]^2 = 0$$

for every i. This is equivalent to

$$a_i(\mathbf{x})(\gamma_i - \sum_{j \in \mathcal{N}_i} \beta_{ij} f_j(x_j)) = 0$$

for every i and, consequently, $\dot{x}_i = 0$ for all i. Hence $\dot{V}(x) = 0$ if, and only if, x is an equilibrium. This finishes the proof of Theorem 1.1. □

Proof of Corollary 1.4 If we assume that the system is dissipative, which is the assumption (D), then all trajectories eventually enter a positively invariant bounded set B. Hence all forward trajectories are bounded.

By LaSalle's invariance principle (see [10]), the omega limit set of the set of bounded trajectories is a subset of the set $\{x \in \mathbf{R}^n | \dot{V} = 0\}$. Since the latter set is the set of equilibria by Theorem 1.1, the Corollary is proved. □

Proof of Corollary 1.5 On the closed positive orthant $\mathcal{O}$ the assumption (H2) is satisfied automatically. The graph of interactions Υ is a tree and there is a Lyapunov function V on the closed positive orthant by Corollary 1.2. Thus every bounded trajectory converges to the set of equilibria. Since the Lyapunov function is defined also on the boundary of the positive orthant we can exclude the existence of the heteroclinic cycles on the boundary. This proves the Corollary. □

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