Large patterns of elliptic systems in infinite cylinders

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Abstract.

We consider systems of elliptic equations $\partial_t^2 u + \Delta_x u + \gamma \partial_t u + f(u) = 0,\ u(t,x) \in \mathbb{R}^N$ in unbounded cylinders $(t,x) \in \mathbb{R} \times \Omega$ with bounded cross-section $\Omega \subset \mathbb{R}^n$ and Dirichlet boundary conditions. We establish existence of bounded solutions $u(t,x)$ with non-trivial dependence on $t \in \mathbb{R}$, $\partial_t u(t,x) \not\equiv 0$. Our main assumptions are dissipativity of the nonlinearity f and the existence of at least two t-independent solutions $w_1(x), w_2(x)$ which solve $\Delta_x w_j + f(w_j) = 0,\ j = 1,2$.

The proof exploits the dynamical systems structure of the equations: solutions can be translated along the axis of the cylinder. We first prove existence and compactness of attractors for the dynamical system induced by this translation. We then compute Conley indices for cross-sectional Galerkin approximations to conclude that the attractor does not consist of only the two solutions $w_j(x),\ j = 1,2$. We also prove existence of solutions converging for $t \to +\infty$ or $t \to -\infty$. If the system possesses a gradient-like structure, in addition, solutions will converge on both sides of the cylinder.

Résumé.

Nous considérons des systèmes d'équations elliptiques $\partial_t^2 u + \Delta_x u + \gamma \partial_t u + f(u) = 0,\ u(t,x) \in \mathbb{R}^N$ dans un cylindre infini $(t,x) \in \mathbb{R} \times \Omega$ avec $\Omega \subset \mathbb{R}^n$ borné et des conditions de bord Dirichlet. Nous établissons l'existence de solutions bornées, dépendant de $t \in \mathbb{R}$ d'une façon non-triviale, $\partial_t u(t,x) \not\equiv 0$. Nous supposons entre autre dissipativité de la fonction f et l'existence de deux solutions $w_1(x), w_2(x)$ de l'équation $\Delta_x w_j + f(w_j) = 0,\ j = 1,2$. Dans la démonstration, nous utilisons la structure d'un système dynamique, engendré par la translation de solutions le long de l'axe du cylindre. Nous démontrons tout d'abord l'existence et la compacité de l'attracteur de ce système dynamique. Nous calculons ensuite des indexes de Conley pour l'approximation de Galerkin afin de déduire que l'attracteur contient des solutions autre que $w_j(x),\ j = 1,2$. Nous démontrons aussi que les solutions $u(t,x)$ convergent pour $t \to +\infty$ ou $t \to -\infty$. Si le système possède une fonction de Lyapunov, en plus, les solutions convergeront des deux côtés du cylindre.

Keywords. attractors, Conley index, traveling waves, elliptic systems

Mots Clés. attracteurs, indexe de Conley, ondes progressives, systèmes elliptiques

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain. We call $Q = \mathbb{R} \times \Omega$ a cylinder. We consider systems of elliptic equations

$$\partial_t^2 u + \Delta_x u + \gamma \partial_t u + f(u) = 0 \ , \qquad (t,x) \in Q. \tag{1.1}$$

Here $u \in \mathbb{R}^N$, $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, γ is a constant real $N \times N-$ matrix and Δ_x is the Laplacian with respect to $x \in \Omega$. We impose Dirichlet boundary conditions at $x \in \partial\Omega$

$$u(t,x) = 0 \ \text{ for } \ (t,x) \in \mathbb{R} \times \partial\Omega. \tag{1.2}$$

Similarly we could impose Neumann, Robin or periodic boundary conditions, with minor adaptations.

For the nonlinearity f and its Jacobian f' we require growth conditions

$$\begin{cases} |f(u)| \leq C_0(1 + |u|^p) \\ |f'(u)| \leq C_1(1 + |u|^{p-1}) \end{cases} \tag{1.3}$$

and a dissipation condition

$$f(u) \cdot u \leq C_2 - C_3 |u|^{2+\sigma} \tag{1.4}$$

with some C_0, C_1, C_2, C_3 and σ positive. Here $1 < p < 1 + \dfrac{4}{n-1}$ if $n > 1$, and $p < \infty$ otherwise.

We restrict ourselves to the above setting for notational simplicity. Generalizations of the results below to other growth and dissipation conditions for $f = f(u)$ and to more general x-dependent second-order elliptic operators replacing Δ_x are straightforward. Similarly, $\gamma = \gamma(x)$ and $f = f(x,u)$ may depend on x. It is crucial to our dynamical systems approach, however, that (1.1) does not depend on "time" t, explicitly. Also, explicit gradient dependence $f = f(u, \nabla u)$ is excluded.

Elliptic systems of the form (1.1) arise, for example, when studying traveling wave solutions of reaction-diffusion systems

$$D\partial_\tau u = \Delta_{t,x} u + f(u),$$

where τ denotes time and again $(t, x) \in Q$. A traveling wave solution is a bounded solution of the special form $u = u(t - c\tau, x)$, and c is called the wave speed. Note that $\gamma = cD$. For a recent comprehensive survey on traveling waves and their applications, see for example the book [42].

We denote by $H^{l,p}_{loc}$, $l = 0, 1, 2$ and $1 \leq p \leq \infty$, the subspace of locally integrable functions u, for which the following semi-norms are finite:

$$||u\,, Q_T||_{l,p} := ||u||_{H^{l,p}(Q_T,\mathbb{R}^N)} = C(T, u) < \infty \quad , \quad T \in \mathbb{R} \tag{1.5}$$

where $Q_T = [T, T+1] \times \Omega$.

The space $H^{l,p}_{loc}$ with this system of semi-norms is a Fréchet space and metrizable. We write

$$H_{loc} := H^{0,2}_{loc} \quad \text{and} \quad H^2_{loc} := H^{2,2}_{loc} \cap \{u = 0 \text{ on } \partial Q\}. \tag{1.6}$$

The space $H^{l,p}_a$ consists of functions $u \in H^{l,p}_{loc}$ with finite norm

$$||u||_{H^{l,p}_a} := \sup_{T \in \mathbb{R}} ||u\,, Q_T||_{l,p} < \infty. \tag{1.7}$$

Throughout we use the abbreviations $H_a := H^{0,2}_a$ and $H^2_a := H^{2,2}_a \cap \{u = 0 \text{ on } \partial Q\}$.

A *solution* $u(t, x)$, $x \in \Omega$, $t \in \mathbb{R}$ of (1.1), (1.2) is always understood to be a weak solution which belongs to the space H^2_{loc}. A solution is said to be *bounded* if it belongs to the space H^2_a. Of course, equation (1.1) is satisfied in H_a for a bounded solution u.

In fact, every solution $u \in H^2_{loc}$ of (1.1), (1.2) is automatically bounded, due to the dissipation condition (1.4); see [43].

Also note that the growth conditions on f ensure the Nemitskii operator

$$\tilde{f} : H^2(Q_T, \mathbb{R}^N) \to H(Q_T, \mathbb{R}^N), \quad \tilde{f}(u)(t, x) = f(u(t, x)), \quad (t, x) \in Q_T \tag{1.8}$$

for every $T \in \mathbb{R}$ to be of class C^1 and compact, by Sobolev embedding and Krasnoselskii's theorem; see [2] for example.

Equilibria are particular solutions of (1.1), (1.2), which do not depend on t, and therefore solve

$$\Delta_x w + f(w) = 0,$$

for $x \in \Omega$, and $w = 0$ on $\partial\Omega$. Equilibria w can be interpreted as functions in $(H^2(\Omega) \cap H_0^1(\Omega))^N$, or as bounded, t-independent solutions of (1.1) in H^2_{loc} or in H^2_a. An equilibrium $w(x)$ is called *hyperbolic* if the formally linearized operator

$$\hat{L}(\lambda) := -\lambda^2 + i\lambda\gamma + \Delta_x + f'(w(x)) \tag{1.9}$$

possesses only trivial kernel on $H^2(\Omega, \mathbb{C}^N) \cap H_0^1(\Omega, \mathbb{C}^N)$, for any $\lambda \in \mathbb{R}$. Note that non-trivial kernel indicates the existence of a bounded solution $e^{i\lambda t} z(x)$ of the linearization of (1.1), (1.2) at $u(t, x) = w(x)$, where $z(\cdot) \in \ker L(\lambda)$.

We call a bounded solution u of (1.1), (1.2) a non-equilibrium solution if it is not an equilibrium. Our main purpose is to find conditions which guarantee the existence of non-equilibrium solutions.

If Ω is just a single point, $n = \dim \Omega = 0$, without boundary conditions, then (1.1) defines a second order system of ordinary differential equations. Global dynamical systems methods like the Conley index have proved to be very useful in detecting bounded solutions [10], [37].

For elliptic systems in a cylinder, $\dim \Omega = n \geq 2$, such global methods have not been developed. Hadamard was the first to notice that the initial value problem for elliptic equations is ill-posed; see [22, Bk. I, Ch. II, §18]. Prescribing u and $\partial_t u$ at $t = 0$, a solution need not exist, even for small times. Nevertheless this difficulty has been overcome in several interesting, particular cases. We first mention the pioneering work by Kirchgässner [25] on small solutions of elliptic equations in infinite cylinders. His idea was to construct invariant manifolds, where the elliptic initial value problem is well-posed and a flow, or at least a semiflow, is defined; see also [15]. This idea was extended to large solutions, later, in the "parabolic", convection dominated limiting case of large wave speeds $\gamma \in \mathbb{R}$; see [7] and [34]. Without such a restriction, Babin and Mielke have treated the case of elliptic equations in a strip, $\Omega = [0, 1]$; see [30] and [3].

We also mention the remarkably early work by Gardner, who used finite difference approximations and applied Conley index to the resulting ODE's [19]. Although his results were restricted to scalar equations $N = 1$, cubic f, and to one dimensional cross-section, $\dim \Omega = 1$, we essentially follow Gardner's idea below. Technically, we replace finite difference discretization by Galerkin projections.

Theorem 1. *Assume $f \in C^1$ satisfies the growth conditions (1.3) and the dissipation condition (1.4). Moreover assume that there exist at least 2κ distinct equilibria which are hyperbolic. Then there exist at least κ distinct bounded non-equilibrium solutions of (1.1), (1.2) in H^2_a.*

The norm in H^2_a, uniform with respect to t, was introduced in (1.7). Of course, solutions $u(t,x)$ which only differ by a constant (time) shift of t are not considered distinct.

In fact, when proving Theorem 1 we obtain slightly more precise information on the bounded non-equilibrium solutions, besides mere existence.

Theorem 2. *Under the assumptions of Theorem 1, for any hyperbolic equilibrium w_j, except possibly one, there is a bounded non-equilibrium solution u_j converging to w_j at one end of the unbounded cylinder:*

$$||u_j - w_j\,, Q_T||_{2,2} \to 0$$

for $T \to +\infty$ or for $T \to -\infty$.

For parabolic equations in bounded domains, a result as in Theorem 1 is far from optimal. In fact, $2\kappa+1$ hyperbolic equilibria then produce at least 2κ non-equilibrium solutions. In the elliptic context, however, our bound κ is optimal. Indeed, fix $\gamma \in \mathbb{R}$ nonzero and consider $\dim \Omega = 0$ again with the dissipative nonlinearity $f(u) = -\varepsilon u + \cos u$, for fixed $\varepsilon > 0$. Then Theorems 1 and 2 also hold. Explicit phase plane analysis shows the count

$$\sharp\{\text{bounded non-equilibrium solutions}\} = \frac{1}{2}(\sharp\{\text{equilibria}\} - 1)$$

for almost all ε.

Note that the number of equilibria is in fact odd in the above example, if all equilibria are hyperbolic. The same observation holds true in our general setting, by dissipativeness and Leray-Schauder degree.

Our notion of hyperbolicity mimics hyperbolicity of equilibria in ordinary differential equations. For example, if $f = \nabla F$, then the Jacobian f' is symmetric. Therefore any equilibrium is hyperbolic in our sense (1.9), if and only if, $\Delta + f'(w(x))$ has trivial kernel. The gradient case is also interesting from another point of view. Let γ^T denote the transpose of the real matrix γ. If $\gamma + \gamma^T > 0$ or $\gamma + \gamma^T < 0$ are strictly definite matrices, then the elliptic system (1.1) possesses a Lyapunov function

$$V(u, \partial_t u) = \int_\Omega [|\partial_t u|^2 - |\nabla_x u|^2 + 2F(u)]dx. \tag{1.10}$$

In particular, any bounded solution converges to the set of equilibria for $t \to +\infty$ and for $t \to -\infty$.

Corollary 1.1. *Assume $\gamma + \gamma^T > 0$ or $\gamma + \gamma^T < 0$ are strictly definite matrices, and $f \in C^1$ is a gradient, $f = \nabla F$, in addition to satisfying growth conditions (1.3) and dissipation conditions (1.4). Moreover, assume there are precisely $2\kappa + 1$ equilibria, all of which are hyperbolic. Then there are at least κ distinct heteroclinic orbits, that is, solutions converging to different equilibria for $t \to \pm\infty$. For any two of these heteroclinics, the equilibria they are converging two are distinct.*

In contrast to this corollary, however, our above theorems neither rely on variational methods nor on comparison principles. Therefore, in general, we cannot claim specific properties of our bounded non-equilibrium solutions like positivity, monotonicity with respect to t, or convergence to cross-sectional equilibria for $t \to \pm\infty$. For some results on non-equilibrium solutions which rely on such additional structure see, for example, [6], [23] and the references therein.

Outline: In Sections 2 and 3, we introduce the concept of global attractors for our particular setting. One of the main tools for our proof of Theorem 1, the Galerkin approximation, is explained and applied to global attractors of elliptic systems (1.1), (1.2). In Section 4 we review Conley index which is the second main tool in our proof. Section 5 is devoted to a detailed study of the neighborhood of a hyperbolic equilibrium. In Section 6 we prove Theorem 1 for the special case $\kappa = 1$ of two hyperbolic equilibria. In Section 7 we extend this result to a proof of Theorems 1 and 2, and we prove Corollary 1.1. We conclude with a brief discussion in Section 8.

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2 Elliptic Attractors

The set of bounded solutions of elliptic equations in infinite cylinders $Q = \mathbb{R} \times \Omega$ has been studied by several authors, from the viewpoint of dynamical systems methods; see for example [4], [7], [15], [25], [30], [40].

We define

$$\mathcal{A} = \{u \in H_a^2 | \quad u \text{ is a solution of } (1.1)\},$$

to be the set of bounded solutions of (1.1), (1.2). We recall that Dirichlet boundary conditions (1.2) are incorporated in the function space H_a^2; see (1.6), (1.7).

In analogy to dissipative evolution equations, the set $\mathcal{A}$ is called the *global attractor* of the elliptic system (1.1), (1.2). We refer to the monographs [5], [21], [26], and [38] for theory and applications of global attractors in dissipative equations; see also [8], [9] for a more recent account.

Though we do not make use of the attractivity property, we now briefly explain in which sense this terminology is justified in our elliptic set-up. Let K^+ denote the set of solutions which are defined only in the half-cylinder $Q_+ = \mathbb{R}_+ \times \Omega$, and which belong to the space $H_a^2(Q_+) := H_a^2\big|_{t \geq 0}$. We can define a semigroup on K^+ by translating solutions

$$(\mathcal{T}_s u)(t,x) := u(t+s,x) \quad , \quad s \geq 0 \tag{2.1}$$

This semigroup $\{\mathcal{T}_s, s \geq 0\}$ acts on K^+, because

$$\mathcal{T}_s K^+ \subset K^+,$$

by translational invariance of equations (1.1), (1.2) and of the norm in H_a^2; see (1.7). In fact, using the dissipation condition (1.4) it can be shown that there exists a global attractor $\mathcal{A}^+$ for the dynamics of $\mathcal{T}_s$ on K^+, with respect to the local topology H_{loc}^2; see [40] and [35]. In addition,

$$\mathcal{A}^+ = \Pi_+ \mathcal{A}$$

where $\Pi_+ : H_a^2(Q) \to H_a^2(Q_+)$ is the restriction operator. It is in this sense that we call $\mathcal{A}$ the 'global attractor'.

Unfortunately very little is known on the set K^+ in general. For the case of large γ, however, the set K^+ is an infinite-dimensional, smooth manifold; see [7] and [34]. We will not refer to the dynamical system structure on K_+ in the present paper.

The main result of this section is an existence result for $\mathcal{A}$.

Theorem 3. *Assume that $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfies the growth conditions (1.3) and the dissipation condition (1.4). Then the global attractor $\mathcal{A} \subset H_a^2$ is bounded and nonempty. Moreover $\mathcal{A} \subset H_{loc}^2$ is compact.*

The proof will be given in Section 3. As our main tool in the proof of Theorem 1, we introduce Galerkin approximations next.

Let $0 < \mu_1 < \mu_2 \leq ...$ denote the eigenvalues of $-\Delta_x$ on $L^2(\Omega)$ with Dirichlet boundary conditions, repeated with multiplicity. Let $e_j(x)$, $j = 1, 2, 3, ...$ be the corresponding complete L^2-orthonormal family of eigenfunctions

$$\begin{cases} -\Delta_x e_j(x) = \mu_j e_j(x)\,, \;\; x \in \Omega \\ e_j(x)\big|_{\partial\Omega} = 0 \end{cases}$$

The projections $P_m : L^2(\Omega)^N \to L^2(\Omega)^N$ are defined as the componentwise orthogonal projection onto span $\{e_1, ..., e_m\}$ in $L^2(\Omega)$.

The *Galerkin approximation* of (1.1) is defined as

$$\partial_t^2 u_m + \gamma \partial_t u_m + \Delta_x u_m + P_m f(u_m) = 0. \tag{2.2}$$

By $\mathcal{A}_m$ we denote the global attractor, alias the set of H_a^2 bounded solutions, of equation (2.2).

Let $u_m \in \mathcal{A}_m$. Then $\bar{u}_m(t, \cdot) = (1 - P_m) u_m(t, \cdot)$ satisfies the linear equation

$$\partial_t^2 \bar{u}_m(t, x) + \gamma \partial_t \bar{u}_m(t, x) + \Delta_x \bar{u}_m(t, x) = 0.$$

We claim $\bar{u}_m \equiv 0$, for large m. Indeed, projecting the above equation onto span $\{e_j\}$, we obtain a linear equation for $y_j(t) = \|P_{j+1}(1 - P_j)\bar{u}_m(t, \cdot)\|_{L^2}$,

$$\partial_t^2 y_j(t) + \gamma \partial_t y_j(t) - \mu_j y_j(t) = 0. \tag{2.3}$$

Solutions are of the form $e^{\lambda t} y_j^0$ with λ such that $\det(\lambda^2 + \gamma\lambda - \mu_j) = 0$. Equivalently, λ satisfies

$$\det\left((\lambda/\sqrt{\mu_j})^2 + \mu_j^{-1/2}\gamma(\lambda/\sqrt{\mu_j}) - 1 \right) = 0.$$

Since $\mu_j \to +\infty$ for $j \to \infty$, the N eigenvalues satisfy $\lambda = \pm\sqrt{\mu_j} + o(1)$ for $j \to \infty$. In particular, for $j \geq m_0$ large enough, all eigenvalues are bounded away from the imaginary axis. In consequence, there do not exist nontrivial bounded solutions of (2.3). Hence $\bar{u}_m(t,\cdot) = (1 - P_m)u_m(t,\cdot) \equiv 0$. This proves our claim.

In other words, the above computation shows that for sufficiently large m our definition of Galerkin approximation coincides with the traditional one, that is solutions $u_m \in \mathcal{A}_m$ of the Galerkin approximation (2.2) really lie in the finite-dimensional range of P_m:

$$u_m(t,\cdot) = \sum_{j=1}^{m} u_m^j(t) e_j(\cdot) = P_m u_m(t,\cdot), \quad \text{for } m \geq m_0, \tag{2.4}$$

where $u_m^j : \mathbb{R} \to \mathbb{R}^N$ are the appropriate vector functions. Note that the range of P_m is in fact a subspace of $(H^2(\Omega) \cap H_0^1(\Omega))^N$ because eigenfunctions are smooth. Moreover, range P_m is closed and has dimension $m \cdot N$.

Proposition 2.1. *The attractors $\mathcal{A}_m$ of the Galerkin approximation (2.2) are uniformly bounded in H_a^2 and compact in H_{loc}^2 for all $m \geq m_0$, with m_0 as in (2.4). Moreover, for every neighborhood $\mathcal{O}(\mathcal{A})$ of the set $\mathcal{A}$ in H_{loc}^2 there exists $m_1 = m_1(\mathcal{O}(\mathcal{A})) \geq m_0$ such that*

$$\mathcal{A}_m \subset \mathcal{O}(\mathcal{A}) \quad \textit{for } m \geq m_1 \tag{2.5}$$

The proof of this proposition and the next one is given in Section 3.

We conclude this exposition on elliptic attractors with a proposition on a homotopy from (2.2) to a linear equation, which is used in Section 6. For a homotopy parameter $0 \leq \vartheta \leq 1$, we consider

$$\partial_t^2 u + \Delta_x u + \vartheta(\gamma\partial_t u + P_m f(u)) = 0 \tag{2.6}$$

We emphasize that the constant m_0 in (2.4) can be chosen uniformly with respect to $\vartheta \in [0,1]$.

Proposition 2.2. *The global attractors $\mathcal{A}_{m,\vartheta}$ of (2.6) are bounded in H^2_a, uniformly for all $\vartheta \in [0,1]$ and $m \geq m_0$.*

We caution our reader that the attractors $\mathcal{A}, \mathcal{A}_m$ and $\mathcal{A}_{m,\vartheta}$ are compact in the H^2_{loc}-topology, but not necessarily in the t-uniform topology of H^2_a!

3 Upper Semicontinuity of Attractors

In this Section we prove Propositions 2.1, 2.2 and Theorem 3 from the previous section. The proofs are merely adaptations of similar proofs in [40], [41], and [4], to the case of our growth conditions (1.3) and (1.4).

Throughout this section C, C' and C'' stand for some positive constants with possibly updated values in different formulae. Moreover, we use the notation $\|u,\Omega\|_{l,p}$ and $\|u,Q\|_{l,p}$ for the Sobolev norms of functions on the the cross-section Ω, or on the cylinder $Q = \mathbb{R}\times\Omega$, respectively; see (1.5).

We begin with the proof of Proposition 2.2 which is prepared by two lemmata. We emphasize here, that both lemmata and the proof of Proposition 2.2 carry over almost verbatim to the case $m = \infty$, that is, to the original equation (1.1) and its global attractor $\mathcal{A}$ instead of the Galerkin approximation (2.6). In the following two lemmata uniform bounds in H_a and then in $H^{1,2}_a$ are derived. Uniform bounds in H^2_a are then established using a bootstrap argument.

Lemma 3.1. *The sets $\mathcal{A}_{m,\vartheta}$ are bounded in H_a, uniformly with respect to $m \geq m_0$ and $\vartheta \in [0,1]$.*

Proof. We introduce the function

$$y(t) = \int_\Omega u_m(t,x)\cdot u_m(t,x)\,dx = (u_m(t,\cdot), u_m(t,\cdot))$$

where $u_m = u_{m,\vartheta}$ is a solution of (2.6). It is not difficult to check that $u_m \in H^2_a$ implies $y''(t) \in H^{0,1}_a(\mathbb{R})$ and this derivative is given by

$$y''(t) = 2\left(\partial_t u_m(t), \partial_t u_m(t)\right) + 2\left(\partial_t^2 u_m(t), u_m(t)\right)$$

Next, we replace the term $\partial_t^2 u_m$ in the above equation by its expression from equation (2.6). We obtain

$$y''(t) - \alpha y(t) = h_u(t), \tag{3.1}$$

where

$$h_u(t) = 2\big((\partial_t u_m(t,\cdot), \partial_t u_m(t,\cdot)) + (\nabla_x u_m(t,\cdot), \nabla_x u_m(t,\cdot)) - \frac{1}{2}\alpha\,(u_m(t,\cdot), u_m(t,\cdot)) -$$
$$- \vartheta\,((\gamma\partial_t u_m(t,\cdot), u_m(t,\cdot)) + (f(u_m(t,\cdot)), u_m(t,\cdot)))\big), \tag{3.2}$$

and α is a chosen to be a sufficiently small positive number.

Using the dissipation condition (1.4), Poincaré's inequality and Hölder's inequality in (3.2), we obtain

$$h_u(t) \geq C(||\partial_t u_m(t,\cdot),\Omega||_{0,2}^2 + ||\nabla_x u_m(t,\cdot),\Omega||_{0,2}^2) - C' \geq -C' \tag{3.3}$$

for some positive constants C and C' not depending on $m \geq m_0$ and $\vartheta \in [0,1]$. Actually, this is the only place where we use the dissipation condition (1.4). The exponent σ is needed in order to compensate for the term $-\vartheta\,(\gamma\partial_t u_m(t,\cdot), u_m(t,\cdot))$.

By the maximum principle, we have $y(t) \leq C''$ for every globally bounded solution y, where the constant C'' depends only on α and C' from (3.3), and not on the solution u. Indeed, we can solve (3.1) for any $h_u \in H_a^{0,1}(\mathbb{R})$ using the explicit Greens function. The unique solution $y \in H_a^{2,1}(\mathbb{R})$ depends continuously on h_u and, by the maximum principle, y is bounded, at least for continuous h, bounded below. We may now approximate h_u in the space $H_a^{0,1}(\mathbb{R})$ by bounded continuous functions h_u^ε with $h_u^\varepsilon \geq -C'$. The solutions y^ε are bounded uniformly in ε and therefore give a uniform upper bound on the limit $y(t)$; see also [40]. This proves Lemma 3.1. ∎

Lemma 3.2. *The sets $\mathcal{A}_{m,\vartheta}$ are bounded in the space $H_a^{1,2}$, uniformly with respect to $m \geq m_0$ and $\vartheta \in [0,1]$.*

Proof. Let $\varphi(\cdot) \in C_0^\infty(\mathbb{R})$ be a cut-off function satisfying $\varphi(t) = 1$ for $t \in [T, T+1]$ and $\varphi(t) = 0$ for $t \notin [T-1, T+2]$. Note that the cut–off functions $\varphi(t) = \varphi_T(t)$ can be chosen such that $|\varphi''(t)| + |\varphi(t)| \leq C$, uniformly with respect to $T \in \mathbb{R}$. We multiply (3.1) with $\varphi(t)$ and integrate over $t \in \mathbb{R}$. Then

$$\begin{aligned} \int_{\mathbb{R}} \varphi(t) h_u(t)\, dt &= \int_{\mathbb{R}} \varphi(t)[y''(t) - \alpha y(t)]\, dt \\ &= \int_{\mathbb{R}} [\varphi''(t) - \alpha\varphi(t)] y(t)\, dt \le C||u_m||^2_{H_a}. \end{aligned} \tag{3.4}$$

Inequalities (3.3) and (3.4) imply that

$$\int_{\mathbb{R}} \varphi(t)[||\partial_t u_m(t,\cdot),\Omega||^2_{0,2} + ||\nabla_x u_m(t,\cdot),\Omega||^2_{0,2}]\, dt \le C(1 + ||u_m||^2_{H_a}),$$

uniformly with respect to $T \in \mathbb{R}$. Hence

$$||\partial_t u_m||^2_{H_a} + ||\nabla_x u_m||^2_{H_a} \le C(1 + ||u_m||^2_{H_a}),$$

which proves Lemma 3.2. ■

Proof of Proposition 2.2. Again, we multiply (2.6) by the cut-off function $\varphi(t)$ defined in the previous lemma, and we rewrite the equation in the following form

$$\partial_t^2(\varphi u_m) + \Delta_x(\varphi u_m) = \varphi'' u_m + 2\varphi'\partial_t u_m - \vartheta\varphi[\gamma\partial_t u_m + P_m f(u_m)] =: \hat{h}$$

It follows from L_2-regularity theory of the Laplacian that

$$||\varphi u_m, Q||_{2,2} \le C||\hat{h}, Q||_{0,2} \le C'(1 + ||u_m||_{H_a^{1,2}} + ||\varphi f(u_m), Q||_{0,2}) \tag{3.5}$$

Due to the growth conditions (1.3),

$$||\varphi f(u_m), Q||^2_{0,2} \le C\left(1 + \int_Q \varphi|u_m|^{2p}\, dxdt\right) \tag{3.6}$$

Hence, it is sufficient to estimate the integral in inequality (3.6).

We first consider the simpler case when $2p \le p_1 = 2(n+1)/(n-1)$, $n > 1$, or when $n = 1$. We then have the embedding $H_{1,2}(Q_{T_1,T_2}) \subset L_{2p}(Q_{T_1,T_2})$ for $T_2 > T_1$, with $Q_{T_1,T_2} = [T_1, T_2] \times \Omega$. Hence

$$\int_Q \varphi|u_m|^{2p}\, dxdt \le C \int_{Q_{T-1,T+2}} |u_m|^{2p}\, dxdt \le C_2||u_m, Q_{T-1,T+2}||^{2p}_{1,2} \le C_3(||u_m||_{H_a^{1,2}})^{2p}$$

Therefore, if $2p \le p_1$, the assertion of the Proposition 2.2 follows from (3.5), (3.6), and Lemma 3.2.

Next, we consider the case $2p > p_1$ and $n > 3$, the case $n \leq 3$ being simpler. Let $p_2 = 2(n+1)/(n-3)$ be the Sobolev embedding exponent such that $H_{2,2}(Q_T) \subset L_{p_2}(Q_T)$. To prepare for an application of Hölder's inequality, we now seek solutions α, β, l and k of the following system of equations

$$\begin{cases} \alpha + \beta = 2p; & \frac{1}{l} + \frac{1}{k} = 1; \\ \alpha l = p_1; & \beta k = p_2. \end{cases} \tag{3.7}$$

A computation shows that

$$\beta = p(n-1) - (n+1) < (1 + \frac{4}{n-1})(n-1) - (n+1) = 2. \tag{3.8}$$

Here we have used the constraint on the growth exponent p in the growth condition (1.3). Since we supposed $2p > p_1$, we obtain $0 < \beta < 2$. Now we check that all numbers α, β, k and l in (3.7)are positive. Indeed from the last equation of (3.7) we obtain $k = p_2/\beta > 2/2 = 1$, so $l > 1$ as well. The positivity of α follows immediately from the first equation of (3.7) and inequality (3.8).

Using Hölder's inequality and (3.7), we can now estimate

$$\begin{aligned} \int_Q \varphi |u_m|^{2p}\, dxdt &= \int_Q (\varphi^{1-\beta}|u_m|^\alpha)(\varphi^\beta |u_m|^\beta)\, dxdt \\ &\leq \left(\int_Q \varphi^{l(1-\beta)}|u_m|^{l\alpha}\, dxdt\right)^{1/l} \left(\int_Q \varphi^{k\beta}|u_m|^{k\beta}\, dxdt\right)^{1/k} \\ &\leq \left(\int_Q \varphi^{l(1-\beta)}|u_m|^{p_1}\, dxdt\right)^{\alpha/p_1} \left(\int_Q \varphi^{p_2}|u_m|^{p_2}\, dxdt\right)^{\beta/p_2} \\ &\leq C\left(||u_m||_{H_a^{1,2}}\right)^\alpha \left(||\varphi u_m, Q||_{2,2}\right)^\beta \end{aligned} \tag{3.9}$$

Putting together the estimates (3.5), (3.6), and (3.9), we obtain

$$||\varphi u_m, Q||_{2,2} \leq C(1 + ||u_m||_{H_a^{1,2}}) + C'\left(||u_m||_{H_a^{1,2}}\right)^{\alpha/2}\left(||\varphi u_m, Q||_{2,2}\right)^{\beta/2}$$

By (3.8), $\beta < 2$ and we conclude that

$$||\varphi u_m, Q||_{2,2} \leq C\left(1 + \left(||u_m||_{H_a^{1,2}}\right)^M\right)$$

for some positive constant $M = M(p)$. This inequality is valid uniformly with respect to $T \in \mathbb{R}$ and therefore

$$||u_m||_{H_a^2} = \sup_{T\in\mathbb{R}} ||u_m, Q_T||_{2,2} \leq C \sup_{T\in\mathbb{R}} ||\varphi_T u_m, Q||_{2,2} \leq C'\left(1 + \left(||u_m||_{H_a^{1,2}}\right)^M\right).$$

In view of Lemma 3.2, this proves Proposition 2.2.

Remark 3.3. *It follows from the estimates of Proposition 2.2 and Leray–Schauder degree theory that all sets $\mathcal{A}$, $\mathcal{A}_m$, $\mathcal{A}_{m,\vartheta}$ are non-empty because the corresponding equations have nonempty sets of equilibria; see [40].*

Proof of Proposition 2.1 and Theorem 3. Uniform estimates for the H^2_a norms of elements of $\mathcal{A}_m$ are obtained in the above proof of Proposition 2.2. The estimate for the H^2_a norms of elements of $\mathcal{A}$ can be obtained almost verbatim in the same way. It remains to prove compactness and upper semicontinuity under Galerkin approximation (2.5) in the topology of H^2_{loc}. Here we only prove upper semicontinuity, the proof of compactness being analogous but simpler.

Since H^2_{loc} is metrizable, it is sufficient to prove the following: from every sequence of solutions $u_m \in \mathcal{A}_m$ we can extract an H^2_{loc}-converging subsequence

$$u_{m_k} \to u \text{ in } H^2_{loc} \quad \text{and } u \in \mathcal{A}. \tag{3.10}$$

We fix an arbitrary $T \in \mathbb{R}$ and rewrite the equations for u_m in the following form

$$\partial_t^2(\varphi u_m) + \Delta_x(\varphi u_m) = \varphi'' u_m + 2\varphi'\partial_t u_m - \varphi[\gamma\partial_t u_m + P_m f(u_m)] =: \hat{h}_m$$

where the cut–of function φ is the same as in the proof of Lemma 3.2.

Due to the m-uniform estimates in Lemmata 3.1, 3.2, and in the proof of Proposition 2.2, the sequence $\{u_m\}$ is bounded in $H^2(Q_{T-1,T+2})$. Hence, there exists a function $u_T \in H^2(Q_{T-1,T+2})$ and a subsequence u_{m_k} — which, for simplicity, we denote again by u_m — such that

$$u_m \rightharpoonup u_T \quad \text{weakly in } H^2(Q_{T-1,T+2}).$$

We next prove that

$$\hat{h}_m \to \hat{h}_T := \varphi'' u_T + 2\varphi'\partial_t u_T - \varphi[\gamma\partial_t u_T + f(u_T)] \quad \text{in } L^2(Q) \tag{3.11}$$

By Sobolev's embedding and our restrictions on the growth exponent p, we have $u_m \to u_T$ in $L^{2p}(Q_{T-1,T+2}) \cap H^{1,2}(Q_{T-1,T+2})$ and therefore

$$\begin{aligned} &||\varphi P_m f(u_m) - \varphi f(u_T), Q_{T-1,T+2}||_{0,2} \\ &\quad \le ||\varphi P_m(f(u_m) - f(u_T)), Q_{T-1,T+2}||_{0,2} + ||(1-P_m)\varphi f(u_T), Q_{T-1,T+2}||_{0,2} \\ &\quad \le C||f(u_m) - f(u_T), Q_{T-1,T+2}||_{0,2} + ||(1-P_m)\varphi f(u_T), Q_{T-1,T+2}||_{0,2}. \end{aligned} \tag{3.12}$$

The right hand side tends to zero for $m \to \infty$. Indeed, the first term in the right hand side of (3.12) tends to zero by Krasnoselskii theorem and the second by Parseval's equality. This proves (3.11). From L_2-regularity theory of the Laplacian we obtain that $\varphi u_m \to \varphi u_T$ in $H^2(Q)$. Consequently

$$u_m \to u_T \quad \text{in } H^2(Q_T)$$

and the function u_T satisfies the equation (1.1) in Q_T. Taking $T \in \mathbb{Z}$ and applying Cantor's diagonalization procedure we can now construct a function $u \in \mathcal{A}$ and a subsequence u_{m_k} of u_m satisfying (3.10). This proves upper semicontinuity.

4 Conley Index

There are several excellent surveys of Conley index theory, both in finite and infinite dimensional dynamical systems. See for example [10], [31], [32], [37]. Here, we only collect some facts relevant to our proofs of Theorems 1 and 2. We note that Conley index theory does not apply, directly and computationally, in our elliptic context. Although our global attractor $\mathcal{A}$ is compact, by Theorem 3, uniqueness of solutions may not hold. Even where uniqueness does hold, it may not be clear, how to compute Conley indices directly within $\mathcal{A}$, in specific cases.

Therefore we use finite-dimensional Galerkin approximations. Accordingly, we consider Conley index for a finite-dimensional flow. Let

$$(t, u) \to u \cdot t \tag{4.1}$$

denote a finite-dimensional continuous flow on $u \in \mathbb{R}^q$; here u itself indicates the initial condition. Without loss of generality, we consider flows which are defined for all real t. In fact, any local flow can be extended to a global flow, possibly modifying the flow outside a large ball.

We call $\mathcal{S} \subseteq \mathbb{R}^q$ (flow) *invariant*, if

$$\mathcal{S} \cdot \mathbb{R} \subseteq \mathcal{S},$$

in the sense of (4.1). Note that invariance is required to hold for both positive and negative times. Any union of invariant sets is invariant. A bounded open subset

$\mathcal{N} \subseteq \mathbb{R}^q$ is called *isolating neighborhood*, if $\mathcal{N}$ contains the maximal invariant subset $\mathcal{S}$ of $\overline{\mathcal{N}} := \operatorname{clos} \mathcal{N}$. The set $\mathcal{S}$ is then compact, and is called *isolated invariant set*; it is isolated by $\mathcal{N}$ and by any open subset of $\mathcal{N}$ which contains $\mathcal{S}$.

An *index pair* $(\overline{\mathcal{N}}_1, \overline{\mathcal{N}}_0)$ of an isolated invariant set $\mathcal{S}$ is defined to consist of two open bounded sets $\mathcal{N}_1 \supseteq \mathcal{N}_0$ such that

(i) $\mathcal{N}_1 \setminus \overline{\mathcal{N}}_0$ is an isolating neighborhood of $\mathcal{S}$;

(ii) $\overline{\mathcal{N}}_0$ is *positively invariant* in $\overline{\mathcal{N}}_1$; and

(iii) $\overline{\mathcal{N}}_0$ is an *exit set* for $\overline{\mathcal{N}}_1$.

Here positive invariance, (ii), means that $u \in \overline{\mathcal{N}}_0,\ u \cdot [0,t] \subset \overline{\mathcal{N}}_1$ implies $u \cdot [0,t] \subseteq \overline{\mathcal{N}}_0$, for $t \geq 0$. The exit set property (iii) means that $u \in \overline{\mathcal{N}}_1, u \cdot t_1 \notin \overline{\mathcal{N}}_1$ for some $t_1 > 0$ imply existence of some $t_0 \in [0, t_1)$ with $u \cdot [0, t_0] \subseteq \overline{\mathcal{N}}_1$ and $u \cdot t_0 \in \overline{\mathcal{N}}_0$. Isolated invariant sets do possess index pairs; see [10].

The *Conley index* $\mathcal{C}(\mathcal{S})$ of an isolated invariant set $\mathcal{S}$ is the homotopy type of the pointed space

$$\mathcal{C}(\mathcal{S}) = (\overline{\mathcal{N}}_1 / \overline{\mathcal{N}}_0, [\overline{\mathcal{N}}_0]),$$

where $(\overline{\mathcal{N}}_1, \overline{\mathcal{N}}_0)$ is an index pair for $\mathcal{S}$. We obtain the homotopy type of the pointed space $(\overline{\mathcal{N}}_1 / \overline{\mathcal{N}}_0, [\overline{\mathcal{N}}_0])$ from $\overline{\mathcal{N}}_1$ by collapsing $\overline{\mathcal{N}}_0$ to a single, distinguished point. It turns out that the Conley index is independent of the particular choice of an index pair for $\mathcal{S}$; see again [10].

For example

$$\mathcal{C}(\{0\}) = \Sigma^l \tag{4.2}$$

is the l-dimensional sphere Σ^l with a distinguished point, if $u = 0$ is a hyperbolic equilibrium of unstable dimension l. In a variational context, where $u \cdot t$ is a gradient flow, l would be called the *Morse index* of the critical point $u = 0$.

In Section 6, we compute

$$\mathcal{C}(\mathcal{A}_m) = \Sigma^{mN}$$

for the set $\mathcal{A}_m$ of bounded trajectories of the Galerkin flow (2.2) in $\mathbb{R}^{2mN}$. Note that $\mathcal{A}_m$ need not, in general, consist of just a single hyperbolic equilibrium of unstable dimension mN.

As a third example, consider an isolated invariant set S which decomposes into two disjoint isolated invariant sets $\mathcal{S}_1$ and $\mathcal{S}_2$. Then

$$\mathcal{C}(\mathcal{S}) = \mathcal{C}(\mathcal{S}_1) \vee \mathcal{C}(\mathcal{S}_2), \tag{4.3}$$

where $\vee$ is the wedge product: the two distinguished points of $\mathcal{C}(\mathcal{S}_1)$ and $\mathcal{C}(\mathcal{S}_2)$ are identified.

Homotopy invariance is one of the most powerful properties of Conley index, from a computational point of view. We only need a rather simple version, which we formulate next.

Proposition 4.1. *Consider a family of flows on $\mathbb{R}^q$ depending continuously on a parameter $\vartheta \in [0,1]$. Let $\mathcal{N} \subset \mathbb{R}^q$ be an isolating neighborhood, for all ϑ, with isolated invariant set $(\mathcal{S}(\vartheta), \vartheta)$. Here $\mathcal{S}(\vartheta) \subseteq \mathbb{R}^q$ denotes the set itself, and the second component ϑ indicates the flow parameter used.*

Then the Conley index does not depend on the flow parameter $\vartheta \in [0,1]$:

$$\mathcal{C}(\mathcal{S}(0), 0) = \mathcal{C}(\mathcal{S}(\vartheta), \vartheta) = \mathcal{C}(\mathcal{S}(1), 1).$$

For a proof, we refer to [10].

5 Hyperbolic Equilibria

The main objective of this section is to show that hyperbolic equilibria in the sense of (1.9) are isolated as bounded solutions of (1.1), (1.2) in H_a^2. In Section 6, this allows us to show that hyperbolic equilibria behave like isolated invariant sets for the Galerkin approximation.

Proposition 5.1. *Suppose w is a hyperbolic equilibrium of (1.1), (1.2). Then w is isolated in H_a^2 as a solution of (1.1), (1.2). That is, there exists a neighborhood $\mathcal{U}$ of w in H_a^2 such that*

$$\mathcal{A} \cap \mathcal{U} = \{w\} \tag{5.1}$$

The proof requires a thorough analysis of the linearization

$$Lu = \partial_t^2 u + \gamma \partial_t u + [\Delta_x + f'(w(x))]u \tag{5.2}$$

and is prepared with several lemmata.

We first prove in Lemma 5.2 that the "time" t Fourier transform

$$\begin{aligned} \hat{L}(\lambda):\ (H^2(\Omega) \cap H_0^1(\Omega))^N &\to L^2(\Omega)^N \\ \hat{u}(\cdot) &\mapsto \left(-\lambda^2 + i\lambda\gamma + [\Delta_x + f'(w)]\right)\hat{u}(\cdot) \end{aligned}$$

is invertible for all λ in a narrow strip $|Im\ \lambda| \leq \delta_0$, by hyperbolicity assumption (1.9). As a second step, we invert L on $L^2(Q)^N$, in Lemma 5.3. With the exponential decay estimates of Lemma 5.4 for $|t| \to \infty$, we then prove surjectivity of $L : H_a^2 \to H_a$ in Lemma 5.5. For injectivity, Lemma 5.6, we make use of the formal adjoint

$$L^* u = \partial_t^2 u - \gamma^* \partial_t u + [\Delta_x + f'(w(x))^*]u, \tag{5.3}$$

and its Fourier transform

$$(\hat{L})^*(\lambda) = -\lambda^2 - i\lambda\gamma^* + [\Delta_x + f'(w(x))^*].$$

Note that L^* is hyperbolic in the sense of (1.9) if, and only if, L itself is hyperbolic. Indeed, $(\hat{L})^*(\lambda)$ is the adjoint operator to $\hat{L}(\lambda)$ in $L^2(\Omega)^N$ and both operators are Fredholm of index zero from $H^2(\Omega)^N \cap H_0^1(\Omega)^N$ into $L^2(\Omega)^N$ as compact perturbations of Δ_x.

In consequence, Lemmata 5.2 – 5.5 also hold with L being replaced by L^*. Finally, Lemma 5.5 for the adjoint L^* is used in Lemma 5.6 to show injectivity of $L : H_a^2 \to H_a$. An application of the inverse function theorem, based on the invertibility of the linearization L then completes the proof of Proposition 5.1.

Lemma 5.2. *Assume L is hyperbolic in the sense of (1.9). Then there exist constants $M, \delta_0 > 0$ such that for all λ in the strip $|Im\ \lambda| \leq \delta_0$ we have*

$$|\lambda^{2-\ell}\hat{L}(\lambda)^{-1}|_{\mathcal{L}(L^2(\Omega)^N, H^\ell(\Omega)^N)} \leq M, \quad \ell = 0, 1, 2, \tag{5.4}$$

and $\hat{L}(\lambda)^{-1}$ is analytic as a function of λ in the strip $|Im\ \lambda| \leq \delta_0$ with values in $\mathcal{L}(L^2(\Omega)^N, H^\ell(\Omega)^N)$.

Setting $\ell = 2$, we note that $\hat{L}(\lambda)$ is invertible for all λ in the strip with uniform bounds.

Proof. As already mentioned, the elliptic operator $\hat{L}(\lambda)$ is Fredholm of index zero from $H^2(\Omega)^N \cap H_0^1(\Omega)^N$ into $L^2(\Omega)^N$, for any fixed λ. By our hyperbolicity assumption, the kernel is trivial and $\hat{L}(\lambda)^{-1} : L^2(\Omega)^N \to H^2(\Omega)^N \cap H_0^1(\Omega)^N$ exists and is bounded, for all real λ. To show analyticity of $\hat{L}(\lambda)^{-1}$, we use the factorization

$$\hat{L}(\lambda+\eta) = \left(\text{id} + \left(\hat{L}(\lambda+\eta) - \hat{L}(\lambda)\right) \hat{L}(\lambda)^{-1}\right) \hat{L}(\lambda),$$

with $\hat{L}(\lambda+\eta) - \hat{L}(\lambda) = \eta(i\gamma - 2\lambda - \eta)$. For $\eta \in \mathbb{C}$ close to zero and $\lambda \in \mathbb{R}$, the first factor is close to identity, and we obtain a Neumann series for $\hat{L}(\lambda+\eta)^{-1}$. In particular $\hat{L}(\lambda)^{-1}$ exists for λ in an open neighborhood of the real axis. This proves (5.4) in any rectangle $|Re\ \lambda| \leq R < \infty$, $|Im\ \lambda| \leq \delta_0'(R)$, with a constant $M = M'(R)$.

For large $|Re\ \lambda|$ we compare $\hat{L}(\lambda)^{-1}$ with the resolvent $(\Delta_x - \lambda^2)^{-1}$ of the Laplacian Δ_x with Dirichlet boundary conditions. In fact, elliptic regularity theory implies that

$$||\lambda^{2-\ell}(\Delta_x - \lambda^2)^{-1}||_{\mathcal{L}(L^2(\Omega)^N, H^\ell(\Omega)^N)} \leq \tilde{M} \tag{5.5}$$

for $\ell = 0, 1, 2$ and $|Im\lambda| \leq \delta_0''$. This proves the required estimate (5.4) for $(\Delta_x - \lambda^2)^{-1}$. In the strip $|Im\lambda| \leq \delta_0''$, we factorize

$$\hat{L}(\lambda) = \left(\text{id} + \left(\hat{L}(\lambda) + \lambda^2 - \Delta_x\right)(\Delta_x - \lambda^2)^{-1}\right)(\Delta_x - \lambda^2),$$

with $\hat{L}(\lambda) + \lambda^2 - \Delta_x = i\lambda\gamma + f'(w(x))$. The first factor is uniformly close to identity in $\mathcal{L}(L^2(\Omega)^N)$. For $|Re\lambda| \geq R_0$ and $|Im\lambda| \leq \delta_0''$ the estimate (5.4) for $\hat{L}(\lambda)^{-1}$ now follows from the corresponding estimate (5.5), again by Neumann series, putting $\delta_0 = \min\{\delta_0'(R_0), \delta_0''\}$. This proves the Lemma. ∎

Lemma 5.3. *The hyperbolic linearization L defined in (5.2) is a bounded linear isomorphism from $H^2(Q)^N \cap H_0^1(Q)^N$ to $L^2(Q)^N$.*

Proof. We solve $Lu = \varphi$, $\varphi \in L^2(Q)^N$, via Fourier transform. Let

$$\hat{\varphi}(\lambda, x) \ = \ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda t} \varphi(t,x) dt \ \in \ L^2(Q)^N \tag{5.6}$$

and define $\hat{u}(\lambda) = \hat{L}(\lambda)^{-1}\hat{\varphi}(\lambda)$

By Lemma 5.2,

$$||\lambda^{2-l}\hat{u}(\lambda, x)||_{L^2(\mathbb{R}, H^l(\Omega)^N)} \leq M||\varphi||_{L^2(Q)^N}.$$

Inverse Fourier transform now proves the Lemma. ■

Lemma 5.4. *Consider hyperbolic L and compactly supported $\varphi \in L^2(Q)^N$ such that $\varphi = 0$ for $t \notin [0, 1]$. Then there exist constants $M_1, \delta_0 > 0$ such that $u = L^{-1}\varphi$ satisfies an exponential decay estimate*

$$||\cosh(\delta_0 t)u||_{H^2(Q)^N} \leq M_1||\varphi||_{L^2(Q)^N}$$

Proof. The Fourier transform $\hat{\varphi}(\lambda)$, defined in (5.6), is globally analytic in $\lambda \in \mathbb{C}$ because φ has compact support in t. Moreover

$$||\hat{\varphi}||_{L^2(\mathbb{R}+i\delta, L^2(\Omega)^N)} \leq e^{|\delta|}||\varphi||_{L^2(Q)^N}$$

for any fixed $\delta \in \mathbb{R}$. By Lemma 5.2, $\hat{L}^{-1}(\lambda)$ is analytic for $\lambda \in \mathbb{R} + i\delta$, $|\delta| \leq \delta_0$. Moreover, $\hat{u} = \hat{L}^{-1}\hat{\varphi}$ satisfies an estimate

$$||\lambda^{2-\ell}\hat{u}||_{L^2(\mathbb{R}+i\delta, H^\ell(\Omega)^N)} \leq Me^{|\delta|}||\varphi||_{L^2(Q)^N} \tag{5.7}$$

for $|\delta| \leq \delta_0$, $\ell = 0, 1, 2$.

Now define the Fourier inversion with shifted integration paths

$$\tilde{u}(t, x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}-i\delta_0} e^{-i\lambda t}\ \hat{u}(\lambda, x)d\lambda & \text{for } t \geq 0 \\ \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}+i\delta_0} e^{-i\lambda t}\ \hat{u}(\lambda, x)d\lambda & \text{for } t < 0. \end{cases}$$

By the estimates (5.7) on $\hat{u}$ in the strip $|Im\lambda| \leq \delta_0$ we have

$$||\cosh(\delta_0 t)\tilde{u}(t, x)||_{H^2(Q)^N} \leq M_1'||\varphi||_{L^2(Q)^N}.$$

It remains to show that $\tilde{u} = u$ is indeed the desired solution of $Lu = \varphi$. We first fix $t \geq 0$; the case $t < 0$ is similar. Define

$$\tilde{u}_R(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-R-i\delta_0}^{R-i\delta_0} e^{-i\lambda t}\ \hat{u}(\lambda, x)d\lambda$$

$$u_R(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-i\lambda t}\ \hat{u}(\lambda, x)d\lambda$$

Clearly $\widetilde{u}_R(t) \to \widetilde{u}(t)$ and $u_R(t) \to u(t)$ in $L^2(\Omega)^N$, for $R \to \infty$.

By Lemma 5.2, the integrand is holomorphic in the rectangle $|Re\lambda| \leq R$, $-\delta_0 \leq Im\lambda \leq 0$. Cauchy's integral formula therefore implies that

$$u_R(t) - \tilde{u}_R(t) = \frac{1}{\sqrt{2\pi}}\left(\int_{R-i\delta_0}^{R} + \int_{-R}^{-R-i\delta_0}\right)e^{-i\lambda t}\hat{u}(\lambda)d\lambda. \tag{5.8}$$

We show that both integrals converge to zero in $L^2(\Omega)^N$, for $R \to +\infty$. Indeed we recall that

$$\hat{\varphi}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^1 e^{i\lambda t}\varphi(t)dt$$

is bounded, uniformly for λ in the strip $|Im\lambda| \leq \delta_0$, with values in $L^2(\Omega)^N$. By (5.7), $l = 0$, the same holds for $\lambda^2\hat{u}(\lambda)$.

Therefore

$$\left\|\int_{R-i\delta_0}^{R} e^{-i\lambda t}\hat{u}(\lambda)d\lambda\right\|_{L^2(\Omega)^N} \leq \sup ||\lambda^2\hat{u}(\lambda)||_{L^2(\Omega)^N}\int_0^{\delta_0} e^{\lambda' t}d\lambda' \cdot \frac{1}{R^2}.$$

A similar bound on the other integral in (5.8) proves that

$$||\tilde{u}_R(t) - u_R(t)||_{L^2(\Omega)^N} \to 0 \quad \text{for } R \to \infty.$$

This proves $\tilde{u}(t) = u(t)$ in $L^2(\Omega)^N$, and the proof of Lemma 5.4 is complete. ■

In the next two Lemmata 5.5 – 5.6, we prove bounded invertibility of L in the t-uniform spaces H_a^2, H_a.

Lemma 5.5. *The hyperbolic operator L is surjective from H_a^2 to H_a. Specifically, there is a bounded linear right inverse $L_0^{-1} : H_a \to H_a^2$, such that $LL_0^{-1} = id$ on H_a.*

Proof. Let $\varphi \in H_a$ be given. We have to find u such that $Lu = \varphi$. We decompose $\varphi = \sum_{j\in\mathbb{Z}} \varphi_j$ with $\varphi_j = \varphi \cdot \chi_{[j,j+1]}(t)$. Here the indicator function $\chi_{[j,j+1]}(t) = 1$ for $t \in [j, j+1]$, and 0 otherwise. Let $u_j := L^{-1}\varphi_j$. Note that $L^{-1}\varphi_j$ is well defined, by Lemma 5.3. From translation invariance of L and of the norms in H_a^2 and H_a with respect to t, together with exponential decay from Lemma 5.4, we conclude that

$$||u_j \cosh(\delta_0(t-j))||_{H^2(Q)^N} \leq M_1||\varphi_j||_{H_a} \leq M_1||\varphi||_{H_a}.$$

Here, again $Q_k = [k, k+1] \times \Omega$. In particular, there is a constant M_2 independent of j, k such that

$$||u_j||_{H^2(Q_k)^N} \leq M_2 e^{-\delta_0|j-k|}||\varphi||_{H_a}$$

Therefore the sum $u = \sum u_j$ converges in $H^2(Q_k)^N$, for any $k \in \mathbb{Z}$, and

$$Lu = L\sum_j u_j = \sum_j Lu_j = \sum_j \varphi_j = \varphi.$$

Moreover we have obtained a bound for the solution $u := L_0^{-1}\varphi$ constructed above:

$$||u||_{H_a^2} \leq M_2'||\varphi||_{H_a}.$$

■

Lemma 5.6. *Assume L is hyperbolic. If $u \in H_a^2$ and $Lu = 0$, then $u = 0$.*

Proof. Consider the unbounded formal adjoint operator L^* of L on $L^2(Q)^N$, defined in (5.3). Recall that L^* is hyperbolic because L is hyperbolic. In particular, Lemma 5.3 implies that L^* is invertible on $L^2(Q)^N$. Decomposing $u_j = u \cdot \chi_{[j,j+1]}(t)$ for $j \in \mathbb{Z}$, as in the proof of Lemma 5.5, we consider $v_j := (L^*)^{-1}u_j \in H^2(Q)^N$.

By Lemma 5.4, applied to L^*, the v_j satisfy exponential estimates

$$||\cosh(\delta_0(t-j))v_j||_{H^2(Q)^N} \leq M_1^*||u||_{H_a}$$

Now, $Lu = 0$ and integration by parts yields

$$0 = \int_{\mathbb{R}}\int_{\Omega} v_j \cdot Lu = \int_{\mathbb{R}}\int_{\Omega} u \cdot L^*v_j = \int_{\mathbb{R}}\int_{\Omega} u \cdot u_j = \int_j^{j+1}\int_{\Omega} |u_j|^2,$$

for all $j \in \mathbb{Z}$. Note that boundary terms of the partial integration with respect to t vanish. Indeed $v_j(t)$ and $\partial_t v_j(t)$ decay exponentially with $e^{-\delta_0|t|}$ in $L^2(\Omega)^N$ and $u(t)$ and $\partial_t u(t)$ are bounded in $L^2(\Omega)^N$, by the Sobolev trace formula

$$\partial_t v_j \in H^1(Q) \hookrightarrow L^2(\Omega)^N \ni \partial_t v_j(t).$$

This proves the lemma. ■

Corollary 5.7. *The hyperbolic operator L is an isomorphism from H_a^2 to H_a.*

Proof. By injectivity, Lemma 5.6, the bounded right inverse L_0^{-1} constructed in Lemma 5.5 is indeed the inverse of L. ■

Remark 5.8. *We emphasize that there is a dynamical interpretation for the set*

$$\operatorname{spec}\hat{L}(\cdot) := \{\lambda \in \mathbb{C} | \hat{L}(\lambda) \text{ possesses non-trivial kernel}\},$$

usually called the spectrum of the operator pencil $\hat{L}(\cdot)$. *Writing the linearized equation* $Lu = 0$ *formally as a first-order differential equation in* t,

$$\partial_t u = v, \quad \partial_t v = -\gamma v - [\Delta_x + f'(w(x))]u, \tag{5.9}$$

we can associate to each $\lambda \in \operatorname{spec}\hat{L}(\cdot)$ *a solution* $(u,v)(t,x) = \exp(i\lambda t)(u_0(x), i\lambda u_0(x))$ *of (5.9). In other words,* $i \cdot \operatorname{spec}\hat{L}(\cdot) = \operatorname{spec}\mathrm{L}$, *where* L *is the operator on the right sides of (5.9),*

$$\begin{array}{rcl} \mathrm{L} : \left(H^2(\Omega)^N \cap H_0^1(\Omega)^N\right) \times H_0^1(\Omega)^N & \to & H_0^1(\Omega)^N \times L^2(\Omega)^N \\ (u,v) & \mapsto & (v, -\gamma v - [\Delta_x + f'(w(x))]u). \end{array}$$

In this setting, γv *and* $f'(w(x))u$ *can be considered as a relatively compact perturbation of the Laplace equation* $\partial_t^2 u + \Delta_x u = 0$, *with spectrum* $\pm\sqrt{\mu_k}$, $k \in \mathbb{N}$, *where again* $0 < \mu_1 < \mu_2 \leq \ldots$ *stand for the eigenvalues of the Laplacian. In particular, we recover the ill-posedness of the initial-value problem in the sense that the spectrum* $\operatorname{spec}\mathrm{L}$ *has unbounded positive and negative real parts. For more general results on operator pencils we refer to [20] and [28, 29].*

We now return to the nonlinear equation (1.1), (1.2) in a neighborhood of the equilibrium w.

Lemma 5.9. *The Nemitskii operator* $\tilde{f} : H_a^2 \to H_a$, $\tilde{f}(u)(t,x) := f(u(t,x))$ *is of class* C^1. *The derivative* $D\tilde{f}(u) \in \mathcal{L}(H_a^2, H_a)$ *is uniformly continuous on bounded subsets of* H_a^2.

Proof. The corresponding Nemitskii operator $\tilde{f}$ from $L^{2p}(Q_0)^N$ to $L^2(Q_0)^N$ is continuously differentiable by the growth assumption (1.3) and the Krasnoselskii lemma [2]. By Sobolev embedding, $\tilde{f}$ is also continuous as a map from $H^2(Q_0)^N$ to $L^2(Q_0)^N$.

By compactness of the embedding, it is uniformly continuous on bounded subsets of $H^2(Q_0)^N$. For $R > 0$, let $\omega_R(\cdot)$ denote the modulus of continuity of $\tilde{f}$ on the ball of radius R centered at the origin in $H^2(Q_0)^N$.

Let again $\mathcal{T}_s$ denote the shift of functions by s along the axis of the cylinder; see (2.1). For $||u||_{H_a^2} \leq R$ and $h \to 0$ in H_a^2 we estimate

$$||\tilde{f}(u+h) - \tilde{f}(u)||_{H_a} = \sup_{s\in\mathbb{R}} ||f(\mathcal{T}_s(u+h)) - f(\mathcal{T}_s u), Q_0||_{0,2} \leq$$
$$\leq \sup_{s\in\mathbb{R}} \omega_{2R}(||\mathcal{T}_s h, Q_0||_{2,2}) \leq \omega_{2R}(||h||_{H_a^2}). \quad (5.10)$$

Following the same type of reasoning for the derivative f', we see that derivatives on H_a^2 exist and are uniformly continuous on bounded subsets of H_a^2. This proves the lemma. ■

Proof of Proposition 5.1. We have to show that any hyperbolic equilibrium w of (1.1), (1.2) is isolated, as a solution in H_a^2. Suppose $u = w + \tilde{u} \in \mathcal{A} \subset H_a^2$, with $\tilde{u}$ small in H_a^2. Then

$$L\tilde{u} = -(\tilde{f}(w+\tilde{u}) - \tilde{f}(w) - D\tilde{f}(w)\tilde{u}) =: R(\tilde{u})$$

holds for the linearization L at w defined in (5.2). By Lemma 5.8, $R \in C^1(H_a^2, H_a)$ and $DR(0) = 0$. On the other hand, $L \in GL(H_a^2, H_a)$ is boundedly invertible. By the inverse function theorem, the solution $\tilde{u} = 0$, alias $u = w$, is therefore unique in a neighborhood $\mathcal{U}$ of w in H_a^2.

We finish this chapter by extending the above result 'continuously' to the Galerkin approximation (2.2).

Proposition 5.10. *Let w be a hyperbolic equilibrium of (1.1), (1.2). Then there exists $m_0 \in \mathbb{N}$ and a neighborhood $\mathcal{U}(w) \subset H_a^2$ of w such that for all $m \geq m_0$ the following holds:*

(i) $\mathcal{U}(w) \cap \mathcal{A}_m = \{w_m\} \subset H_a^2$;

(ii) w_m are equilibria of the Galerkin approximation (2.2). Moreover $w_m \to w$ in H_a^2, for $m \to \infty$;

(iii) The linearization L_m of (2.2) at w_m is invertible with m-uniform bounds, as a map from H_a^2 to H_a.

Proof. We consider the left hand sides of (1.1) and (2.2) as nonlinear operators from H_a^2 to H_a. By Lemma 5.8, these operators are of class C^1. We show that they depend continuously on m. We then complete the proof by invoking an implicit function theorem with respect to the "parameter" m.

We first claim that the difference $(1-P_m)\tilde{f} : H_a^2 \to H_a$ converges to zero with respect to uniform C^1-convergence on bounded subsets of H_a^2. Let us prove convergence in C^0 first. We argue by contradiction. Suppose

$$||(1-P_m)\tilde{f}(u_m)||_{H_a} \geq \varepsilon > 0$$

for some bounded sequence $u_m \in H_a^2$. Possibly shifting the u_m in t, by t'_m, we may then assume

$$||(1 \ - P_m)\tilde{f}(u_m), Q_0||_{0,2} \geq \varepsilon/2 > 0$$

By compactness of the embedding $H^2(Q_0)^N \hookrightarrow L^{2p}(Q_0)^N$ we may assume $u_m \to u$ in $L^{2p}(Q_0)^N$, possibly for a subsequence. Therefore

$$||(1-P_m)\tilde{f}(u), Q_0||_{0,2} \geq \varepsilon/2 - ||\tilde{f}(u_m) - \tilde{f}(u), Q_0||_{0,2} \geq \varepsilon/4$$

for m large, by continuity of the Nemitskii operator $\tilde{f}_{loc} : L^{2p}(Q_0)^N \to L^2(Q_0)^N$. This clearly contradicts the strong convergence $(1 \ - P_m)\tilde{f}(u) \to 0$ in $L^2(Q_0)^N$ for $m \to \infty$. For the derivative $(1-P_m)D\tilde{f} : H_a^2 \to H_a$, the arguments are similar and we omit the details.

We now consider the Galerkin approximation (2.2) together with the limit (1.1) as a family of equations with "parameter" m. To equation (1.1) we naturally associate the parameter value $m = \infty$. The parameter space then becomes a metric space with discrete metric for finite m and distance $d(m, \infty) = \frac{1}{m}$. Having established continuous dependence on the parameter m in this sense, we invoke the implicit function theorem; see for example [36, Ch.III,Thm.25]. This yields a locally unique family of solutions $w_m \in H_a^2$ of the Galerkin approximation (2.2) such that $w_m \to w$ in H_a^2 for $m \to \infty$. This proves (i). By uniqueness, w_m is translation invariant and thereby an equilibrium. This proves (ii). The last assertion, (iii), follows simply from convergence of the equilibria w_m in H_a^2 and of the derivatives $(1 - P_m)D\tilde{f}(w_m)$ in $\mathcal{L}(H_a^2, H_a)$. This proves the proposition. ∎

6 Existence of Non-Equilibrium Solutions

As a first step towards Theorem 1 we prove the following crucial proposition:

Proposition 6.1. *Suppose w_1 and w_2 are two equilibria of (1.1), (1.2), both hyperbolic in the sense of (1.9). Then at least one of these two equilibria is not isolated in $\mathcal{A} \subset H^2_{loc}$.*

From this result it is easy to conclude the existence of a non-equilibrium solution by the following central argument. For equilibrium solutions, convergence in the space H^2_{loc} coincides with convergence in the t-uniform space H^2_a. By the above Proposition 6.1, at least one of the two equilibria, say w_1, *is not isolated* in $\mathcal{A}$ — with respect to the topology of H^2_{loc}. By hyperbolicity, Proposition 5.1, on the other hand, w_1 *is isolated* in $\mathcal{A}$ — with respect to the t-uniform topology of H^2_a. In particular, w_1 is isolated within the set of equilibria, even with respect to the topology of H^2_{loc}. Therefore w_1, not being H^2_{loc}-isolated in $\mathcal{A}$, must be an accumulation point, in H^2_{loc}, of non-equilibrium solutions in $\mathcal{A}$. As we will see in the next section, these non-equilibrium solutions can in fact be chosen to belong to a single non-equilibrium trajectory in $\mathcal{A}$.

We outline our proof of Proposition 6.1. The proof is based on Conley index theory for the Galerkin approximation (2.6) with homotopy parameter $0 \le \vartheta \le 1$. Recall that bounded solutions of the Galerkin approximation lie in the finite-dimensional subspace range P_m for $m \ge m_0$ and any fixed $t \in \mathbb{R}$; see (2.4). Therefore, instead of (2.6), we may consider the following system of ordinary differential equations

$$\begin{aligned} \frac{du}{dt} &= v \\ \frac{dv}{dt} &= -\Delta_x u - \vartheta(\gamma v + P_m f(u)). \end{aligned} \tag{6.1}$$

Here the pair $\xi = (u, v)$ belongs to the phase space

$$V_m := P_m L^2(\Omega)^N \times P_m L^2(\Omega)^N \cong \mathbb{R}^{2mN}.$$

See Lemma 6.2 for this reduction to an ordinary differential equation. The right–hand side of (6.1) is of class C^1, because f is of class C^1. Hence (6.1) defines a local C^1-flow on V_m. We write

$$\xi_0 \cdot t := \xi(t) \tag{6.2}$$

where $\xi(t) = (u(t), v(t))^T$ is a solution of (6.1) and $\xi(0) = \xi_0 \in V_m$.

Let $\mathcal{A}'_{m,\vartheta} \subset V_m$ denote the initial values of global orbits which are bounded in V_m for all positive and negative times. We abbreviate $\mathcal{A}'_m := \mathcal{A}'_{m,1}$. Note that $\mathcal{A}'_m$ is an isolated invariant set in the sense of Section 4. It is invariant under the flow of (6.2) and, by boundedness and maximality, it is isolated in any sufficiently large ball. Our strategy of proof for Proposition 6.1 is as follows. We compute the Conley index of $\mathcal{A}'_m$ using the homotopy parameter ϑ. For $\vartheta = 0$, the differential equation is linear and the Conley index that of a hyperbolic equilibrium. Using the a priori estimates from Propositions 2.1 and 2.2, and the continuation property of Proposition 4.1, we have thus calculated the Conley index of $\mathcal{A}'_m = \mathcal{A}'_{m,1}$; see Lemma 6.3.

The proof of Proposition 6.1 is then completed indirectly, as follows. We suppose the two equilibria w_1 and w_2 were isolated. We could then write $\mathcal{A}'_m$ as a disjoint union of two hyperbolic equilibria, and a compact complement. Using the wedge formula (4.3) for Conley index we then reach a contradiction to the assumption of w_1 and w_2 being hyperbolic.

As a first step, we relate the dynamics on $\mathcal{A}_{m,\vartheta} \subset H^2_{loc}$ and $\mathcal{A}'_{m,\vartheta} \subset V_m$. Recall that the dynamics on $\mathcal{A}_{m,\vartheta} \subset H^2_{loc}$ is defined by the shift $\mathcal{T}_s$ of bounded solutions; see (2.1). The dynamics on $\mathcal{A}'_{m,\vartheta}$, on the other hand, is induced by the ordinary differential equation (6.1). Time orbits in $\mathcal{A}_{m,\vartheta}$ or $\mathcal{A}'_{m,\vartheta}$ are always understood as trajectories with respect to the so-defined dynamics.

Lemma 6.2. *Assume $m \geq m_0$. Then there is a homeomorphism $\Pi_0 = \Pi_0(m, \vartheta)$: $\mathcal{A}_{m,\vartheta} \to \mathcal{A}'_{m,\vartheta}$, such that*

$$\xi \cdot s := \Pi_0 \, \mathcal{T}_s \, {\Pi_0}^{-1} \, \xi, \quad \text{for all} \quad \xi \in \mathcal{A}'_{m,\vartheta} \quad \text{and all} \quad s \in \mathbb{R}. \tag{6.3}$$

In particular, all $\mathcal{A}'_{m,\vartheta}$ are compact and bounded in V_m, uniformly with respect to $0 \leq \vartheta \leq 1$.

Proof. We define Π_0 as the trace operator

$$\begin{aligned} \Pi_0 : H^2_{loc} &\rightarrow L^2(\Omega)^N \times L^2(\Omega)^N \\ u &\mapsto \Pi_0 u := \{u\big|_{t=0}, \, \partial_t u\big|_{t=0}\}. \end{aligned} \tag{6.4}$$

We claim

$$\mathcal{A}'_{m,\vartheta} = \Pi_0 \mathcal{A}_{m,\vartheta} \tag{6.5}$$

Indeed let $u \in \mathcal{A}_{m,\vartheta}$. Then due to (2.4), u coincides with its Galerkin projection $P_m u$, and $\xi(t) \in V_m$ for all $t \in \mathbb{R}$. By definition of $\mathcal{A}_{m,\vartheta}$, the function $\xi(t) = \Pi_0(\mathcal{T}_t u)$ solves the system of ordinary differential equations (6.1). But since $u \in H^2_a$, $\xi(t)$ is bounded by continuity of the trace embedding. Therefore, $\xi(0) = \Pi_0 u \in \mathcal{A}'_{m,\vartheta}$.

Conversely, let $\xi_0 \in \mathcal{A}'_{m,\vartheta}$ and $\xi(t) = (u(t), v(t))$ be the corresponding bounded solution of (6.1). From the second equation in (6.1), we obtain that $\partial_t^2 u(t) = \partial_t v(t)$ is also bounded. From (2.4) and from the smoothness of eigenfunctions $e_i(x)$ of the Laplace operator we conclude that $u \in \mathcal{A}_{m,\vartheta}$ with $\xi_0 = \Pi_0 u$, by definition. This proves equation (6.5).

By Proposition 2.2 the set $\mathcal{A}_{m,\vartheta}$ is compact in H^2_{loc}. By continuity of the trace operator Π_0, $\mathcal{A}'_{m,\vartheta}$ is also compact. Since the initial value problem for the system of ordinary differential equations (6.1) possesses a unique solution, the trace operator Π_0 is injective and therefore defines a continuous bijection between $\mathcal{A}_{m,\vartheta}$ and $\mathcal{A}'_{m,\vartheta}$. But a continuous bijection between compact sets is in fact a homeomorphism. This proves that $\mathcal{A}_{m,\vartheta}$ and $\mathcal{A}'_{m,\vartheta}$ are homeomorphic.

The remaining assertions of the lemma now follow easily. The conjugacy (6.3) of the flows is an immediate consequence of the explicit expression (6.4) for Π_0. Compactness and uniform boundedness of the sets $\mathcal{A}'_{m,\vartheta}$ follow immediately from Proposition 2.2. This proves Lemma 6.2. ∎

Lemma 6.3. *Let $m \geq m_0$ be sufficiently large. The "global attractor" $\mathcal{A}'_{m,\vartheta}$ is an isolated invariant set of the ordinary differential equation (6.1) in the sense of Section*

4. Its Conley index is independent of $\vartheta \in [0,1]$, *and is given by*

$$\mathcal{C}(\mathcal{A}'_{m,\vartheta}) = \Sigma^{mN}$$

Proof. The attractors $\mathcal{A}_{m,\vartheta}$ are bounded, uniformly for $m \geq m_0$, $\vartheta \in [0,1]$; see Proposition 2.2. Likewise the "attractors" $\mathcal{A}'_{m,\vartheta}$ are uniformly bounded in $V_m \cong \mathbb{R}^{2mN}$; see Lemma 6.2. Any sufficiently large ball in V_m is therefore an isolating neighborhood for all $\mathcal{A}'_{m,\vartheta}$. Indeed, all the $\mathcal{A}'_{m,\vartheta}$, $\vartheta \in [0,1]$, are invariant sets, contained in a fixed, chosen large ball. They are the maximal invariant sets in this ball because any orbit outside $\mathcal{A}'_{m,\vartheta}$ is unbounded, by definition of $\mathcal{A}'_{m,\vartheta}$. In particular, all $\mathcal{A}'_{m,\vartheta}$ are isolated invariant sets. By homotopy invariance of Conley index, Proposition 4.1, the Conley index $\mathcal{C}(\mathcal{A}_{m,\vartheta})$ does not depend on ϑ.

It remains to compute $\mathcal{C}(\mathcal{A}_{m,\vartheta=0})$ for the flow of (6.1) on V_m, with $\vartheta = 0$. The flow is linear with eigenvalues $\pm\sqrt{\mu_l}$, $1 \leq l \leq m$, each of multiplicity N. Here $0 < \mu_1 < \mu_2 \leq \ldots \leq \mu_m$ denote the first m eigenvalues of $-\Delta_x$ on Ω with Dirichlet boundary conditions. We conclude that the origin is a hyperbolic equilibrium with mN-dimensional unstable eigenspace.

Therefore

$$\mathcal{C}(\mathcal{A}'_{m,0}) = \mathcal{C}(\{0\}, \vartheta = 0) = \Sigma^{mN}$$

where Σ^{mN} denotes the mN-dimensional pointed sphere; see (4.2). This proves the lemma. ∎

From the above lemma, we see that the set $\mathcal{A}'_{m,\vartheta}$ is a hyperbolic set of increasing unstable dimension, rather than an attractor, if we consider arbitrary initial conditions for the dynamical system in V_m.

Proof of Proposition 6.1. We argue by contradiction. Suppose the hyperbolic equilibria w_1, w_2 are isolated in $\mathcal{A} \subset H^2_{loc}$. Then the attractor $\mathcal{A}$ decomposes disjointly into two equilibria w_1, w_2 and their (possibly empty) H^2_{loc}-closed complement

$$\mathcal{A} = \{w_1\} \dot{\cup} \{w_2\} \dot{\cup} \mathcal{A}^c \subset\subset H^2_{loc}$$

Upper semicontinuity under Galerkin approximations yields corresponding decompositions

$$\mathcal{A}_m = (\mathcal{A}_m \cap \mathcal{U}_\varepsilon(w_1)) \dot{\cup} (\mathcal{A}_m \cap \mathcal{U}_\varepsilon(w_2)) \dot{\cup} \mathcal{A}^c_m \subset H^2_{loc}$$

into compact disjoint sets, for all m sufficiently large; see Proposition 2.1. Here, $\mathcal{U}_\varepsilon(w)$ denotes the ε-neighborhood of w in H^2_{loc}. Increasing m_0 we may choose $\varepsilon > 0$ arbitrarily small.

By hyperbolicity, Proposition 5.10 (i),(ii), for any $\varepsilon > 0$ small enough we can fix m sufficiently large such that

$$\mathcal{A}_m \cap \mathcal{U}_\varepsilon(w_i) = \{w_{i,m}\}, \quad i = 1, 2,$$

are unique hyperbolic equilibria of (2.6). We have thus obtained a decomposition

$$\mathcal{A}_m = \{w_{1,m}\}\dot\cup\{w_{2,m}\}\dot\cup\mathcal{A}^c_m \subset\subset H^2_{loc} \tag{6.6}$$

of $\mathcal{A}_m$ into disjoint compact subsets, for some large m. By the flow equivalence of Lemma 6.2, the decomposition (6.6) yields an analogous decomposition

$$\mathcal{A}'_m = \{\underline{w}_{1,m}\}\dot\cup\{\underline{w}_{2,m}\}\dot\cup\mathcal{A}^{c'}_m \subset\subset V_m \tag{6.7}$$

into compact isolated invariant sets.

By flow invariance of $\mathcal{A}'_m$, the isolated points $\underline{w}_{j,m} = (w_{j,m}, 0) \in V_m$ of $\mathcal{A}'_m$ are equilibria of the ordinary differential equation (6.1), for $\vartheta = 1$. By Proposition 5.10 these equilibria are in fact hyperbolic in the sense that the linearization of (6.1) at $\underline{w}_{j,m}$ does not possess eigenvalues on the imaginary axis (for sufficiently large fixed m).

The Conley index of $\underline{w}_{j,m}$ is, in consequence, a pointed sphere of dimension l_j. Using the wedge product formula (4.3) for the index and the decomposition (6.7), we obtain

$$\mathcal{C}(\mathcal{A}'_m) = \mathcal{C}(\{\underline{w}'_{1,m}\}) \vee \mathcal{C}(\{\underline{w}'_{2,m}\}) \vee \mathcal{C}(\mathcal{A}^{c'}_m) = \Sigma^{l_1} \vee \Sigma^{l_2} \vee \mathcal{C}(\mathcal{A}^{c'}_m).$$

This contradicts the previous calculation from Lemma 6.2, where we have shown that

$$\mathcal{C}(\mathcal{A}'_m) = \Sigma^{mN}.$$

Indeed let us compute the total dimensions of homology groups in both cases:

$$\dim H_*(\Sigma^{mN}) = 1$$

but

$$\dim H_*(\Sigma^{l_1} \vee \Sigma^{l_2} \vee \mathcal{C}(\mathcal{A}^{c'}_m)) \geq \dim H_*(\Sigma^{l_1}) + \dim H_*(\Sigma^{l_2}) = 2.$$

By homotopy invariance, the two dimensions have to coincide. This contradiction proves that w_1 and w_2 are not both isolated in $\mathcal{A}$, in the H^2_{loc}-topology.

Remark 6.4. *It can be checked that both l_1 and l_2, are non-zero for large m and, in fact, $l_j \to \infty$, $j = 1, 2$, for $m \to \infty$; see also Remark 5.8 and the discussion in Section 8.*

7 Convergent Non-Equilibrium Solutions

We complete the proof of Theorem 1 and we prove Theorem 2 in this section. In addition to Proposition 6.1 on non-isolated, but hyperbolic equilibria, the following lemma is the main ingredient.

Lemma 7.1. *Suppose the equilibrium w is hyperbolic in the sense of (1.9), and not isolated in $\mathcal{A} \subset H^2_{loc}$. Then there exists a non-equilibrium solution $u \in \mathcal{A} \setminus \{w\}$ such that the shifted solution $\mathcal{T}_t u$ converges to w,*

$$\mathcal{T}_t u \to w \qquad \text{in } H^2_{loc},$$

for $t \to +\infty$ or for $t \to -\infty$.

Proof. We first construct a solution $u \in \mathcal{A}$ whose time orbit stays in a small H^2_a-neighborhood $\overline{\mathcal{U}}_\varepsilon(w)$ of the hyperbolic equilibrium w, say, for all negative times. We then argue that this solution must converge to w for $t \to -\infty$. Proceeding indirectly, we show that otherwise we could construct a solution $\bar{u}$ in the attractor $\mathcal{A}$, different from w, which remains close to w for all times $t \in \mathbb{R}$. However, by Proposition 5.1, the hyperbolic equilibrium w is isolated in the global attractor $\mathcal{A}$ with respect to the t-uniform H^2_a-topology, and we have reached a contradiction.

To start, let us fix $\varepsilon > 0$ small such that $\mathcal{A} \cap \overline{\mathcal{U}}_\varepsilon(w) = \{w\}$, by hyperbolicity of w and Proposition 5.1. Here, $\overline{\mathcal{U}}_\varepsilon$ denotes the closed ε-ball in H^2_a. By assumption, w is not isolated in $\mathcal{A}$ with respect to the H^2_{loc}-topology. Hence there exists a sequence $u_\ell \in \mathcal{A}$, $u_\ell \neq w$ such that $u_\ell \to w$ in H^2_{loc}, for $\ell \to \infty$. By definition of the topology in H^2_{loc}, this is equivalent to

$$||u_\ell - w, Q_T||_{2,2} \to 0 \quad \text{for } \ell \to \infty \qquad \text{and every fixed } T \in \mathbb{R}. \tag{7.1}$$

We recall the notation $Q_T := [T, T+1] \times \Omega \subset Q$. Since $\overline{\mathcal{U}}_\varepsilon(w) \cap \mathcal{A} = \{w\}$ in H^2_a, we have $u_\ell \notin \overline{\mathcal{U}}_\varepsilon(w)$. Therefore, there exists a sequence $T_\ell \in \mathbb{R}$ such that

$$||u_\ell - w, Q_{T_\ell}||_{2,2} = \varepsilon > 0. \tag{7.2}$$

For T_ℓ, we pick the first positive or negative exit times from the ε-ball in H^2 around w, that is,

$$||u_\ell - w, Q_T||_{2,2} < \varepsilon \quad \text{for } |T| < |T_\ell|. \tag{7.3}$$

From (7.1), (7.2) we conclude that $|T_\ell| \to \infty$ for $\ell \to \infty$. Possibly after passing to a subsequence, we may therefore assume $T_\ell \to +\infty$, or $T_\ell \to -\infty$. We henceforth consider the case $T_\ell \to +\infty$, the other case being completely analogous.

Let us consider the shifted sequence $\hat{u}_\ell = \mathcal{T}_{T_\ell} u_\ell \in \mathcal{A}$. Formulae (7.2) and (7.3) now are equivalent to

$$||\hat{u}_\ell - w, Q_0||_{2,2} = \varepsilon \quad \text{and} \quad ||\hat{u}_\ell - w, Q_T||_{2,2} < \varepsilon \quad \text{for } T \in (-2T_\ell, 0) \tag{7.4}$$

By Theorem 3 of Section 2, $\mathcal{A}$ is compact in H^2_{loc}. We may therefore assume without loss of generality that $\hat{u}_\ell \to u \in \mathcal{A}$ in H^2_{loc}. Passing to the limit in (7.4) we obtain

$$||u - w, Q_0||_{2,2} = \varepsilon \quad \text{and } ||u - w, Q_T||_{2,2} \leq \varepsilon \quad \text{for } T \leq 0$$

In particular, $u \neq w$. We claim that

$$\mathcal{T}_{-\ell} u \to w \quad \text{for } \ell \to \infty \quad \text{in } H^2_{loc},$$

revealing u to be the non-equilibrium solution sought for in the lemma. Indeed, let us consider any H^2_{loc} convergent subsequence

$$\mathcal{T}_{-\ell_k} u \to \bar{u} \quad \text{in } H^2_{loc} \quad \text{for } \ell_k \to \infty$$

in the precompact set $\{\mathcal{T}_{-\ell} u, \ell \in \mathbb{R}_+\} \subset \mathcal{A}$. From (7.4) we conclude

$$||\mathcal{T}_{-\ell_k} u - w, Q_T||_{2,2} \leq \varepsilon \quad \text{for } T < \ell_k,$$

and, passing to the limit, $||\bar{u} - w||_{H^2_a} \leq \varepsilon$. By assumption, $\mathcal{A} \cap \overline{\mathcal{U}}_\varepsilon(w) = \{w\}$, and therefore $\bar{u} = w$. ■

Proof of Theorems 1 and 2, for $\kappa = 1$. By Proposition 6.1, at least one of the two hyberbolic equilibria w_1, w_2 is not isolated in $\mathcal{A}$ with respect to the "local" H^2_{loc}-topology. Then, by Lemma 7.1, there is a non-equilibrium solution, converging to this equilibrium for $t \to +\infty$ or for $t \to -\infty$, just as claimed in Theorem 2.

Proof of Theorems 1 and 2, for $\kappa \geq 1$. By Proposition 6.1, at most one of the 2κ hyberbolic equilibria is isolated in the H^2_{loc}-topology on $\mathcal{A}$. By Lemma 7.1 we can construct $2\kappa - 1$ non-equilibrium solutions which converge to these equilibria for $t \to +\infty$ or $t \to -\infty$. These $2\kappa - 1$ solutions can be labeled by the equilibria they are converging to. Our labeling considers time-shifted solutions as identical. If ever the same solution carries two different equilibrium labels, it must be heteroclinic between these two equilibria. Therefore there exist at least κ distinct bounded non-equilibrium solutions. This completes the proofs of Theorems 1 and 2.

We finish this section with the variational case, $f(u) = \nabla_u F(u)$ and $\gamma + \gamma*$ strictly definite.

Proof of Corollary 1.1. For every $u \in \mathcal{A}$ we construct the Lyapunov function

$$V_u(t) := \big(\partial_t u(t,\cdot), \partial_t u(t,\cdot)\big) - \big(\nabla_x u(t,\cdot), \nabla_x u(t,\cdot)\big) + 2\big(F(u(t,\cdot)), 1\big)$$

Here again $(\cdot,\cdot)$ denotes the scalar product in the cross section $L^2(\Omega)^N$ and $f(u) = \nabla_u F(u)$.

Since $u \in H^2_a$, we have $V_u(\cdot) \in C^1_b(\mathbb{R})$ and a calculation shows that

$$\frac{d}{dt} V_u(t) = -2(\gamma \partial_t u(t,\cdot), \partial_t u(t,\cdot)) \tag{7.5}$$

Since $\gamma + \gamma*$ is strictly definite, the right-hand side of (7.5) is non-zero for all $t \in \mathbb{R}$, along non-equilibrium solutions in the global attractor $\mathcal{A}$. Therefore, the function V_u is monotone. Because the continuous functional V is bounded on the compact global attractor $\mathcal{A}$, the limits

$$\lim\nolimits_{t \to \pm\infty} V_u(t) = V_\pm \tag{7.6}$$

exist. Following the standard definition, we define the ω-limit set $\omega(u)$ of the point $u \in \mathcal{A} \subset H^2_{loc}$ as the set of accumulation points of $\{\mathcal{T}_s u, s \geq 0\}$ in H^2_{loc}. Since $\mathcal{A}$

is compact and $\mathcal{T}_s$-invariant, the set $\omega(u)$ is a non-empty, compact, and connected subset of $\mathcal{A} \subset H^2_{loc}$. Moreover, for every point $\bar{u} \in \omega(u)$ there is a sequence $s_l \to +\infty$ such that

$$\bar{u} = \lim_{l\to\infty} \mathcal{T}_{s_l} u \quad \text{in } H^2_{loc}. \tag{7.7}$$

Using the identity $V_{\mathcal{T}_s u}(t) = V_u(t+s)$, we obtain

$$V_u(t_2 + s_l) - V_u(t_1 + s_l) = -2\int_{t_1}^{t_2} (\gamma \partial_t \mathcal{T}_{s_l} u(t,\cdot), \partial_t \mathcal{T}_{s_l} u(t,\cdot))\, dt$$

for arbitrary but fixed $t_1, t_2 \in \mathbb{R}$. Using (7.6) and (7.7) we obtain

$$0 = \int_{t_1}^{t_2} (\gamma \partial_t \bar{u}(t), \partial_t \bar{u}(t))\, dt$$

in the limit $s_l \to \infty$. Hence $\partial_t \bar{u}(t) \equiv 0$ and $\bar{u}$ is an equilibrium solution of the problem (1.1), (1.2). This proves that the ω-limit set $\omega(u)$ consists of equilibria only. But the ω-limit set $\omega(u)$ must be connected, and, by assumption, there are only finitely many equilibria. Therefore $\omega(u)$ consists of a single point $\{w_+\}$ and $\mathcal{T}_s u \to w_+$ in H^2_{loc} for $s \to +\infty$.

The case $s \to -\infty$ can be treated in the same way and, in consequence, $\mathcal{T}_s u \to w_-$ in H^2_{loc} for $s \to -\infty$. It remains to show that $w_+ \neq w_-$ for the non-equilibrium solution $u \in \mathcal{A}$. Indeed integrating (7.5) and using (7.6) we obtain that

$$V_+ - V_- = \int_{\mathbb{R}} (\gamma \partial_t u(t), \partial_t u(t))\, dt \neq 0 \tag{7.8}$$

for the non-equilibrium solution u. On the other hand, continuity of V implies $V_{w_\pm}(t) \equiv V_\pm$. Therefore, equation (7.8) shows that $w_+ \neq w_-$. This proves Corollary 1.1.

8 Concluding Remarks

Our Theorems 1 and 2 are just small steps towards a more specific investigation of the global dynamics on the global attractor $\mathcal{A}$ of an elliptic system (1.1), (1.2). For one-dimensional cross-section, dim $\Omega = 1$, of the cylinder $Q = \mathbb{R} \times \Omega$, and a single scalar equation, $N = 1$, much more information is available. We summarize some of these results below. As already mentioned in the introduction, the attractor $\mathcal{A}$ then lies inside a finite-dimensional, locally flow-invariant manifold. In particular, $\mathcal{A}$ has finite Hausdorff-dimension.

For $\gamma \to +\infty$, that is, for convection dominated problems, the elliptic dynamics in the strip limits onto a parabolic semigroup

$$\partial_t u = \partial_{xx} u + f(u). \tag{8.1}$$

The global attractors $\mathcal{A}$ for these gradient-like systems are rather well understood. In particular, information on the hyperbolic equilibrium set alone determines which equilibria possess a heteroclinic connection, and which do not. See [13], [14] for recent accounts of this theory, which is based on nodal properties of Sturm oscillation type. The Morse-Smale property of (8.1), and thereby the structure of the global attractor $\mathcal{A}(\gamma)$, both persist for large $\gamma \geq \gamma_0$. For explicit bounds on γ_0; see [7]. The gradient-dependent case $f = f(x, u, \partial_x u)$ was treated in [34].

Even in the phase plane of $\dim \Omega = 0$, $N = 1$, non-generic saddle-saddle connections can occur as the wave speed parameter γ decreases through positive $\gamma_* < \gamma_0$; see for example [10], Example II. 7.3 and also [39]. This observation was the starting point of Gardner's result for scalar cubic f, $N = 1$, and $\dim \Omega = 1$ under Dirichlet boundary conditions. Today his result can be recovered by reduction to inertial manifolds $\mathcal{M}(\gamma)$ of fixed finite dimension and a direct application of Conley index and transition matrices [17],[18] within $\mathcal{M}(\gamma)$. Indeed, after finite-dimensional reductions to inertial manifolds $\mathcal{M}(\gamma)$, $\gamma \neq 0$, Conley index theory applies within $\mathcal{M}(\gamma)$, directly. An additional Galerkin discretization is not necessary — albeit, more elementary — in those cases.

In the variational case $f = \nabla F$ and $\gamma \in \mathbb{R} \setminus \{0\}$, an easy computation shows that

$$\mathcal{C}(\underline{w}_m) = \Sigma^{i(\underline{w})+mN}$$

for the Galerkin approximation $\underline{w}_m \in \mathcal{A}_m \subset \mathbb{R}^{2mN}$ of a hyperbolic equilibrium $\underline{w} \in \mathcal{A}$; $m \geq m_0$. Here $i(\underline{w})$ is a suitably chosen constant, independent of m. In particular, the appropiately shifted homology of the Conley index

$$\widetilde{CH}_*(\underline{w}_m) := CH_{*-mN}(\underline{w}_m) \tag{8.2}$$

stabilizes, for $m \to \infty$. It is therefore tempting to define

$$\widetilde{CH}_*(\mathcal{S}) := CH_{*-mN}(\mathcal{S}_m),$$

as the Conley homology index of an arbitrary isolated invariant set $\mathcal{S} \subset \mathcal{A}$. Morse decompositions, connection matrices, and connection graphs seem to stabilize under this Galerkin approximation. In the present paper, we have verified (8.2) for hyperbolic equilibria and for $\mathcal{A}$ itself; see Lemma 6.3 and Remarks 5.8, 6.4.

Definition (8.2) for an elliptic Conley homology index is reminiscent of Floer homology. See [1] for a Floer homology construction associated to a strongly indefinite variational problem describing an elliptic system on a bounded domain. For Floer's original construction see [16] and also [24], [33]. The original applications to periodic solutions of Hamiltonian systems differ from our approach in important technical details. First, we do not assume a variational structure of our elliptic system in the cylinder $(t, x) \in \mathbb{R} \times \Omega$. Even where we do, as in Corollary 1.1, our Lyapunov functional V, given in (1.10), is bounded below on a cross-section, if we set $\partial_t u = 0$. The strong indefiniteness of V is, in our problem, generated by the unbounded ∂_t-component. Notwithstanding those two differences, our original equation is elliptic — like the equations for the pseudo-holomorphic curves which constitute the ill-defined gradient-"flow" to the action functional in the elliptic context. We are therefore cautiously optimistic towards (8.2) becoming a viable, more direct definition of Conley homology for elliptic systems in cylinder domains.

For wave speed $\gamma = 0$, the elliptic system (1.1), (1.2) becomes "time" reversible under the reflection $t \mapsto -t$. If $f(u) = \nabla F(u)$ is a gradient, the system is in addition formally Hamiltonian with respect to the strongly indefinite energy functional V defined in (1.10). Reductions to finite-dimensional symplectic manifolds $\mathcal{M}(\gamma = 0)$ with Hamiltonian flows are available, both locally [25] and – under spectral gap conditions on Δ – globally [30]. Families of nontrivial periodic traveling waves $u(t)$ occur in a Hamiltonian context. For example, a local minimum of the Hamiltonian on $\mathcal{M}(\gamma)$ is surrounded by families of solutions, which are periodic with respect to t. This fact is known as the Lyapunov center theorem and requires certain non-degeneracy conditions.

Although this may not be obvious in the present paper, applications to traveling waves in reaction diffusion systems and in semilinear hyperbolic systems are a driving motivation of our work. In the introduction we have pointed out the relevance of our results to reaction diffusion systems. A specific example is the Fitz-Hugh-Nagumo model for propagation of electric impulses in the giant squid axon:

$$\begin{cases} \partial_\tau u_1 = \Delta_{t,x} u_1 + g(u_1) - u_2 \\ \partial_\tau u_2 = \delta \Delta_{t,x} u_2 + a u_1 - b u_2. \end{cases} \tag{8.3}$$

Here $\delta > 0$ is small, a, b are positive and $g(u_1) = -u_1(u_1 - \beta)(u_1 - 1)$ is a negative cubic, $0 < \beta < 1/2$. See [37], [42], for some background. Remarkably, system (8.3) is gradient-like for $\delta = 0$ and $b^2 \geq a$ on bounded domains; see [11] for an explicit Lyapunov function. The traveling wave ansatz $u = u(t + c\tau, x)$ leads to the elliptic system

$$\begin{cases} c\partial_t u_1 = \Delta_{t,x} u_1 + g(u_1) - u_2 \\ c\partial_t u_2 = \delta \Delta_{t,x} u_2 + a u_1 - b u_2. \end{cases} \tag{8.4}$$

Existence of equilibria for this equation has been studied in [11] for the case $\dim \Omega = 1$. We expect hyperbolicity of equilibria to hold for generic lengths of the interval Ω. The growth conditions (1.3) are satisfied for dimensions of the cross-section $n = \dim \Omega \leq 2$. Though the dissipation condition is only satisfied with $\sigma = 0$, our results apply. In particular Theorem 3, and Propositions 2.1 and 2.2 remain true for (8.4) and its Galerkin approximation. Indeed, exploiting the diagonal structure of the matrix $\gamma = \operatorname{diag}(c, c/\delta)$, the proofs of Lemmata 3.1 and 3.2 can be easily adapted.

Damped semilinear hyperbolic systems

$$-a^2 \partial_{\tau\tau} u - D\partial_\tau u + \Delta_{t,x} u + f(u) = 0 \tag{8.5}$$

in cylindrical domains $x \in \Omega$, $t \in \mathbb{R}$ are yet another source of inspiration. Here $u \in \mathbb{R}^N$, and the damping matrix D is assumed to be strictly positive definite. The scalar case $N = 1$ corresponds to models from quantum electrodynamics; see [27] and the references therein. The Ginzburg-Landau equations for $u \in \mathbb{C} \simeq \mathbb{R}^2$, with cubic nonlinearity $f(u) = u \cdot \varphi(|u|^2)$ arise in nonlinear optics. Traveling waves $u = u(t + \tilde{c}\tau, x)$ satisfy

$$(1 - (a\tilde{c})^2)\partial_t^2 u - \tilde{c} D \partial_t u + \Delta_x u + f(u) = 0, \tag{8.6}$$

where $\Delta = \Delta_x$ acts on the cross section Ω of $(t, x) \in Q = \mathbb{R} \times \Omega$, as before. For $|a\tilde{c}| < 1$, system (8.6) is elliptic of the form (1.1), (1.2) studied in the present paper. Rescaling t, the "wave velocity" c discussed in the introduction takes the "relativistic" form

$$c = -\tilde{c}(1 - (a\tilde{c})^2))^{-1/2}$$

in terms of the wave velocity $\tilde{c}$ of system (8.5).

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