Grow-Up Solutions and Heteroclinics to Infinity for Scalar Parabolic PDEs

by

Nitsan Ben-Gal

B. S., University of Michigan, 2004

Sc. M., Brown University, 2005

A Dissertation submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Division of Applied Mathematics at Brown University

Providence, Rhode Island

May 2010

This dissertation by Nitsan Ben-Gal is accepted in its present form
by the Division of Applied Mathematics as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date ____________ ______________________________

Bernold Fiedler, Director

Recommended to the Graduate Council

Date ____________ ______________________________

Constantine M. Dafermos, Reader

Date ____________ ______________________________

Björn Sandstede, Reader

Approved by the Graduate Council

Date ____________ ______________________________

Sheila Bonde, Dean of the Graduate School

The Vita of Nitsan Ben-Gal

Born on September 6, 1982 in Haifa, Israel.

Education

- Sc.M. in Applied Mathematics, Brown University, May 2005.
- B.S. in Honors Mathematics, with Highest Honors, University of Michigan, May 2004.

Awards and Honors

- US Junior Oberwolfach Fellow, 2009.
- NSF Graduate Research Fellowship, Honorable Mention, 2006.
- NSF VIGRE Fellowship, Brown University, 2005-2007.
- Brown University Fellowship, Division of Applied Mathematics, 2004-2005.
- Phi Beta Kappa, University of Michigan, inducted 2003.

Research Interests

- Infinite-Dimensional Dynamical Systems, particularly global attractors for scalar parabolic partial differential equations.
- Fučik Spectra for ordinary differential equations and partial differential equations.
- Nonlinear spring-mass system modeling.

Publications

- Ben-Gal, N., *Asymptotics of Grow-Up Solutions and Global Attractors of Non-Dissipative PDEs*, Oberwolfach Reports, 6 (2009), no. 2, 1434-1436.
- Ben-Gal, N., Shaw-Krauss, A., Strichartz, R.S., and Young, C., *Calculus on the Sierpinski gasket II: Point singularities, eigenfunctions, and normal derivatives of the heat kernel*, Transactions of the American Mathematical Society, 358 (2006), no. 9, 3883-3936.

- Ben-Gal, N., Moore, K.S., *Bifurcation and stability properties of periodic solutions to two nonlinear spring-mass systems*, Nonlinear Analysis: Theory, Methods and Applications, 61 (2005), no. 6, 1015-1030.

Research Experience

- Global attractors and grow-up for scalar parabolic partial differential equations, Collaborative Research Center for Space-Time-Matter, Freie Universität Berlin, Germany, January 2009-present.
- Global attractors and grow-up for scalar parabolic partial differential equations, Collaborative Research Center for Complex Nonlinear Processes, Freie Universität Berlin, Germany, May 2007-January 2008, May 2008-January 2009.
- Global attractors, Fučik spectra, and grow-up for scalar parabolic partial differential equations, Division of Applied Mathematics, Brown University, 2005-present.

Presentations

- Workshop in Nonlinear Elliptic PDEs, Université Libre de Bruxelles, *Asymptotics of Grow-Up Solutions and Global Attractors of Slowly Non-Dissipative PDEs*, September 2009.
- Nonlinear Dynamics Seminar, Institut für Mathematik, Freie Universität Berlin, *Asymptotics of Grow-Up Solutions and the Connection Problem for Slowly Non-Dissipative Parabolic PDEs*, July 2009.
- Workshop on Topological and Variational Methods for Partial Differential Equations, Oberwolfach Research Institute, Germany, *Asymptotics of Grow-Up Solutions and the Connection Problem for Slowly Non-Dissipative Parabolic PDEs*, May 2009.
- Dynamical Systems Seminar, University of Minnesota, School of Mathematics, *Asymptotics of Grow-Up Solutions and Global Attractors of Non-Dissipative PDEs*, April 2009.
- Oberseminar Nonlinear Dynamics, Weierstrass Institute for Applied Analysis and Stochastics, Berlin, *Asymptotics of Grow-Up Solutions and Global Attractors of Non-Dissipative PDEs*, November 2008.
- Nonlinear Dynamics Seminar, Institut für Mathematik, Freie Universität Berlin, *Global Attractors for Nondissipative Scalar Parabolic PDEs*, January 2008.

- Mathematical Tools for Complex Systems in Science and Technology Ph.D-Course, Danish Center for Applied Mathematics and Mechanics, Copenhagen, *Sturm Attractors; a Tutorial*, June 2007.
- Mathematical Tools for Complex Systems in Science and Technology Ph.D-Course, Danish Center for Applied Mathematics and Mechanics, Copenhagen, *Global Attractors and Fučik Spectra for Scalar Parabolic Differential Equations with Jump Nonlinearities*, June 2007.
- Transatlantic Seminar, Brown University, Division of Applied Mathematics, *Spectral Theory of the Laplacian, a Tutorial*, March 2007.
- Connections For Women: Dynamical Systems, MSRI, *Bifurcation and Stability Properties of Two Nonlinear Spring-Mass Systems*, January 2007.
- Dynamics Days US 2007, *Bifurcation and Stability Properties of Two Nonlinear Spring-Mass Systems*, January 2007.
- Transatlantic Seminar, Brown University, Division of Applied Mathematics, *Characterizing the Fučik Spectrum Through Early Theory and New Results*, March 2006.
- Transatlantic Seminar, Brown University, Division of Applied Mathematics, *Bifurcation and Stability Properties of Two Nonlinear Spring-Mass Systems*, December 2005.

Academic Experience

- Teaching Assistant, Guest Lecturer: *Methods of Applied Mathematics I*, Division of Applied Mathematics, Brown University, Spring 2006 and 2009.
- Teaching Assistant, Guest Lecturer: *Methods of Applied Mathematics II, Honors*, Division of Applied Mathematics, Brown University, Spring 2008.
- Teaching Assistant, Guest Lecturer: *Nonlinear Dynamical Systems: Theory and Applications II*, Division of Applied Mathematics, Brown University, Spring 2007.
- Teaching Assistant: *Nonlinear Dynamical Systems: Theory and Applications I*, Division of Applied Mathematics, Brown University, Fall 2006.
- Teaching Assistant: *Methods of Applied Mathematics II*, Division of Applied Mathematics, Brown University, Fall 2006.
- Instructor: *Introductory Mechanics*, Science Learning Center, University of Michigan, Spring 2004.

"I protest against the use of infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a figure of speech, the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction."

-C.F. Gauss

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds. "

Georg Cantor

Dedicated to my loving family, Dorit, Joseph, Itai, Melanie, and Ilan Ben-Gal.

Acknowledgements

First and foremost, I would like to extend great thanks to my advisor, Professor Dr. Bernold Fiedler, for giving me the honor of being his student. He introduced me to the wonders of infinite-dimensional dynamical systems, and for that I will be eternally grateful. I am a better mathematician for his guidance and insight, and I count myself extremely lucky to have had a mentor who challenged me to reach new heights.

For their guidance and support during my time at Brown University, I wish to extend special thanks to my advisory committee, Professors Björn Sandstede, Constantine Dafermos, and John Mallet-Paret. Their guidance and insight have oftentimes been invaluable, and the opportunity to discuss research with them has been a true pleasure.

I wish to thank Professor Almut Burchard for our many discussions on the topic of slowly non-dissipative systems, as well as for her invaluable advice. Her questions allowed me to take my research to realms I had never before considered. I would like to thank Professor George Sell both for his insight and for his warmth. Our discussions on inertial manifolds and infinite-dimensional dynamics were informative, inspiring, and a pleasure to have.

My thanks go out to Professor Kristen Moore for taking me on as her researcher so many years ago, and introducing me to the wonders I could find in mathematics beyond the walls of classrooms. Her mentorship was invaluable to me, and without it I may never have found this path which I love so much.

I would like to thank my friends and colleagues in the Division of Applied Mathematics and the Insitut für Mathematik for their friendship and assistance over the years. Most especially, I wish to thank Stefan Liebscher and Juliette Hell for their insight and open doors.

Last, and certainly not least, I would like to thank my wonderful family. Mom, Dad, Itai, and Melanie, without your love, support, and advice this accomplishment would not have been possible. Your pride means the world to me. This work is dedicated to you.

Contents

List of Tables

List of Figures

Abstract of "Grow-Up Solutions and Heteroclinics to Infinity for Scalar Parabolic PDEs"

by Nitsan Ben-Gal, Ph.D., Brown University, May 2010

In recent years, there has been a great deal of interest surrounding the study of the asymptotics and global attractor structure for scalar parabolic PDEs which are either dissipative or undergo finite-time blow-up. This thesis presents solutions to the asymptotics and connection problems for slowly non-dissipative scalar PDEs, i.e. the final remaining class of scalar parabolic reaction-diffusion equations. Such PDEs produce solutions that neither blow up nor are dissipative. These "grow-up" solutions grow to infinite norm in infinite time, and it is the added challenges they introduce that are overcome in this thesis.

In the pursuit of this result, a number of new concepts are defined, including slowly non-dissipative PDEs, non-compact global attractors, and the completed inertial manifold. Many of the underlying assumptions used in the study of dissipative PDEs are inapplicable to slowly non-dissipative PDEs. Thus, concepts must be redefined and techniques updated or extended. The effects of a slowly non-dissipative nonlinearity on the global bifurcation diagram are investigated, and the y-map, a technique critical to solving the connection problem for dissipative systems, is extended to slowly non-dissipative PDEs and a wider range of boundary conditions.

The completed inertial manifold is introduced and proven to exist for certain classes of slowly non-dissipative equations. The development of this new structure and its advantageous characteristics provide the tools necessary to prove convergence for grow-up solutions which form heteroclinic connections to infinity. The asymptotics of grow-up solutions are determined, and via combining the various expanded techniques, a full decomposition of the non-compact global attractor is produced for a generic choice of nonlinearity. The results are studied for a selection of interesting cases and are shown to hold for both Neumann and Dirichlet boundary conditions.

CHAPTER 1

Introduction

There has been a great deal of work in recent decades devoted to understanding the asymptotic behavior of scalar parabolic partial differential equations. For clarifying the properties of those equations whose solutions remain bounded, we have Ladyzhenskaya, Hale, Smoller, Matano, Chafee and Infante, Henry, Brunovský, Fiedler and many others to thank. For those equations which experience finite time blow-up, we can likewise thank Fila, Matano, Giga, Kohn and many others. In both of these regions, there has also been a significant amount of study on the global attractor of those solutions which remain bounded. But these two categories do not encompass the full scope of behaviors which such equations may produce.

There is a middle ground which has yet to be addressed: partial differential equations for which some solutions exist for all forward time, are boundable for any fixed positive time $t > 0$, and whose norm becomes infinite as $t \to +\infty$. Such behavior arises, for instance, from equations of the form

$$\begin{aligned} u_t &= u_{xx} + bu + g(u) \\ u(0, \cdot) &= u_0, \end{aligned} \tag{1}$$

where the usual nonlinearity $f(u) = bu + g(u)$ is comprised of a positive linear growth term and a bounded, "well-behaved" nonlinear term. In this thesis we will address such solutions, henceforth referred to as *grow-up solutions*, determining when they occur and their asymptotic behaviors. In addition, we will address what it means for such systems to have a non-compact global attractor, and shall provide a method for explicitly describing all elements of the unbounded attractor for a generic class of linearly growing nonlinearities. Such a non-compact attractor will include elements connecting to or contained within infinity. We will describe what it means to be an "equilibrium at infinity", in what sense a bounded stationary solution can connect to such equilibria, and how these equilibria and their heteroclinic connections provide the missing pieces of the connection problem for linearly growing scalar parabolic PDEs.

This research ties together a number of different techniques in infinite-dimensional dynamical systems, as well as expanding many of these techniques beyond the restriction of dissipative systems. Classical semigroup theory in regards to equations of the form (1) provides for the existence of immortal solutions, i.e. solutions which exist globally for all forward time. Further, we can guarantee the existence of eternal solutions (those which exist globally for all time both backwards and forwards) for solutions in the unstable manifolds of equilibria, and it is these very solutions that require our attention.

We introduce the concept of a non-compact global attractor, an object comprised of eternal solutions to (1), which attracts all bounded sets in the underlying Hilbert space. For a dissipative reaction-diffusion system we are guaranteed the existence of a classical global attractor: the minimal compact, connected, invariant attracting set for the entirety of the Hilbert Space. But the linearly growing nonlinearity given in (1) straddles the line between dissipativity and finite-time blow-up. The semigroup that Equation (1) generates is neither point dissipative nor compact dissipative. Thus, the minimal attracting set in $H^2 \cap \{Boundary\ Conditions\}$ is non-compact. But the linear growth term is sufficiently slow that all solutions are guaranteed to exist in forward time, i.e. all solutions are immortal. It is in this situation that we refer to the dynamical system generated by (1) as a "grow-up" system, for while no solutions achieve infinite norm in finite time, some solutions will do so in infinite time. Thus, while there exists some minimal invariant set which attracts all solutions and informs the behavior of the overarching system, this global attractor is non-compact. We henceforth refer to dynamical systems displaying these characteristics as slowly non-dissipative systems, in contrast to the fast non-dissipativity displayed in systems with finite-time blow-up.

Due to the non-dissipativity of the system and the non-compactness of the attractor, we cannot obtain convergence results through only those techniques used in the dissipative form of the problem. The stumbling block that arises in the slowly non-dissipative system is the inability to prevent a change in nodal properties for unbounded solutions at $t = \infty$ using only the techniques required in the dissipative case. Thus, we prove the existence of inertial manifolds in order to bridge this gap. Inertial manifolds are finite-dimensional Lipschitz manifolds which contain all invariant sets of their corresponding evolutionary equation and exponentially attract all solutions. The existence of such manifolds reduces the study of a global attractor from an infinite-dimensional problem to a finite-dimensional problem, with its attendant advantages. Thus, the inertial manifold is of great use to the study of nonlinear evolutionary equations, and we are immensely grateful for the work produced by Sell, Temam, Foias, Chow, Lu, Mallet-Paret, Robinson, and others.

Until now, inertial manifolds have only been shown to exist for dissipative systems; in this thesis we shall prove the existence of continuously differentiable inertial manifolds for a general class of slowly non-dissipative equations. While the inertial manifold is a versatile tool, for a desirably broad class of nonlinearities we are only able to construct an inertial manifold which is Lipschitz with values in L^2. But L^2 allows for a great deal of irregularity, thus we proceed to prove that the inertial manifold is Lipschitz with values in C^1 as well. Since inertial manifolds contain all invariant sets for a dynamical system, including the unstable manifolds of equilibria, we are able to study the limit of a grow-up solution in the C^1-norm. The use of the C^1-norm ensures that the nodal properties of our grow-up solutions are unchanged at $t = \infty$.

We thus parlay our work on inertial manifolds into the final step necessary to determining the asymptotics of grow-up solutions and solving the connection problem for the non-compact global attractor. The first main result, Theorem 7.4, proves that such grow-up solutions connect to those equilibria at infinity which are not blocked by a bounded equilibrium. Theorem 8.1 and its corollary then combine the implications of Theorem 7.4, classical Conley index theory, and results on the existence of heteroclinics between bounded equilibria in a slowly non-dissipative system to provide a complete decomposition of the attractor. In Theorem 8.1 we prove that every bounded hyperbolic equilibrium connects to all those bounded equilibria and equilibria at infinity which are not blocked, and determine explicitly which connections are blocked and which exist. In the corollary, we detail all elements of the non-compact attractor, including those within infinity.

In order to prove these results, we proceed as follows: In Chapters 2 and 3 we will lay the groundwork, introducing the standard tools which will be used most frequently in the rest of the thesis: the Lyapunov functional, lap number, zero number, time map, bifurcation diagram, and Morse index. We then prove that the class of linearly growing nonlinearities will provide for the existence of grow-up solutions and non-compact attractors.

In Chapter 3 we tackle the ODE problem resulting from the study of stationary solutions to (1). We study the time map and the behavior of these solutions in the phase plane in order to produce the global bifurcation diagram for a given choice of $g(u)$. The bifurcation diagram provides a major tool for the determination and depiction of connections between bounded equilibria.

In Chapter 4 we return to the original PDE and restrict our focus to the nodal properties of a certain subset of solutions to (1). We revisit a result of Fiedler and Brunovský, the y-map, a functional which provides information on the nodal properties of solutions limiting in backwards time to a stationary solution. We extend their work to the study of scalar parabolic PDEs with linearly growing nonlinearities.

Chapter 5 builds upon the results of Chapters 2, 3, and 4. It is here that we prove the lemmas that determine when a heteroclinic from a given equilibrium is a grow-up solution, as well as a quartet of blocking lemmas. These blocking lemmas describe under what conditions a heteroclinic connection between two equilibria is prevented, and are crucial to the full decomposition of the non-compact global attractor. Finally, we reduce the possible asymptotic behavior of any grow-up solution to a finite number of equilibria at infinity.

In Chapter 6 we extend inertial manifold theory beyond the confines of dissipative systems. We prove the existence of an unbounded inertial manifold for a class of slowly non-dissipative systems, prove that this manifold is Lipschitz with values in C^1 for Equation (1), and in so doing, provide the final tool necessary to the proof of Theorem 7.4. We are able to determine the number of distinct types of heteroclinics connecting to infinity which originate at a given bounded stationary solution through analysis of the time map, phase plane, and bifurcation diagrams. The y-map allows us to determine the nodal properties on each heteroclinic, and the existence of an inertial manifold in C^1_{Lip} ensures that these nodal properties hold even at time $t = \infty$. This observation provides the key to studying the strong limits of grow-up solutions at infinity.

In Chapter 7 we combine all these previous results in order to determine uniquely the object to which an unbounded heteroclinic connects. We then establish and exclude connections between bounded equilibria, which leads us to Theorem 8.1. Our second main result, Theorem 8.1, provides a complete description of the connecting orbit structure of the unbounded global attractor for the entire class of linearly growing nonlinearities $f(u)$. For this result we must introduce and explicitly define the concept of equilibria at infinity, which we do in the next chapter.

In Chapter 8, we make use of the Poincaré compactification, a crucial tool in the study of behavior at infinity for partial differential equations. We apply the thesis work of Juliette Hell on the Conley index at infinity to provide the connecting orbit structure within this extended region of the global attractor. Additionally, the Poincaré compactification provides a method to depict the unbounded portions of the global attractor. This provides us with the final piece of the explicit decomposition of the entire structure of the non-compact global attractor.

In Chapter 9, we will illustrate the results for a specific choice of nonlinearity: $f(u) = bu + asin(u)$. We present the results derived in the previous chapters for two distinct choices of coefficients b and a, and illustrate the bifurcation diagrams, phase portraits, time maps, and global attractors for these choices. In chapters 2 through 9 we primarily address the case of scalar reaction-diffusion equations with Neumann boundary conditions. In Chapter 10 we shall present the modifications which allow the extension of Theorems 7.4 and 8.1 to the Dirichlet case. We show that the necessary modifications are minimal and that the theorems, as well as the majority of minor results, carry over to Dirichlet boundary conditions. Finally, in Chapter 11 we shall summarize our previous results, address applications therein and open questions for further exploration.

CHAPTER 2

Grow-Up and Its Groundwork

1. Setting

The central equation of study in this thesis is the scalar reaction-diffusion equation with Neumann boundary conditions

$$
\begin{aligned}
&u_t = u_{xx} + f(u), \quad x \in [0,\pi] \\
&u_x(t,0) = u_x(t,\pi) = 0
\end{aligned}
\tag{2}
$$

where $f(u)$ is linearly growing and well-behaved, specifically

$$
\begin{gathered}
f(u) = bu + g(u), \\
b > 0, \quad g(u) \in C^2, \quad g(u) \text{ bounded} \\
g \text{ globally Lipschitz with values in } L^2.
\end{gathered}
\tag{3}
$$

Standard theory states that for nonlinearities $f = f(x, u, u_x)$ of class C^2 we are provided with a local solution semigroup $u(t,\cdot) = \mathcal{S}(t)u_0,\ t \geq 0$, on initial conditions $u_0 \in X$ [**18, 24**]. For the choice of nonlinearity presented in (3) this semigroup is a compact, global C^1-semigroup [**18**]. Presently we choose the underlying Banach space X to be the Sobolev Space H^2 intersected with Neumann boundary conditions. The results derived in this thesis can also be proven for Dirichlet boundary conditions, although there are notable changes in the bifurcation diagrams. This will be addressed later, in Chapter 10, at which time we shall use $H^2 \cap H_0^1$ for the underlying Banach space instead. In addition, let $\|\cdot\|_{1/2}$ denote the H^1-norm on H^2, $\|\cdot\|_1$ denote the H^2-norm on H^2, and $\|\cdot\|_0$ denote the L^2-norm on H^2. The notation here is chosen to reflect the fractional power space concerned, i.e. the $\|\cdot\|_\alpha$ norm corresponds to the αth power space of $-\frac{d^2}{dx^2}$.

One of the key tools we shall use to study the dynamics of (2) is the existence of a Lyapunov functional

$$
\begin{aligned}
V(u) &:= \int_0^{\pi} \frac{1}{2}u_x^2 - F(u)dx \\
F(u) &:= \int_0^{u} f(s)ds
\end{aligned} \tag{4}
$$

on X with the property that

$$\frac{d}{dt}V(u(t)) < 0$$

along solutions $u(t) \in X$ of Equation (2) and

$$\frac{d}{dt}V(u(t)) = 0$$

at equilibria of Equation (2).

A second set of crucial tools are the lap number and zero number, which are used to study nodal properties. The lap number, which was invented and studied by Hiroshi Matano [**22**], is defined as follows: given a function $u(x)$ on an interval $I = [0, \pi]$, we call u piecewise monotone if I can be divided into a finite number of non-overlapping subintervals $I_1, I_2, ...I_m$ upon each of which u is monotone. There exists a least value of the number m for which we can divide I as above. This value is called the lap number of u, denoted $l(u)$. If $u(x)$ is a constant function, $l(u) = 0$.

For a function $u = u(t, x)$ with $x \in I$ and $t \in \mathbb{R}$, the lap number is defined for each fixed t, and thus $l(u(t, \cdot))$ is now an integer-valued function of t. As proven in [**22**], for homogeneous Neumann boundary conditions, $l(u(t, \cdot))$ is nonincreasing in forward time.

The zero number, as its name would indicate, determines the number of interior zeros of a function. For any continuous function $u(x)$ on an interval I, the zero number $z(u) = n$ for a nonnegative n which is the maximal element of $\mathbb{N}_0 \cup \{\infty\}$ such that there is a strictly increasing sequence $0 \leq x_0 < x_1 < \ldots < x_n \leq \pi$ with $u(x_j)$ of alternating signs, i.e.

$$u(x_j) \cdot u(x_{j+1}) < 0 \quad for\ 0 \leq j < n.$$

We note that the zero number of a constant function is set to be zero [**4**]. As with the lap number, for a function $u = u(t, x)$ with $x \in I$ and $t \in \mathbb{R}$, the zero number is defined for each fixed t, and thus $z(u(t, \cdot))$ is now an integer-valued function of t. Further, the lap number and zero number are related by the equation $l(u) = z(u_x) + 1$ for any continuously differentiable function u. As proven in [**22, 23**], the zero number of a solution $u(t, x)$ to Equation (2) with Neumann boundary conditions is nonincreasing in forward time.

It is with these tools that we may determine when our infinite-dimensional dynamical system leaves the well-studied realm of dissipative semigroups and draws our attention towards infinity. At this point we remind the reader of the definition of dissipative dynamical systems, and introduce definitions for slowly non-dissipative and fast non-dissipative dynamical systems.

Dissipative dynamical systems are those dynamical systems for which the semigroup possesses a compact absorbing set B; i.e. for any bounded set Y there exists a time $t_0(Y)$ such that

$$\mathcal{S}(t)Y \subset B \quad for\ all \quad t \geq t_0(Y).$$

A dynamical system is considered to be fast non-dissipative, i.e. to experience finite-time blow-up, if for some initial condition u_0, the maximal time t^* such that $\mathcal{S}(t)u_0$ exists and is unique for $t \in [0, t^*)$ is finite, i.e. $t^* < \infty$. A dynamical system need only have one solution that experiences finite-time blow-up in order to be considered a fast non-dissipative system.

A slowly non-dissipative dynamical system, or “grow-up” system, is a dynamical system wherein all solutions are immortal, but there does not exist a compact absorbing set. In other words, in a slowly non-dissipative dynamical system, the maximal time t^* such that $\mathcal{S}(t)u_0$ exists and is unique for $t \in [0, t^*)$ is $t^* = \infty$ for all $u_0 \in X$, but for at least one initial condition v_0 there does not exist any t_v such that $\mathcal{S}(t)v_0$ can be bounded for $t \geq t_v$ in any appropriate norm. Heuristically, this means that at least one solution experiences unchecked growth, yet it exists for all forward time.

To summarize, at least one solution achieves infinite norm, but only in infinite time. Such a solution is called a "grow-up solution".

Because the linear operator in Equations (2) and (3), $A = \frac{d^2}{dx^2} + bI$, is a sectorial operator, and the Nemitskii operator g is globally Lipschitz on L^2 and bounded in the L^2-norm, it follows that all solutions to Equations (2) and (3) are immortal [**18**]. Thus, the dynamical system defined by Equations (2) and (3) is either dissipative or slowly non-dissipative, but it cannot experience finite-time blow-up.

2. Non-boundedness

Due to the existence of a Lyapunov functional V, it is obvious that any initial condition on which the Lyapunov functional has a lower value than at any equilibria will not encounter any bounded equilibrium along its trajectory. But this alone is not enough to ensure that the orbit progressing from such an initial condition will not remain bounded. It is critical to determine what choices of b and $g(u)$ will lead to a semigroup $\mathcal{S}$ that is neither dissipative nor undergoes blow-up. The first step is to show that trajectories which do not terminate at a bounded equilibrium cannot be bounded for all time.

Lemma 2.1. *Given an initial condition u_0 and corresponding solution $u(t, \cdot)$ to Equation (2) which does not limit to any bounded equilibrium, and any $r,\ T \in \mathbb{R}^+$, the solution $u(t, \cdot)$ will leave the ball B_r of radius r in X at some time $t^* > T$.*

The implication of this lemma is that no matter how big a ball in X we construct, nor how long a time we wait, a solution which does not limit to a bounded equilibrium will always leave this ball at some later time. If $u(t, \cdot)$ cannot be bounded by any set, no matter how large, it must eventually achieve infinite norm.

Proof. The existence of such an initial condition is easy to derive, and is done in the next section, so for now we shall simply assume we already have such an initial condition.

For b either positive, negative, or zero, the semigroup $\mathcal{S}$ generated by Equations (2) and (3) is compact. One can clearly see that the semigroup $\mathcal{S}$ is a continuous gradient system, as it contains a strict Lyapunov functional for all $t \in [0, +\infty]$, and $\mathcal{S}$ is a continuous, compact semigroup. Thus, the LaSalle Invariance Principle applies. We shall now proceed with a proof by contradiction.

Let us assume that we have an initial condition u_0 whose corresponding solution $u(t, \cdot)$ does not limit to any bounded equilibrium. Let us also assume there exists some T and r such that for $t > T$, $u(t, \cdot) \subset B_r$. Since we can contain the forward trajectory of $u(t, \cdot)$ in B_r for $t > T$, there exists some $\widetilde{r}(r, T)$ such that $u(t, \cdot) \subset B_{\widetilde{r}}$ for all $t \geq 0$. We may now apply the LaSalle Invariance Principle, which implies that $u(t, \cdot)$ must limit to some equilibrium contained in $B_{\widetilde{r}}$. But this contradicts our initial assumption on u_0. Therefore, there exists some time $t^* > T$ such that $u(t^*, \cdot) \notin B_r$. □

3. Non-dissipativity

In order to ensure the existence of grow-up solutions for

$$u_t = u_{xx} + bu + g(u), \quad x \in [0, \pi]$$

$$u_x(t, 0) = u_x(t, \pi) = 0 \tag{5}$$

$$g(u) \in C^2, \textit{ bounded}, \textit{ g globally Lipschitz with values in } L^2,$$

we must first determine in what cases the semigroup $\mathcal{S}$ is non-dissipative, i.e. where it fails to be point dissipative [**16**].

Lemma 2.2. *There is at least one spatially homogeneous equilibrium of Equation (5). If there is only one spatially homogeneous equilibrium, we shall denote it by η^*, the unique value at which $bu = -g(u)$ for $u \in \mathbb{R}$. If there is more than one spatially homogeneous equilibrium, we shall denote the equilibrium with smallest norm (all considered higher norms being equivalent for spatially homogeneous solutions) as η^*.*

PROOF. A constant function $u(x) \equiv \eta$ is a spatially homogeneous equilibrium for Equation (5) if and only if $b\eta + g(\eta) = 0$ or $b\eta = -g(\eta)$. We recall that $g(u)$ is uniformly bounded, i.e. $\underline{\gamma} \leq g(u) \leq \overline{\gamma}$. Thus we may conclude that

$$
\begin{aligned}
b\eta + g(\eta) &< -\overline{\gamma} + g(\eta) \leq -\overline{\gamma} + \overline{\gamma} \leq 0 \quad for\ \eta < \frac{-\overline{\gamma}}{b} \\
b\eta + g(\eta) &> -\underline{\gamma} + g(\eta) \geq -\underline{\gamma} + \underline{\gamma} \geq 0 \quad for\ \eta > \frac{-\underline{\gamma}}{b}.
\end{aligned}
\tag{6}
$$

Therefore, by the Intermediate Value Theorem, the function $b\eta + g(\eta)$ must have value equal to 0 at some point $-\frac{\overline{\gamma}}{b} \leq \eta \leq -\frac{\underline{\gamma}}{b}$, therefore there is at least one function $u(x) \equiv \eta$ which is a spatially homogeneous equilibrium solution to Equation (5) $\square$

LEMMA 2.3. *Given a scalar parabolic equation of the form (5), the corresponding semigroup $\mathcal{S}$ is not point dissipative if $b > 0$.*

PROOF. For a given $b > 0$ and g as defined in (3) where $\underline{\gamma} \leq g(u) \leq \overline{\gamma}$, we are able to use the time map and bifurcation diagram to determine an ordering of the equilibria. We extract from this set those equilibria which are spatially homogeneous solutions to Equation (5). As stated in Lemma 2.2, there will always be at least one spatially homogeneous equilibrium for Equation (5). Let us denote the spatially homogeneous equilibrium with maximal norm as $v(x) = v$. If there exist a positive and negative equilibrium with the same norm which is maximal, we let $v(x)$ denote the positive equilibrium.

The set of spatially homogeneous stationary solutions $u(x) = \eta$ will always be bounded by $\frac{-\overline{\gamma}}{b} \leq \eta \leq \frac{-\underline{\gamma}}{b}$, thus the choice of $v(x) = v$ is well defined. We choose a spatially homogeneous initial condition u_0 such that $|u_0| > |v| + \frac{2\Gamma}{b}$ where $\Gamma = max\left\{|\underline{\gamma}|, |\overline{\gamma}|\right\}$. The value of the Lyapunov functional at the spatially homogeneous u_0, $V(u_0)$, is then restricted to be below not only $V(v)$, the Lyapunov functional at the "maximal" spatially homogeneous equilibrium, but below the value of the Lyapunov functional at all spatially homogeneous equilibria. To illustrate this, let $w(x) = w$ be any spatially homogeneous equilibrium, and necessarily $-|v| \leq w \leq |v|$. Then,

$$V(u_0) = \int_0^\pi \frac{1}{2}(u_0)_x^2 - \frac{b}{2}u_0^2 - G(u_0)dx = \int_0^\pi 0 - \frac{b}{2}u_0^2 - G(u_0)dx$$

$$-G(u_0) = -\int_0^{u_0} g(s)ds \Rightarrow -\Gamma|u_0| \leq -G(u_0) \leq \Gamma|u_0|$$

$$-G(w) = -\int_0^{w} g(s)ds \Rightarrow -\Gamma|w| \leq -G(w) \leq \Gamma|w|$$

$$V(u_0) = \int_0^\pi -\frac{b}{2}u_0^2 - G(u_0)dx \leq \int_0^\pi -\frac{b}{2}u_0^2 + \Gamma|u_0|$$

$$= \pi(-\frac{b}{2}u_0^2 + \Gamma|u_0|) = -\frac{\pi b}{2}u_0^2 + \pi\Gamma|u_0|$$

$$|u_0| > |v| + \frac{2\Gamma}{b} \Rightarrow |u_0| - \frac{\Gamma}{b} > |v| + \frac{\Gamma}{b} > 0$$

$$\Rightarrow (|u_0| - \frac{\Gamma}{b})^2 > (|v| + \frac{\Gamma}{b})^2 \Rightarrow -\frac{\pi b}{2}(|u_0| - \frac{\Gamma}{b})^2 < -\frac{\pi b}{2}(|v| + \frac{\Gamma}{b})^2$$

$$\Rightarrow -\frac{\pi b}{2}|u_0|^2 + \pi\Gamma|u_0| < -\frac{\pi b}{2}|v|^2 - \pi\Gamma|v|$$

$$\Rightarrow V(u_0) < -\frac{\pi b}{2}|v|^2 - \pi\Gamma|v| = \int_0^\pi -\frac{b}{2}|v|^2 - \Gamma|v|dx$$

$$\leq \int_0^\pi -\frac{b}{2}|w|^2 - G(w)dx = V(w)$$

$$\Rightarrow V(u_0) < V(w) \text{ for } |u_0| > |v| + \frac{2\Gamma}{b}.$$

The value of the Lyapunov functional along the orbit beginning at u_0 must necessarily decrease as time moves forward. But due to the nonincrease of the lap number [**22**], this orbit cannot contain any solutions which are non-spatially homogeneous for $t \geq 0$. Since the Lyapunov functional has higher value at all of the bounded spatially homogeneous equilibria, they cannot be in the omega limit set of u_0. Thanks to Lemma 2.1, we know that an orbit not limiting to any bounded equilibrium cannot remain bounded for all forward time, thus we have determined that the trajectory $u(t, \cdot)$ corresponding to the initial condition u_0 does not remain in any bounded set for all time. As we have now discovered at least one trajectory which does not remain bounded for all time, it follows that the semigroup $\mathcal{S}$ is not point dissipative, and thus is not compact dissipative, which is a stronger restriction.

Furthermore, we can explicitly determine the growth of the solution u_0, because it is a spatially homogeneous solution. As all solutions $u(t, \cdot)$ on the trajectory beginning at u_0 must be spatially homogeneous solutions, we may conclude that $u_{xx}(t, \cdot) = 0$ for $t \geq 0$. In addition, we have chosen u_0 in a specific range $|u_0| > |v| + \frac{2\Gamma}{b}$. Let us first address $u_0 > 0$. Recalling that $|v|$ is the absolute value of the height of the spatially homogeneous solution of maximal constant norm, this implies that for $u > |v|$ and $u < -|v|$ there do not exist any spatially homogeneous solutions, therefore $bu + g(u) < 0$ for $u < -|v|$ and $bu + g(u) > 0$ for $u > |v|$.

Therefore, if $u_0 > |v|$, $u_t = bu + g(u) > 0$ for the corresponding solution $u(t, \cdot)$, thus $u(t, \cdot)$ will remain a positive spatially homogeneous solution, growing for all time, reaching an infinitely large positive spatially homogeneous solution in infinite time. If $u_0 < -|v|$, $u_t = bu + g(u) < 0$ for the corresponding solution $u(t, \cdot)$, and therefore $u(t, \cdot)$ will remain a negative spatially homogeneous solution, growing in magnitude for all time, reaching an infinitely large negative spatially homogeneous solution in infinite time.

Thus, for an arbitrary nonlinearity $g(u)$ which is bounded and C^2 and $b > 0$, the semigroup $\mathcal{S}$ is non-dissipative. $\square$

REMARK 2.4. *For certain choices of nonlinearities g, b=0 also yields non-dissipative semigroups. An example of such a nonlinearity, $g(u) = a\ sin(u)$, is addressed in Chapter 9.*

CHAPTER 3

The Time Map and Bifurcation Diagrams

Three well known tools, the lap number, zero number, and Morse index, are key to determining which heteroclinic connections are made and which are blocked. This information is easily determined from the stationary global bifurcation diagram of Equation (5), which is itself determined by the time map. In this chapter we discuss the methods necessary to produce the global bifurcation diagram and discuss a number of results on the time map and bifurcation diagram which are characteristic of slowly non-dissipative systems.

1. The Time Map

The time map $T(\eta, f)$ determines the period of a periodic solution to the second order equation

$$u_{xx} + f(u) = 0 \tag{7}$$

identified by either the initial condition $(u(0), u_x(0)) = (\eta, 0)$ when (7) is combined with Neumann boundary conditions, or the initial condition $(u(0), u_x(0)) = (0, \eta)$ when (7) is combined with Dirichlet boundary conditions. It is referred to as the time map because it measures the "time" in x it takes for a solution in the phase plane to travel from a point on the u-axis (or u_x-axis respectively) to its first subsequent intersection with the u-axis (or u_x-axis) [**3, 30, 37**].

In Figures 1 and 2 we present examples of typical phase portraits of

$$\begin{gathered} u_{xx} + bu + g(u) = 0 \\ g(u) \textit{ bounded, } C^2, \textit{ globally Lipschitz with values in } L^2. \end{gathered} \tag{8}$$

Due to the restrictions we have placed on our nonlinearity, our dynamical system comes equipped with a Hamiltonian,

$$H(u, u_x) = \frac{1}{2}u_x^2 + \frac{b}{2}u^2 + G(u)$$
$$G(u) = \int_0^u g(u) \tag{9}$$

which is a first integral of Equation (8). Thus, to obtain the phase portrait, we simply plot the level curves of (9).

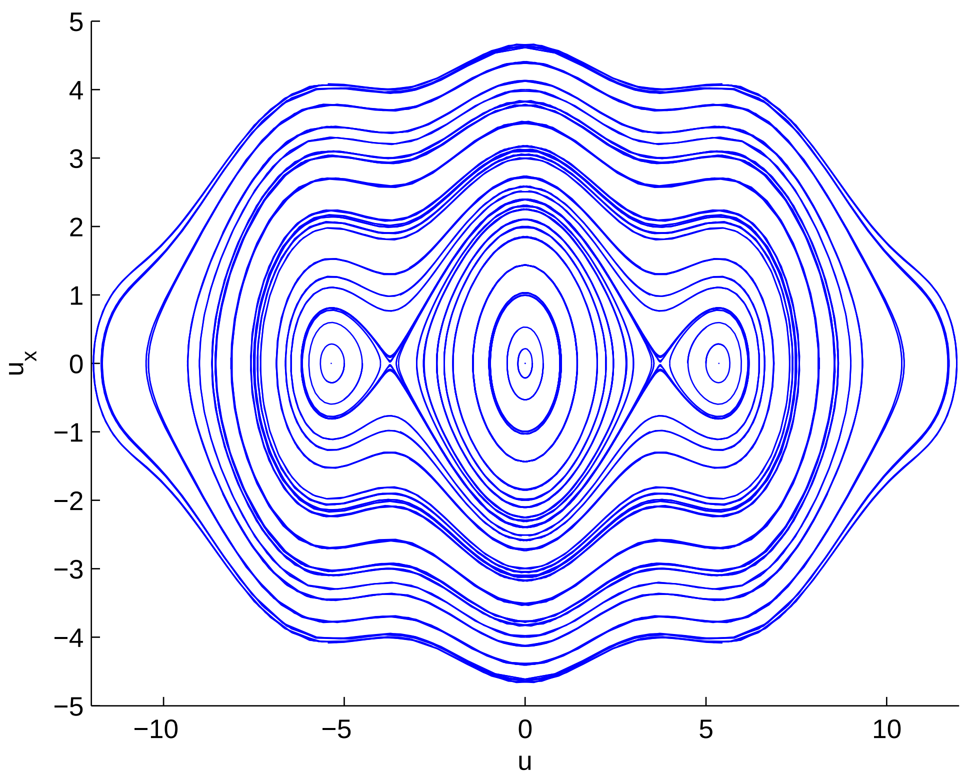

FIGURE 1. Phase portrait for $b = .15$, $g(u) = sin(u)$

We further define the nth time map $T_n(\eta, f)$ for any function $f(u)$ as follows: for Neumann boundary conditions, it is the "time", measured in x, necessary for a function $v(x)$ to reach its nth intersection with the u-axis (excluding the starting point), where $v(x)$ is a solution to Equation (7) with $v(0) = \eta, v_x(0) = 0$. An alternate definition is that $T_n(\eta, f)$ is the nth positive zero of the function $v_x(x)$ corresponding

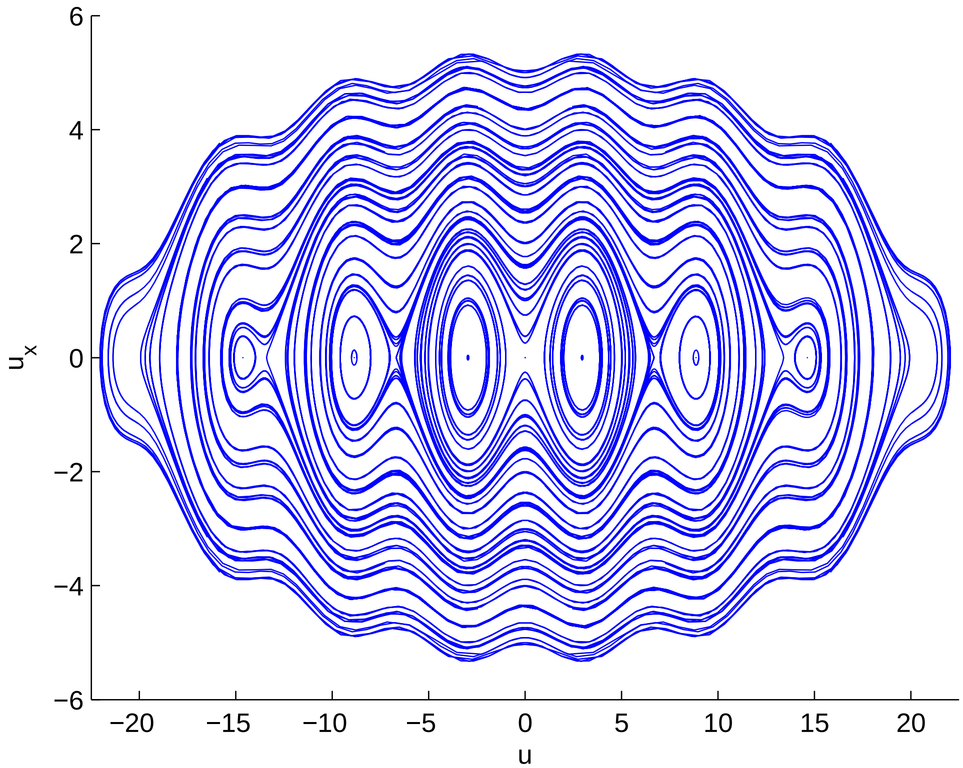

FIGURE 2. Phase portrait for $b = .06$, $g(u) = -sin(u)$

to a solution $v(x)$ of Equation (7) which satisfies

$$v(0) = \eta, \quad v_x(0) = 0, \tag{10}$$

whenever this zero exists.

Once we have determined the first intersection with the u-axis, the second one is derived as follows: for Neumann boundary conditions, $T_2(\eta, f) = 2T(\eta, f)$. This follows from the fact that (7) (and thus (8)) defines a conservative system. Thus, the phase portrait reflects across the u-axis, and any curve which intersects the axis twice in finite time will return along it's mirrored trajectory to its starting point in the same amount of time. As it has now returned to its point of origin, $T_3(\eta, f) = 3T(\eta, f)$ and $T_4(\eta, f) = 2T_2(\eta, f) = 4T(\eta, f)$. As one can quickly see, each nth time map is derived from the first time map in the case of Neumann boundary conditions. This does not hold for Dirichlet boundary conditions; we shall address these differences in Chapter 10.

Thus, we need to derive a formula which will allow us to explicitly determine $T(\eta, f)$. Luckily, for systems with a first integral, the formula is straightforward to derive, though not necessarily to evaluate:

$$T(\eta, f) = \frac{1}{\sqrt{2}} \int_{\eta}^{\alpha(\eta)} \frac{du}{\sqrt{F(\eta) - F(u)}} \tag{11}$$

with $F(u)$ derived as in (4). In our specific case, keeping in mind that trajectories move to the left if $\eta > \alpha(\eta)$,

$$\begin{aligned} T(\eta, b, g) &= \frac{1}{\sqrt{2}} \int_{\eta}^{\alpha(\eta)} \frac{du}{\sqrt{\frac{b}{2}\eta^2 - \frac{b}{2}u^2 + G(\eta) - G(u)}}, \qquad && \eta < \alpha(\eta) \\ T(\eta, b, g) &= \frac{1}{\sqrt{2}} \int_{\eta}^{\alpha(\eta)} \frac{-du}{\sqrt{\frac{b}{2}\eta^2 - \frac{b}{2}u^2 + G(\eta) - G(u)}}, \qquad && \eta > \alpha(\eta) \end{aligned} \tag{12}$$

where $\alpha(\eta)$ is derived from the equation $H(\eta, 0) = H(\alpha(\eta), 0)$ with $\eta \neq \alpha(\eta)$ and $\alpha(\eta)$ existing on the same trajectory as η. Note that $\alpha(\eta)$ is unique for η corresponding to solutions of Equation (8) with Neumann boundary conditions. When $T(\eta, b, g) = \infty$ it is possible that there is more than one trajectory originating from the point $(\eta, 0)$, such as two separate homoclinic orbits.

Although evaluating the time map analytically is possible only for a small number of bounded nonlinearities, it is simple to create a numerical solver that can evaluate it for any given input. All solutions of Equation (8) with Neumann boundary conditions, which are equivalently stationary solutions of Equation (5), must be either spatially homogeneous solutions or oscillatory solutions with lap number $n \geq 1$. The spatially homogeneous solutions will be fixed points in the phase portrait, while the oscillatory solutions must take "time" $\frac{\pi}{n}, n \in \mathbb{N}$, to travel from the u-axis back to itself, thus fulfilling the boundary conditions.

Brunovský and Chow [**3**] and Smoller and Wasserman [**37**] proved a number of very useful properties of the time map. A number of theorems on the domain of definition, monotonicity properties, and relations between the time map and stationary solutions are proven therein. We shall reiterate some of these results which are useful to note.

Firstly, the time map for a saddle point in the phase portrait defined by the stationary solution $u(x) \equiv \overline{\eta}$ is $T(\overline{\eta}, b, g) = \infty$ and $\lim_{\eta \to \overline{\eta} \pm 0} T'(\eta, b, g) = \mp\infty$. Secondly, let us define a δ-point in the phase portrait as the non-fixed point on a homoclinic orbit where the homoclinic intersects the u-axis.

We introduce here some notation for phase plane analysis with Neumann boundary conditions which will become useful in relating lap numbers to zero numbers. Because of the reflective nature of the phase portrait across the u-axis and the structure of the nonlinearity $f(u) = bu + g(u)$, all homoclinic orbits in the phase portrait will intersect the u-axis exactly twice, once at a saddle point, denoted by $\overline{\eta}$ and once at another point which is the δ-point of the homoclinic, and between $\overline{\eta}$ and the δ-point lies a center in the phase portrait, which is referred to as a β-point of the homoclinic. These points are illustrated in Figure 3.

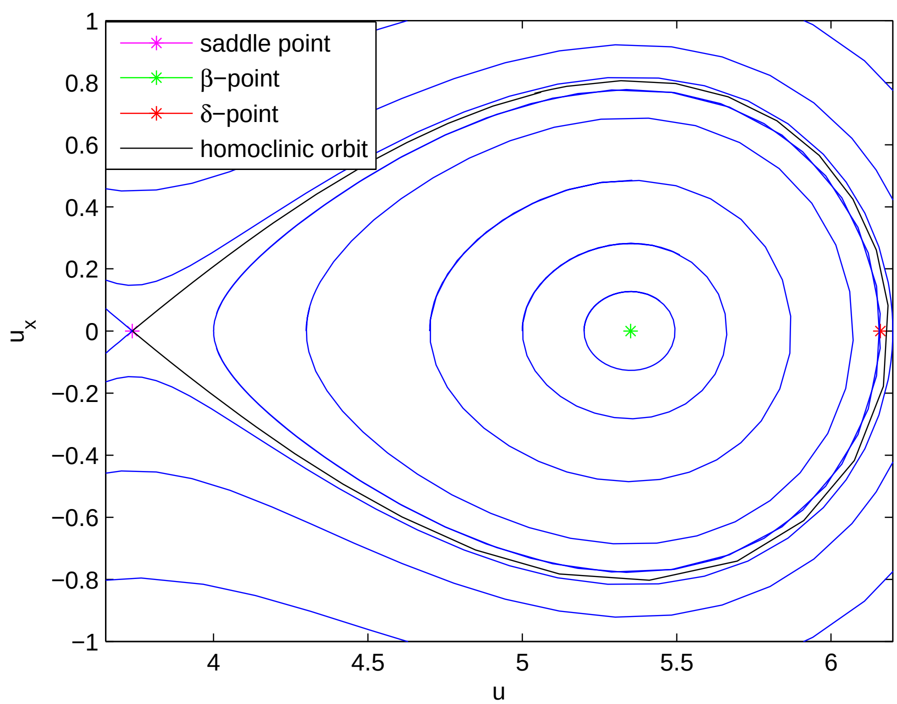

FIGURE 3. Close-up of a single homoclinic orbit for $b = .15$, $g(u) = sin(u)$ and its saddle-point, β-point and δ-point

We note that on any given homoclinic in the phase portrait, $\lim_{\eta \to \delta - 0} T'(\eta, b, g) = +\infty$. In addition, if a homoclinic contains only one fixed point within, the time map decreases monotonically from the saddle point $\overline{\eta}$ to the β-point, and increases monotonically from the β-point to the δ-point, with the minimum inside the homoclinically bounded region being achieved at the β-point, which is a center fixed point. It is possible to have more complicated behavior of the time map between $\overline{\eta}$ and the δ-point if there exists more than one fixed point within.

We further define the β-function of any stationary solution of Equation (5) as follows. Any non-spatially homogeneous stationary solution $v(x)$ must have lap number $n \geq 1$, i.e. it traces n halves of a periodic orbit in the phase plane. For any such periodic orbit there exists a spatially homogeneous stationary solution $u(x) \equiv \eta_v$ which corresponds to the fixed point $(\eta_v, 0)$ in the phase plane around which $v(x)$ rotates. We define $\beta(v) = \eta_v$. If there is more than one fixed point encompassed within the periodic orbit of v there are two possibilities, either v is contained within a homoclinic which contains other homoclinics, or v is not contained in any homoclinics, and thus is nested within periodic orbits. In the first case we define the β-function of v to be the middle fixed point, for there will always be an odd number of fixed points within a homoclinic in order to ensure periodic orbits. In the second case we define the β-function of v to be the trivial solution, $\beta(v) = \eta^*$.

If v is a spatially homogeneous stationary solution such that $v \equiv \eta$, then $\beta(v) = \eta$. For any v constrained to lie within a homoclinic orbit in the phase plane it follows that $\beta(v)$ is equal to the β-point of the immediately constraining homoclinic orbit. For any nested solutions v^1 and v^2 for which there do not lie any fixed points between v^1 and v^2 in the phase plane, it follows that $\beta(v^1) = \beta(v^2)$.

For periodic orbits more complicated than those shown in Figure 3, the time map is no longer necessarily monotonic, but can oscillate depending on the choice of $g(u)$. In the example studied in Chapter 9, wherein $g(u) = a\sin(u),\ a \neq 0$, the time map oscillates around the value $\frac{\pi}{\sqrt{b}}$. Thus, when $b = n^2$ we have an infinite number of orbits in the phase plane which fulfill both Equation (8) and the Neumann boundary conditions, and thus we have an infinite number of equilibria to our boundary value problem, with increasingly large norm. This corresponds to the sinusoidal nonlinearity having infinitely many points wherein $g(u) = 0$, and thus is not a behavior found for many choices of g.

Before we continue, we will note a few specific behaviors of the time map which play a part in the construction of the bifurcation diagram. Recall that the choice of η corresponds to the left endpoint $u(0)$ of an equilibrium solution of Equation (5).

LEMMA 3.1. *As $u(0)$ approaches $\pm\infty$, the time map for $\eta = u(0)$ for any choice of $g(u)$ and b fulfilling conditions (3) will approach the value $\frac{\pi}{\sqrt{b}}$, i.e.*

$$\lim_{\eta \to \pm\infty} T(\eta, b, g) = \frac{\pi}{\sqrt{b}}.$$

PROOF. We choose $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$, where $\overline{\eta}_+$ is defined as the value $\eta > 0$ such that $T(\eta, b, g) < \infty$ for all $\overline{\eta}_+ < \eta < \infty$, and $\overline{\eta}_-$ is defined as the value $\eta < 0$ such that $T(\eta, b, g) < \infty$ for all $-\infty < \eta < \overline{\eta}_-$. Essentially, we choose to start with η outside of all separatrices in the phase plane, as these contribute the only points where the time map is infinite [**3**]. The dominance of the linear part of $f(u) = bu + g(u)$ ensures that the region of discontinuities of T is bounded. As we have chosen η outside of the separatrices, it is ensured that $\alpha(\eta)$ is outside of the separatrices as well. This is due to $\alpha(\eta)$ being determined by the solution to the equation

$$\frac{b}{2}\eta^2 + G(\eta) = \frac{b}{2}\alpha^2(\eta) + G(\alpha(\eta)) \tag{13}$$

where $(\alpha(\eta), 0)$ is the unique point on the phase plane trajectory containing the point $(u(0), u_x(0)) = (\eta, 0)$ such that $\alpha(\eta) \neq \eta$. By definition of $\overline{\eta}_\pm$, it is clear that $b\eta + g(\eta) \neq 0$ for all $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$, and $b\alpha(\eta) + g(\alpha(\eta)) \neq 0$ for all $\alpha(\eta)$ corresponding to $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$. As $\frac{d\alpha(\eta)}{d\eta} = \frac{b\eta + g(\eta)}{b\alpha(\eta) + g(\alpha(\eta))}$, it is clear that $\frac{d\alpha(\eta)}{d\eta}$ is defined everywhere in the regions $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$.

Studying the Hamiltonian (9), and recalling that η and $\alpha(\eta)$ refer to points in the phase plane where $u_x = 0$, it is clear that solutions to $0 = u_{xx} + bu + g(u)$ which are outside of all separatrices must be nested. Therefore, $\frac{d\alpha(\eta)}{d\eta} < 0$ and $\lim_{\eta \to \pm\infty} \alpha(\eta) = \mp\infty$.

Thus, $\lim\limits_{\eta\to\pm\infty} T(\eta,b,g) = \lim\limits_{\eta\to\pm\infty} \int_{\eta}^{\alpha(\eta)} \frac{\mp du}{\sqrt{b\eta^2-bu^2+2G(\eta)-2G(u)}}$. Recalling that $\underline{\gamma} \leq g(u) \leq \overline{\gamma}$ and thus $\underline{\gamma}\eta \leq G(\eta) \leq \overline{\gamma}\eta$, and applying a change of variables $x\eta = u$ for $\eta \to \infty$ and $-x\eta = u$ for $\eta \to -\infty$ enables the following calculations:

$$\lim_{\eta\to\infty} \int_{\eta}^{\alpha(\eta)} \frac{-du}{\sqrt{b\eta^2 - bu^2 + 2G(\eta) - 2G(u)}}$$

$$= \lim_{\eta\to\infty} \int_{1}^{\frac{\alpha(\eta)}{\eta}} \frac{-\eta dx}{\sqrt{b\eta^2 - b\eta^2x^2 + 2G(\eta) - 2G(x\eta)}}$$

$$= \lim_{\eta\to\infty} \int_{1}^{\frac{\alpha(\eta)}{\eta}} \frac{-dx}{\sqrt{b - bx^2 + 2\frac{G(\eta)}{\eta^2} - 2\frac{G(x\eta)}{\eta^2}}} = \int_{1}^{-1} \frac{-dx}{\sqrt{b-bx^2}}$$

$$= \frac{1}{\sqrt{b}} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{\sqrt{b}}$$

$$\lim_{\eta\to-\infty} \int_{\eta}^{\alpha(\eta)} \frac{du}{\sqrt{b\eta^2 - bu^2 + 2G(\eta) - 2G(u)}}$$

$$= \lim_{\eta\to-\infty} \int_{-1}^{-\frac{\alpha(\eta)}{\eta}} \frac{-\eta dx}{\sqrt{b\eta^2 - b\eta^2x^2 + 2G(\eta) - 2G(-x\eta)}}$$

$$= \lim_{\eta\to-\infty} \int_{-1}^{\frac{-\alpha(\eta)}{\eta}} \frac{-\eta dx}{|\eta|\sqrt{b - bx^2 + 2\frac{G(\eta)}{\eta^2} - 2\frac{G(-x\eta)}{\eta^2}}}$$

$$= \lim_{\eta\to-\infty} \int_{-1}^{\frac{-\alpha(\eta)}{\eta}} \frac{dx}{\sqrt{b - bx^2 + 2\frac{G(\eta)}{\eta^2} - 2\frac{G(-x\eta)}{\eta^2}}} = \int_{-1}^{1} \frac{dx}{\sqrt{b-bx^2}}$$

$$= \frac{1}{\sqrt{b}} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{\sqrt{b}}.$$

It is clear that as long as $g(u)$ is bounded and C^2, the limit of the time map as $u(0)$ approaches infinity remains the same. It is entirely dependent on the asymptotic linearity, rather than the specific nonlinearity. $\square$

2. Time Map for a Fixed Nonlinearity

Let us suppose we have chosen a nonlinearity $g(u)$ fulfilling conditions (3) and wish to produce its bifurcation diagram. We must first observe certain aspects of its behavior with regard to the time map. Before plotting the time map, we must study the phase portrait to see where solutions are capable of existing. Decomposing our standard second order differential equation $u_{xx} + bu + g(u) = 0$ into a system of differential equations

$$
\begin{aligned}
u_x &= v \\
v_x &= -bu - g(u),
\end{aligned}
\tag{14}
$$

it becomes obvious that the first graph of interest is that of bu versus $g(u)$. As $g(u)$ is bounded, there exist values $\underline{\gamma}$ and $\overline{\gamma}$ such that $\underline{\gamma} \leq g(u) \leq \overline{\gamma}$ for all choices of $u(x)$. We let η_- denote the value of the leftmost intersection of bu and $g(u)$ and η_+ denote the value at the rightmost intersection. Due to the boundedness of $g(u)$, it follows that $\eta_- \geq -\frac{\overline{\gamma}}{b}$ and $\eta_+ \leq -\frac{\underline{\gamma}}{b}$. In Figure 4, $\eta_- = -51.3337$ and $\eta_+ = 40.6173$, and are indicated by squares.

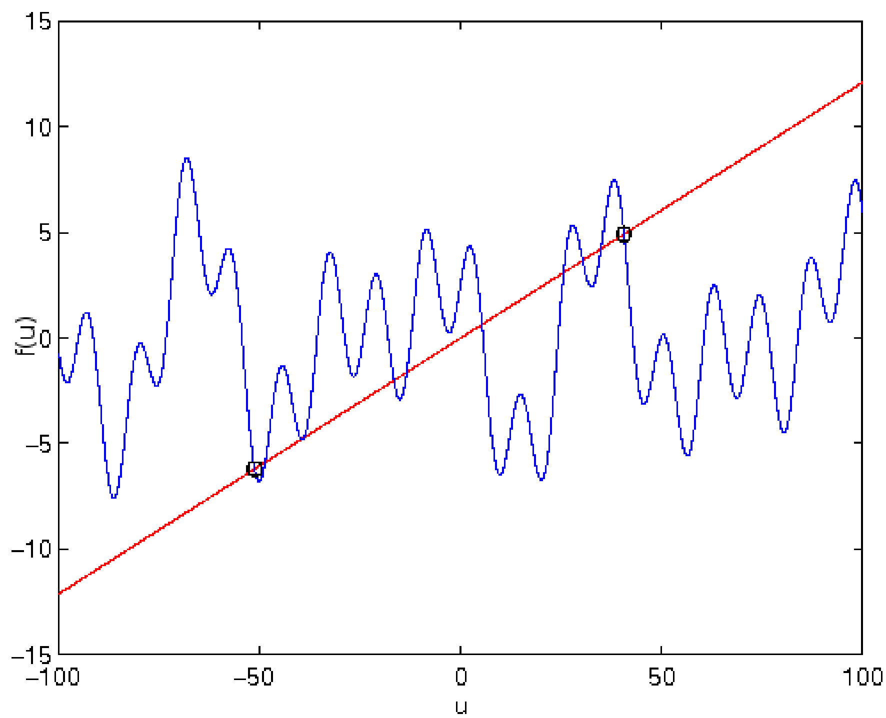

FIGURE 4. Graph of bu and $-g(u)$ plotted simultaneously

All intersections of bu with $-g(u)$ correspond to spatially homogeneous stationary solutions of Equation (5). By evaluating the Jacobian at these fixed points, we can determine which of these are centers in the phase portrait, and which are saddle points. This is also made evident by studying the graph of bu versus $-g(u)$, as those regions within which $bu > -g(u)$ correspond to $v_x < 0$, while those regions in which $bu < -g(u)$ correspond to $v_x > 0$. Thus, it becomes clear which intersections correspond to which type of fixed point in the phase portrait.

We now introduce two results characterizing the behavior and organization of stationary solutions of (5) in the phase plane. In order to clarify the next lemma, let us denote degenerate fixed points and lines of degenerate fixed points as follows. We refer to a degenerate fixed point or line of fixed points as "simply degenerate" if v_x has the same sign to the left of the fixed point or line of fixed points as to the right. We refer to a line of degenerate fixed points as a degenerate saddle if $v_x < 0$ to the left of the line of fixed points and $v_x > 0$ to the right of the line of fixed points. Finally, we refer to a line of degenerate fixed points as a degenerate center if $v_x > 0$ to the left of the line of fixed points and $v_x < 0$ to the right of the line of fixed points.

PROPOSITION 3.2. *The fixed points in the phase plane for*

$$0 = u_{xx} + bu + g(u) \tag{15}$$

are either degenerate, saddles, or centers, with the outermost fixed points, at η_- and η_+ always being either centers or degenerate. In addition, if any non-degenerate fixed point or line of degenerate fixed points is a saddle, the next non-degenerate fixed point or line of degenerate fixed points will be a center, and vice versa.

PROOF. The Jacobian at any fixed point u^* in the phase plane of Equation (15) is given by $J\mid_{u^*} = \begin{bmatrix} 0 & 1 \\ -b - g'(u^*) & 0 \end{bmatrix}$. Thus, the eigenvalues of the Jacobian at u^* are $\lambda = \pm\sqrt{-b - g'(u^*)}$. For $u^* = \eta_-$ and $u^* = \eta_+$, $b \geq -g'(u^*)$, implying the eigenvalues of the Jacobian are either both imaginary or both zero, and thus η_- and η_+ are either centers or degenerate. Since $g(u)$ is bounded, C^2, and Lipschitz with values in L^2, we know that $-g'(u)$ can only equal b for a limited time, let us say for $\epsilon_i^- \leq u \leq \epsilon_i^+$ for a fixed number of values i. In the phase portrait, all trajectories in this region will travel horizontally left or right (depending on whether they are above or below the u-axis). Thus, in determining the ordering of fixed points in the phase portrait, we may truncate any line of fixed points into one saddle or one center or ignore it entirely if it is simply degenerate, depending on the evaluation of v_x to the left and right of the line of fixed points, for the purpose of determining its nondegenerate neighbors. Thus do we, without loss of generality, assume that $b \neq -g'(u)$ for all $\eta_- \leq u \leq \eta_+$.

Let us assume that there exists a u^* such that u^* is a saddle fixed point. Since u^* is a saddle, there exist values κ_- and κ_+ such that $v_x(u) < 0$ and thus $bu > -g(u)$ for $\kappa_- \leq u < u^*$ and $v_x(u) > 0$ and thus $bu < -g(u)$ for $u^* < u \leq \kappa_+$. Therefore, the graph of bu on $[u^*, \kappa_+]$ is below the graph of $-g(u)$. But we know that for $u > \eta_+$ the graph of bu is above the graph of $-g(u)$. It follows that there must exist at least one point κ^* such that $\eta_+ \geq \kappa^* > \kappa_+$ and $b\kappa^* = -g(\kappa^*)$. If $\eta_+ = \kappa^*$, then either η_+ is degenerate and there does not exist a next nondegenerate fixed point, or else η_+ is a center, and we are finished. If $\kappa^* < \eta_+$, then $v_x(\kappa^* - \varepsilon) > 0$ and $v_x(\kappa^*) = 0$. Since we have assumed that $b \neq -g'(u)$, this implies that $v_x(\kappa^* + \varepsilon) < 0$ and therefore κ^* is a center. We may conduct this same analysis to the left of u^* with respect to η_- to obtain the same results.

Now, suppose we choose a fixed point u^* which is a center. Since u^* is a center, there exist values κ_- and κ_+ such that $v_x(u) > 0$ and thus $bu < -g(u)$ for $\kappa_- \leq u < u^*$, and $v_x(u) < 0$ and thus $bu > -g(u)$ for $u^* < u \leq \kappa_+$. Therefore, the graph of bu on $[u^*, \kappa_+]$ is above the graph of $-g(u)$. Now, let us assume that there exists a fixed point to the right of u^*. Then $u^* \neq \eta_+$ by definition. It follows that there must exist a point κ^* such that $\eta_+ \geq \kappa^* > \kappa_+$ and $b\kappa^* = -g(\kappa^*)$. Then $v_x(\kappa^* - \varepsilon) < 0$ and $v_x(\kappa^*) = 0$. And since we have assumed that $b \neq -g'(u)$, this implies that $v_x(\kappa^* + \varepsilon) > 0$ and therefore κ^* is a saddle. We may conduct this same analysis to the left of u^* with respect to η_- to obtain the same results. $\square$

Lemma 3.3. *All non-spatially homogeneous stationary solutions to*

$$u_t = u_{xx} + bu + g(u)$$

$$x \in [0, \pi], \quad \underline{\gamma} \le g(u) \le \overline{\gamma}, \quad g \in C^2 \tag{16}$$

$$u_x(t,0) = u_x(t,\pi) = 0$$

are either nested about each other, or separated along the u-axis by a fixed point in the phase portrait, i.e. a spatially homogeneous equilibrium solution to (16). If they are separated by a fixed point, each non-spatially homogeneous stationary solution is bound within either a homoclinic or pairs of heteroclinics in the phase portrait.

Proof. The introduction of Neumann boundary conditions requires that a solution both begins and terminates on the u-axis in the phase portrait. The equation $u_{xx} + f(u) = 0$ defines a conservative system regardless of the choice of f, therefore all trajectories in the phase plane which intersect the u-axis twice must trace half or all of a closed curve in the phase plane. Let us consider two equilibrium solutions to (16), denoted v and w, with $|v(0)| > |w(0)|$.

Since $v(x)$ and $w(x)$ are two distinct periodic solutions, they cannot intersect in the phase plane. Let us assume that $bv(0) + g(v(0)) > 0$. We may do so without loss of generality, as the proof proceeds identically for $bv(0) + g(v(0)) < 0$ with signs and orders reversed. This implies that the phase plane trajectory for the solution v at time 0 is pointed downwards, and therefore the trajectory travels to the left in the phase plane for as long as it remains below the u-axis. Therefore, the next point of intersection with the u-axis is to the left of $v(0)$ at the point $(v_1, 0)$. At $(v_1, 0)$, the phase plane trajectory must be pointed upwards, as we have excluded homoclinics and heteroclinics by the boundary conditions and exclusion of fixed points. Therefore $bv_1 + g(v_1) < 0$. Since $-|v(0)| < w(0) < |v(0)|$, there are three possibilities as to the location of $w(0)$ with respect to v_1. If $v(0) > 0$ then either $v_1 < w(0) < v(0)$ or $w(0) < v_1 < v(0)$. If $v(0) < 0$, then there is only one possibility: $v_1 < v(0) < w(0)$.

If $v(0) > 0$ and $v_1 < w(0) < v(0)$, then the w-curve begins inside the closed v-curve, and because the trajectories may not cross, it must remain inside for all time.

Thus, $w(x)$ is nested inside $v(x)$. If $v(0) > 0$ and $w(0) < v_1 < v(0)$, then the w-curve begins outside the closed v-curve, and must remain outside for all time. Because of this, the next intersection of the w-curve with the u-axis, at the point $(w_1, 0)$, cannot be in the region $[v_1, v(0)]$.

If $w(0) < w_1 < v_1$, it follows that $bw(0) + g(w(0)) < 0$ and $bw_1 + g(w_1) > 0$. As $f(u) = bu + g(u)$ is a continuous function of u, there must be some point $u^* \in (w_1, v_1)$ such that $bu^* + g(u^*) = 0$. The point $(u^*, 0)$ in the phase plane separates the w-curve and the v-curve, and by definition corresponds to a spatially homogeneous equilibrium solution to (16) of the form $u(x) \equiv u^*$.

If $w_1 < w(0) < v_1$, it follows that $bw(0) + g(w(0)) > 0$ and $bw_1 + g(w_1) < 0$. As before, there must be some point $u^* \in (w(0), v_1)$ such that $bu^* + g(u^*) = 0$, and we then have a spatially homogeneous equilibrium separating the two curves once more. If $w(0) < v_1 < w_1$, then $v(x)$ is nested inside $w(x)$.

If $v(0) < 0$ and $v_1 < v(0) < w(0)$, again we have the w-curve originating outside the closed v-curve. We may repeat the procedure as in the previous case and determine, depending on the location of w_1 in relation to v_1 and $v(0)$, that either the v-curve is nested inside the w-curve, or there is some fixed point separating the two in the phase plane.

Let us presume we are in one of the cases wherein the two curves are separated by a fixed point in the phase portrait. In each case, at $(u, u_x) = (u^*, 0)$ we have a fixed point such that $b(u^* - \varepsilon) + g(u^* - \varepsilon) > 0$ and $b(u^* + \varepsilon) + g(u^* + \varepsilon) < 0$ for ε small. This implies that the fixed point at $(u^*, 0)$ is a saddle point. Earlier we showed that $bu + g(u) > 0$ for $u > \eta_+$ and $bu + g(u) < 0$ for $u < \eta_-$ and that the linear portion will increasingly dominate outside of these regions since the nonlinearity is bounded. Thus we know that the unstable manifolds of the saddle point in the phase plane will eventually return to the u-axis, and, due to the reversible nature of system, form either a homoclinic or a heteroclinic between two fixed points. Since trajectories may not cross in the phase plane, the w-curve and v-curve must be contained in said homoclinic or paired-heteroclinic boundaries. $\square$

3. The Bifurcation Diagram

Knowing the time map for a given value of η provides us with other information as well, specifically the lap number. Given that Equation (15) defines a conservative system, a solution $u(x)$ to Equation (15) starting at $u(0) = \eta, u_x(0) = 0$ with $T(\eta, b, g) = \frac{\pi}{n}$ changes from regions having $u_x > 0$ to regions having $u_x < 0$ and vice versa n times. Thus, by definition, $l(u) = n$. By plotting the time map, we can easily discern all values η for which $T(\eta, b, g) = \frac{\pi}{n}$, and from this, all values for a given b and $g(u)$ for which there exist equilibria of our dynamical system (16), as well as their lap number. The natural ordering of values $\eta = u(0)$ for stationary solutions leads to a logical ordering of solutions, and thus is the vertical axis in our bifurcation diagram. By varying b, we can construct the global bifurcation diagram of stationary solutions. Given boundary conditions and a fixed nonlinearity $g(u)$, the pair (b, η) uniquely determines any stationary solution to the boundary value problem (16).

We designate curves in the bifurcation diagram as $n - branches$ when said curves stretch from $\eta = \infty$ to $\eta = -\infty$, and all solutions on said curve fulfill $T_n(b, \eta, g) = \pi$. For Neumann boundary conditions, this is equivalent to having lap number equal to n (this is not the case for Dirichlet boundary conditions). We note here that the 0-branch can have a discontinuity if $g(0) \neq 0$. In such a case, the 0-branch will limit to $-sign(g(0)) \cdot \infty$ from above and $sign(g(0)) \cdot \infty$ from below. It is only at the horizontal axis that this discontinuity may occur; as we move away from the axis, the 0-branch is well-behaved. Such a discontinuity is only possible for the 0-branch; all other branches are well-behaved. By definition, the β-function of a stationary solution on a nonzero n-branch is always the trivial solution η^*.

In addition to the n-branches, there is one other important curve: the curve of trivial solutions, which is parametrized over b. Recalling Lemma 2.2, the trivial solution for any b is the spatially homogeneous equilibrium solution η^* with minimal H^2-norm. We note that if the 0-branch suffers a discontinuity at $\eta = 0$, so will the curve of trivial solutions. The uniqueness of solutions to Equation (15) with defined left intercept and boundary conditions ensures that n-branches in the (b, η)-plane are globally parametrized by η. Thus, they do not intersect each other. Due to the parametrization in η, an n-branch cannot intersect another branch. This does not preclude the bifurcation of a curve to temporarily higher lap number in special cases.

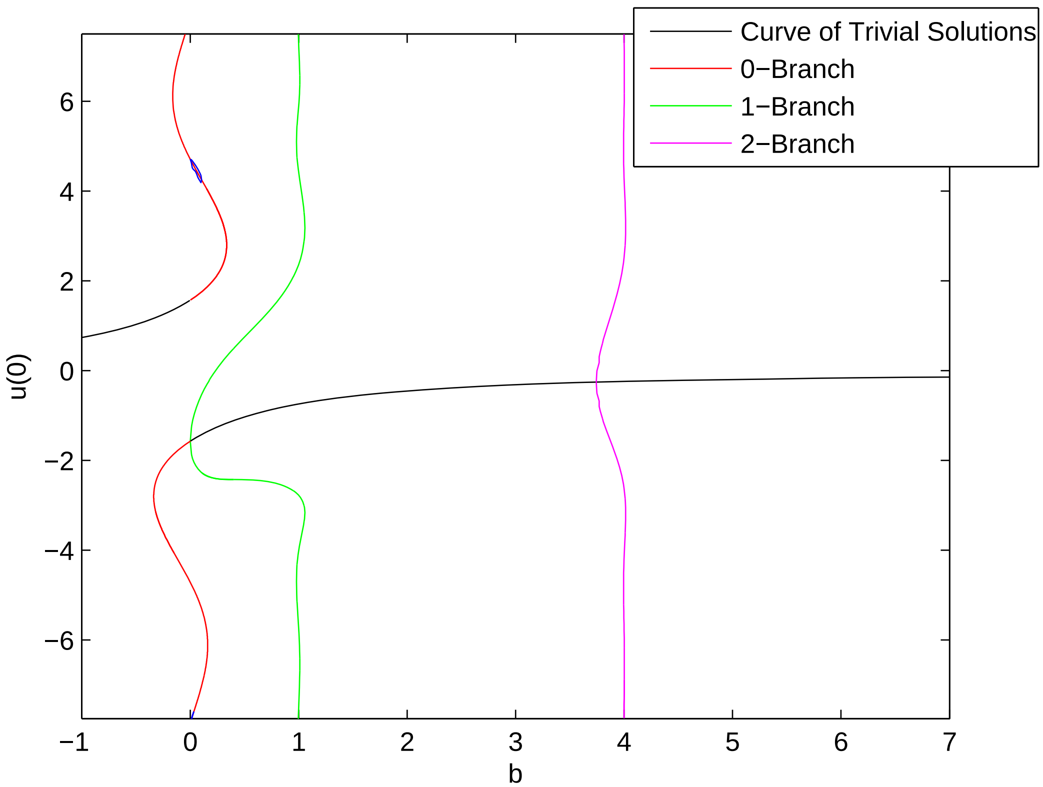

FIGURE 5. Bifurcation diagram for $g(u) = cos(u)$

Figure 5 provides an example of the behavior of the bifurcation diagram arising from the nonlinearity $g(u) = cos(u)$. As the reader can see, there is a discontinuity in both the curve of trivial solutions and the 0-branch. Furthermore, the 0-branch and curve of trivial solutions do not intersect transversely. This is a common behavior for nonlinearities where the curve of trivial solutions is not the b-axis. As the nonlinearity is not an odd function, the bifurcation diagram is not symmetric. The lack of symmetry between stationary solutions above and below the curve of trivial solutions is more pronounced for smaller b. As b increases, it outweighs the influence of the nonlinearity. Thus, while the 0-branch is noticeably non-symmetric, the 2-branch appears to be symmetric at a casual glance. Furthermore, the three n-branches depicted in Figure 5 asymptotically approach the lines $b = n^2$ as they grow further from the curve of trivial solutions. This is a property of all bifurcation diagrams for all equations of the form (16), as is proved in the following lemma.

LEMMA 3.4. *An n-branch will asymptotically approach the line $b = n^2$ in the bifurcation diagram.*

PROOF. As proven in Lemma 3.1, $\lim_{\eta\to\pm\infty} T(\eta, b, g) = \frac{\pi}{\sqrt{b}}$ for b fixed, g as in (16). As we have defined an n-branch, all solutions on an n-branch are stationary solutions of (16) with $u(0) = \eta$ and time map $T(\eta, b, g) = \frac{\pi}{n}$. For a fixed $b \neq n^2$, as $\eta \to \pm\infty$, $\lim_{\eta\to\pm\infty} T(\eta, b, g) = \frac{\pi}{\sqrt{b}} \neq \frac{\pi}{n}$. As the time map is continuous for η sufficiently large, this implies that there exists an η_b such that for $|\eta| > |\eta_b|$, $T(\eta, b, g) \neq \frac{\pi}{n}$ and therefore, the n-branch does not intersect a line at this fixed value of b. Equivalently, there are no equilibria with left boundary value at larger $u(0)$, and correspondingly larger norm, for $b \neq n^2$. Only for $b = n^2$ will this not hold. The n-branch is continuous, defined by the intersections of the time map with the line $\frac{\pi}{n}$ for variable b. Thus, as $\eta \to \infty$, the n-branch must approach $b = n^2$. The 0-branch is determined not by the time map, but via the function $b = -\frac{g(\eta)}{\eta}$. Since $g(\eta)$ is bounded for all η, it follows that $\lim_{\eta\to\pm\infty} b(\eta) = 0$. Thus the 0-branch asymptotically approaches the line $b = 0$. $\square$

Although we have only discussed the lap number on n-branches so far, it is useful to note that there is a related property of the zero number for n-branches.

LEMMA 3.5. *For any fixed b in Equation (16), let v be any stationary solution on an n-branch, v not being the bifurcation point of the n-branch from the curve of trivial solutions, i.e. $v \neq \eta^*$. Then*

$$z(v - \eta^*) = n = l(v).$$

PROOF. Note that if $v = \eta^*$, then $z(v - \eta^*) = l(v) = 0$. A solution v fulfilling the assumptions of this lemma solves

$$v_{xx} + bv + g(v) = 0. \tag{17}$$

If v is on the 0-branch, it follows that v must be a spatially homogeneous stationary solution. Therefore, $v(x) - \eta^*$ is also a spatially homogeneous function, although not necessarily a solution of (16). Since the zero number of all spatially homogeneous functions is defined to be 0, it follows that $z(v - \eta^*) = 0$.

If v is not on the 0-branch nor is it the trivial solution, it follows that $l(v) = n > 0$ and $T(v(0), b, g) = \frac{\pi}{n}$. We may plot the phase portrait of Equation (16). Since v exists on an n-branch, it follows that v is a periodic solution oscillating around the value η^*. Since the lap number of v is n, it follows that tracing $v(x)$ from 0 to π the trajectory will cross the line $u = \eta^*$ n times, once for each time it hits the u-axis again. Therefore, $v(x) - \eta^*$ is equal to zero n times, once for each half circle of the trajectory. Therefore the function $u(x) = v(x) - \eta^*$ has n interior zeros (and no exterior zeros since $v(x) \neq \eta^*$) and $z(v - \eta^*) = n$. $\square$

4. Bifurcations on the 0-Branch

Every branch undergoes saddle-node bifurcations (possibly infinitely many for nonlinearities such as $g(u) = \sin(u)$ or $g(u) = \cos(u)$ which oscillate about $u = 0$) as we increase through b [**3, 37**]. The 0-branch is unique in that it has the potential to produce pitchfork bifurcations as well, corresponding to Hopf bifurcations in the phase plane. The portion of the pitchfork bifurcation which exists before and after the bifurcation point is the 0-branch. The portions which come into existence and annihilate are referred to as the pitchfork branches. In Figure 5, the pitchfork branches are depicted in blue, and do not stray far from the 0-branch.

LEMMA 3.6. *The 0-branch of a bifurcation diagram for stationary solutions of Equation (16) undergoes paired supercritical pitchfork bifurcations at the values of* b *and* $u(0)$ *where* $b + g'(u(0)) = n^2$. *The equilibria solutions* $u(x)$ *on these new curves in the bifurcation diagram have lap number* $l(u) = n$ *and Morse index* $i(u) = n$. *Additionally, all pitchfork bifurcations are nested, such that pitchfork branches with higher lap number and Morse index are contained inside of pitchfork branches with lower lap number and Morse index, the outermost having* $l(u) = i(u) = 1$.

PROOF. First we shall address why it is impossible for such bifurcations to occur on any branch except the 0-branch. In order to obtain a pitchfork bifurcation in the phase plane, this requires two new equilibria to bifurcate from a previously existing equilibrium. Let us assume that this preexisting equilibrium is a solution on a nonzero n-branch. It then represents a unique periodic orbit in the phase plane which crosses the u-axis n times, taking time $\frac{\pi}{n}$ for each crossing. In order for a new equilibrium to bifurcate from the preexisting one, which we shall denote $u^*(x)$, it requires that changing b to $b \pm \delta$ will cause a new solution to spring into existence at initial conditions $u^*(0) \pm \varepsilon$, with $\varepsilon \to 0$ as $\delta \to 0$. In order to fulfill the Neumann boundary conditions, the new solutions which are located on the pitchforks must either be periodic orbits or fixed points in the phase portrait.

A solution which bifurcates from a periodic orbit cannot be a fixed point, as all possible fixed points are already included in either the 0-branch or the curve of trivial solutions, being derived from solutions of $bu + g(u) = 0$. Therefore, a bifurcating solution must be a periodic orbit with either the same lap number as the originating solution, or a different lap number. But $u^*(x)$ cannot have a different lap number than the bifurcated solution, as the time map is continuous. Thus, we cannot have a solution whose time map remains $\frac{\pi}{m}$ as it approaches a solution with time map $\frac{\pi}{n}$, for $n \neq m$. It follows that the only possibility is that $u^*(x)$ is a periodic solution with the same lap number as solutions on the originating n-branch. But the time map is not a multi-valued function, thus the only possible solutions infinitesimally close to a given equilibrium solution on an n-branch with lap number n must be those solutions on the n-branch itself.

Now that we have eliminated the possibility of nonzero n-branches undergoing pitchfork bifurcations, we shall address the circumstances allowing for such bifurcations on the 0-branch. Spatially homogeneous equilibria, or fixed points in the phase plane, do not have a period. Instead, their time map is defined as the limit of the time map of periodic solutions approaching these equilibria as discussed in [**29**]. As such, it is entirely possible that the time map of a spatially homogeneous equilibrium may pass through values $\frac{\pi}{n},\ n \in \mathbb{N}$.

It is easy to see that for spatially homogeneous equilibria $u(x) \equiv u(0)$ corresponding to centers in the phase plane, the quantity $b+g'(u(0))$ is positive, while equilibria corresponding to saddles in the phase plane have the property that $b+g'(u(0)) < 0$. Additionally, a degenerate solution which bifurcates into a saddle and a center in the phase plane fulfills $b+g'(u(0)) = 0$. Thus, in the bifurcation diagram the saddle node bifurcations on the 0-branch correspond to solutions $u(x) = u(0)$ wherein $b+g'(u(0)) = 0$ [**3, 37**]. The time map at a center fixed point $u(x) = \beta$ is equal to $\frac{\pi}{\sqrt{f'(\beta)}} = \frac{\pi}{\sqrt{b+g'(\beta)}}$. Notice that the limit of this formula as we approach a saddle-node bifurcation point (and thus the portion of the 0-branch corresponding to saddle fixed points) is infinity. This corresponds to our knowledge of the time map approaching a saddle, even though the time map is not well-defined at a saddle fixed point.

We may easily plot the 0-branch for any nonlinearity $g(u)$ fulfilling the conditions of (16) by recalling that solutions on the 0-branch solve the equation $bu+g(u) = 0$. Since solutions on the 0-branch are spatially homogeneous, for any such solution $u(x) = u(0)$ for all $x \in [0, \pi]$. Therefore, we may parametrize the curve in the plane $(b, u(0))$ by $b = -\frac{g(u(0))}{u(0)}$. For regions on the 0-branch where $b+g'(u) > 0$, all solutions are centers with a well-defined time map, and it is in this region that we have the potential for pitchfork bifurcations.

Since the 0-branch may be parametrized over the vertical axis, we can determine the slope in the regions in which all solutions are centers and as such, determine which regions of the bifurcation diagram these will be. Let us temporarily denote the vertical axis by η rather than $u(0)$ for ease of notation. We shall address the 0-branch in two regions: where $\eta(b) > \eta^*(b)$ and where $\eta(b) < \eta^*(b)$ (with $\eta = \eta^*(b)$ being a bifurcation point and thus degenerate). Let us perform the change of variables $v = u - \eta^*$ to simplify our calculations. Then, solutions on the 0-branch are equilibria solutions which solve

$$v_{xx} + bv + \underbrace{b\eta^* + g(v+\eta^*)}_{\widetilde{g}(v)} = 0. \tag{18}$$

The curve of trivial solutions $\eta^*(b)$ is now the horizontal axis, i.e. $\eta^*(b) = 0$ for all b, and the 0-branch is comprised of solutions fulfilling

$$b\eta + \widetilde{g}(\eta) = 0. \tag{19}$$

Finally, $b + g'(\eta) = b + \widetilde{g}'(\eta)$ as well. Recall that $b + g'(\eta) > 0$ on centers, and that for all solutions on the 0-branch, $b\eta + g(\eta) = 0$. We first consider solutions above the curve of trivial solutions, which in the change of variables is $\eta > 0$.

$$b\eta + \widetilde{g}(\eta) = 0 \Rightarrow b + \frac{\widetilde{g}(\eta)}{\eta} = 0, \quad b + \widetilde{g}'(\eta) > 0 \Rightarrow b + \widetilde{g}'(\eta) > b + \frac{\widetilde{g}(\eta)}{\eta}$$
$$\Rightarrow \widetilde{g}'(\eta) > \frac{\widetilde{g}(\eta)}{\eta} \Rightarrow \eta\widetilde{g}'(\eta) > \widetilde{g}(\eta) \Rightarrow \eta\widetilde{g}'(\eta) - \widetilde{g}(\eta) > 0 \Rightarrow \frac{\eta\widetilde{g}'(\eta) - \widetilde{g}(\eta)}{\eta^2} > 0$$
$$\Rightarrow \frac{-\eta\widetilde{g}'(\eta) + \widetilde{g}(\eta)}{\eta^2} < 0 \Rightarrow \frac{d}{d\eta}(\frac{-\widetilde{g}(\eta)}{\eta}) < 0 \Rightarrow \frac{d}{d\eta}(b(\eta)) < 0.$$

Therefore, for the portion of the 0-branch above the curve of trivial solutions, the branch moves to the left as it moves upwards, i.e. b decreases as η increases.

For $\eta < 0$ in the new variable, we recall that $\frac{d}{d(-\eta)} = -\frac{d}{d\eta}$. Then

$$b + \widetilde{g}'(\eta) > 0 \Rightarrow b + \widetilde{g}'(\eta) > b + \frac{\widetilde{g}(\eta)}{\eta} \Rightarrow \widetilde{g}'(\eta) > \frac{\widetilde{g}(\eta)}{\eta} = \frac{\widetilde{g}(\eta)}{-|\eta|}$$
$$\Rightarrow \eta\widetilde{g}'(\eta) = -|\eta|\widetilde{g}'(\eta) < \widetilde{g}(\eta) \Rightarrow \eta\widetilde{g}'(\eta) - \widetilde{g}(\eta) < 0 \Rightarrow \frac{\eta\widetilde{g}'(\eta) - \widetilde{g}(\eta)}{\eta^2} < 0$$
$$\Rightarrow \frac{d}{d\eta}(\frac{\widetilde{g}(\eta)}{\eta}) < 0 \Rightarrow \frac{d}{d(-\eta)}(-\frac{\widetilde{g}(\eta)}{\eta}) < 0 \Rightarrow \frac{d}{d(-\eta)}(b(\eta)) < 0.$$

Therefore, for the portion of the 0-branch below the curve of trivial solutions, b decreases as $|\eta|$ increases, i.e. the branch moves to the left as it moves downwards. These results both hold when we return to our normal frame of reference in the bifurcation diagram. On these portions of the 0-branch the solutions will all be centers and $b + g'(\eta) > 0$.

Combining Proposition 3.2 and Lemma 3.3 clearly implies that all non-trivial center fixed points will be contained within homoclinics or mirrored heteroclinics in the phase plane. This property also extends to the trivial solution, so long as it is not the only fixed point in the phase plane.

As we decrease b below a threshold, saddle-node bifurcations appear which produce the saddle and center portions of the 0-branch. As the center regions require that $b + g'(\eta) > 0$, and saddle-node bifurcations only occur for $b + g'(\eta) = 0$ [**37**], it follows that the value of $b + g'(\eta)$ must increase on the center portion immediately after the bifurcation. Thus, the time map decreases from infinity immediately along the center portion of the bifurcation. Recalling that the time map is continuous through centers, we decrease b and study the behavior along the center branch.

As we decrease b, the distance between the center and the saddle increases, and the value of the time map at a center fixed point $u(x) = \eta$ decreases from $+\infty$ at the bifurcation point. When the time map for a center spatially homogeneous equilibrium is greater than π, this implies that all periodic orbits in a small bounded neighborhood of the center will also have period greater than π. But if the time map at the center decreases to less than π, the time map of periodic orbits in this small neighborhood will also pass through the value π. For any point $\eta + \varepsilon$ for which $T(\eta + \varepsilon, b, g) = \pi$, there must be a corresponding $\alpha(\eta + \varepsilon)$ for which $T(\alpha(\eta + \varepsilon), b, g) = \pi$. Thus, two solutions now fulfill the Neumann boundary conditions with lap number 1.

When $b + g'(\eta) > 1$ for a spatially homogeneous solution $u(x) = \eta$, due to the continuity of the time map between singularities, the time map must cross the line at π in two spots between the saddle and its δ-point, allowing for periodic solutions in this region to fulfill Neumann boundary conditions. It is in this instance that we have a pitchfork bifurcation with two curves of solutions, both having lap number equal to 1, bifurcating into existence from the 0-branch. The pitchfork must be supercritical, as it is at the point where $b + g'(\eta) = 1$ that the Morse index on the 0-branch changes from $i(u(x)) = 1$ to $i(u(x)) = 2$.

We recall that at saddle-node bifurcation points of the 0-branch, the quantity $b + g'(\eta) = 0$, and thus the Morse index is 0. The continuity of $b + g'(\eta)$ implies that its value must eventually drop back below 1 on this portion of the 0-branch, and thus the bifurcated curves with lap number 1 solutions must return to the 0-branch via another supercritical pitchfork and annihilate. This corresponds to the value of b at which the time map will have increased in the neighborhood of the center to the point where $T(\eta, b, g) > \pi$ again.

Similarly, if there exist any values of η such that $b + g'(\eta) > n^2,\ n > 1$ on the 0-branch, then as above the time map will drop below $\frac{\pi}{n}$ and two new periodic solutions with lap number n will bifurcate out from this point. As $b + g'(\eta) > n^2$ obviously implies $b + g'(\eta) > (n-1)^2$, it is clear that pitchforks with higher lap number must be nested within pitchforks of lower lap number. The continuity of the quantity $b + g'(\eta)$ implies that each bifurcation point is unique. Finally, the Morse index on a pitchfork branch of lap number n must be equal to n, as it arises from a supercritical pitchfork of the 0-branch at a point where the Morse index on the 0-branch increases to $n + 1$. □

Remark 3.7. *As these extra bifurcations can only occur on the 0-branch, it follows that they can only occur when one chooses Neumann boundary conditions. If one were to study an equation of the form (16) with Dirichlet boundary conditions, the only possible spatially homogeneous solution would be the trivial solution, i.e. the 0-branch would cease to exist.*

Remark 3.8. *Note that the curve of trivial solutions also undergoes pitchfork bifurcations, but does not fall under the edicts of Lemma 3.6 as it is not an n-branch.*

Lemma 3.9. *The n-branches bifurcate from the curve of trivial solutions at the b value fulfilling $b + g'(\eta^*) = n^2$. The bifurcation will be a supercritical pitchfork if $g'(\eta^*) > 0$ and a subcritical pitchfork if $g'(\eta^*) < 0$.*

Proof. Recall that the curve of trivial solutions is parametrized over b, so we may write it as $\eta^*(b)$. As η^* is by definition a spatially homogeneous equilibrium, the eigenvalues and eigenvectors at η^* may be explicitly determined by

$$(20) \qquad \begin{aligned} \lambda_k &= -k^2 + b + g'(\eta^*) \\ \varphi_k &= cos(kx). \end{aligned}$$

For b sufficiently small or negative, $b+g'(\eta^*(b)) < 0$ and thus $0 > \lambda_0 > \ldots > \lambda_k > \ldots$. But as we increase b, the quantity $b + g'(\eta^*(b))$ will eventually increase to the point where $b + g'(\eta^*) > 0$, then greater than 1, 4, 9, and so forth. Let us denote b_k as the value at which $b_k + g'(\eta^*(b_k)) = k^2$. For $b_{k-1} < b < b_k$, solutions $\eta^*(b)$ on the curve of trivial solutions will have Morse index $i(\eta^*(b)) = k$, while solutions at values of $b < b_0$ will be asymptotically stable.

Recall that the time map on this curve is defined by $T(\eta^*(b), b, g) = \frac{\pi}{\sqrt{b+g'(\eta^*(b))}}$. This implies that for $b = b_k$ the time map at η^* will be equal to $\frac{\pi}{k}$, and for $b > b_k$, the value of $T(\eta^*(b), b, g)$ will fall below $\frac{\pi}{k}$. Therefore, as b passes through each b_k, two new periodic solutions with lap number k bifurcate from the curve of trivial solutions. These b_k we will refer to as the origination points of the k-branch.

If $g'(\eta^*(b)) > 0$, then η^* forms a local minimum of the time map. Thus, as b increases towards b_k, the time map descends towards the line $T = \frac{\pi}{k}$, intersecting it for $b = b_k$ and only dropping below it for $b > b_k$. Thus, increasing b through b_k causes two new solutions on the k-branch to bifurcate into existence above and below η^*, while decreasing b through b_k causes their annihilation. Thus, for $g'(\eta^*(b)) > 0$, there is a supercritical pitchfork bifurcation from the trivial solution curve at $b = b_k$.

If $g'(\eta^*(b)) < 0$, then η^* is a local maximum of the time map, and thus for $b < b_k$, a sufficiently small local neighborhood of η^* is above the line $T = \frac{\pi}{k}$, while the global minimum of the time map may be well below this line. Increasing b through b_k causes periodic solutions with lap number k to enter this neighborhood of η^*, i.e. the intersection of the time map and the line $T = \frac{\pi}{k}$ moves closer to η^*, and eventually these solutions annihilate at η^* for $b = b_k$. Thus, for $g'(\eta^*(b)) < 0$ there is a subcritical pitchfork bifurcation of the k branch from the trivial curve at $b = b_k$.

We refer to the intersection points of the k-branch with the curve of trivial solutions as origination points, but the k-branch may exist for lower values of b even when

$g'(\eta^*)$ is positive, or higher values when $g'(\eta^*)$ is negative. This is all dependent on where $g(u)$ achieves its maximum and the relative strength of bu at this point. The larger b is, the more the linear term bu will dominate the nonlinearity, but for small b the nonlinearity has the chance to win for some time. $\square$

Now that we have discussed all forms of pitchfork bifurcations present in the bifurcation diagram and where they occur, we may introduce a result which allows us to relate the lap number on the pitchfork branches to the zero number of a related function. Recall that the primary branch of the pitchfork is a portion of the 0-branch, and therefore both the lap number and zero number are equal to zero there.

LEMMA 3.10. *For any fixed b in Equation (16), let v be any stationary solution on any portion of a connected pair of pitchfork branches such that $l(v) = n$. Then*

$$z(v - \beta(v)) = n.$$

PROOF. We call any two pitchfork branches with lap number n a connected pair if, for any fixed b and stationary solution $v(x)$ represented on one branch by $v(0)$, the point $v(\frac{\pi}{n})$ identifies the stationary solution on the other branch. We note that it is impossible that $v \equiv \eta^*$. This follows from the fact that for the value of b such that the curve of trivial solutions and the 0-branch intersect, the quantity $b + g'(\eta^*) = 0$, and the pitchfork bifurcations only occur when $b+g'(\eta) \geq 1$. We remind the reader that for some nonlinearities the zero branch may have a discontinuity, owing to the possibility that $b = \frac{-g(\eta)}{\eta} = \pm\infty$. If v is a solution on the 0-branch itself, i.e. on the primary branch of a supercritical pitchfork, then v must be a spatially homogeneous solution. In this case it follows that $\beta(v) \equiv v$. We know that for a spatially homogeneous stationary solution the zero number is defined to be zero, and further, that the zero number of the function $u(x) \equiv 0$ is zero as well. Thus, since v is on the 0-branch, $l(v) = 0 = z(v) = z(v - \beta(v)) = z(0)$.

Now we may assume that v lives on one of the pitchfork branches. If v lives on a branch which bifurcates from the 0-branch at a point where $b + g'(\beta(v)) = n^2 \geq 1$, then $l(v) = n$ and $T(v(0), b, g) = \frac{\pi}{n}$. We may consider the phase portrait of Equation

(16). Since v exists on a pitchfork branch with lap number n, it follows that v is a periodic solution oscillating around the point $(\beta(v), 0)$. Since the lap number of v is n, it follows that in tracing $v(x)$ from 0 to π, the trajectory will cross the line $u = \beta(v)$ in the phase portrait n times, once for each time it hits the u-axis again. Therefore, $v(x) - \beta(v)$ is equal to zero n times, once for each half cycle of the trajectory. It follows that the function $u(x) = v(x) - \beta(v)$ has n interior zeros, and no exterior zeros since $l(v) \neq 0$, and thus $v(x) \neq \beta(v)$. Thus $z(v - \beta(v)) = n$. $\square$

5. Index on the Bifurcation Diagram

The bifurcation diagram provides both pieces of necessary information for determining connections between equilibria, i.e. the zero or lap number and the Morse index [**6, 7**]. This section addresses the relation between these nodal properties and the Morse index, as well as providing a method for determining this information from the appearance of the bifurcation diagram.

LEMMA 3.11. *The Morse index $i(v)$ and lap number $l(v)$ of a hyperbolic non-spatially homogeneous stationary solution to Equation (16) are related by*

$$i(v) \in \{l(v), l(v) + 1\}. \tag{21}$$

PROOF. We consider v, a hyperbolic stationary solution of (16) with Morse index $i(v) = i$. The linearization of (16) at v

$$\begin{gathered} Lu := u_{xx} + f'(v(x))u = u_{xx} + bu + g'(v(x))u \\ u_x(0) = u_x(\pi) = 0 \end{gathered} \tag{22}$$

allows us to determine the corresponding eigenvalues and eigenfunctions of v, denoted by λ_k and φ_k with $k \geq 0$. As we have noted that $i(v) = i$, this implies that

$$\begin{gathered} \lambda_0 > \ldots > \lambda_{i-1} > 0 > \lambda_i \\ l(\varphi_{i-1}) = z(\varphi_{i-1}) = i - 1, \quad l(\varphi_i) = z(\varphi_i) = i \\ (\varphi_k)_x(0) = (\varphi_k)_x(\pi) = 0 \ for\ all\ k. \end{gathered} \tag{23}$$

Now recall that while the eigenfunctions φ_k solve the equation $Lu = \lambda u$ with $\lambda = \lambda_k$, the function $w = v_x$ solves $Lu = 0$, i.e. $w(x)$ solves the eigenvalue problem with $\lambda = 0$. The Sturm-Liouville Comparison Theorem implies that between any two zeros of φ_{i-1} there is at least one zero of v_x, and between any two zeros of v_x there is at least one zero of φ_i. As stated above, $z(\varphi_{i-1}) = i-1$, and every φ_k satisfies Neumann boundary conditions, thus there are only $i-1$ total zeros of φ_{i-1}. Therefore $z(v_x) \geq i-2$. We must recall though, that because v satisfies Neumann boundary conditions, v_x satisfies Dirichlet boundary conditions, so the total number of zeros in $v_x = z(v_x)+2$. Therefore, $z(\varphi_i) \geq z(v_x)+2-1 \geq i-1$. Recalling that $l(v) = z(v_x)+1$, we now have the inequality

$$i = z(\varphi_i) \geq z(v_x) + 1 \geq i - 1 \Rightarrow i \geq l(v) \geq i - 1$$

i.e. $l(v) \in \{i-1, i\}$, or in other words, $i(v) \in \{l(v), l(v)+1\}$. □

Remark 3.12. *Note that if we define $i(v)$ for a non-hyperbolic stationary solution to be the number of positive eigenvalues, then Lemma 3.11 holds even for nonhyperbolic stationary solutions.*

Corollary 3.13. *The Morse index $i(v)$ and the zero number $z(v)$ of a hyperbolic non-spatially homogeneous stationary solution to Equation (16) are related by*

$$i(v) \in \{z(v - \beta(v)), z(v - \beta(v)) + 1\}.$$

Proof. Lemma 3.5 proved that for stationary solutions v on a nonzero n-branch we have $z(v - \eta^*) = n = l(v)$. Further, since v is on an n-branch rather than a pitchfork branch, v oscillates around the trivial solution in the phase portrait. In fact, $\beta(v) = \eta^*$ by definition. Thus, $l(v) = z(v - \eta^*) = z(v - \beta(v))$ and $l(v) + 1 = z(v - \eta^*) + 1 = z(v - \beta(v)) + 1$ for solutions on an n-branch. Thus $i(v) \in \{l(v), l(v)+1\} = \{z(v - \beta(v)), z(v - \beta(v)) + 1\}$.

Now let us assume that v is a solution on a pitchfork branch. Then by Lemma 3.10, it follows that $l(v) = z(v - \beta(v))$ and $l(v) + 1 = z(v - \beta(v)) + 1$. Thus
$i(v) \in \{l(v), l(v)+1\} = \{z(v - \beta(v)), z(v - \beta(v)) + 1\}$. □

Indeed, it has been shown that for a non-spatially homogeneous stationary solution v on an n-branch in the (b,η)-plane, v is hyperbolic if and only if $\frac{dT(\eta,b,g)}{d\eta} \neq 0$ for $\eta = v(0)$ [**34, 37**]. Since a point wherein $\frac{dT}{d\eta} \neq 0$ in the time map plot corresponds to a point wherein $\frac{db}{d\eta} \neq 0$ in the bifurcation diagram, it follows that v is hyperbolic if and only if $\frac{db}{d\eta} \neq 0$ on said n-branch [**34**]. Equivalently, one may interpret points where $\frac{db}{d\eta} = 0$ and (b,η) is on an n-branch as being points on the graph of the time map where $v(0) = \eta$, $T(\eta,b,g) = \frac{\pi}{n}$, and $\frac{dT}{d\eta} = 0$. We now present a result which allows us to determine the Morse index at a point in the bifurcation diagram simply by its location on an n-branch.

LEMMA 3.14. *For v a hyperbolic stationary solution to (16) on a nonzero n-branch with $v(0) = \eta_0 \neq \eta^*$,*

$$(24)\qquad \begin{aligned} (\eta_0 - \eta^*) \cdot \frac{db}{d\eta}(\eta_0) > 0 &\Rightarrow i(v) = l(v) = z(v - \eta^*) \\ (\eta_0 - \eta^*) \cdot \frac{db}{d\eta}(\eta_0) < 0 &\Rightarrow i(v) = l(v) + 1 = z(v - \eta^*) + 1. \end{aligned}$$

PROOF. In [**28, 34, 37**] it was proven that $T'(\eta_0,b,g) \neq 0$ for v hyperbolic and $v(0) = \eta_0$, and that

$$(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) > 0 \Rightarrow i(v) = l(v)$$
$$(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) < 0 \Rightarrow i(v) = l(v) + 1.$$

We recall that the n-branch is determined by the points where, for a given value of b, the function $T(\eta,b,g)$ intersects the line at $\frac{\pi}{n}$. Further, for fixed η in the vicinity of an n-branch point, $\frac{dT}{db}(\eta) < 0$. Thus, let us consider a fixed $\eta = \eta_0$ where $(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) > 0$. Then as b is increased, $T(\eta_0)$ decreases. If $\eta_0 > \eta^*$, then the value of η for which the new time map will equal $\frac{\pi}{n}$ increases, since $(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) > 0$. If $\eta_0 < \eta^*$, then the value of η for which the new time map equals $\frac{\pi}{n}$ decreases, since $(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) > 0$. In the bifurcation diagram, this is equivalent to $(\eta_0 - \eta^*) \cdot \frac{db}{d\eta}(\eta_0) > 0$.

We now consider a fixed $\eta = \eta_0$ wherein $(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) < 0$. Then as b is increased, $T(\eta_0)$ decreases. If $\eta_0 > \eta^*$, then the value of η for which the new time map will equal $\frac{\pi}{n}$ decreases, since $(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) < 0$. If $\eta_0 < \eta^*$, then the value of η for which the new time map equals $\frac{\pi}{n}$ increases, since $(\eta_0 - \eta^*) \cdot \frac{dT}{d\eta}(\eta_0) < 0$. In the bifurcation diagram, this is equivalent to $(\eta_0 - \eta^*) \cdot \frac{db}{d\eta}(\eta_0) < 0$. Finally, we recall from Lemma 3.5 that $l(v) = z(v - \eta^*)$ on an n-branch.

Combining this with the results from [**28, 34**], it follows that

$$(\eta_0 - \eta^*) \cdot \frac{db}{d\eta}(\eta_0) > 0 \Rightarrow i(v) = l(v) = z(v - \eta^*)$$
$$(\eta_0 - \eta^*) \cdot \frac{db}{d\eta}(\eta_0) < 0 \Rightarrow i(v) = l(v) + 1 = z(v - \eta^*) + 1.$$

□

Thus, we may determine the degeneracy of a stationary solution, its lap number, its shifted zero number, and its Morse index (assuming it is hyperbolic) just from its location in the bifurcation diagram.

CHAPTER 4

The Y-Map

In 1988, Fiedler and Brunovský devised a tool for establishing connections in semilinear parabolic equations [**6**]. For a given partial differential equation, boundary conditions, and initial condition, the y-map completely described the behavior of the zero number in forward time for a solution beginning at the initial condition. Unfortunately, the y-map was only constructed to deal with Dirichlet and mixed boundary conditions, and not pure Neumann boundary conditions. More importantly, the properties of the y-map were only proven for dissipative scalar parabolic PDEs. In this chapter we shall update the y-map to deal with scalar parabolic PDEs with linearly growing nonlinearities and Neumann boundary conditions, and expand some of the implications proved in [**6**]. In so doing we will gain insight into the nodal properties of our grow-up solutions, an element crucial to the determination of their asymptotic behavior.

As the y-map was initially studied for an unspecified Banach space, we will broaden our focus temporarily and study the semigroup of our equation on a general Banach space X. Additionally, properties of the y-map may be proven for nonlinearities dependent on both u and x, so we shall prove this more general case. We primarily use the methodology which was applied to the dissipative case, but as the original construction was more restrictive as well as excluding Neumann boundary conditions, all the necessary derivations will be included here.

Throughout this chapter we consider the equation

$$\begin{gathered} u_t = u_{xx} + \underbrace{bu + g(x,u)}_{f(x,u)}, \quad x \in [0,\pi] \\ u_x(t,0) = u_x(t,\pi) = 0. \end{gathered} \tag{25}$$

We denote by $\mathcal{G}$ the set of nonlinearities $g(x, u)$ satisfying

$$\begin{aligned} &g(x,u) \in C^2, \quad g \textit{ uniformly Lipschitz continuous with values in } L^2 \\ &g(x,u) \textit{ bounded uniformly in } x \in [0,\pi], \textit{ bounded uniformly in } u. \end{aligned} \tag{26}$$

Additionally, we shall denote by $\mathcal{F}$ the set of nonlinearities $f(x, u)$ of the form $f(x, u) = bu + g(x, u)$ with $g \in \mathcal{G}$. We endow $\mathcal{G}$ and $\mathcal{F}$ with the weak Whitney topology [**20**]. We construct the y-map as a continuous mapping

$$y : \{u_0 \in X | z(u_0) \leq n, u_0 \neq 0\} \to S^n \tag{27}$$

where S^n denotes the standard n-sphere in $\mathbb{R}^{n+1}$. Knowledge of $y(u_0)$ provides us with knowledge of $z(u(t, \cdot))$ on the forward trajectory of $u(t, \cdot)$ over $[0, \infty)$ for Equation (25) with initial condition u_0.

We could have chosen to define the y-map on initial conditions wherein $l(u_0) \leq n$ in order to obtain knowledge of $l(u(t, \cdot))$ for the forward trajectory. All lemmas in the following sections can be proven with minor changes for the lap number, but this leads to a difficulty later on, most notably in Corollary 4.7. As yet, it has not been proven that the difference of two solutions to a scalar parabolic partial differential equation has nonincreasing lap number over time. This is a crucial property necessary to the application of the lap number to determining the asymptotics of heteroclinic trajectories. By studying the behavior of the zero number, which has the property of the nonincrease over time for the difference of two solutions, we may, with some minor adjustments, obtain all the knowledge of asymptotic behaviors that we seek. In addition, the y-map has continuous dependence on the linearly growing nonlinearity, $f \in \mathcal{F}$. The restriction of y to an n-dimensional sphere Σ^n contained in the unstable manifold of the trivial solution provides an essential mapping of spheres. This essential mapping allows us to establish the existence of connections between equilibria.

1. Constructing the Mapping

We first introduce a restriction on the broader set of nonlinearities in $\mathcal{F}$ and $\mathcal{G}$ which are dependent on x as well as u. We define $\mathcal{F}_0$ and $\mathcal{G}_0$ as follows:

$$\begin{aligned} \mathcal{F}_0 &:= \{f \in \mathcal{F} | f(x,0) = 0 \ for \ all \ x\} \\ \mathcal{G}_0 &:= \{g \in \mathcal{G} | g(x,0) = 0 \ for \ all \ x\}. \end{aligned} \tag{28}$$

Proposition 4.1. *Any equation of the form (25) where $g(x,u) = g(u)$, i.e. where g and f are independent of x, may be rewritten into an equivalent equation in $\widetilde{u}$ where $\widetilde{g} \in \mathcal{G}_0$ and $\widetilde{f} \in \mathcal{F}_0$.*

Proof. We introduce the change of variables $\widetilde{u} = u - \eta^*$. Applying this change of variables transforms Equation (25) into

$$\begin{aligned} \widetilde{u}_t &= \widetilde{u}_{xx} + b\widetilde{u} + g(\widetilde{u} + \eta^*) - g(\eta^*), \quad x \in [0,\pi] \\ &\widetilde{u}_x(t,0) = \widetilde{u}_x(t,\pi) = 0 \\ \widetilde{g}(\widetilde{u}) &= g(\widetilde{u} + \eta^*) - g(\eta^*), \quad \widetilde{f}(\widetilde{u}) = b\widetilde{u} + \widetilde{g}(\widetilde{u}). \end{aligned} \tag{29}$$

It is clear that $\widetilde{g} \in \mathcal{G}$ is a function of $\widetilde{u}$ alone, and additionally that $\widetilde{g}(x,0) = \widetilde{g}(0) = 0$, and therefore $\widetilde{f}(x,0) = \widetilde{f}(0) = 0$. Thus $\widetilde{f} \in \mathcal{F}_0$ and $\widetilde{g} \in \mathcal{G}_0$, and therefore Equation (29) is a scalar parabolic partial differential equation of the form (25). Furthermore, it is possible to rewrite Equation (29) to see that it is linear in $\widetilde{u}$. As nonlinearities in $\mathcal{G}$ are twice continuously differentiable in u, we may rewrite $g(\widetilde{u} + \eta^*) - g(\eta^*)$ as $\widetilde{g}(\widetilde{u}) = \int_0^1 g'(\eta^* + \theta\widetilde{u})d\theta \cdot \widetilde{u}$.

Finally, let us note that we could have substituted any spatially homogeneous stationary solution of Equation (25) in place of η^* in this proof and the conclusion would still hold. Thus, the change of variables where $\widetilde{u} = u - \beta(v)$ for any stationary solution v of Equation (25) is equally valid. □

When f (and therefore g) are dependent on u alone, Proposition 4.1 shows that $f(0) = 0$ need not hold. But for nonlinearities dependent on x we cannot dispense with this requirement. In the case where $f(0) = 0$, it follows that $u \equiv 0$ is a stationary solution of Equation (25), and thus $\eta^* = 0$.

Finally, we note that for Equations of the form (25) where the nonlinearity $g \in \mathcal{G}_0$, the zero number of solutions $u(t,\cdot)$ is nonincreasing in forward time [**1, 23**]. Further, if we restrict the nonlinearity g to be independent of x, it follows that the zero number of the difference of any two solutions is also nonincreasing in time [**1**].

We now construct the y-map below, following the methodology laid out in [**6**]. Henceforth, for ease of notation, we shall assume that we have already made whatever changes of variables are necessary, and replace $\widetilde{u}$ with u in our notation. In other words, we will reference Equation (25) with the understanding that we have already made the change of variables indicated in Proposition 4.1 if necessary. Let $u_0 \in X$, $u_0 \neq 0,\ z(u_0) \leq n$ with corresponding trajectory $u(t,\cdot)$ forward in time.

We define the *dropping times* $t_k \in [0,\infty]$ as the first time that the zero number $z(u(t,\cdot))$ drops below the value k:

$$\begin{aligned} t_k &:= \inf\{t \geq 0 |\ z(u(t,\cdot)) \leq k\} \\ \tau_k &:= tanh(t_k) \in [0,1]. \end{aligned} \tag{30}$$

Because the zero number is nonincreasing for homogeneous Neumann boundary conditions and nonlinearities $g \in \mathcal{G}_0$ [**1**], it follows that $0 = t_n \leq t_{n-1} \leq \cdots \leq t_0$, and $0 = \tau_n \leq \tau_{n-1} \leq \cdots \leq \tau_0$. We define the sign of each element of the y-map by

$$\sigma_k := \begin{cases} sign\ u(t,0)\ for\ some\ t \in (t_k, t_{k-1}), & if\ t_k < t_{k-1} \\ 0, & otherwise. \end{cases} \tag{31}$$

The components of the y-map, $y = (y_0, y_1, \cdots, y_n)$, are defined by

$$\begin{aligned} y_0 &:= \sigma_0(1-\tau_0)^{1/2} \\ y_k &:= \sigma_k(\tau_{k-1}-\tau_k)^{1/2}, \qquad 1 \leq k \leq n. \end{aligned} \tag{32}$$

The σ_k are well-defined if $u(t,0) \neq 0$ for $t \in (t_k, t_{k-1})$. Thus, we provide a lemma to ensure this.

Lemma 4.2. *Given $f \in \mathcal{F}_0$ and $z(u(0,\cdot)) < \infty$, define the dropping times t_k of $z(u(t,\cdot))$ as in (30). The set of times $t > 0$ such that $x \to u(t,x)$ has only simple* *zeros is open and dense in* $\mathbb{R}^+$*. Further, if* $t_k < t_{k-1}$*, then*

$$u(t,0) \neq 0 \quad for\ all\ t \in (t_k, t_{k-1}).$$

Lemma 4.2 relies on properties proven previously in [**1, 22**] being applicable to the equations we study. The interested reader is directed to these works to discover why such properties hold for more general equations.

Proof. For the length of this proof we return to the notation wherein $u(t,x)$ solves Equation (25) while $\widetilde{u}(t,x)$ is a solution of the shifted Equation (29). For $f \in \mathcal{F}_0$, it follows that either $g = g(x,u)$ and $g(x,0) = 0$, i.e. the trivial solution is $\eta^* \equiv 0$, or $g = g(u)$ and we work in the shifted system when $g(0) \neq 0$. Returning to Equation (29), we recall that we may rewrite $\widetilde{g}(\widetilde{u})$ as $\int_0^1 g'(\eta^* + \theta\widetilde{u})d\theta \cdot \widetilde{u}$, thus we may rewrite Equation (29) in the form $u_t = u_{xx} + c(x,t)u$. Equation (25) may also be rewritten in this form, where $c(x,t) = \int_0^1 g_u(x,u)d\theta \cdot u$, recalling that here $\eta^* = 0$.

As proven in [**1, 22**], if $(t,x) = (t^*, x^*)$ provides a multiple zero of a solution $u(t,\cdot)$ for an equation of the form

$$u_t = u_{xx} + c(x,t)u$$

with $c \in L^\infty$, then t^* is a dropping time. Let us first set $x^* = 0$. In both possible cases, our equations (25) and (29) fulfill the required conditions, thus "Matano's Principle" applies. Therefore, for all $t \in (t_k, t_{k-1})$ where $t_k \neq t_{k-1}$, the left endpoint cannot be a multiple zero, and thus to conform to Neumann boundary conditions, $u(t,0) \neq 0$ for all $t \in (t_k, t_{k-1})$. If we let x^* be any point in $[0,\pi]$, then if there exists a multiple zero at time t^*, t^* is a dropping time. Since the zero number of $u(t,\cdot)$ is finite, it follows that the number of dropping times is also finite. Therefore, the set of times $t > 0$ such that $x \to u(t,x)$ has only simple zeros is open and dense in $\mathbb{R}^+$. □

2. Properties of the Y-Map

It is clear from construction, and easily verified via simple calculation, that y maps into S^n. Now, let us suppose that for a given u_0, we already know $y(u_0)$. We are able to then reconstruct the dropping times t_k and signs σ_k, as they are uniquely determined by $y(u_0)$. The case where $y(u_0) = \sigma e_k$, where e_k denotes the k^{th} unit vector and $\sigma \in \{1, -1\}$, is of particular importance. When $y(u_0) = \sigma e_k$, this implies that $0 = t_n = \ldots = t_k$ and $t_{k-1} = \infty$, i.e. $z(u_0) = k$, and $z(u(t, \cdot)) = k$ for all finite forward time. In addition, from [**1**] we know that the sign of $u(t, 0)$ cannot change for $t \in (t_k, t_{k-1})$, and thus $\sigma \cdot u(t, 0) > 0$ for all non-dropping times t.

LEMMA 4.3. *The y-map (32) depends continuously on $f \in \mathcal{F}_0$ and $u_0 \in X - \{0\}$ with $z(u_0) \leq n$.*

PROOF. We use the fact that a solution $u(t, \cdot)$ of Equation (25) viewed as a C^1-function of x depends continuously on f, u_0, and t. Specifically, let us rewrite the solution $u(t, x)$ as $u(t, x; f, u_0)$ to remind the reader of its dependence on the nonlinearity f and the initial condition u_0. Then the map

$$\mathcal{F}_0 \times X \times [0, \infty) \to C^1(\mathbb{R}, [0, \pi])$$

$$(f, u_0, t) \mapsto (x \mapsto u(t, x; f, u_0))$$

is continuous owing to it being the composition of the analogous map

$$\mathcal{F}_0 \times X \times [0, \infty) \to H^2(\mathbb{R}, [0, \pi]) \cap \{\textit{Neumann Boundary Conditions}\},$$

which is continuous [**18**], and the continuous Sobolev embedding

$$H^2(\mathbb{R}, [0, \pi]) \hookrightarrow C^1(\mathbb{R}, [0, \pi]).$$

The reader is reminded that we use the weak Whitney topology on $\mathcal{F}_0$ [**20**] here. This is sufficient as continuous dependence on initial conditions is a local property.

The first step in proving the continuous dependence of the y-map on f and u_0 is to show that τ_k depends continuously on $(f, u_0) \in \mathcal{F}_0 \times X$. To that end, we first show lower semicontinuity. Recall that the zero number

$$z : C^0(\mathbb{R}, [0, \pi]) \to \mathbb{Z}$$

$$u \mapsto z(u)$$

is lower semicontinuous by definition [**6**]. Combining this lower semicontinuity with the continuity of $u(t, \cdot; f, u_0)$ and the definition (30) of τ_k implies that given any

$\varepsilon > 0$ such that $\tau_k - \varepsilon > 0$, and given t such that $tanh(t) = \tau_k - \varepsilon$, there exists a neighborhood N of (f, u_0) in $\mathcal{F}_0 \times X$ such that for any $(\widehat{f}, \widehat{u}_0) \in N$,

$$z(u(t, \cdot; \widehat{f}, \widehat{u}_0)) \geq z(u(t, \cdot; f, u_0)) > k,$$

and thus

$$\tau_k(\widehat{f}, \widehat{u}_0) \geq \tanh(t) = \tau_k(f, u_0) - \varepsilon.$$

Therefore, τ_k is lower semicontinuous.

Next, we assume that $t_k < \infty$, otherwise the upper semicontinuity of τ_k is obvious. For $g \in \mathcal{G}_0$ and $z(u(0, \cdot)) < \infty$ the set of times $t > 0$ such that $x \to u(t, x)$ has only simple zeros is dense in $\mathbb{R}^+$ [**4, 6, 22**]. This implies that for any positive ε there exists some t such that $\tau_k < \tanh(t) < \tau_k + \varepsilon$ and all zeros of the map $x \to u(t, x; f, u_0)$ are simple. Using the continuity of $u(t, \cdot; f, u_0) \in C^1$ and recalling the definition (30) of τ_k, we obtain the existence of a neighborhood N of $(f, u_0) \in \mathcal{F}_0 \times X$ such that for any $(\widehat{f}, \widehat{u}_0) \in N$,

$$z(u(t, \cdot; \widehat{f}, \widehat{u}_0)) = z(u(t, \cdot; f, u_0)) \leq k$$

$$\tau_k(\widehat{f}, \widehat{u}_0) \leq \tanh(t) < \tau_k(f, u_0) + \varepsilon.$$

Thus, τ_k is upper semicontinuous, and therefore continuous.

The next step to proving the continuous dependence of the y-map on $(f, u_0) \in \mathcal{F}_0 \times X$ is to prove that each component y_k individually is continuously dependent on (f, u_0). As k was not specified above, we know that each of $\tau_0, ..., \tau_{n-1}$ is continuously dependent on (f, u_0). By definition, τ_n is always equal to 0, and therefore its dependence is arbitrarily guaranteed. For $\tau_k(f, u_0) < \tau_{k-1}(f, u_0)$, Lemma 4.2 implies that $u(t, 0; f, u_0) \neq 0$ for all t such that $tanh(t) \in (\tau_k, \tau_{k-1})$. Fixing one such t, there exists a neighborhood N of (f, u_0) in $\mathcal{F}_0 \times X$ such that $u(t, 0; \widehat{f}, \widehat{u_0}) \neq 0$ for any $(\widehat{f}, \widehat{u_0}) \in N$, due to the continuous dependence of $u(t, 0; f, u_0) \in C^1$ on (f, u_0). Thus, σ_k is constant on N, and therefore the continuity of y_k follows from the continuity of τ_k and τ_{k-1}.

If $\tau_k = \tau_{k-1}$, then $y_k(f, u_0) = 0$ by definition and the continuity of τ implies that $|y_k| < \varepsilon$ for all $(\widehat{f}, \widehat{u_0}) \in N$, where N is some neighborhood of $(f, u_0) \in \mathcal{F}_0 \times X$, regardless of the value of σ_k in N. Therefore, in either case every component of y is continuously dependent on f and u_0, and so y itself is continuously dependent on f and u_0. $\square$

Remark 4.4. *We have been forced to ignore the point $u_0 \equiv 0 \in X$ (or $u_0 = \eta^*$ before the change of variables if one was necessary) because at this point the construction of the y-map falls apart. Rather than mapping to $S^0 \equiv \{+1, -1\}$, the point $u_0 \equiv 0$ is always mapped to the value 0. The reader is now reminded of the proof of Proposition 4.1. We are able to shift to a different but equivalent system, wherein the point $u_0 \equiv 0$ becomes a new point $w_0 \neq 0$. The y-map is now well-defined at such a point, and will produce the same implications for w_0 as would be provided by focused analysis on $u_0 \equiv 0$ for the original system. Thus, although the y-map is not defined at this point, the implications of the y-map may still be obtained with a little extra work.*

Now that we have proven the continuous dependence of the y-map on $f \in \mathcal{F}_0$ and $u_0 \in X - \{0\}$, we seek to prove the surjectivity of the y-map for any such choice of f. Of course, it is significantly simpler to study y for f linear. We must show that using a homotopy to deform f from linear to nonlinear, i.e. to deform f from bu to $bu + g(u)$, does not destroy surjectivity.

First, we consider the linear case $f(x, u) = b(x)u$ and show that in this case, $y : \Sigma^n \to S^n$ is an *essential* mapping, where Σ^n is an n-dimensional sphere in the unstable manifold of the trivial solution. In other words, there is no homotopy from y to the constant map. By definition, the property of being essential is invariant under homotopies to nonlinear f. This implies that y remains surjective under the homotopy; otherwise, the image of y would miss some point in S^n and could be contracted to a single point, contradicting the fact that y is essential. For $n = 0$, surjective implies essential. This does not generally hold for $n > 0$.

Let $f(x,u) = b(x)u, \; b \in C^2$. We denote the Sturm-Liouville eigenvalues and eigenfunctions of

$$
\begin{aligned}
&u_{xx} + b(x)u = \lambda u \\
&u_x(0) = u_x(\pi) = 0
\end{aligned}
\tag{33}
$$

by $\lambda_0 > \lambda_1 > \ldots$ and $\varphi_0, \varphi_1, \ldots$, and take $\varphi_i(x)$ renormalized to unit length in the X-norm, choosing the sign convention $\varphi_i(0) > 0$. Let us assume that $\lambda_n > 0$, i.e. the Morse index $i(u \equiv 0) \geq n+1$. We denote

$$
W_n = span\left\{\varphi_0, \; \ldots, \varphi_n\right\}. \tag{34}
$$

Sturm-Liouville theory states that $z(w) \leq n$ for $w \in W_n$ [**2**].

LEMMA 4.5. *For $\lambda_n > 0$, the y-map restricted to Σ^n is essential and, in particular, surjective.*

PROOF. The proof of Lemma 4.5 in the Dirichlet case can be found in [**6**]. We need simply redefine σ_k to be the sign of $\varphi_k(0)$ rather than $\varphi_k'(0)$ to update this proof to the Neumann case. As such, the proof itself will not be copied here, but can be found in [**6**]. We conclude by noting that the property of being an essential mapping implies surjectivity [**9**]. $\square$

We now address the nonlinear case, and prove that surjectivity carries over. It is at this point where the distinctions between the original equation (25) in u and the shifted equation (29) in $\widetilde{u}$ become pertinent, and so we will dispense with the assumption of a pre-shifted system which was used to simplify notation in the previous pages.

LEMMA 4.6. *Let $v \equiv \eta$ be a spatially homogeneous hyperbolic stationary solution of (25) with unstable manifold W^u of dimension $i(v) > 0$. Let $\Sigma \subset W^u \backslash \{v\}$ be homotopic in $W^u \backslash \{v\}$ to a small sphere in W^u centered at v of dimension $n = i(v) - 1$.*

For any finite sequence

$$
\begin{aligned}
&0 = \delta_n \le \delta_{n-1} \le \ldots \le \delta_0 \le \infty \\
&s_k \in \{1, -1\}, \qquad 0 \le k \le n,
\end{aligned} \tag{35}
$$

there exists a point $u_0 \in \Sigma$ corresponding to an initial condition $u(0, \cdot) \in X$ such that the graph $t \to z(u(t, \cdot) - \eta)$ is determined by (δ_k). In other words, for any $0 \le t < \infty$,

$$
\begin{aligned}
&t \ge \delta_k \Leftrightarrow z(u(t, \cdot) - \eta) \le k \\
&\delta_k < t < \delta_{k-1} \Rightarrow sign(u(t, 0) - \eta) = s_k.
\end{aligned} \tag{36}
$$

Proof. We apply the change of variables $\widetilde{u} = u - \eta$ to Equation (25), thus transforming it into

$$
\begin{aligned}
\widetilde{u}_t &= \widetilde{u}_{xx} + \underbrace{b\widetilde{u} + b\eta + g(x, \widetilde{u} + \eta)}_{f(x, \widetilde{u}+\eta)}, \quad x \in [0, \pi] \\
\widetilde{u}_x(t, 0) &= \widetilde{u}_x(t, \pi) = 0 \\
\widetilde{g}(x, \widetilde{u}) &:= b\eta + g(x, \widetilde{u} + \eta) = g(x, \widetilde{u} + \eta) - g(x, \eta) \\
\widetilde{f}(x, \widetilde{u}) &:= f(x, \widetilde{u} + \eta) = b\widetilde{u} + \widetilde{g}(x, \widetilde{u}),
\end{aligned} \tag{37}
$$

and noting that if $g(x, u) \in \mathcal{G}_0$, then $\widetilde{g}(x, \widetilde{u}) \in \mathcal{G}_0$. In the dynamical system determined by Equation (37), $\widetilde{v} \equiv 0$ is a stationary solution corresponding to $v \equiv \eta$ in the original system. Thus, any solution $\widetilde{u}(t, \cdot)$ in the unstable manifold of $\widetilde{v} \equiv 0$ corresponds to a solution $u(t, \cdot)$ in the unstable manifold of $v \equiv \eta$ in the original system (25). It further follows that while the lap numbers and Morse indices of $\widetilde{u}(t, \cdot)$ and $u(t, \cdot)$ are identical, the zero numbers need not be. The eigenfunctions and eigenvalues of $\widetilde{v} \equiv 0$ and $v \equiv \eta$ are exactly the same, and therefore the Morse index is identical as well. Since all spatially homogeneous stationary solutions have the same zero number and lap number, we may therefore focus solely on the spatially homogeneous stationary solution $\widetilde{v} \equiv 0$ in order to understand the implications of the y-map in the unstable manifold of any spatially homogeneous stationary solution to (25).

We first assume that the restricted y-map, $y : \Sigma \to S^n$, is essential. Therefore, y is surjective. We now define the vector ς exactly as y was defined in (30 - 32), replacing t_k with δ_k and σ_k with s_k. By the surjectivity of y, there exists an initial datum $\widetilde{u}_0 \in \Sigma$ such that $y(\widetilde{u}_0) = \varsigma$. But as we noted earlier, knowing $y(\widetilde{u}_0)$ uniquely determines the dropping times t_k and signs σ_k of the solution $\widetilde{u}(t, \cdot)$ corresponding to $\widetilde{u}_0$. Thus, it is determined that $t_k = \delta_k$ and that $\sigma_k = s_k$ whenever $\delta_k < \delta_{k-1}$.

We must now prove that the restricted y-map is indeed essential. In order to prove that y is essential, we must homotopically deform our nonlinearity $\widetilde{f}$ from the corresponding linear form. We define

$$\widetilde{f}_\vartheta(x, \widetilde{u}) := \vartheta \widetilde{f}(x, \widetilde{u}) + (1-\vartheta)\widetilde{f}_{\widetilde{u}}(x, 0) \cdot \widetilde{u} \tag{38}$$

or, recalling the types of nonlinearities f over which we are interested,

$$\begin{aligned} \widetilde{f}_\vartheta(x, \widetilde{u}) = b\widetilde{u} + \widetilde{g}_\vartheta(x, \widetilde{u}) := b\widetilde{u} + \vartheta \widetilde{g}(x, \widetilde{u}) + (1-\vartheta)\widetilde{g}_{\widetilde{u}}(x, 0) \cdot \widetilde{u} \\ \Rightarrow \widetilde{g}_\vartheta(x, \widetilde{u}) := \vartheta \widetilde{g}(x, \widetilde{u}) + (1-\vartheta)\widetilde{g}_{\widetilde{u}}(x, 0) \cdot \widetilde{u} \end{aligned} \tag{39}$$

with the homotopy parameter $0 \leq \vartheta \leq 1$. As we deform $\widetilde{f}$, the unstable manifold of the stationary solution $\widetilde{v} \equiv 0$ of (37) with a specific nonlinearity $\widetilde{f}_\vartheta$ is simultaneously deformed. The linearization at $\widetilde{v} \equiv 0$ in the homotopically deformed system

$$0 = \widetilde{u}_{xx} + b\widetilde{u} + \vartheta \widetilde{g}_{\widetilde{u}}(x, 0)\widetilde{u} + (1-\vartheta)\widetilde{g}_{\widetilde{u}}(x, 0)\widetilde{u} = \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}_{\widetilde{u}}(x, 0)\widetilde{u} \tag{40}$$

is entirely unchanged. Additionally, $\widetilde{f}_\vartheta \in \mathcal{F}_0$ depends continuously on ϑ as $\mathcal{F}_0$ supports the weak Whitney topology.

We denote the cut-off tangent space of $W^u(\widetilde{f}_\vartheta)$ at $\widetilde{v} \equiv 0$ for $\vartheta = 0$ by

$$W^u_{loc}(\widetilde{f}_0) := span\left\{\varphi_0, \ldots, \varphi_n\right\} \cap \left\{\widetilde{u}_0 \in X \middle| |\widetilde{u}_0| < 2\varepsilon\right\}. \tag{41}$$

The local unstable manifolds with respect to an $\widetilde{f}_\vartheta$ are then parametrized by diffeomorphisms

$$\rho_\vartheta : W^u_{loc}(\widetilde{f}_0) \to W^u_{loc}(\widetilde{f}_\vartheta) \tag{42}$$

where ρ_ϑ^{-1} is induced by the orthogonal projection onto $span\left\{\varphi_0, \ldots, \varphi_n\right\}$. We observe that ρ_ϑ depends continuously on ϑ in the uniform C^0 topology.

Now we fix a sphere

$$\Sigma^n := \left\{ \widetilde{u} \in W^u_{loc}(\widetilde{f}_0) \middle| \, |\widetilde{u}| < \varepsilon \right\} \tag{43}$$

in the cut-off unstable manifold of $\widetilde{v} \equiv 0$ and let y^ϑ denote the restriction to $\rho_\vartheta(\Sigma^n)$ of the y-map associated to $\widetilde{f}_\vartheta$. After a homotopy we may assume that $\Sigma = \rho_1(\Sigma^n)$. Finally, we define

$$\overline{y}_\vartheta := y^\vartheta \cdot \rho_\vartheta : \Sigma^n \to S^n. \tag{44}$$

This mapping is well-defined, as $z(\widetilde{u}) \leq n$ on $W^u(\widetilde{f}_\vartheta)$. The mapping is continuous, and depends continuously on ϑ thanks to Lemma 4.3. Lemma 4.5 implies that $\overline{y}^0 = y_0 \cdot \rho_0 = y_0 : \Sigma^n \to S^n$ is essential. By the homotopy invariance of this property, $\overline{y}^1 = y_1 \cdot \rho_1 = y \cdot \rho_1$ is essential, and therefore y is essential.

Thus, choosing a sequence of δ_k and s_k we have shown that for any $0 \leq t < \infty$

$$t \geq \delta_k \Leftrightarrow z(\widetilde{u}(t,\cdot)) = z(u(t,\cdot) - v(\cdot)) = z(u(t,\cdot) - \eta) \leq k$$

$$\delta_k < t < \delta_{k-1} \Rightarrow sign(\widetilde{u}(t,0)) = sign(u(t,0) - v(0)) = sign(u(t,0) - \eta) = s_k$$

recalling that $v(0) = v(x) = \eta$ by assumption. □

COROLLARY 4.7. *Let v be a non-spatially homogeneous hyperbolic stationary solution of*

$$\begin{aligned} &u_t = u_{xx} + bu + g(u), \quad x \in [0,\pi] \\ &u_x(t,0) = u_x(t,\pi) = 0, \quad g(u) \in \mathcal{G} \end{aligned} \tag{45}$$

with Morse index $i(v) = n + 1 > 0$. Let $\Sigma \subset W^u \setminus \{v\}$ be homotopic in $W^u \setminus \{v\}$ to a small sphere centered at v in W^u of dimension n. For any finite sequence

$$\begin{aligned} &0 = \delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_0 \leq \infty \\ &s_k \in \{1,-1\}, \qquad 0 \leq k \leq n, \end{aligned} \tag{46}$$

there exists a point $u_0 \in \Sigma$ corresponding to an initial condition $u(0,\cdot) \in X$ such that the graph $t \to z(u(t,\cdot) - v(\cdot))$ is determined by (δ_k). In other words, for any

$0 \le t < \infty$,

$$
\begin{aligned}
&t \ge \delta_k \Leftrightarrow z(u(t,\cdot) - v(\cdot)) \le k \\
&\delta_k < t < \delta_{k-1} \Rightarrow sign(u(t,0) - v(0)) = s_k.
\end{aligned}
\tag{47}
$$

PROOF. We now extend the results of Lemma 4.6 to any non-spatially homogeneous hyperbolic stationary solution v of (45). This implies that v is a bounded stationary solution. Let u be a solution of (45) with $g(u) \in \mathcal{G}$. Then $\widetilde{u} := u - v$ satisfies

$$
\begin{aligned}
&\widetilde{u}_t = \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}(x,\widetilde{u}) \\
&\widetilde{g}(x,\widetilde{u}) := g(\widetilde{u} + v(x)) - g(v(x)),
\end{aligned}
\tag{48}
$$

noting that $\widetilde{g}(x,0) = 0$. The eigenvalue problem of (48) at a hyperbolic stationary solution $\widetilde{w} = w - v$ is

$$
\lambda u = u_{xx} + bu + \widetilde{g}_u(x,\widetilde{w})u = u_{xx} + bu + g_u(w)u. \tag{49}
$$

If we assume that the stationary solution w in Equation (45) has Morse index j and therefore $\lambda_0 > \lambda_1 > \ldots > \lambda_{j-1} > 0 > \lambda_j > \ldots$, with corresponding eigenfunctions $\varphi_0, \ldots, \varphi_{j-1}, \ldots$, it is clear that $\widetilde{w}$ must have the same eigenfunctions and eigenvalues, and therefore the same Morse index as w. This is because the eigenvalue problem for a stationary solution w of (45) is the same as the eigenvalue problem for the corresponding stationary solution $\widetilde{w}$ of (48).

The arguments within the proof of Lemma 4.6 hold in the shifted system, but regarding initial datum $\widetilde{u}_0 = u_0 - v$ and corresponding solutions $\widetilde{u} = u - v$. This is owing to the fact that the zero number of the difference of two solutions is nonincreasing in time [**22**]. Thus, Lemma 4.6 asserts that there exists an initial datum $\widetilde{u}_0$ such that for $0 \le t < \infty$

$$
\begin{aligned}
&t \ge \delta_k \Leftrightarrow z(\widetilde{u}(t,\cdot)) \le k \\
&\delta_k < t < \delta_{k-1} \Rightarrow sign(\widetilde{u}(t,0)) = s_k
\end{aligned}
$$

for any choice of sequences

$$0 = \delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_0 \leq \infty$$

$$s_j \in \{1, -1\}, \quad 0 \leq j \leq n.$$

Since $\widetilde{u}(t,0) = u(t,0) - v(0)$ and $\widetilde{u}(t,\cdot) = u(t,\cdot) - v(\cdot)$, the corollary is proven. $\square$

In order to make Corollary 4.7 more useful, we prove a property of the zero number for stationary solutions. We adopt some useful notation introduced by Brunovský and Fiedler in [**7**]. Let E be the set of stationary solutions for a given fixed b and $g(u) \in \mathcal{G}$. For a given interval $J \in \mathbb{R}$, we use EJ to denote the set of those stationary solutions $w \in E$ for which $w(0) \in J$. For a stationary solution v with lap number $l(v) = n > 0$ we define the set J_v by $J_v = (v(\frac{\pi}{n}), v(0))$ when $v(\frac{\pi}{n}) < v(0)$ or $J_v = (v(0), v(\frac{\pi}{n}))$ for $v(0) < v(\frac{\pi}{n})$. For a spatially homogeneous stationary solution v, $J_v = v(0)$.

LEMMA 4.8. *Consider an equation*

$$\begin{gathered} u_t = u_{xx} + f(u) \\ x \in [0,\pi],\ t \geq 0,\ f \in C^2 \\ u_x(t,0) = u_x(t,\pi) = 0. \end{gathered} \tag{50}$$

Let v^1 and v^2 be two distinct stationary solutions existing on n-branches in the bifurcation diagram and $z(v^2 - \eta^*) = m$, in other words, v^2 is found on the m-branch. If $v^1(0) \notin EJ_{v^2}$ and $l(v^1) = i \leq m$, then

$$z(v^1 - v^2) = z(v^1 - \eta^*) = l(v^1). \tag{51}$$

PROOF. As v^1 and v^2 are both stationary solutions of (50), it follows that

$$0 = v^1_{xx} + f(v^1)$$

$$0 = v^2_{xx} + f(v^2)$$

and therefore $w = v^1 - v^2$ solves

$$0 = w_{xx} + \underbrace{f(w + v^2(x)) - f(v^2(x))}_{\widetilde{f}(x,w)}. \tag{52}$$

As in the preceding proof of Corollary 4.7, the eigenvalue problem for the stationary solution $w = v^1 - v^2$ of the shifted equation is identical to the eigenvalue problem for the stationary solution v^1 in the original equation (50). Therefore

$$i(v^1 - v^2) = i(v^1). \tag{53}$$

We note here that as a consequence of Corollary 3.13, it follows that if v^1 is a non-spatially homogeneous stationary solution to (50) then

$$\begin{gathered} i(v^1 - v^2) \in \left\{z(v^1 - v^2), z(v^1 - v^2) + 1\right\} \\ i(v^1) \in \left\{z(v^1 - \eta^*), z(v^1 - \eta^*) + 1\right\}. \end{gathered} \tag{54}$$

This implies that one of the following equalities holds:

$$z(v^1 - v^2) = z(v^1 - \eta^*)$$

$$z(v^1 - v^2) = z(v^1 - \eta^*) + 1$$

$$z(v^1 - v^2) = z(v^1 - \eta^*) - 1.$$

If $m = 0$, then $z(v^2 - \eta^*) = 0 = z(v^2)$ and therefore $z(v^1 - \eta^*) = 0 = z(v^1)$. For both v^1 and v^2 spatially homogeneous stationary solutions, it follows that $z(v^1 - v^2) = 0$ as well. Thus we may assume $m > 0$.

If v^1 lives on the 0-branch and $v^1 \notin EJ_{v^2}$, it then follows that $v^1(x) - v^2(x) = v^1(0) - v^2(x) > 0$ for all $x \in [0, \pi]$, as solutions may not intersect at any point in the phase portrait. Therefore $z(v^1 - v^2) = 0 = z(v^1) = z(v^1 - \eta^*)$, since v^1 is a spatially homogeneous stationary solution if it lives on the 0-branch.

If v^1 is not on the 0-branch and $v^1 \notin EJ_{v^2}$, it follows that v^2 is nested inside v^1 in the phase portrait, since trajectories cannot cross, both v^1 and v^2 are on n-branches, and both stationary solutions have lap numbers (and therefore zero numbers shifted by η^*) greater than 0. We recall that for equations of the form (50)

$$z(v^1 - v^2) = \begin{cases} l(v^1) \geq 1 & if\ range(v^2) \subset range(v^1) \\ 0 & if\ range(v^2) \cap range(v^1) = \emptyset. \end{cases} \tag{55}$$

Thus, in the case where v^1 and v^2 are on two nonzero n-branches (possibly the same branch), it follows that $z(v^1 - v^2) = l(v^1) = z(v^1 - \eta^*)$, since $l(v^1) = z(v^1 - \eta^*)$ on

n-branches by Lemma 3.5. Thus we have shown that in all possible cases introduced in the lemma, $z(v^1 - v^2) = z(v^1 - \eta^*) = l(v^1)$. We remind the reader that for v^1 on a nonzero n-branch, $\beta(v^1) = \eta^*$, and for v^1 on the 0-branch, $z(v^1 - \eta^*) = z(v^1 - \beta(v^1)) = z(v^1) = l(v^1)$. $\square$

Lemma 4.8 only addresses the relation between $z(v^1 - v^2)$ and $l(v^1)$ for v^1 and v^2 on n-branches in the bifurcation diagram. We now introduce a corollary to address the situation wherein v^2 is on a pitchfork bifurcation from the 0-branch. If v^2 is on the portion of the 0-branch encased in pitchfork bifurcations, it follows that for any stationary solution v^1 on one of the pitchfork branches surrounding v^2, that $v^2 = \beta(v^1)$ and thus $z(v^1 - v^2) = l(v^1)$ by Lemma 3.10. Additionally, recall that the β-function of any stationary solution on a single set of nested pitchforks will be the same regardless of which pitchfork branch the solution exists on. All solutions on nested pitchforks oscillate around the same spatially homogeneous stationary solution, which forms a center in the phase portrait.

Corollary 4.9. *Let v^1 and v^2 be two distinct stationary solutions to Equation (50) existing on nested pitchfork branches of the* 0*-branch, i.e. oscillating around the same center fixed point in the phase plane, and let $z(v^2 - \beta(v^2)) = l(v^2) = m$. If $v^1 \notin EJ_{v^2}$ and $l(v^1) = i \leq m$ then*

$$z(v^1 - v^2) = z(v^1 - \beta(v^1)) = l(v^1). \tag{56}$$

Proof. We first recall that pitchforks of higher lap number are nested within pitchforks of lower lap number. Thus it follows that in the phase plane v^2 will be nested within v^1 or on the same orbit, or in other words, $range(v^2) \subseteq range(v^1)$. Thus $z(v^1 - v^2) = l(v^1)$. Recalling Lemma 3.5, this implies that $z(v^1 - v^2) = l(v^1) = z(v^1 - \beta(v^1))$. $\square$

Applying Lemma 4.8 and Corollary 4.9 to Corollary 4.7, we see that if $\widetilde{u}(t, \cdot)$ limits to a stationary solution $v^1(\cdot)$ as $t \to \infty$, where v^1 is either on an n-branch or within the same nested pitchfork bundle as v, then $z(\widetilde{u}(\infty, \cdot)) = z(u(\infty, \cdot) - v(\cdot)) =$

$z(v^1(\cdot) - v(\cdot)) = z(v^1(\cdot) - \beta(v^1)) = l(v^1)$. This will become quite useful in the coming chapters. While we have not addressed the relation between lap numbers where v^1 is on a pitchfork branch and v^2 is not within the same nested pitchfork bundle or the portion of the 0-branch encased; we shall see later that this relation is unnecessary.

CHAPTER 5

Heteroclinic Connections and the Asymptotic Behavior of Grow-Up Solutions

1. Existence of Heteroclinic Connections Originating from Bounded Equilibria

In Chapter 4 we established elements of the behavior of the zero number of trajectories in the unstable manifolds of bounded hyperbolic equilibria. We note that the class of nonlinearities with only hyperbolic equilibria is generic in $\mathcal{G}$ [**3, 19, 26**]. Furthermore, the subset of nonlinearities dependent on u alone with only hyperbolic equilibria are generic in the set of nonlinearities dependent on u alone. It is in this chapter that we seek to determine not only the rough shape solutions have, but their specific behavior as we let time tend towards infinity.

We wish to determine when connections between bounded equilibria do or do not exist, i.e. when trajectories are "caught" by one specific bounded equilibrium as opposed to another. Further, we seek to determine when trajectories eventually escape all bounded regions and travel to infinity, i.e. are grow-up solutions. Toward this end, we wish to know when certain heteroclinic connections are blocked, a priori. There are a number of methods with which a heteroclinic connection may be blocked. These variously depend on the nodal properties of the equilibria in question, the unstable dimensions of the equilibria in question, and the existence of another equilibrium to which a heteroclinic will link instead. Such blocking properties are crucial to determining when a solution may grow to infinity. We say that an equilibrium v has a heteroclinic connection to another equilibrium w if there exists some solution $u(t,\cdot)$ to Equation (45) such that $\lim_{t\to\infty} u(t,\cdot) = w$ and $\lim_{t\to-\infty} u(t,\cdot) = v$. We may also say that v connects to some function $\overline{u}(0,\cdot)$ which solves Equation (45) if there exists some solution $u(t,\cdot)$ to (45) such that $u(T,\cdot) = \overline{u}(0,\cdot)$ for some time $T \geq 0$ and $\lim_{t\to-\infty} u(t,\cdot) = v$. Thus, v may connect to $\overline{u}$ at some intermediary point along a heteroclinic. We now introduce a lemma which will be crucial in determining when heteroclinic connections are blocked.

LEMMA 5.1. (Finite Blocking Lemma)
Let v and w be two distinct stationary solutions of Equation (45), v hyperbolic, and let $\overline{w}(0,\cdot)$ be a function which solves Equation (45), such that $w(0)$ lies strictly between $v(0)$ and $\overline{w}(0,0)$. Then

$$z(v-w) \leq z(\overline{w}-w)$$

implies that v does not connect to $\overline{w}$.

PROOF. This proof is very similar to the proof of the Blocking Lemma for dissipative systems introduced in [**6**]. We proceed by contradiction. Assume that v connects to $\overline{w}$ via a trajectory $u(t,\cdot)$, $t \in (-\infty, T]$. Then $\widetilde{u} = u - w$ satisfies an equation of the form

$$\begin{aligned} \widetilde{u}_t &= \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}(x,\widetilde{u}) \\ \widetilde{g}(x,\widetilde{u}) &:= g(\widetilde{u} + w(x)) - g(w(x)), \quad \widetilde{g}(x,0) = 0. \end{aligned} \tag{57}$$

Via the results of Lemma 4.6 and Corollary 4.7, we may assume that $w = 0$ without loss of generality by working in the shifted system (57). Thus, either $\widetilde{v}(0) < 0 < \widetilde{\overline{w}}(0,0)$ or $\widetilde{\overline{w}}(0,0) < 0 < \widetilde{v}(0)$. The nonincrease of the zero number $z(u(t,\cdot))$ on the trajectory connecting v to $\overline{w}$ and the concomitant nonincrease of $z(\widetilde{u}(t,\cdot))$ on the trajectory connecting $\widetilde{v}$ to $\widetilde{\overline{w}}$ imply that

$$z(v-w) = z(\widetilde{v}) \geq z(\widetilde{\overline{w}}) = z(\overline{w}-w).$$

Recalling Lemma 4.2, we know that $z(\widetilde{v}) \neq z(\widetilde{\overline{w}})$ as $\widetilde{v}(0)$ and $\widetilde{\overline{w}}(0,0)$ have opposite sign. Therefore, $z(v-w) > z(\overline{w}-w)$ if v connects to $\overline{w}$, and the lemma is proved by contraposition. $\square$

Remark 5.2. *If we let $\overline{w}(0,\cdot)$ be a stationary solution of Equation (45), then the Finite Blocking Lemma 5.1 still holds with $T = \infty$. This makes the Finite Blocking Lemma crucial in determining when heteroclinic connections are a priori blocked between bounded equilibria.*

We have given Lemma 5.1 in this form rather than explicitly defining $\overline{w}$ as a stationary solution because this generality provides a great deal of usefulness. It can not only be applied to determine connection blocking among bounded equilibria, but may also be used to block connections when solutions undergo grow-up or even finite-time blow-up.

The Finite Blocking Lemma is additionally crucial to the a priori blocking of heteroclinic connections to infinity. We say that an equilibrium v connects to infinity if there exists a trajectory $u(t,\cdot)$ in the unstable manifold of v such that $u(t,\cdot)$ is a grow-up solution, as defined in Chapter 2.

Lemma 5.3. (Infinite Blocking Lemma)
Let v and w be two distinct stationary solutions to (45), v hyperbolic. Let $\sigma = sign(w(0)-v(0))$. Let $u(t,\cdot)$ be any trajectory in the unstable manifold of v such that $sign(u(t,0)-v(0)) = \sigma$ and $z(u(t,\cdot)-v(\cdot)) = j \geq k$ for all $t \in (t_j,\infty]$, with $t_j < \infty$. If

$$z(v-w) \leq k,$$

then $u(t,\cdot)$ remains bounded. In other words, v does not contain any heteroclinic connections to objects at infinity with zero number greater than or equal to k, i.e. $z(v-w) \leq k$ implies that w blocks certain types of heteroclinics to infinity.

We may consider v to have a heteroclinic connection to a smooth object at infinity denoted by Φ^∞ if there exists a grow-up solution $u(t,\cdot)$ in the unstable manifold of v such that $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_{L^2}}$ limits in C^1_{loc} to a bounded function $\Phi(x)$ which has only simple zeros and unit norm in L^2. Then Φ^∞ is the projection of $\Phi(x)$ onto the sphere at infinity, i.e. it is the limit of $t \cdot \Phi(x)$ as time grows to infinity. In other words

$$\lim_{t\to\infty} \left\| \frac{u(t,\cdot)}{\| u(t,\cdot) \|_{L^2}} - \Phi(x) \right\|_{C^1} = 0,$$

so if the normalized shape profiles of $u(t,\cdot)$ and a $\Phi(x)$ with simple zeros align as $u(t,\cdot)$ grows to infinite norm, $u(t,\cdot)$ is considered to connect to the corresponding Φ^∞.

PROOF. We proceed by contradiction. Assume that v connects to Φ^∞ via a given trajectory $u(t,\cdot),\ t \in \mathbb{R}$. We require that the renormalized Φ^∞ (i.e. $\Phi(x)$) have only simple zeros, which will be shown to hold for all relevant objects at infinity later in the chapter. Thus, the zero number of Φ^∞ is well-defined; let $z(\Phi^\infty) = z(\Phi^\infty - v) = j$ and $sign(\Phi^\infty(0)) = \sigma$. This implies that $u(t,\cdot)$ limits to infinity in the C^1-norm (and thus all lower norms) as t goes to infinity, i.e. $u(t,\cdot)$ is a grow-up solution. Let $\overline{w}(0,\cdot)$ be any function such that $w(0)$ lies strictly between $v(0)$ and $\overline{w}(0,0)$ and $z(\overline{w} - w) = j$. By Lemma 5.1, v does not connect to $\overline{w}$, since $j \geq k$.

If v connects to any Φ^∞ with $z(\Phi^\infty) = j$ via a trajectory $u(t,\cdot)$, then in the shifted equation v must connect to $\Phi^\infty - w = \Phi^\infty$ via a trajectory $\widetilde{u}(t,\cdot)$. It then follows that $z(u(t,\cdot) - w(\cdot)) \geq j$ for all time $t \in \mathbb{R}$. This implies that at some time $T < \infty$, the value of $w(0)$ must lie between $u(T,0) = \overline{w}(0,0)$ and $v(0)$, since it lies between $\Phi^\infty(0)$ and $v(0)$. But this leads to a contradiction, as the Finite Blocking Lemma prevents $u(t,\cdot)$ from ever crossing w in its left intercept. Therefore, v cannot connect to any object at infinity with zero number greater than or equal to k. Thus, any trajectories in the unstable manifold of v where the shifted zero number never drops below k must remain bounded. $\square$

In other words, for any unstable hyperbolic equilibrium v of

$$\begin{gathered} u_t = u_{xx} + \underbrace{bu + g(u)}_{f(u)}, \quad x \in [0,\pi] \\ u_x(t,0) = u_x(t,\pi) = 0 \\ b > 0, \quad g(u) \in \mathcal{G}, \end{gathered} \tag{58}$$

we may block entire classes of heteroclinic connections to infinity given the existence of appropriate bounded equilibria. Suppose there exists an equilibrium w of (58) such that $z(v-w) \leq z(\Phi^{\infty})$ and $w(0)$ lies between $v(0)$ and $\Phi^{\infty}(0)$ for an infinitely large function $\Phi^{\infty}(x)$. Then v will not connect to Φ^{∞}. In short, w blocks v from connecting to all objects at infinity above a specified zero number.

Lemma 5.3 thus leads us to the following conclusion: whatever is not blocked connects to infinity in a specific way. We may consider this an Infinite Liberalism Lemma, as it relates to the Liberalism Lemmas of dissipative dynamical systems wherein bounded connections exist whenever they are not blocked.

LEMMA 5.4. *Let v be a hyperbolic stationary solution to (58) such that $i(v) = n+1$. Fix k such that $0 \leq k \leq n$ and $\sigma \in \{1, -1\}$. If v is not blocked from connecting to infinity in the sense of Lemma 5.3, or equivalently, if there does not exist any stationary solution w to (58) such that $l(w) = k$, $w \notin EJ_v$, and $sign(w(0)-v(0)) = \sigma$, and when $k = n = l(v)$, there additionally does not exist any $w \in EJ_v$ such that $l(w) \leq n$, then there exists an initial condition $u_0 \in W^u(v)$ and a corresponding solution $u(t,\cdot)$ to (58) such that the following hold:*

$$\begin{gathered} z(u(t,\cdot) - v(\cdot)) = k \; for \; all \; 0 \leq t < \infty \\ sign(u(t,0) - v(0)) = \sigma \\ \lim_{t\to-\infty} u(t,\cdot) = v \\ \lim_{t\to\infty} \parallel u(t,\cdot) \parallel_{L^2} = \infty. \end{gathered} \tag{59}$$

PROOF. We may apply Lemma 4.6 for spatially homogeneous v, or Corollary 4.7 for non-spatially homogeneous v to the set of solutions of (58), choosing

$$\delta_j := \begin{cases} 0 & for \quad j \geq k \\ \infty & for \quad j < k \end{cases}$$

$$s_k := \sigma.$$

By Lemma 4.6 or Corollary 4.7 there exists an initial condition $u_0 \in W^u(v)$ corresponding to our choice of k and σ, and the lemma or corollary asserts that for the solution $u(t,\cdot)$ corresponding to $u(0,\cdot) = u_0$, the properties $z(u(t,\cdot) - v(\cdot)) = k$ and $sign(u(t,0) - v(0)) = \sigma$ hold for all $0 \leq t < \infty$. Since u_0 is in the unstable manifold of v, this implies that $\lim_{t\to-\infty} u(t,\cdot) = v$.

Further, $u(t,\cdot)$ cannot connect to some bounded equilibrium $\overline{w} \notin EJ_v$ with lap number less than k. If $l(\overline{w}) < k$ it follows that $l(\overline{w}) = z(\overline{w} - \beta(\overline{w})) < k$ by Lemmas 3.5 and 3.10. For $u(t,\cdot)$ to connect to $\overline{w}$, $\lim_{t\to\infty} z(u(t,\cdot) - v(\cdot)) = z(\overline{w}(\cdot) - v(\cdot)) < k$. Thus, the zero number in the shifted system $z(\widetilde{u}(t,\cdot)) = z(u(t,\cdot) - v)$ must drop at $t = \infty$. To show why this is not possible, let us assume that $\lim_{t\to\infty} u(t,\cdot) = \overline{w}$ with $l(\overline{w}) < k$. Then $\lim_{t\to\infty} \widetilde{u}(t,\cdot) = \overline{w} - v = \widetilde{\overline{w}}$. Because $\widetilde{\overline{w}}(t,\cdot) \neq 0$, it follows that $\widetilde{\overline{w}}$ must have only simple zeros, as it solves the ordinary differential equation $0 = \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}(x,\widetilde{u})$. Any solution in a small neighborhood of $\widetilde{\overline{w}}$ must also have simple zeros. Thus, the shifted zero number is constant over some small neighborhood of $\widetilde{\overline{w}}$, and therefore in this neighborhood $z(\widetilde{u}) = z(\widetilde{\overline{w}}) = z(\overline{w} - v) < k$.

But if this is the case, then $z(u(t,\cdot) - v(\cdot))$ would have to drop at some finite time as $u(t,\cdot)$ approaches $\overline{w}$. Therefore, if $u(t,\cdot)$ connects to a bounded equilibrium, this equilibrium must fulfill $z(w - v) = k$, which occurs when $l(w) = k$ for $w \notin EJ_v$, or when $k = l(v)$ for $w \in EJ_v$. Since $k = l(v)$ implies that $k = n$, and $u(t,\cdot)$ will only limit to stationary solutions $w \in EJ_v$ if $i(w) < i(v)$ and therefore $l(w) \leq n$ by Lemma 5.6, it follows that the only bounded stationary solutions to which $u(t,\cdot)$ may connect are those listed in the lemma. Additionally, Lemma 4.6 and Corollary 4.7 imply that the sign of $(u(t,0) - v(0))$ remains always positive or always negative. Therefore, if $u(t,\cdot)$ were to limit to any bounded equilibrium w, it would have to be one such that $w(0) - v(0) = \sigma$.

Since there does not exist any bounded equilibrium w fulfilling these conditions, we may conclude that $u(t,\cdot)$ cannot limit to any bounded stationary solution. But as Lemma 2.1 states, $u(t,\cdot)$ can then not be bounded in any ball, no matter how large, and therefore $\lim_{t\to\infty} ||u(t,\cdot)|| = \infty$ in any appropriate norm in H^2. Since the L^2-norm of $u(t,\cdot)$ must be less than or equal to the H^1, H^2, and C^1 norms, we choose it for the formulation of our lemma, to guarantee the infiniteness of the other two norms.

Finally, knowing that $u(t, \cdot)$ grows to be infinitely large, we may conclude that there exist infinitely many times $t^l > 0$ at which $z(u(t^l, \cdot) - v(\cdot)) = z(u(t^l, \cdot))$. $\square$

REMARK 5.5. *We bring the reader's attention to the effect the existence of a blocking equilibrium (in the sense of Lemma 5.3) has on the possible connections to infinity. To wit: if there exists a blocking equilibrium with a given lap number, there will always exist additional blocking equilibria with higher lap numbers, up to the lap number of the original equilibrium. Furthermore, the lack of a blocking equilibrium of a given lap number will ensure the lack of blocking equilibria of lower lap numbers.*

Continuing with our focus on an individual stationary solution v, *let* $l(v) = n$. *We presume that for a given* $k \leq n$ *we do not have any bounded stationary solution* w_k^+ *defined such that* w_k^+ *lives on the* k*-branch,* $w_k^+(0) > v(0)$ *for Equation (58) with fixed* $b = b^*$. *Then there are no blocking solutions above* v *with lap number lower than* k, *i.e. there do not exist any bounded stationary solutions* w_i^+ *for* $i < k$ *wherein* $w_i^+(0) > v(0)$. *By replacing* u, v, *and* w *with* $-u$, $-v$, *and* $-w$, *we arrive at the analogous statement for* w_k^- *and the corresponding* w_i^-, *with the minus denoting* $w_k^-(0) < v(0)$.

This follows from the fact that n*-branches may not cross and the origination point of an* n*-branch is always to the left of the origination point of an* $(n+1)$*-branch. If there does not exist any* w_k^+, *then in the bifurcation diagram the* k*-branch does not intersect the line at* $b = b^*$ *above the point* $v(0)$ *on the vertical axis. Since* i*-branches for* $i < k$ *must be to the left of the* k*-branch, they may never intersect this line either, which implies that there are no stationary solutions* w_i^+ *with lap number* $i < k$ *for Equation (58). The same holds for the* w_i^- *in relation to the* i*-branches intersecting* $b = b^*$ *below* $v(0)$. *This property does not hold for pitchfork branches, as there may be pitchforks of lower lap number without pitchforks of higher lap number ever coming into existence. Note that this does not nullify the statement, since either* $l(v) < l(w)$, *or there must exist intermediary* k*-branches between the* n*-branch and the* 0*-branch.*

There are other circumstances which allow for the blocking of connections, which we shall introduce at this point. For any stationary solution $w \in E$ which is not hyperbolic, $i(w)$ denotes the number of strictly positive eigenvalues.

LEMMA 5.6. *If* $v,\ w \in E$ *satisfy* $i(w) \geq i(v)$*, then* v *does not connect to* w*.*

PROOF. In [**19**], Henry proved that for v and w stationary solutions to (50), both not necessarily hyperbolic, the stable and unstable manifolds of v and w intersect transversely if they intersect at all. Since $dim\ W^u(v) = i(v) \leq i(w) = \ codim\ W^s(w)$, it follows that $dim\ W^u(v) \cap W^s(w) \leq 0$. Since $v \neq w$, it follows that v cannot connect to w. □

LEMMA 5.7. *Let* $v \in E$ *be a hyperbolic stationary solution of Equation (58) and let* $w \in E$ *be a second stationary solution,* $w \neq v$*, such that* $z(v - w) \geq i(v)$*. Then* v *does not connect to* w*.*

PROOF. Let us assume that v connects to w, i.e. that there exists an initial condition $u_0 \in W^u(v)$ such that $\lim_{t \to -\infty} u(t, \cdot) = v$ and $\lim_{t \to \infty} u(t, \cdot) = w$ for the corresponding solution $u(t, \cdot)$. Then Lemma 5.6 implies that $i(w) < i(v)$. Further, Fiedler and Brunovský proved in [**4**] that $z(u - v) \geq i(v)$ for $u \in W^s(v) \backslash \{v\}$ and $z(u - v) < i(v)$ for $u \in W^u(v) \backslash \{v\}$.

By Lemma 4.6 or Corollary 4.7, any stationary solution to which v connects must satisfy $z(w - v) = k$ and $sign(w(0) - v(0)) = \sigma$ for some choice of $0 \leq k < i(v)$ and $\sigma \in \{1, -1\}$, due to the property that $z(u_0 - v) < i(v)$ for all initial datum u_0 in the unstable manifold of v. Thus $k = z(v - w) < i(v)$, and the lemma is proved by contraposition. □

Now that we have shown what circumstances allow a connection to be blocked, we introduce a useful property of stationary solutions which provides information on the stationary solutions that block other connections.

Lemma 5.8. *Given any non-trivial stationary solution v where $i(v) = l(v) + 1$, let w be the nearest stationary solution such that $w \notin EJ_v$ and $l(w) \leq l(v)$. If w exists, then $l(w) = i(w)$.*

We define any stationary solution in a set to be the "nearest" stationary solution to v if it minimizes $|w(0) - v(0)|$ over all stationary solutions w in the given set.

Proof. If v is a stationary solution such that $i(v) = l(v) + 1$, then v cannot live on a set of nested pitchforks. Thus, v must live on an n-branch. Due to the spacing of n-branch origination points and the fact that they cannot intersect, it follows that the $n - 1$-branch, if it exists, is between the n-branch and all lower branches, and the $n + 1$-branch is between the n-branch and all higher branches. By Lemma 3.14, we know that for $v(0) = \eta$, $(\eta - \eta^*) \cdot \frac{db}{d\eta} < 0$. It follows that the nearest branch, both above and below EJ_v, must be either the $n + 1$-branch or the n-branch if any stationary solutions exist either above or below EJ_v. Let us assume there exists a stationary solution $w \notin EJ_v$ such that $l(w) \leq l(v)$, and let us fix $l(w) = k$.

For $i(w) = k + 1$, we would need $(w(0) - \eta^*) \cdot \frac{db}{d\eta}(w(0)) < 0$. But since lower branches always exist to the left of higher branches, it follows that we must first have a region where $(\eta - \eta^*) \cdot \frac{db}{d\eta} > 0$ on the $n + 1$-branch if it crosses the line at b, then on the n-branch, and so forth for all intermediary branches, or else the k-branch could not cross the line at b. But the stationary solutions in these regions have $i(w_{n+1}) = l(w_{n+1})$, $i(w_n) = l(w_n)$, ... Thus, there exists a stationary solution w_j between v and all solutions w such that $l(w) = k < j \leq n = l(v)$. Furthermore, the continuity of the k-branch to its origination point implies that the k-branch itself must first cross the line at b with $(\eta - \eta^*) \cdot \frac{db}{d\eta} > 0$. Therefore, excluding solutions in EJ_v and those with $l(w) > l(v)$, it follows that the nearest stationary solutions, both above and below v, such that $l(w) = k$ must fulfill $l(w) = i(w) = k$ if they exist. If they do not exist, then there are no stationary solutions with $l(w) \leq k$, as all lower lap number stationary solutions are to the left of the k-branch. □

The combination of these blocking lemmas plays a crucial role in determining which connections are excluded. Thus, for a given heteroclinic connection, the trajectory either limits to a bounded equilibrium, of which we can ascertain a good deal of information, or an object at infinity, which we have not yet completely defined. Let us now study the asymptotic behavior on heteroclinics to infinity, which serve as our guide to the behavior of general grow-up solutions.

2. Asymptotic Behavior of Grow-Up Solutions

In this section we shall study more closely the explicit behavior of grow-up solutions in the unstable manifolds of bounded equilibria, i.e. those grow-up solutions which form heteroclinics to infinity. Although we have determined a number of properties on these heteroclinic orbits through the combined implications of the y-map and the blocking lemmas in the previous section, we have not yet detailed exactly what objects we limit to. We proceed in this section to study the grow-up solutions themselves.

Let us fix a hyperbolic stationary solution v to

$$\begin{aligned} &u_t = u_{xx} + bu + g(u), \quad x \in [0, \pi] \\ &u_x(t,0) = u_x(t,\pi) = 0 \qquad\qquad (60) \\ &b > 0,\ g(u) \in \mathcal{G} \end{aligned}$$

with $i(v) = n + 1$. Recall that Equation (60) induces in any solution a Fourier eigendecomposition based on the Neumann boundary value modes $cos(kx)$. We here introduce some notation which will be of great use in the coming chapters. Loosely speaking, we let K^+ denote the number of $\cos(kx)$ modes which have "escaped to infinity" and have positive left intercept. The related term K^- denotes the number of modes with negative left intercept which "escape to infinity".

They are more rigorously defined as follows: let K^+ be the smallest integer $k < n + 1$ such that there exists at least one bounded equilibrium w with $l(w) = k$, $sign(w(0) - v(0)) = 1$, and $w \notin EJ_v$. If there exist no bounded equilibria w such that $w(0) - v(0) > 0$, then we define $K^+ = i(v)$. If there exist no such bounded equilibria

$w \notin EJ_v$, but $l(v) = n$ and there exist any bounded equilibria $w \in EJ_v$ such that $w(0) > v(0)$ and $l(w) \leq l(v)$, then $K^+ = i(v) - 1 = n$. If $K^+ < i(v)$, we define w_{K^+} as the stationary solution w with $l(w) = K^+$ and $sign(w(0) - v(0)) = 1$ which minimizes $w(0) - v(0)$. In the case of a dissipative system, $K^+ = 0$. Correspondingly, we let K^- be the smallest integer $k < n + 1$ such that there exists at least one bounded equilibrium w with $l(w) = k$, $sign(w(0) - v(0)) = -1$, and $w \notin EJ_v$. If there exist no bounded equilibria w such that $w(0) - v(0) < 0$, then we define $K^- = i(v)$. If there exist no such bounded equilibria $w \notin EJ_v$, but $l(v) = n$ and there exist any bounded equilibria $w \in EJ_v$ such that $w(0) < v(0)$ and $l(w) \leq l(v)$, then $K^- = i(v) - 1 = n$. If $K^- < i(v)$, we define w_{K^-} as the stationary solution w with $l(w) = K^-$ and $sign(w(0) - v(0)) = -1$ which minimizes $-w(0) + v(0)$. Again, for a dissipative system $K^- = 0$. Lemma 3.4 implies that if w_{K^+} is a non-pitchfork stationary solution and $(K^+)^2 > b$, then K^+ is the next integer after $\sqrt{b}$, and if w_{K^-} is a non-pitchfork stationary solution and $(K^-)^2 > b$, then K^- is the next integer after $\sqrt{b}$.

Remark 5.5 ensures that, outside of pitchfork branches, there do not exist any bounded equilibria not in EJ_v with left boundary value greater than $v(0)$ and lap number less than K^+. Similarly, it ensures that, again excepting pitchfork branches, there do not exist any bounded equilibria not in EJ_v with left boundary value less than $v(0)$ and lap number less than K^-. Then Lemma 5.4 on infinite liberalism states that there exist heteroclinics in $W^u(v)$ which grow to infinity with $K^+ + K^-$ distinct behaviors denoted by sign and asymptotic zero number. In the vast majority of cases, there will be infinitely many heteroclinics of each type, in a small minority of situations (such as when the difference between the Conley index of v and its limiting object differs by only 1), there will be only one heteroclinic of a given type.

Let us consider a specific grow-up solution in $W^u(v)$. Let $u_k^{\pm}(t, \cdot)$ denote the solution with $z(u_k^{\pm}(t, \cdot) - v(\cdot)) = k$ for $0 \leq t < \infty$ with $u(t, 0) > v(0)$ for u_k^+ and $u(t, 0) < v(0)$ for u_k^-. For u_k^+ it is clear that $k < K^+$. For u_k^- it is clear that $k < K^-$.

Since L^2, H^1, and H^2 are all Hilbert spaces with the standard inner products, each has a countable orthonormal basis. We fix a related orthonormal basis for each, denoted $\{\Phi_j\}_{j\in\mathbb{N}_0}$, as follows: we define Φ_j as the jth eigenfunction of the operator $A = -\Delta - bI$, i.e.

$$
\begin{gathered}
-\Delta\Phi_j - b\Phi_j = \mu_j\Phi_j \\
\frac{d}{dx}\Phi_j(0) = \frac{d}{dx}\Phi_j(\pi) = 0 \\
\mu_j = j^2 - b
\end{gathered}
\tag{61}
$$

and normalize it in the appropriate norm:

$$
\begin{gathered}
L^2 : \Phi_0(x) = \frac{1}{\sqrt{\pi}}, \quad \Phi_j(x) = \sqrt{\frac{2}{\pi}}\cos(jx), \quad j > 0 \\
H^1 : \Phi_0(x) = \frac{1}{\sqrt{\pi}}, \qquad \Phi_j(x) = \sqrt{\frac{2}{\pi(1+j^2)}}\cos(jx), \quad j > 0 \\
H^2 : \Phi_0(x) = \frac{1}{\sqrt{\pi}}, \qquad \Phi_j(x) = \sqrt{\frac{2}{\pi(1+j^2+j^4)}}\cos(jx), \quad j > 0.
\end{gathered}
\tag{62}
$$

Thus, we can write any point on a trajectory $u(t,\cdot) \in X = H^2 \cap \{\textit{Neumann Boundary Conditions}\}$ in terms of the L^2 basis: $u(t,\cdot) = \sum\limits_{j=0}^{\infty} \widehat{u}_j(t)\Phi_j(x)$, where $\widehat{u}_j(t) = \langle u, \Phi_j\rangle_0$.

This also implies that

$$
\begin{gathered}
||u(t,\cdot)||_0 = \sum_{j=0}^{\infty} \widehat{u}_j^2(t) \\
||u(t,\cdot)||_{1/2} = \sum_{j=0}^{\infty}(1+j^2)\widehat{u}_j^2(t) \\
||u(t,\cdot)||_1 = \sum_{j=0}^{\infty}(1+j^2+j^4)\widehat{u}_j^2(t).
\end{gathered}
\tag{63}
$$

Lemma 5.4 proved that $u_k^+(t,\cdot)$ grows to infinity in the $\|\cdot\|_0$ norm, and thus in all three above norms and any equivalent norms. To investigate what object at infinity the solution limits to, we must study the growth of each individual mode. Taking the inner product of Equation (60) with the basis element Φ_j yields

$$\frac{d}{dt}\widehat{u}_j = (b-j^2)\widehat{u}_j + \langle g(u(t,\cdot)), \Phi_j \rangle_0 . \tag{64}$$

Recalling that $g(u)$ is uniformly bounded, this implies that the inner product of $g(u)$ and any basis element must also be bounded. Let us denote $\langle g(u(t,\cdot)), \Phi_j \rangle_0$ by $g_j(t)$. Equation (64) then becomes

$$\frac{d}{dt}\widehat{u}_j(t) = (b-j^2)\widehat{u}_j(t) + g_j(t), \tag{65}$$

which is simply a first order inhomogeneous ordinary differential equation, and easily solved:

$$\widehat{u}_j(t) = e^{(b-j^2)t}\widehat{u}_j(0) + \int_0^t e^{(b-j^2)(t-s)} g_j(s)ds. \tag{66}$$

Let us return to the specific solution we wish to study. For ease of notation, let us simply refer to $u_k^{\pm}(t,\cdot)$ as $u(t,\cdot)$. If $j^2 > b$, then $b - j^2 < 0$. Therefore, all modes j with $j^2 > b$ lose strength as time goes to infinity, and in fact each of these modes must be bounded for all time. This follows explicitly from the following calculations: Let us choose initial time $t = 0$ in a small neighborhood of a bounded equilibrium v, ensuring that $\widehat{u}_l(0)$ is bounded for all l. Let $j^2 > b$. Then,

$$\int_0^t e^{(b-j^2)(t-s)}(-\Gamma)ds \leq \int_0^t e^{(b-j^2)(t-s)}\widehat{g}_j(s)ds \leq \int_0^t e^{(b-j^2)(t-s)}(\Gamma)ds$$

$$\int_0^t e^{(b-j^2)(t-s)}(\Gamma)ds = \frac{\Gamma}{j^2-b}(1-e^{(b-j^2)t})$$

$$\Rightarrow \frac{-\Gamma}{j^2-b}(1-e^{(b-j^2)t}) \leq \int_0^t e^{(b-j^2)(t-s)}\widehat{g}_j(s)ds \leq \frac{\Gamma}{j^2-b}(1-e^{(b-j^2)t})$$

$$j^2 > b \Rightarrow \frac{\Gamma}{j^2-b}(1-e^{(b-j^2)t}) \leq \frac{\Gamma}{j^2-b} \text{ and } \frac{-\Gamma}{j^2-b}(1-e^{(b-j^2)t}) \geq \frac{-\Gamma}{j^2-b}$$

$$\Rightarrow -\frac{\Gamma}{j^2-b} \leq \int_0^t e^{(b-j^2)(t-s)}\widehat{g}_j(s)ds \leq \frac{\Gamma}{j^2-b}.$$

Further, $e^{(b-j^2)t}\widehat{u}_j(0) \leq \widehat{u}_j(0)$, thus it is clear that all modes where $j > \sqrt{b}$ remain bounded for all time.

Therefore, denoting $\Phi_{+,k}^{\infty} = \lim_{t\to\infty} u_k^+(t,\cdot)$ and $\Phi_{-,k}^{\infty} = \lim_{t\to\infty} u_k^-(t,\cdot)$ for any $k < K^+$ or $k < K^-$ respectively, it follows that $\Phi_{\pm,k}^{\infty} \in P_{\lfloor\sqrt{b}\rfloor}X$ with P_N the orthogonal projection onto the $N+1$ lowest modes. We recall that Φ^∞ is the projection of the normalized limit, which we study explicitly in this and the succeeding chapters.

Let us study exactly how large a role any particular basis element plays in the construction of $u_k^+(t,\cdot)$ in the limit. Setting $u(t,\cdot) = u_k^\pm(t,\cdot)$ and $\Phi_j^\pm(x) = \pm\Phi_j(x)$,

$$||\frac{u(t,\cdot)}{||u(t,\cdot)||_0} - \Phi_j^\pm(\cdot)||_0^2 = 1 - 2\left\langle \frac{u(t,\cdot)}{||u(t,\cdot)||_0}, \pm\Phi_j(\cdot)\right\rangle_0 + 1$$
$$= 2 - 2\left\langle \frac{u(t,\cdot)}{(\sum_{l=0}^\infty \widehat{u}_l^2(t))^{1/2}}, \pm\Phi_j(\cdot)\right\rangle_0 = 2 \mp 2\int_0^\pi \frac{u(t,x)}{(\sum\limits_{l=0}^\infty \widehat{u}_l^2(t))^{1/2}}\Phi_j(x)dx$$
$$= 2 \mp \frac{2}{(\sum\limits_{l=0}^\infty \widehat{u}_l^2(t))^{1/2}}\int_0^\pi \sum_{m=0}^\infty \widehat{u}_m(t)\Phi_m(x)\Phi_j(x)dx = 2 \mp 2\frac{\widehat{u}_j(t)}{(\sum\limits_{l=0}^\infty \widehat{u}_l^2(t))^{1/2}}$$
$$\Rightarrow \lim_{t\to\infty} ||\frac{u(t,\cdot)}{||u(t,\cdot)||_0} - \Phi_j^\pm(\cdot)||_0^2 = 2 \mp 2\lim_{t\to\infty}\frac{\widehat{u}_j(t)}{(\sum\limits_{l=0}^\infty \widehat{u}_l^2(t))^{1/2}}.$$

LEMMA 5.9. *The rescaled trajectory* $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0}$ *can only limit to one particular* $\Phi_j^\pm$ *in* L^2*, and will limit to* $\Phi_j^\pm$ *for a given* j *if and only if*

$$\lim_{t\to\infty}\frac{\widehat{u}_j^2(t)}{\sum_{l=0}^\infty \widehat{u}_l^2(t)} = 1$$

and $u(t,0)$ *has the same sign as* $\Phi_j^\pm(0)$ *for all* $t \in [t^*,\infty)$*,* t^* *finite.*

PROOF. If $u(t,0)$ and $\Phi(0)$ have opposite sign for all $t \geq t^*$, then $\mp 2\lim\limits_{t\to\infty}\frac{\widehat{u}_j(t)}{(\sum\limits_{l=0}^\infty \widehat{u}_l^2(t))^{\frac{1}{2}}}$ ≥ 0, and therefore $\lim\limits_{t\to\infty} \| \frac{u(t,\cdot)}{\|u(t,\cdot)\|_0} - \Phi_j(\cdot) \|_0^2 \geq 2$, thus it is clear that $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0}$ can only limit to a $\Phi_j^\pm$ strongly if they have the same sign at their left intercepts past some finite time.

If the norm of $u(t,\cdot)$ grows infinitely large, it follows that at least one mode $\widehat{u}_j(t)$ must grow infinitely large as well. This is due to the fact that for every $j \geq \sqrt{b}$, $|\ \widehat{u}_j\ | \leq \frac{C}{j^2}$. Let us assume that more than one mode $\widehat{u}_j$ grows infinitely large. If not, then all $\widehat{u}_m$ for $m \neq j$ must remain bounded. Thus $\lim\limits_{t\to\infty}\sum\limits_{l=0}^\infty \widehat{u}_l^2 = \lim\limits_{t\to\infty}\widehat{u}_j^2$ and it clearly follows that

$$\lim_{t\to\infty}\frac{\widehat{u}_j^2(t)}{\sum\limits_{l=0}^\infty \widehat{u}_l^2(t)} = 1$$

and

$$\lim_{t\to\infty} \frac{\widehat{u}_m(t)}{(\sum\limits_{l=0}^{\infty} \widehat{u}_l^2(t))^{1/2}} = 0$$

for all $m \neq j$.

Let us denote the minimal j for which the jth mode grows infinitely large by the subscript i, i.e. we denote the mode that escapes to infinity with lowest subscript by $\widehat{u}_i$. Let $\widehat{u}_j$ be any other infinitely growing mode. Then $j \geq i+1$. Since the nonlinearity g is bounded by $-\Gamma \leq g(u) \leq \Gamma$, it follows that $0 \leq \sum\limits_{l=0}^{\infty} \widehat{g}_l^2(t) \leq \Gamma^2$ and thus $-\Gamma \leq \widehat{g}_l(t) \leq \Gamma$ for any l. Let us first assume that $b \neq l^2$ for any integer l. Then $b \neq i^2$ and $b \neq j^2$.

Taking the L^2-inner product of Equation (60) with the ith and jth modes respectively yields equations for $\widehat{u}_j(t)$ and $\widehat{u}_i(t)$ as before:

$$\frac{d}{dt}\widehat{u}_i(t) = (b-i^2)\widehat{u}_i(t) + \widehat{g}_i(t) \tag{67}$$

$$\frac{d}{dt}\widehat{u}_j(t) = (b-j^2)\widehat{u}_j(t) + \widehat{g}_j(t). \tag{68}$$

As both Equations (67) and (68) are first order linear inhomogeneous equations, we may construct their general solutions via combining their homogeneous solutions with particular solutions. Thus, we may write the corresponding solutions to Equations (67) and (68) as follows:

$$\widehat{u}_i(t) = \underbrace{e^{(b-i^2)t}\widehat{u}_i^h(0)}_{\widehat{u}_i^h(t)} + \underbrace{\int_{\infty}^{t} e^{(b-i^2)(t-s)} g_i(s)ds}_{\widehat{u}_i^p(t)} \tag{69}$$

$$\widehat{u}_j(t) = \underbrace{e^{(b-j^2)t}\widehat{u}_j^h(0)}_{\widehat{u}_j^h(t)} + \underbrace{\int_{\infty}^{t} e^{(b-j^2)(t-s)} g_j(s)ds}_{\widehat{u}_j^p(t)} \tag{70}$$

where $\widehat{u}_i^h(0) = \widehat{u}_i(0) - \int_{\infty}^{0} e^{(i^2-b)s} g_i(s)ds$ and $\widehat{u}_j^h(0) = \widehat{u}_j(0) - \int_{\infty}^{0} e^{(j^2-b)s} g_j(s)ds$. Despite our defining $\widehat{u}_i(t)$ and $\widehat{u}_j(t)$ as grow-up modes, both particular solutions are bounded. More specifically, $-\frac{\Gamma}{b-i^2} \leq \widehat{u}_i^p(t) \leq \frac{\Gamma}{b-i^2}$ and $-\frac{\Gamma}{b-j^2} \leq \widehat{u}_j^p(t) \leq \frac{\Gamma}{b-j^2}$. It is clear that the growth to infinity is determined by the homogeneous term in each equation. Thus, the ith mode $\widehat{u}_i(t)$ grows exponentially faster than jth mode $\widehat{u}_j(t)$ (to the order of $e^{(j^2-i^2)t}$), and therefore

$$\lim_{t\to\infty} \frac{\widehat{u}_j(t)}{\widehat{u}_i(t)} = 0.$$

Due to this exponential staggering, $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0}$ can only limit to one $\Phi_j^{\pm}$. But we let j be any arbitrary infinitely growing mode with less than minimal index, therefore this holds true for all modes growing to infinity with index greater than i. Thus, for any $m < b^2$,

$$\lim_{t\to\infty} \frac{\widehat{u}_m(t)}{\widehat{u}_i(t)} = \lim_{t\to\infty} \frac{\widehat{u}_m(t)}{(\sum\limits_{l=0}^{\infty} \widehat{u}_l^2(t))^{1/2}} = 0$$

and

$$\lim_{t\to\infty} \frac{\widehat{u}_i^2(t)}{\widehat{u}_i^2(t)} = \lim_{t\to\infty} \frac{\widehat{u}_i^2(t)}{\sum\limits_{l=0}^{\infty} \widehat{u}_l^2(t)} = 1.$$

Finally, let us presume that $l^2 = b$ for one particular l. Then, returning to Equation (65), it follows that the growth of $\widehat{u}_l(t)$ is bounded linearly, i.e. $-\Gamma t \leq \widehat{u}_l(t) \leq \Gamma t$. If there exist any unbounded modes $i \leq l$, then $\frac{\widehat{u}_l(t)}{\widehat{u}_i(t)}$ clearly limits to zero as time goes to infinity, and thus we may repeat the above calculations for all other growing modes. Therefore, $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0}$ may only limit in the L^2-norm to one particular $\Phi_j^{\pm}$, which will be designated by the lowest index i for which $\widehat{u}_i(t)$ grows to infinity, and then $\lim\limits_{t\to\infty} \frac{\widehat{u}_i^2(t)}{\sum_{l=0}^{\infty} \widehat{u}_l^2} = 1$. $\square$

Therefore, as t grows to infinity, the lowest mode which does not remain bounded must win, and the shape profile of a grow-up solution $u(t,\cdot)$ must approach that of one of the basis functions $\Phi_j = c_j \cos(jx)$ as it grows to infinity. This is the motivation behind our previous choice of notation Φ^{∞} for the limiting objects. Recall that we have only explicitly stated that the modes $j > \sqrt{b}$ must remain bounded. Now, let us consider any $k < j < \sqrt{b}$. If $\lim\limits_{t\to\infty} \| \frac{u(t,\cdot)}{\|u(t,\cdot)\|_0} - \Phi_j^{\sigma}(\cdot) \|_0^2 = 0$, then $\lim\limits_{t\to\infty} \frac{\widehat{u}_j^2(t)}{\sum_{l=0}^{\infty} \widehat{u}_l^2} = 1$ by Lemma 5.9. But then j must be the lowest mode which does not remain bounded. If this were true, then the kth mode would have to be bounded while the jth grew to infinity. Thus, at some time t^* the zero number of the shifted equation would be greater than k. But this contradicts a fundamental property of scalar parabolic PDEs. Thus, if $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0} = \frac{u_k^\sigma(t,\cdot)}{\|u_k^\sigma(t,\cdot)\|_0}$ limits to $\Phi_j^\sigma(\cdot)$, then $j \leq k$.

Because each of the Φ_j are distinctly bounded away from the others, it follows that if $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0}$ limits to a given Φ_j in the C^1-norm, it must limit to the same Φ_j in the H^2-norm. The question now becomes to which object we limit, or in other words, which $\widehat{u}_j$ wins and controls the outcome. For anything but the most ideal nonlinearities, it becomes impossible to explicitly define the $\widehat{u}_j(t)$. Thus, we can exclude modes from winning at infinity (those for which $j^2 > b$), but we cannot determine which mode wins with only the calculations above. This is why we introduced the study of nodal properties in previous chapters. It is these nodal properties which allow us to narrow down and eventually determine which mode wins.

Unfortunately, we are unable to study the limit in the H^1-norm or the H^2-norm, as this would require the use of either the H^1 inner product $\langle u, v\rangle_{1/2} = \int_0^\pi u\overline{v} + u_x\overline{v}_x dx$ or the H^2 inner product $\langle u, v\rangle_1 = \int_0^\pi u\overline{v} + u_x\overline{v}_x + u_{xx}\overline{v}_{xx} dx$. Taking the inner product of Equation (60) with any basis element Φ_j will yield a $\frac{d}{dx}g(u(t,x))\frac{d}{dx}\Phi_j(x)$ term in either inner product. For example, in the H^1-norm we arrive at the equation

$$
\begin{gathered}
\frac{d}{dt}\widehat{u}_j(t) = (b - j^2)\widehat{u}_j(t) + \frac{g_j'(t)j^2}{1+j^2} + \frac{g_j(t)}{1+j^2} \\
g_j(t) = \langle g(u(t,\cdot)), \Phi_j(\cdot)\rangle_0 , \quad g_j'(t) = \left\langle u(t,\cdot)\frac{dg}{du}(u(t,\cdot)), \Phi_j(\cdot)\right\rangle_0 ,
\end{gathered}
\tag{71}
$$

where $\widehat{u}_j$ for the H^1-norm is equal to $\frac{1}{\sqrt{1+j^2}}\widehat{u}_j$ for the L^2-norm. Even for a fixed nonlinearity $g(u)$ we are unable to determine $g_j'(t)$ as it is nonlinearly dependent on the function $u(t,\cdot)$, which frequently cannot be written explicitly. Admittedly, the influence of the bounded and Lipschitz nonlinearity must wane as $u(t,\cdot)$ grows large, since for any fixed time T and large $||u(T,\cdot)|| = U$, it follows that $\frac{u(t,\cdot)}{U}$ solves

$$
w_t = w_{xx} + bw + \frac{1}{U}g(Uw) \tag{72}
$$

at time T. Yet the higher norms of the $g(u)$ term become untenable as they contain multiple instances of u_x^2, a quantity which grows to infinity for non-spatially homogeneous grow-up solutions. As we can see in the explicit formulae

$$
\begin{aligned}
&||g(u(t,\cdot))||_{1/2}^2 = \int_0^{\pi} [g^2(u(t,x)) + g_u^2(u(t,x))u_x^2(t,x)]dx \\
(73) \qquad &||g(u(t,\cdot))||_1^2 = \int_0^{\pi} [g^2(u(t,x)) + g_u^2(u(t,x))u_x^2(t,x) \\
&+g_{uu}^2(u(t,x))u_x^4 + g_u^2(u(t,x))u_{xx}^2 + 2g_u(u(t,x))g_{uu}(u(t,x))u_x^2 u_{xx}]dx,
\end{aligned}
$$

direct calculation becomes useless. If we were unable to prove the limit in higher norms, we could not determine whether a trajectory with shifted zero number k for $0 \leq t < \infty$ limits to an object at infinity with zero number k or an object with zero number less than k, as the y-map alone does not prohibit a drop in zero number at infinity.

We illustrate this conundrum as follows. Let us consider a grow-up solution $u(t,\cdot)$ with initial condition u_0 in the unstable manifold of a stationary solution v such that $z(u(t,\cdot) - v(\cdot)) = 3$ for $0 \leq t < \infty$ and $u(0) - v(0) > 0$. Based on such information, it is possible that $\frac{u(t,\cdot)}{||u(t,\cdot)||_0}$ limits to $\frac{\Phi_3^+}{\|\Phi_3^+\|_0}$ in the H^2-norm and therefore all lower norms, and that there is no drop in the shifted zero number at infinity. But based on the same information, it is also possible that $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_0}$ limits to $\frac{\Phi_1^+}{\|\Phi_1^+\|_0}$, for example if the shape of $u(t,\cdot)$ is that of $cos(x)$ plus a small squiggle at $x = \frac{\pi}{2}$, as in Figure 6. As $u(t,\cdot)$ grows to infinity, the size of the perturbation in $\frac{u(t,\cdot)}{||u(t,\cdot)||_0}$decreases, but slowly enough so that $u(t,\cdot)$ only truly matches $cos(x)$ and NOT $cos(3x)$ at time infinity, i.e. the zero number does not drop in finite time. In such a case, $\frac{u(t,\cdot)}{||u(t,\cdot)||_0}$ will still be L^2-close to $\frac{\Phi_1^+}{\|\Phi_1^+\|_0}$ as time increases to infinity, but not H^2-close or C^1-close. It is only in higher norms that we are able to distinguish our limiting objects. We must therefore introduce a new tool in order to truly ascertain to what objects at infinity these heteroclinic trajectories will connect.

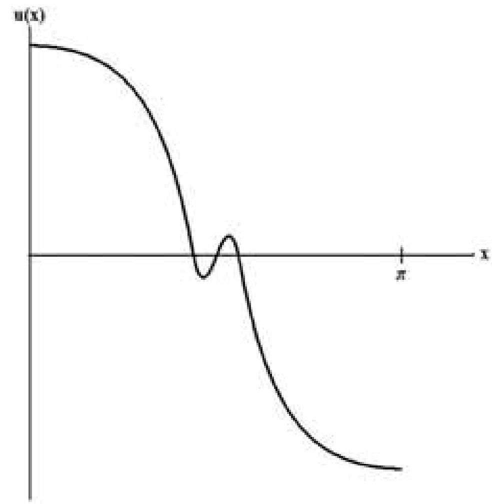

FIGURE 6. A solution with 3 zeros which might limit to $\mathbf{\Phi}_1^+$

CHAPTER 6

The Completed Inertial Manifold

Beginning in 1985 with the work of Foias, Sell, and Temam [**10, 11**], the inertial manifold was introduced as a tool for the study of the long-time behavior of solutions to dissipative nonlinear evolutionary equations. The classical inertial manifold is a finite-dimensional Lipschitz manifold which is positively invariant and attracts all trajectories exponentially [**27**]. The classical inertial manifold, whether studied for scalar parabolic PDEs [**11, 27, 39**] or higher-dimensional equations [**32**], was restricted to dissipative systems of the form

$$u_t + Au = R(u), \tag{74}$$

on a Hilbert space H. For the classical inertial manifold, A is restricted to be a positive, linear, unbounded, self-adjoint operator in H, with domain $D(A) \subset H$ and compact inverse [**39**]. In addition, $R(u)$ is a bounded operator which is restricted to be "dominated" by A such that solutions of Equation (74) are attracted to some finite absorbing set. In particular, there exists a compact global attractor for the dynamical system corresponding to Equation (74) [**11, 27, 39**]. In fact, the classical inertial manifold theory involves the construction of a "prepared" form of the equation which uses a mollifier to eliminate noxious behavior at large $\|u\|$ and smoothes the nonlinearity sufficiently such that the "prepared nonlinearity" is globally Lipschitz from one fractional power space $D(A^{\alpha})$ into another, $D(A^{\beta})$. The consequences of this mollification are minimal because the attractor is bounded and one may always extend the mollifier to be large enough that it leaves untouched any fixed bounded region.

Herein we will introduce a method for constructing an inertial manifold for slowly non-dissipative scalar reaction-diffusion equations of the form

$$
(75) \qquad \begin{gathered}
u_t + \widetilde{A}u - bu = g(u), \quad x \in [0, \pi] \\
u_x(t, 0) = u_x(t, \pi) = 0 \\
b > 0, \quad g(u) \in C^2,\ g(u) \textit{ bounded}, \\
g \textit{ globally Lipschitz with values in } H,
\end{gathered}
$$

where $\widetilde{A}$ is a nonnegative, linear, unbounded, self-adjoint operator with compact inverse acting on the Hilbert space H, and $D(\widetilde{A}) \subset H$. In order that (75) define a slowly non-dissipative dynamical system, we require that $b > \widetilde{\mu}_0$, where $\widetilde{\mu}_0$ is the smallest eigenvalue of the nonnegative operator $\widetilde{A}$. In the case of $\widetilde{A} = -\frac{d^2}{dx^2}$, this is equivalent to $b > 0$. An operator $\widetilde{A}$ fulfilling these conditions is a sectorial operator, and thus the operator $A = \widetilde{A} - bI$ is also a sectorial operator [**18, 33**].

The completed inertial manifold will retain the crucial reductive powers of the classical inertial manifold. This inertial manifold will not require the use of a mollifier, thus it will have no constraints preventing its existence arbitrarily far from the origin. In this sense, the inertial manifold is completed, as it is a manifold defined on the complete Hilbert space rather than on a bounded subset. The reductive powers which inertial manifolds provide will allow us to uniquely determine the limiting object of any grow-up heteroclinic orbit.

While the existence of a completed inertial manifold for Equation (75) with $\widetilde{A} = -\frac{d^2}{dx^2}$ is interesting in its own right, and reduces the equation to a finite-dimensional ODE, this in itself is not enough to resolve the difficulty presented at the end of Chapter 5. It is our ability to prove that such an inertial manifold not only exists, but is Lipschitz with values in C^1 that is responsible for our focus on the theory of inertial manifolds and our extension of these objects to slowly non-dissipative systems. Due to Lemma 4.2, we know that shifted grow-up solutions $u(t, \cdot) - v(\cdot)$ have simple zeros for t in an open dense subset of $\mathbb{R}^+$. Furthermore, the times when a shifted solution will not have simple zeros are the dropping times. Thus, for $t \in [t^*, \infty)$ where t^* is the largest finite dropping time, the solution $u(t, \cdot) - v(\cdot)$ has only simple zeros, as do all possible limiting objects $\mathbf{\Phi}_i^{\pm}$. It follows that a grow-up solution with simple zeros cannot be both C^1-close to a $\mathbf{\Phi}_i^{\pm}$ with simple zeros and have a different zero number from $\mathbf{\Phi}_i^{\pm}$. Thus, we are able to prevent a drop in the zero number at infinity for grow-up solutions. Combining this with our knowledge of the zero number for all finite forward time, thanks to the y-map, we are able to uniquely determine the limiting object on heteroclinic orbits which escape to infinity.

1. The Classical Inertial Manifold

The classical inertial manifold for scalar nonlinear evolutionary equations of the type described above was first derived in [**11**]. For the class of equations described in the previous section, Foias, Sell, and Temam proved the existence of a manifold $\mathcal{M}$ with the following properties [**11**]:

- $\mathcal{M}$ is a finite-dimensional Lipschitz manifold with values in H.
- $\mathcal{M}$ is positively invariant, i.e. $\mathcal{S}(t)\mathcal{M} \subseteq \mathcal{M}$ for all $t \geq 0$.
- $\mathcal{M}$ attracts exponentially all solutions of (74).

The manifold $\mathcal{M}$ is constructed as the graph of a Lipschitz function which maps PH into QH, where P is an orthogonal projection with finite-dimensional range in the Hilbert Space H and $Q = I - P$. In the classical notation, Ψ was called Φ, but we use Ψ here to distinguish the function used in inertial manifold construction from basis vectors and limits of solutions at infinity. One of the major requirements on the evolutionary equation was that the operator A fulfill a "Spectral Gap Condition". The spectral gap condition has a variety of forms, the most general of which we introduce here:

Given that the prepared nonlinearity R_θ is Lipschitz from one fractional power space $D(A^\alpha)$ into another, $D(A^\beta)$, with $\beta \leq \alpha$, so that $\kappa = \alpha - \beta < 1$, and given that λ_n is the nth eigenvalue of the operator A, there exists some integer $n \geq 1$ and constant C such that

$$\lambda_{n+1}^{1-\kappa} > C \qquad (76)$$

$$\lambda_{n+1} - \lambda_n \geq C(\lambda_{n+1}^\kappa + \lambda_n^\kappa).$$

If R_θ is Lipschitz from $D(A^\beta)$ into $D(A^\beta)$, the spectral gap condition reduces to

$$\lambda_{n+1} - \lambda_n \geq C. \tag{77}$$

It is the spectral gap condition which determines what finite-dimensional subset the operator P projects onto, and therefore determines the dimension of the inertial manifold $\mathcal{M}$. One might be inclined to guess that the dimension of the inertial manifold ought to be the highest Morse index among the equilibria of the evolutionary equation, but this only provides a lower bound. In order to satisfy the spectral gap condition, one may increase the number of modes onto which the operator P projects, and in so doing ensure the attractive properties $\mathcal{M}$ must have in order to fulfill the definition of an inertial manifold.

The function Ψ is derived as the fixed point of a mapping $\mathcal{J}$, which is a Lyapunov-Perron transformation. The semigroup $\mathcal{S}(t)$ of Equation (74) has both the Squeezing Property and the Cone Property, also known as the Strong Squeezing Property [**8, 12, 27**]. Thanks to these squeezing properties, the finite-dimensional manifold $\mathcal{M}$ attracts exponentially all trajectories, and thus contains the universal attractor of (74).

2. Constructing a Completed Inertial Manifold

We must first define a number of elements at the core of the construction of the inertial manifold, as the original definitions do not apply for systems which exhibit grow-up. Let $b > 0$ be fixed. Rather than requiring our operator A to be positive, we instead define

$$A := \widetilde{A} - bI, \tag{78}$$

with the operator $\widetilde{A}$ defined as above. We refer to A as the "shifted $\widetilde{A}$ operator". This substitution transforms Equation (75) into $u_t + Au = g(u)$. Note that for $\widetilde{A} = -\Delta$, A is the operator introduced in Section 5.2. By the Spectral Theorem, H is equipped with an orthonormal basis comprised of the eigenfunctions $\{\Phi_i\}_{\mathbb{N}_0}$ of the operator A with Neumann boundary conditions. We remind the reader that while the eigenfunctions of A form an orthogonal basis for all fractional power spaces $D(A^\alpha)$, they must be normalized differently depending on the power space. This was illustrated in (63) for the choice of $A = -\Delta - bI$. Henceforth we use $\langle\cdot\rangle_0$ and $||\cdot||_0$ to denote the inner product and norm corresponding to $H = D(A^0)$. For $A = -\Delta - bI$, these correspond to the inner products and norms defined in Chapter 2. Thus, we are provided with a Fourier decomposition of u:

$$u(t,\cdot) = \sum_{j=0}^{\infty} \widehat{u}_j(t)\Phi_j(x), \quad \widehat{u}_j(t) = \langle u(t,\cdot), \Phi_j(\cdot)\rangle_0 . \tag{79}$$

Owing to the fact that the basis we have fixed for H is the set of eigenfunctions of A, with the set of eigenvalues of A denoted by μ_j as before, we may define a fractional power of A via a Fourier decomposition as well. For $0 \leq \alpha \leq 1$

$$A^\alpha u(t,\cdot) = \sum_{\mu_j \geq 0} \mu_j^\alpha \widehat{u}_j(t)\Phi_j(x) + \sum_{\mu_j < 0} (-1)^\alpha(-\mu_j)^\alpha \widehat{u}_j(t)\Phi_j(x). \tag{80}$$

It is clear that $D(A^\alpha) = D((\widetilde{A})^\alpha)$. We recall that if A is a sectorial operator on a Banach space X, for each $\alpha \geq 0$ we may define the corresponding fractional power space via

$$X^\alpha = D(A_1^\alpha) \text{ with the graph norm}$$

$$\| x \|_\alpha = \| A_1^\alpha x \|_0, \quad x \in X^\alpha$$

for $A_1 = A + aI$, with a chosen such that the real part of the spectrum of A_1 is positive [**18**]. Thus $\langle u, v\rangle_\alpha = \langle A_1^\alpha u, A_1^\alpha v\rangle_0$. We define $a = b+1$ in order to circumvent any difficulties originating if $\mu_j = 0$. For $\mu_0 > 0$ the definition of $A^\alpha u(t,\cdot)$ holds for $-1 \leq \alpha \leq 1$.

2.1. The Strong Squeezing Property. The idea behind the various squeezing properties is that for the Hilbert space over which a nonlinear evolutionary equation acts, there is a natural splitting into a finite-dimensional subspace and its infinite-dimensional orthogonal complement through which the finite-dimensional component of the solution dominates. There are a number of useful texts which detail the construction of this splitting and use varying but related methods for proving the squeezing property. For consistency we adhere with the methods found in [**27**].

Without specifying a particular value for N, we define the projections P_N and Q_N as follows, noting the similarity to the basis elements referenced in Chapter 5. Let P_N be the orthogonal projection onto the first $N+1$ Fourier eigenfunctions Φ_j of the operator $A = \widetilde{A} - bI$ with Neumann boundary conditions (which are also the eigenfunctions of the operators $\widetilde{A}$ and A_1). We denote the jth eigenvalues corresponding to the operators A, $\widetilde{A}$, and A_1 by μ_j, $\widetilde{\mu}_j$, and $\mu_{j,1}$ respectively. We order the eigenfunctions so that $\mu_{j+1} \geq \mu_j$ for the eigenvalues μ_j of A, and thus $\widetilde{\mu}_{j+1} \geq \widetilde{\mu}_j$ and $\mu_{j+1,1} \geq \mu_{j,1}$. For the specific case of $\widetilde{A} = -\Delta$,

$$\Phi_j(x) = c_j \cos(jx), \quad \mu_j = j^2 - b, \quad \widetilde{\mu}_j = j^2, \quad \mu_{j,1} = j^2 + 1, \tag{81}$$

where the c_j are the constants given in (62). We define $Q_N = I - P_N$ as the projection onto the orthogonal complement of P_N. Thus, we may divide any solution $u \in D(A) \subset H$ into two components p and q such that $u = p + q = P_N u + Q_N u$, where

$$\begin{aligned} p &= P_N u = \sum_{j=0}^{N} \langle u, \Phi_j \rangle_0 \, \Phi_j \\ q &= Q_N u = \sum_{j=N+1}^{\infty} \langle u, \Phi_j \rangle_0 \, \Phi_j. \end{aligned} \tag{82}$$

By applying P_N and Q_N to Equation (75) we obtain the system

$$\begin{aligned} p_t &= -\widetilde{A}p + bp + P_N g(p+q) \\ q_t &= -\widetilde{A}q + bq + Q_N g(p+q). \end{aligned} \tag{83}$$

The portion of a solution $u(t, \cdot)$ in $P_N H$ is referred to as the "low modes" of u and the portion in $Q_N H$ as the "high modes" [**27**]. Our goal is to express the Fourier coefficients of the high modes in terms of the Fourier coefficients of the low modes. We may always construct such a definition when allowing for error, i.e.

$$\widehat{u}_j(t) = \psi_j(\widehat{u}_0, \dots, \widehat{u}_N) + error \quad for\ all\ j > N. \tag{84}$$

This is generally referred to as a "slaving rule", as the higher modes are enslaved to the lower [**14**]. We require that any ψ_j be Lipschitz continuous from $P_N H$ into $Q_N H$ for all j, noting that $P_N H = PD(A)$. We are looking for a specific ψ such that the error becomes zero. Although it is simple to find an initial condition for which (84) does not hold with zero error, i.e. where the initial condition is not on the manifold defined by the graph of ψ, the goal is to show that the error will then decay to zero exponentially with time. Additionally, we must prove that if (84) holds with zero error at some initial time, it must then hold without error for all forward time. By applying the definitions of p and q to (84) we may rewrite (84) as

$$q = \psi(p) + error, \tag{85}$$

where $\psi : P_N H \to Q_N H$. The graph of ψ

$$\mathbb{G}[\psi] \equiv \{u : u = p + \psi(p),\ p \in P_N H\} \tag{86}$$

defines an $(N+1)$-dimensional manifold $\mathcal{M}$ in the Hilbert space H. Our requirement that (85) remains an equality in forward time is equivalent to requiring $\mathcal{M}$ to be positively invariant. The requirement that the error decay exponentially to zero is equivalent to $\mathcal{M}$ being exponentially attracting, or

$$||q(t) - \psi(p(t))|| \leq C(||u_0||)e^{-kt} \tag{87}$$

for some $k > 0$. As discussed in Section 6.1, the spectral gap condition is based on the extent to which the nonlinearity is Lipschitz continuous, i.e. the level of regularity the evaluation operator g provides. We shall only require that g is globally Lipschitz with values in H. This may seem insufficient to our ultimate goals, in view of the C^1 target set required by nodal properties and the fact that $H = L^2$ for $\widetilde{A} = -\Delta$. But once the Lipschitz property in H is established, we will prove that this is indeed sufficient to prove convergence in C^1. We prove the strong squeezing property outright as our choices of operator A and nonlinearities $g(u) \in \mathcal{G}$ are sufficiently well-behaved to allow us to forgo the need for a weaker version to prove the stronger.

There are multiple equivalent formulations of the squeezing property in the literature; as stated before we follow the one provided by [**27**]:

The strong squeezing property holds if, for any two solutions $u(t)$ and $\overline{u}(t)$, we have the cone invariance property:

$$||q(0) - \overline{q}(0)||_0 \leq ||p(0) - \overline{p}(0)||_0 \tag{88}$$

implies that for all $t > 0$

$$||q(t) - \overline{q}(t)||_0 \leq ||p(t) - \overline{p}(t)||_0, \tag{89}$$

and the decay property: if, for some $t > 0$

$$||q(t) - \overline{q}(t)||_0 \geq ||p(t) - \overline{p}(t)||_0, \tag{90}$$

then

$$||q(t) - \overline{q}(t)||_0 \leq ||q(0) - \overline{q}(0)||_0 e^{-kt} \tag{91}$$

for some $k > 0$.

It is at this point that we introduce the form of the spectral gap condition which we shall use for the construction of our unconstrained inertial manifold. There are many formulations of the spectral gap condition in various works on the subject of inertial manifolds. This is owing to the fact that there is a minimum necessary gap dependent on the particulars of the evolutionary equation in question, but the author need not define their spectral gap condition to state the minimum gap if the existence of a wider gap is both provable and advantageous. For equations of the form (75), the "Spectral Gap Condition" is given by the following lemma.

LEMMA 6.1. *If there exists an N such that the eigenvalues μ_N and μ_{N+1} of A satisfy*

$$\begin{gathered} \mu_N > 0 \\ \mu_{N+1} - \mu_N > 4C_1, \end{gathered} \tag{92}$$

where C_1 is the Lipschitz coefficient in H of $g(u)$, then the strong squeezing property holds with the k in (91) bounded below by

$$k > \mu_N + 2C_1. \tag{93}$$

PROOF. Let us define the difference of two solutions u and $\overline{u}$ by $w(t) = u(t) - \overline{u}(t)$. We study what happens on the boundary of the "cone"

$$\{(u, \overline{u}) \mid ||Q_N(u - \overline{u})||_0 \leq ||P_N(u - \overline{u})||_0\}. \tag{94}$$

To show that trajectories are unable to leave this cone we show that

$$\frac{d}{dt}(||Qw||_0 - ||Pw||_0) < 0 \; for \; ||Qw||_0 = ||Pw||_0, \tag{95}$$

which illustrates why the cone invariance property is named as such. The equation of the difference of two solutions $w(t, \cdot)$ is

$$\frac{dw}{dt} + Aw = g(u) - g(\overline{u}), \tag{96}$$

and we write

$$p = P_N w, \quad q = Q_N w, \quad w = p + q.$$

Since the operator A commutes with P_N, we may take the inner product of Equation (96) with p to obtain

$$\frac{1}{2}\frac{d}{dt}||p||_0^2 + \langle Ap, p\rangle_0 = \langle g(u) - g(\overline{u}), p\rangle_0. \tag{97}$$

It is obvious that $\langle Ap, p\rangle_0 \leq \mu_N ||p||_0^2$ for all $u \in H$. Additionally, we may derive bounds on $\langle g(u) - g(\overline{u}), p\rangle_0$ as follows:

$$||g(u) - g(\overline{u})||_0 \leq C_1||u - \overline{u}||_0 = C_1||w||_0$$

$$\Rightarrow \langle g(u) - g(\overline{u}), p\rangle_0 \leq ||g(u) - g(\overline{u})||_0 \cdot ||p||_0 \leq C_1||u - \overline{u}||_0 \cdot ||p||_0$$

$$= C_1||w||_0 \cdot ||p||_0.$$

Plugging these inequalities into Equation (97) yields

$$\frac{1}{2}\frac{d}{dt}||p||_0^2 \geq -\mu_N||p||_0^2 - C_1||w||_0 \cdot ||p||_0, \tag{98}$$

or when $||q(0)||_0 = ||p(0)||_0$, denoting the right-hand time derivative as $\frac{d}{dt_+}$

$$\left(\frac{d}{dt_+}||p||_0\right)_{t=0} \geq -\mu_N||p||_0 - C_1||w||_0 = -\mu_N||p||_0 - C_1||p + q||_0$$

$$\geq -\mu_N||p||_0 - C_1(||p||_0 + ||q||_0) = -\mu_N||q||_0 - 2C_1||q||_0,$$

since we are studying the boundary of the cone where $||q||_0 = ||p||_0$. We may repeat these calculations with the inner product of Equation (96) with q and, noting that $\langle Aq, q\rangle_0 \geq \mu_{N+1}||q||_0^2$, we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}||q||_0^2 \leq -\mu_{N+1}||q||_0^2 + C_1||w||_0 \cdot ||q||_0. \tag{99}$$

When $||p(0)||_0 = ||q(0)||_0$ this inequality is equivalent to

$$(\frac{1}{2}\frac{d}{dt}||q||_0^2)_{t=0} \leq -\mu_{N+1}||q||_0^2 + 2C_1||q||_0^2, \tag{100}$$

and therefore we obtain the inequality

$$(\frac{d}{dt_+}||q||_0)_{t=0} \leq -\mu_{N+1}||q||_0 + 2C_1||q||_0 = -(\mu_{N+1} - 2C_1)||q||_0. \tag{101}$$

Therefore, at $t = 0$

$$\frac{d}{dt_+}(||q||_0 - ||p||_0)_{t=0} \leq -(\mu_{N+1} - \mu_N - 4C_1)||q(0)||_0 \tag{102}$$

which is negative provided that our spectral gap condition (92) holds, and thus we obtain the cone invariance property.

To obtain the decay property we proceed as follows. For exponential squeezing outside the cone we plug $||q||_0 \geq ||p||_0$ into inequality (99), which yields

$$\frac{1}{2}\frac{d}{dt}||q||_0^2 \leq -\mu_{N+1}||q||_0^2 + 2C_1||q||_0^2. \tag{103}$$

We recall that C_1 is positive and our spectral gap condition implies that $-\mu_{N+1} + 2C_1 < 0$. Recalling that our use of q herein was actually the Q-component of $w(t) = u(t) - \overline{u}(t)$, the exponential decay in (90 - 91) then follows with $k = \mu_{N+1} - 2C_1$ via Gronwall's Inequality. The lower bound (93) follows then from the spectral gap condition. □

2.2. The Formal Operator. Due to our choice of operator A and nonlinearity $g(u) \in \mathcal{G}$, the initial value problem defined by Equation (75) with initial condition $u(0) = u_0$ is guaranteed the existence of a unique mild solution $u(t, \cdot) = u(t)$ which satisfies

$$u(t) = \mathcal{S}(t-0)u_0 + \int_0^t \mathcal{S}(t-s)g(u(s))ds, \tag{104}$$

as stated in Pazy [**24**]. As $-A$ is the infinitesimal generator of the C^0 semigroup $\mathcal{S}(t)$ on the Hilbert Space $D(A)$, we may rewrite (104) as

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}g(u(s))ds, \tag{105}$$

the variation of constants formula. It further follows that for any $u(t)$ solving (75) which exists and is bounded on $(-\infty, 0]$, there exists a $u_0 \in P_N D(A)$ such that

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}P_N g(u(s))ds + \int_{-\infty}^t e^{-A(t-s)}Q_N g(u(s))ds \tag{106}$$

as proved in [**15**]. We recall that the nature of the operator A and projections P_N and Q_N imply that

$$\begin{gathered}
\| e^{-AP_N t} \|_{\mathscr{L}(H)} \leq e^{-\mu_N t}, \quad t \leq 0 \\
\| e^{-AQ_N t} \|_{\mathscr{L}(H)} \leq e^{-\mu_{N+1} t}, \quad t > 0 \\
\| (AQ_N)^{\alpha} e^{-AQ_N t} \|_{\mathscr{L}(H)} \leq t^{-\alpha} e^{-\mu_{N+1} t}, \quad t > 0
\end{gathered} \tag{107}$$

where $\| \cdot \|_{\mathscr{L}(H)}$ is the operator norm on the bounded linear operators $e^{-AP_N t}$ and $e^{-AQ_N t}$ for t negative or positive respectively [**18, 39**].

Recalling that $u_0 \in P_N D(A)$ and A commutes with P_N, it follows that the first two terms in (106) map to $P_N D(A)$ and the final term maps to $Q_N D(A)$. Returning to the notation defined in (82) this means that for $u(t) = p(t) + q(t)$ we may further define

$$\begin{gathered}
p(t) = P_N u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-s)}P_N g(u(s))ds \\
q(t) = Q_N u(t) = \int_{-\infty}^t e^{-A(t-s)}Q_N g(u(s))ds.
\end{gathered} \tag{108}$$

We now recall our definition of the function $\psi : P_N H \to Q_N H$ as an arbitrary Lipschitz function from the finite-dimensional subspace $P_N H = P_N D(A)$ to its complement. We define the set $\mathcal{R}_{c,\, l}$ as the class of functions ψ from $P_N H$ into $Q_N H$ such that

$$\begin{aligned}
\| \psi(p) \|_0 &\leq c, && \text{for all } p \in P_N H \\
\| \psi(p_1) - \psi(p_2) \|_0 &\leq l \, \| p_1 - p_2 \|_0 && \text{for all } p_1, p_2 \in P_N H.
\end{aligned} \tag{109}$$

For the sake of convenience we set $l = 1$, but it is possible to restrict l to any fixed positive value less than 1. The effect of choosing $l < 1$ is that the coefficient 4 in the spectral gap condition must be increased. The exact relation will be shown later. It is clear that $u(t) = p(t) + \psi(p(t))$ is a solution of (75) if and only if $p(t)$ and $q(t) = \psi(p(t))$ satisfy (83) with $u = p + \psi(p)$. Let us fix an arbitrary such choice of ψ. It is clear that $p = p(t) = p(t; \psi, p(0))$ since $p(t)$ is dependent on our choice of ψ in order to fulfill $u(t) = p(t) + \psi(p(t))$ with $u(t)$ solving (75). We now define the Lyapunov-Perron transformation $\widehat{\psi} = \mathcal{J}\psi$ by

$$\widehat{\psi}(p(0)) = \mathcal{J}\psi(p(0)) = \int_{-\infty}^{0} e^{AQ_N s} Q_N g(p(s) + \psi(p(s)))ds \tag{110}$$

for $p(0) \in P_N D(A) = P_N H$. We see that the Lyapunov-Perron transformation at a point $p(0)$ is dependent on $\psi \in \mathcal{R}_{c,\ l}$ and $p_0 \in P_N D(A)$. We have thus defined a formal mapping $\psi \to \mathcal{J}\psi$ on the set of functions mapping $P_N H$ to $Q_N H$ which are Lipschitz with values in H, where ψ is the function $p_0 \to \psi(p_0)$ and $\mathcal{J}\psi$ is the function which maps p_0 to $\int_{-\infty}^{0} e^{AQ_n s} Q_N g(p(s) + \psi(p(s)))ds$.

We shall now prove a number of properties of the operator $\mathcal{J}$. The set $\mathcal{R}_{c,\ l}$ is a metric space when endowed with the metric

$$\| \psi_1 - \psi_2 \| := \sup_{p \in P_N H} \| \psi_1(p) - \psi_2(p) \| . \tag{111}$$

The mapping $\mathcal{J}$ associates a function on $P_N H$ defined by $\mathcal{J}\psi(p_0) = \int_{-\infty}^{0} e^{AQ_N s} Q_N g(u(s))ds$ to each function $\psi \in \mathcal{R}_{c,\ l}$.

LEMMA 6.2. *For every $p_0 \in P_N H$, $\mathcal{J}\psi(p_0)$ belongs to $Q_N H$ and*

$$\| \mathcal{J}\psi(p_0) \|_0 \leq \Gamma \sqrt{\pi} \mu_{N+1}^{-1}. \tag{112}$$

PROOF. Since $e^{AQ_N s} = Q_N e^{AQ_N s}$, it follows that $Q_N \mathcal{J}\psi(p_0) = \mathcal{J}\psi(p_0)$ and therefore $\mathcal{J}\psi(p_0) \in Q_N H$ for $p_0 \in P_N H$. Furthermore,

$$\begin{aligned} \| \mathcal{J}\psi(p_0) \|_0 &= \| \int_{-\infty}^{0} e^{AQ_N s} Q_N g(p(s) + \psi(p(s)))ds \|_0 \\ &\leq \int_{-\infty}^{0} \| e^{AQ_N s} Q_N g(p(s) + \psi(p(s))) \|_0 \, ds \end{aligned}$$

$$\leq \int_{-\infty}^{0} \| e^{AQ_N s} \|_{\mathscr{L}(H)} \| Q_N g(p(s) + \psi(p(s))) \|_0 \, ds$$

$$\leq \int_{-\infty}^{0} e^{\mu_{N+1} s} \| Q_N g(p(s) + \psi(p(s))) \|_0 \, ds$$

$$\leq \int_{-\infty}^{0} e^{\mu_{N+1} s} \| g(p(s) + \psi(p(s))) \|_0 \, ds \leq \int_{-\infty}^{0} e^{\mu_{N+1} s} \Gamma \sqrt{\pi} ds$$

$$= \Gamma \sqrt{\pi} \mu_{N+1}^{-1},$$

where we recall that $\Gamma = \max\{| \underline{\gamma} |, | \overline{\gamma} |\}$. $\square$

Thus, we may set the c in $\mathcal{R}_{c,\ l}$ to be $c = \Gamma\sqrt{\pi}\mu_{N+1}^{-1}$. We proceed to show that under the restrictions imposed by the spectral gap condition (92), $\mathscr{J}$ is a Lipschitz mapping of $\mathcal{R}_{c,\ l}$ into $\mathcal{R}_{c,\ l}$ with a strict contraction.

First let us fix $\psi \in \mathcal{R}_{c,\ l}$. Let us choose $p_1(0), p_2(0) \in P_N D(A)$ and let $p_1 = p_1(t),\ p_2 = p_2(t)$ be the corresponding solutions of the first part of Equation (83). For ease of notation, we set $\widetilde{p} = p_1 - p_2$. It follows that $\widetilde{p}$ solves the evolutionary equation

$$\frac{d\widetilde{p}}{dt} + A\widetilde{p} = P_N g(u_1) - P_N g(u_2) \tag{113}$$

where $u_1 = p_1 + \psi(p_1)$ and $u_2 = p_2 + \psi(p_2)$, with u_i solving (75). Taking the scalar product of Equation (113) with $\widetilde{p}$ yields

$$\frac{1}{2}\frac{d}{dt} \| \widetilde{p} \|_0^2 + \langle A\widetilde{p}, \widetilde{p}\rangle_0 = \langle P_N(g(u_1) - g(u_2)), \widetilde{p} \rangle_0 .$$

Recalling from Section 6.2.1 that $\langle A\widetilde{p}, \widetilde{p}\rangle_0 \leq \mu_N \| \widetilde{p} \|_0^2$ and $\langle g(u_1) - g(u_2), \widetilde{p}\rangle_0 \geq - \| g(u_1) - g(u_2) \|_0 \cdot \| \widetilde{p} \|_0$ it follows that

$$\frac{1}{2}\frac{d}{dt} \| \widetilde{p} \|_0^2 \geq -\mu_N \| \widetilde{p} \|_0^2 - \| g(u_1) - g(u_2) \|_0 \cdot \| \widetilde{p} \|_0 .$$

We recall that by our choice of $g(u) \in \mathcal{G}$

$$\| g(u_1) - g(u_2) \|_0 \leq C_1 \| u_1 - u_2 \|_0 .$$

Using the fact that $u_1 - u_2 = p_1 - p_2 + (\psi(p_1) - \psi(p_2))$ and the definition (109) of $\mathcal{R}_{c,\ l}$, we obtain the inequality

$$\| u_1 - u_2 \|_0 \leq \| \widetilde{p} \|_0 + l \| \widetilde{p} \|_0 = (1 + l) \| \widetilde{p} \|_0 \tag{114}$$

and thus

$$\| g(u_1) - g(u_2) \|_0 \leq C_1(1+l) \| \widetilde{p} \|_0 \, . \tag{115}$$

Plugging this into our calculations, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \| \widetilde{p} \|_0^2 &\geq -\mu_N \| \widetilde{p} \|_0^2 - C_1(1+l) \| \widetilde{p} \|_0^2 \\
\Rightarrow \frac{d}{dt} \| \widetilde{p} \|_0 &\geq -(\mu_N + C_1(1+l)) \| \widetilde{p} \|_0,
\end{aligned}$$

which implies

$$\| \widetilde{p}(t) \|_0 \leq \| \widetilde{p}(0) \|_0 \, e^{-(\mu_N + C_1(1+l))t} \quad \text{for all } t \leq 0. \tag{116}$$

This leads us to our next property of the Lyapunov-Perron operator $\mathcal{J}$:

Lemma 6.3. *Given that the spectral gap condition (92) holds, for any choice of $\psi \in \mathcal{R}_{c,\, l}$ and $p_1(0), p_2(0) \in P_N D(A)$ we have*

$$\| \mathcal{J}\psi(p_1(0)) - \mathcal{J}\psi(p_2(0)) \|_0 \leq L \| p_1(0) - p_2(0) \|_0, \tag{117}$$

where $L = \frac{C_1(1+l)}{\mu_{N+1} - \mu_N - C_1(1+l)}$.

Proof. Recall that the spectral gap condition requires that $\mu_{N+1} - \mu_N > 4C_1$. This implies that $\mu_{N+1} - \mu_N - C_1(1+l) \geq \mu_{N+1} - \mu_N - 2C_1 > 2C_1 > 0$. Incorporating properties (107), (109), (115), and (116) yields

$$\begin{aligned}
\| \mathcal{J}\psi(p_1(0)) - \mathcal{J}\psi(p_2(0)) \|_0 &\leq \int_{-\infty}^{0} \| e^{AQ_N s} Q_N(g(u_1(s)) - g(u_2(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^{0} \| e^{AQ_N s} \|_{\mathscr{L}(H)} \| Q_N(g(u_1(s)) - g(u_2(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^{0} e^{\mu_{N+1}s} \| Q_N(g(u_1(s)) - g(u_2(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^{0} e^{\mu_{N+1}s} \| g(u_1(s)) - g(u_2(s)) \|_0 \, ds \\
&\leq \int_{-\infty}^{0} e^{\mu_{N+1}s} C_1(1+l) \| p_1(s) - p_2(s) \|_0 \, ds \\
&\leq C_1(1+l) \int_{-\infty}^{0} e^{\mu_{N+1}s} \| p_1(0) - p_2(0) \|_0 \, e^{-(\mu_N + C_1(1+l))s} ds
\end{aligned}$$

$$= C_1(1+l) \parallel p_1(0) - p_2(0) \parallel_0 \int_{-\infty}^{0} e^{(\mu_{N+1}-\mu_N-C_1(1+l))s} ds$$

$$= C_1(1+l) \parallel p_1(0) - p_2(0) \parallel_0 \frac{1}{\mu_{N+1}-\mu_N-C_1(1+l)} = L \parallel p_1(0) - p_2(0) \parallel_0 .$$

□

Thus, we have shown that $\mathcal{J}$ maps $\mathcal{R}_{c,\ l}$ into $\mathcal{R}_{c,\ L}$. We shall now show that $\mathcal{J}$ is a Lipschitz mapping on these spaces. Let us consider two functions $\psi_1, \psi_2 \in \mathcal{R}_{c,\ l}$ and a single initial condition $p(0) \in P_N D(A)$. Let $p_1 = p(t; \psi_1, p(0))$ and $p_2 = p(t; \psi_2, p(0))$, and $u_1 = u_1(t) = p_1(t) + \psi_1(p_1(t)),\ u_2 = u_2(t) = p_2(t) + \psi_2(p_2(t))$. Using calculations similar to those we have just completed, we demonstrate that $\mathcal{J}$ is a Lipschitz operator with values in $\mathcal{R}_{c,\ l}$.

$$\begin{aligned}
&\parallel \mathcal{J}\psi_1(p(0)) - \mathcal{J}\psi_2(p(0)) \parallel_0 \\
&= \parallel \int_{-\infty}^{0} e^{AQ_N s} Q_N(g(u_1(s)) - g(u_2(s))) ds \parallel_0 \\
&\leq \int_{-\infty}^{0} \parallel e^{AQ_N s} Q_N(g(u_1(s)) - g(u_2(s))) \parallel_0 ds \\
&\leq \int_{-\infty}^{0} \parallel e^{AQ_N s} \parallel_{\mathscr{L}(H)} \parallel Q_N(g(u_1(s)) - g(u_2(s))) \parallel_0 ds \\
&\leq \int_{-\infty}^{0} e^{\mu_{N+1} s} \parallel Q_N(g(u_1(s)) - g(u_2(s))) \parallel_0 ds \\
&\leq \int_{-\infty}^{0} e^{\mu_{N+1} s} \parallel g(u_1(s)) - g(u_2(s)) \parallel_0 ds \\
&\leq \int_{-\infty}^{0} e^{\mu_{N+1} s} C_1 \parallel u_1(s) - u_2(s) \parallel_0 ds.
\end{aligned}$$

We here note that $u_1 - u_2 = p_1 - p_2 + \psi_1(p_1) - \psi_2(p_2) = p_1 - p_2 + \psi_1(p_1) - \psi_1(p_2) + \psi_1(p_2) - \psi_2(p_2)$. Plugging this into our calculations produces

$$\begin{aligned}
&\parallel \mathcal{J}\psi_1(p(0)) - \mathcal{J}\psi_2(p(0)) \parallel_0 \\
&\leq C_1 \int_{-\infty}^{0} e^{\mu_{N+1} s} (\parallel p_1(s) - p_2(s) \parallel_0 + \parallel \psi_1(p_1(s)) - \psi_1(p_2(s)) \parallel_0 \\
&\qquad + \parallel \psi_1(p_2(s)) - \psi_2(p_2(s)) \parallel_0) ds \\
&\leq C_1 \int_{-\infty}^{0} e^{\mu_{N+1} s} ((1+l) \parallel p_1(s) - p_2(s) \parallel_0 + \parallel \psi_1 - \psi_2 \parallel) ds.
\end{aligned}$$

Recalling our notation of $p_1(s)-p_2(s)=\widetilde{p}(s)$, we take the scalar product of Equation (113) with $\widetilde{p}$ again and recall that $\langle A\widetilde{p},\widetilde{p}\rangle_0 \leq \mu_N \parallel \widetilde{p} \parallel_0^2$ and $\langle g(u_1)-g(u_2),\widetilde{p}\rangle_0 \geq - \parallel g(u_1)-g(u_2) \parallel_0 \cdot \parallel \widetilde{p} \parallel_0$. This yields

$$\frac{1}{2}\frac{d}{dt} \parallel \widetilde{p} \parallel_0^2 + \langle A\widetilde{p},\widetilde{p}\rangle_0 = \langle P_N(g(u_1)-g(u_2)),\widetilde{p}\rangle_0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt} \parallel \widetilde{p} \parallel_0^2 \geq -\mu_N \parallel \widetilde{p} \parallel_0^2 - \parallel g(u_1)-g(u_2) \parallel_0 \cdot \parallel \widetilde{p} \parallel_0$$

$$\geq -\mu_N \parallel \widetilde{p} \parallel_0^2 - C_1 \parallel u_1 - u_2 \parallel_0 \cdot \parallel \widetilde{p} \parallel_0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt} \parallel \widetilde{p} \parallel_0^2 \geq -\mu_N \parallel \widetilde{p} \parallel_0^2 - C_1((1+l) \parallel \widetilde{p} \parallel_0 + \parallel \psi_1 - \psi_2 \parallel) \parallel \widetilde{p} \parallel_0$$

$$\geq -(\mu_N + C_1(1+l)) \parallel \widetilde{p} \parallel_0^2 - C_1 \parallel \psi_1 - \psi_2 \parallel \cdot \parallel \widetilde{p} \parallel_0$$

$$\Rightarrow \frac{d}{dt} \parallel \widetilde{p} \parallel_0 \geq -(\mu_N + C_1(1+l)) \parallel \widetilde{p} \parallel_0 - C_1 \parallel \psi_1 - \psi_2 \parallel$$

$$\Rightarrow \parallel \widetilde{p}(t) \parallel_0 \leq -\frac{C_1 \parallel \psi_1 - \psi_2 \parallel}{\mu_N + C_1(1+l)} + (\parallel \widetilde{p}(0) \parallel_0 + \frac{C_1 \parallel \psi_1 - \psi_2 \parallel}{\mu_N + C_1(1+l)})e^{-(\mu_N + C_1(1+l))t}$$

for all $t \leq 0$. Recalling that $\widetilde{p}(0) = p_1(0) - p_2(0) = p(0) - p(0) = 0$, it follows that for all $t \leq 0$

$$\parallel \widetilde{p}(t) \parallel_0 \leq = -\frac{C_1 \parallel \psi_1 - \psi_2 \parallel}{\mu_N + C_1(1+l)} + \frac{C_1 \parallel \psi_1 - \psi_2 \parallel}{\mu_N + C_1(1+l)} e^{-(\mu_N + C_1(1+l))t}$$

$$= \frac{C_1 \parallel \psi_1 - \psi_2 \parallel}{\mu_N + C_1(1+l)}(e^{-(\mu_N + C_1(1+l))t} - 1) \leq \frac{C_1 \parallel \psi_1 - \psi_2 \parallel}{\mu_N + C_1(1+l)} e^{-(\mu_N + C_1(1+l))t}.$$

Plugging this into our calculations on the Lipschitz coefficient of $\mathcal{J}$ yields

$$\parallel \mathcal{J}\psi_1(p(0)) - \mathcal{J}\psi_2(p(0)) \parallel_0$$

$$\leq C_1 \int_{-\infty}^{0} e^{\mu_{N+1}s}((1+l) \parallel p_1(s) - p_2(s) \parallel_0 + \parallel \psi_1 - \psi_2 \parallel)ds$$

$$\leq C_1 \int_{-\infty}^{0} e^{\mu_{N+1}s}((1+l)\frac{C_1 \parallel \psi_2 - \psi_2 \parallel}{\mu_N + C_1(1+l)} e^{-(\mu_N + C_1(1+l))s} + \parallel \psi_1 - \psi_2 \parallel)ds$$

$$= C_1 \parallel \psi_1 - \psi_2 \parallel \int_{-\infty}^{0} \frac{C_1(1+l)}{\mu_N + C_1(1+l)} e^{(\mu_{N+1} - \mu_N - C_1(1+l))s} + e^{\mu_{N+1}s} ds$$

$$= C_1 \parallel \psi_1 - \psi_2 \parallel (\int_{-\infty}^{0} \frac{C_1(1+l)}{\mu_N + C_1(1+l)} e^{(\mu_{N+1} - \mu_N - C_1(1+l))s} ds + \frac{1}{\mu_{N+1}})$$

$$= C_1 \parallel \psi_1 - \psi_2 \parallel (\frac{C_1(1+l)}{\mu_N + C_1(1+l)} \frac{1}{\mu_{N+1} - \mu_N - C_1(1+l)} + \frac{1}{\mu_{N+1}})$$

$$\Rightarrow \parallel \mathcal{J}\psi_1(p(0)) - \mathcal{J}\psi_2(p(0)) \parallel_0 \leq L' \parallel \psi_1 - \psi_2 \parallel$$

for all $p(0) \in P_N D(A)$, where

$$L' = \frac{C_1^2(1+l)}{(\mu_N + C_1(1+l))(\mu_{N+1} - \mu_N - C_1(1+l))} + \frac{C_1}{\mu_{N+1}}.$$

Now that we have completed the necessary calculations, we may proceed to prove that $\mathcal{J}$ maps $\mathcal{R}_{c,\ l}$ into itself and is a strict contraction on $\mathcal{R}_{c,\ l}$. This is equivalent to ensuring conditions on μ_{N+1} and μ_N such that

$$L < l \text{ and } L' \leq r < 1$$

for some r, let us say $r = \frac{3}{4}$. Recalling the spectral gap condition (92), it follows that $\mu_{N+1} - \mu_N > 4C_1 > 2(1+l)C_1$. Since we have set $l = 1$, we obtain $4 = (1+l)(1+\frac{1}{l})$, with the quantity $(1+l)(1+\frac{1}{l})$ arising from the calculations below. It is here that we see how the choice of l affects the spectral gap condition. Choosing l smaller, we would have to replace the $4C_1$ in the second part of the spectral gap condition with $(1+l)(1+\frac{1}{l})$, which gets larger as l decreases towards 0. For $\widetilde{A} = -\Delta$, our spectral gap condition may always be satisfied for any nonzero l for sufficiently large N since $\mu_{N+1} - \mu_N = 2N+1$ for any b and $g(u) \in \mathcal{G}$. Applying the spectral gap condition and recalling that $l = 1 > 0$ and $C_1 > 0$, we have

$$\mu_{N+1} - \mu_N > 4C_1 = (1+l)(1+\frac{1}{l})C_1 = (1+l)C_1 + \frac{1+l}{l}C_1$$
$$\Rightarrow \mu_{N+1} - \mu_N - C_1(1+l) > \frac{1+l}{l}C_1 \Rightarrow l > \frac{C_1(1+l)}{\mu_{N+1} - \mu_N - C_1(1+l)} = L.$$

Furthermore, by applying both portions of the spectral gap condition we obtain

$$\begin{aligned} L' &= \frac{C_1^2(1+l)}{(\mu_N + C_1(1+l))(\mu_{N+1} - \mu_N - C_1(1+l))} + \frac{C_1}{\mu_{N+1}} \\ &= \frac{2C_1^2}{(\mu_N + 2C_1)(\mu_{N+1} - \mu_N - 2C_1)} + \frac{C_1}{\mu_{N+1}} < \frac{2C_1^2}{(\mu_N + 2C_1)(\mu_{N+1} - \mu_N - 2C_1)} \\ &+ \frac{C_1}{\mu_N + 4C_1} < \frac{2C_1^2}{(\mu_N + 2C_1)(2C_1)} + \frac{C_1}{\mu_N + 4C_1} = \frac{C_1}{\mu_N + 2C_1} + \frac{C_1}{\mu_N + 4C_1} \\ &< \frac{C_1}{2C_1} + \frac{C_1}{4C_1} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1. \end{aligned}$$

Thus we have shown that $\mathcal{J}$ is a contraction mapping from $\mathcal{R}_{c,\ l}$ into itself and is strictly contracting. Therefore, by the Banach Fixed Point Theorem it follows that $\mathcal{J}$ must have a fixed point. Let us denote this fixed point by $\mathbf{\Psi}$. It follows that

$$\mathbf{\Psi}(p_0) = \int_{-\infty}^{0} e^{AQ_N s} Q_N g(p(s) + \mathbf{\Psi}(p(s)))ds \tag{118}$$

where $p(s) = p(s; p_0)$, and we may define

$$\mathbf{\Psi}(p(t)) = \int_{-\infty}^{t} e^{-AQ_N(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s)))ds. \tag{119}$$

By definition, $\mathbf{\Psi}(p(t)) \in Q_N H$ for any $p(t) \in P_N H$. Then

$$\begin{aligned} &e^{-At}(p_0) + \int_0^t e^{-A(t-s)} P_N g(p(s) + \mathbf{\Psi}(p(s)))ds + \mathcal{J}\mathbf{\Psi}(p(t)) \\ &= e^{-At}(p_0) + \int_0^t e^{-A(t-s)} P_N g(p(s) + \mathbf{\Psi}(p(s)))ds \\ &+ \int_{-\infty}^{t} e^{-AQ_N(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s)))ds = p(t) + \mathbf{\Psi}(p(t)) \end{aligned} \tag{120}$$

where $p(t) \in P_N H$, $\mathbf{\Psi}(p(t)) \in Q_N H$, $\int_0^t e^{-A(t-s)} P_N g(p(s) + \mathbf{\Psi}(p(s)))ds \in P_N H$, $\int_{-\infty}^t e^{-AQ_N(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s)))ds \in Q_N H$. But any solution $u(t) \in D(A)$ which exists and is bounded on $(-\infty, 0]$ may be defined as in (120) with $u(t) = p(t)+\mathbf{\Psi}(p(t))$. Furthermore, by differentiating

$$\mathbf{\Psi}(p(t)) = \int_{-\infty}^{t} e^{-AQ_N(t-s)} Q_N g(p(s, p_0) + \mathbf{\Psi}(p(s, p_0)))ds$$

with respect to t, it is clear that $q(t) = Q_N u(t) = \mathbf{\Psi}(p(t))$ and $q(t) = \mathbf{\Psi}(p(t))$ solves

$$q_t = q_{xx} + bq + Q_N g(p + q), \quad q(0) = \mathbf{\Psi}(p_0) \ for \ p_0 \in P_N D(A) \tag{121}$$

whenever $p(t)$ solves

$$p_t = p_{xx} + bp + P_N g(p + \mathbf{\Psi}(p)), \quad p(0) = p_0. \tag{122}$$

We may now define a manifold $\mathcal{M} = Graph[\mathbf{\Psi}]$ with values in H and norm $\| \cdot \|_0$, where the domain of the graph is $P_N H = P_N D(A)$.

LEMMA 6.4. *$\mathcal{M}$ is an inertial manifold for Equation (75), i.e. it is a positively invariant Lipschitz manifold with the following properties:*

- *$\mathcal{M}$ is a finite-dimensional Lipschitz manifold with values in H.*
- *$\mathcal{M}$ is positively invariant, i.e. $\mathcal{S}(t)\mathcal{M} \subseteq \mathcal{M}$ for all $t \geq 0$.*
- *$\mathcal{M}$ attracts exponentially all solutions of (75).*

PROOF. Owing to the previously stated fact that $(p(t), \mathbf{\Psi}(p(t)))$ is a solution of (83) whenever $u(t) = p(t) + \mathbf{\Psi}(p(t))$ is a solution of (75), it follows that $\mathcal{S}(t)\mathcal{M} \subseteq \mathcal{M}$ for all $t \geq 0$, i.e. $\mathcal{M}$ is positively invariant. Since $\mathcal{M}$ is the graph over an $(N+1)$-dimensional subspace of the Hilbert space H and equivalently a graph over an $(N+1)$-dimensional subspace of the Hilbert Space $D(A)$, it follows that $\mathcal{M}$ is an $(N+1)$-dimensional manifold.

In order to prove that $\mathcal{M}$ is a Lipschitz manifold with values in H, we proceed with calculations similar to those enacted in the proof of Lemma 6.3. Let us choose two arbitrary functions $p_1(t)$ and $p_2(t)$ in $P_N H$. We recall that the Lipschitz coefficient of g is denoted by C_1 and that since $\mathbf{\Psi} \in \mathcal{R}_{c,\, l}$, the Lipschitz coefficient of $\mathbf{\Psi}$ in H is l.

$$\| \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \|_0 = \| \mathcal{J}(\mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t))) \|_0$$
$$= \left\| \int_{-\infty}^{t} e^{-AQ_N(t-s)} Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) - g(p_2(s) + \mathbf{\Psi}(p_2(s)))]ds \right\|_0$$
$$\leq \int_{-\infty}^{t} \| e^{-AQ_N(t-s)} Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds$$
$$\leq \int_{-\infty}^{t} \| e^{-AQ_N(t-s)} \|_{\mathscr{L}(H)} \| Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds$$
$$\leq \int_{-\infty}^{t} e^{-\mu_{N+1}(t-s)} \| Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds$$
$$\leq \int_{-\infty}^{t} e^{-\mu_{N+1}(t-s)} \| g(p_1(s) + \mathbf{\Psi}(p_1(s))) - g(p_2(s) + \mathbf{\Psi}(p_2(s))) \|_0 \, ds$$
$$\leq \int_{-\infty}^{t} e^{-\mu_{N+1}(t-s)} C_1 \| p_1(s) + \mathbf{\Psi}(p_1(s)) - p_2(s) - \mathbf{\Psi}(p_2(s)) \|_0 \, ds$$
$$\leq \int_{-\infty}^{t} e^{-\mu_{N+1}(t-s)} C_1(1+l) \| p_1(s) - p_2(s) \|_0 \, ds.$$

Recalling our earlier calculations towards bounding $\| p_1(t) - p_2(t) \|_0$, we use the property that

$$\| p_1(s) - p_2(s) \|_0 \leq \| p_1(t) - p_2(t) \|_0 \, e^{(\mu_N + C_1(1+l))(t-s)}$$

for all $s \leq t$. Thus

$$\begin{aligned} &\| \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \|_0 \\ &\leq C_1(1+l) \int_{-\infty}^{t} e^{-\mu_{N+1}(t-s)} \| p_1(t) - p_2(t) \|_0 \, e^{(\mu_N + C_1(1+l))(t-s)} ds \\ &= C_1(1+l) \| p_1(t) - p_2(t) \|_0 \int_{-\infty}^{t} e^{-(\mu_{N+1} - \mu_N - C_1(1+l))(t-s)} ds. \end{aligned}$$

By the spectral gap condition, $\mu_{N+1} - \mu_N - C_1(1+l) > 2C_1 > 0$, therefore

$$\| \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \|_0 \leq \frac{C_1(1+l)}{\mu_{N+1} - \mu_N - C_1(1+l)} \| p_1(t) - p_2(t) \|_0 .$$

Thus we have shown that $\mathbf{\Psi}$ is Lipschitz continuous with values in H, with Lipschitz coefficient $C_2 = \frac{C_1(1+l)}{\mu_{N+1} - \mu_N - C_1(1+l)}$. Furthermore, $\mathbf{\Psi}$ is bounded in $Q_N H$ as well. Lemma 6.2 applies to the Lipschitz map $\mathbf{\Psi}$, thus

$$\| \mathbf{\Psi}(p) \|_0 \leq \Gamma \sqrt{\pi} \mu_{N+1}^{-1}$$

for any $p \in P_N H$. Thus $\mathcal{M}$ is bounded in the infinite-dimensional subspace of H, but is unbounded in the finite-dimensional subspace.

To show that $\mathcal{M}$ is exponentially attracting, choose an initial condition $u_0 \in H$ and its image after time t given by $\mathcal{S}(t)u_0 = u(t) = p(t) + q(t)$. Now consider the point $\overline{u} \in \mathcal{M}$ given by $\overline{u} = p + \mathbf{\Psi}(p)$. It follows that

$$||Q_N u - Q_N \overline{u}||_0 \geq 0 = ||P_N u - P_N \overline{u}||_0. \tag{123}$$

Applying the decay property (91) implies that

$$||u(t) - \overline{u}(t)||_0 = ||q(t) - \overline{q}(t)||_0 \leq ||Q_N u_0 - \mathbf{\Psi}(P_N u_0)||_0 e^{-kt} \tag{124}$$

and

$$dist(\mathcal{S}(t)u_0, \mathcal{M}) \leq ||u(t) - \overline{u}(t)||_0 \leq ||Q_N u_0 - \mathbf{\Psi}(P_N u_0)||_0 e^{-kt}. \tag{125}$$

Therefore any solution $u(t)$ to (75) is tracked exponentially by a solution on $\mathcal{M}$. $\square$

As the properties put forth in Lemma 6.4 comprise the definition of an inertial manifold, it therefore follows that, choosing N for projections P_N and $Q_N = I - P_N$ such that the spectral gap (92) holds, and therefore the strong squeezing property holds, we are guaranteed the existence of an inertial manifold for our slowly non-dissipative evolutionary equation (75). In particular, we are guaranteed an inertial manifold for Equation (75) with $\widetilde{A} = -\Delta$.

Furthermore, because $\mathcal{M}$ is positively invariant and attracts all solutions exponentially, this implies that any bounded set $B \in H$ limits in forward time to a subset of $\mathcal{M}$ and any invariant set must be contained in $\mathcal{M}$. Therefore, the minimal set which attracts every bounded set in $D(A)$ must be contained in $\mathcal{M}$. But grow-up solutions in the unstable manifold of a bounded equilibrium are on invariant sets, and therefore must be contained in $\mathcal{M}$. Therefore it follows that not only can we not prove a bound on $\mathcal{M}$ as is done for dissipative evolutionary equations, but for slowly non-dissipative equations of the form (75), $\mathcal{M}$ must be unbounded.

We may now prove a higher degree of smoothness for the inertial manifold $\mathcal{M}$ for our original equation

$$\begin{gathered} u_t = u_{xx} + bu + g(u), \quad x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0 \\ b > 0, \quad g(u) \in C^2,\ g(u) \textit{ uniformly bounded}, \\ g(u) \textit{ globally Lipschitz with values in } L^2. \end{gathered} \tag{126}$$

Lemma 6.5. *The inertial manifold $\mathcal{M} = Graph[\mathbf{\Psi}]$ for Equation (126) is a finite-dimensional Lipschitz manifold with values in $D(A^\alpha)$, with α-norm, for $\frac{3}{4} < \alpha < 1$. Moreover, it is a Lipschitz manifold with values in C^1, with the C^1-norm.*

Proof. Setting $\widetilde{A} = -\frac{d^2}{dx^2}$, Lemma 6.4 ensures the existence of a finite-dimensional Lipschitz manifold for Equation (126), where $\mathcal{M}$ is Lipschitz with values in $H = L^2$. In order to prove that $\mathcal{M}$ is a Lipschitz manifold in C^1, we must prove that it is the graph of a function $\mathbf{\Psi}$ that is Lipschitz with values in C^1. This can be done by proving that $\mathbf{\Psi}$ is in fact not only Lipschitz with values in H, but with values in $D(A^\alpha)$ for $\frac{3}{4} < \alpha < 1$. Sobolev embedding and the finite-dimensionality of $P_N H$ then prove that the manifold $\mathbf{\Psi} \in C^1_{Lip}$. Let us choose two arbitrary functions $p_1(t)$ and $p_2(t)$ in $P_N D(A)$. We recall that the Lipschitz coefficient of g in H is denoted by C_1 and that since $\mathbf{\Psi} \in \mathcal{R}_{c,\ l}$, l is an upper bound for the Lipschitz coefficient of $\mathbf{\Psi}$ in H.

We note that the eigenvalues of the operator A_1 as defined at the beginning of this chapter are $\mu_{j,1} = \mu_j + b + 1 = \widetilde{\mu}_j + 1$, and thus $\mu_{j,1} \geq 1$ for all j. Furthermore, $\mu_{N+1} - \mu_N = \mu_{N+1,1} - \mu_{N,1}$ and $D(A_1) = D(A)$. Then

$$
\begin{aligned}
\| \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \|_\alpha &= \| A_1^\alpha(\mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t))) \|_0 \\
&= \| (A_1 Q_N)^\alpha (\mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t))) \|_0 \\
&= \| \int_{-\infty}^{t} (A_1 Q_N)^\alpha e^{-AQ_N(t-s)} Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] ds \|_0 \\
&= \| \int_{-\infty}^{t} (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)Q_N(t-s)} Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) \\
&\qquad - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] ds \|_0 \\
&\leq \int_{-\infty}^{t} \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)Q_N(t-s)} Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) \\
&\qquad - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds \\
&= \int_{-\infty}^{t} \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)(t-s)} Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) \\
&\qquad - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds \\
&\leq \int_{-\infty}^{t} \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)(t-s)} \|_{\mathscr{L}(H)} \| Q_N[g(p_1(s) + \mathbf{\Psi}(p_1(s))) \\
&\qquad - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds \\
&= \int_{-\infty}^{t} e^{(b+1)(t-s)} \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} \|_{\mathscr{L}(H)} \| [g(p_1(s) + \mathbf{\Psi}(p_1(s))) \\
&\qquad - g(p_2(s) + \mathbf{\Psi}(p_2(s)))] \|_0 \, ds
\end{aligned}
$$

$$\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-(\mu_{N+1,1}-b-1)(t-s)} \parallel Q_N[g(p_1(s)+\mathbf{\Psi}(p_1(g(p_2(s)+\mathbf{\Psi}(p_2(s)))] \ \parallel_0 ds$$

$$= \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-(\mu_{N+1})(t-s)} \parallel Q_N[g(p_1(s)+\mathbf{\Psi}(p_1(s))) - g(p_2(s)+\mathbf{\Psi}(p_2(s)))] \ \parallel_0 ds$$

$$\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} \parallel g(p_1(s)+\mathbf{\Psi}(p_1(s))) - g(p_2(s)+\mathbf{\Psi}(p_2(s))) \ \parallel_0 ds$$

$$\leq C_1 \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} (\parallel p_1(s) - p_2(s) \parallel_0 + \parallel \mathbf{\Psi}(p_1(s)) - \mathbf{\Psi}(p_2(s)) \ \parallel_0) ds$$

$$\leq C_1 \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} (1+l) \parallel p_1(s) - p_2(s) \parallel_0 ds.$$

Recalling that

$$\parallel p_1(s) - p_2(s) \parallel_0 \leq \parallel p_1(t) - p_2(t) \parallel_0 e^{(\mu_N + C_1(1+l))(t-s)}$$

for all $s \leq t$, it follows that

$$\parallel \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \parallel_{D(A^\alpha)} = \parallel \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \parallel_\alpha$$

$$\leq C_1(1+l) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} \parallel p_1(t) - p_2(t) \parallel_0 e^{(\mu_N + C_1(1+l))(t-s)} ds$$

$$= C_1(1+l) \parallel p_1(t) - p_2(t) \parallel_0 \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-(\mu_{N+1} - \mu_N - C_1(1+l))(t-s)} ds.$$

By the spectral gap condition, $\mu_{N+1} - \mu_N - C_1(1+l) > 2C_1 > 0$, therefore

$$\parallel \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \parallel_\alpha$$

$$\leq C_1(1+l) \cdot (1-\alpha)^{-1} e^{-\alpha} \frac{1}{(\mu_{N+1} - \mu_N - C_1(1+l))^\alpha} \parallel p_1(t) - p_2(t) \parallel_0 .$$

Finally, we recall that on $\mathbb{R}^{N+1}$ all norms are equivalent. Thus,

$$\parallel p_1(t) - p_2(t) \parallel_\alpha = \parallel A_1^\alpha p_1(t) - A_1^\alpha p_2(t) \parallel_0$$

$$\geq \mu_{0,1} \parallel p_1(t) - p_2(t) \parallel_0 \geq \parallel p_1(t) - p_2(t) \parallel_0 .$$

Therefore

$$\parallel \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \parallel_\alpha \leq$$

$$C_1(1+l) \cdot (1-\alpha)^{-1} e^{-\alpha} \frac{1}{(\mu_{N+1} - \mu_N - C_1(1+l))^\alpha} \parallel p_1(t) - p_2(t) \parallel_\alpha,$$

and we have shown that $\mathbf{\Psi}$ is Lipschitz continuous from $D(A^\alpha)$ to $D(A^\alpha)$ with Lipschitz coefficient $C_2 = C_1(1+l) \cdot (1-\alpha)^{-1} e^{-\alpha} \frac{1}{(\mu_{N+1} - \mu_N - C_1(1+l))^\alpha}$. Thus, it follows that

$\mathcal{M} = Graph[\mathbf{\Psi}]$ is a finite-dimensional Lipschitz manifold with values in $D(A^\alpha)$. Since $D(A^\alpha)$ compactly embeds into C^1 via Sobolev embedding, and $P_N H = P_N D(A)$ is a finite-dimensional subspace isomorphic to $\mathbb{R}^{N+1}$, it follows that $\mathbf{\Psi} \in C^1_{Lip}(P_N D(A))$. Therefore, $\mathcal{M} = Graph[\mathbf{\Psi}]$ is a finite-dimensional Lipschitz manifold with values in C^1. Further, $\mathbf{\Psi}(p)$ is bounded for all $p \in P_N D(A)$ in the α-norm and C^1-norm as well. Recalling that $D(A^\alpha)$ embeds into C^1, it follows that

$$\| \mathbf{\Psi}(p(t)) \|_{C^1} \leq \| \mathbf{\Psi}(p(t)) \|_\alpha = \| A_1^\alpha \mathbf{\Psi}(p(t)) \|_0 . \tag{127}$$

Thus,

$$\begin{aligned}
\| \mathbf{\Psi}(p(t)) \|_{C^1} &\leq \| A_1^\alpha \mathbf{\Psi}(p(t)) \|_0 = \| (A_1 Q_N)^\alpha \mathbf{\Psi}(p(t)) \|_0 \\
&= \| \int_{-\infty}^t (A_1 Q_N)^\alpha e^{-AQ_N(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s))) ds \|_0 \\
&= \| \int_{-\infty}^t (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)Q_N(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s))) ds \|_0 \\
&\leq \int_{-\infty}^t \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)Q_N(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&= \int_{-\infty}^t \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)(t-s)} Q_N g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^t \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} e^{(b+1)(t-s)} \|_{\mathscr{L}(H)} \| Q_N g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&= \int_{-\infty}^t e^{(b+1)(t-s)} \| (A_1 Q_N)^\alpha e^{-A_1 Q_N(t-s)} \|_{\mathscr{L}(H)} \| Q_N g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^t (t-s)^{-\alpha} e^{-(\mu_{N+1,1} - b - 1)(t-s)} \| Q_N g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&= \int_{-\infty}^t (t-s)^{-\alpha} e^{-(\mu_{N+1})(t-s)} \| Q_N g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^t (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} \| g(p(s) + \mathbf{\Psi}(p(s))) \|_0 \, ds \\
&\leq \int_{-\infty}^t (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} \sqrt{\pi} \Gamma ds \\
&\leq \sqrt{\pi} \Gamma \int_{-\infty}^t (t-s)^{-\alpha} e^{-\mu_{N+1}(t-s)} ds \\
&\leq \sqrt{\pi} \Gamma (1-\alpha)^{-1} e^{-\alpha} \frac{1}{\mu_{N+1}^\alpha} = C_3.
\end{aligned}$$

Thus, $\| \mathbf{\Psi}(p(t)) \|_{C_1} \leq \| \mathbf{\Psi}(p(t)) \|_{\alpha} \leq C_3$ and therefore $\mathbf{\Psi}(p)$ is bounded in $D(A^{\alpha})$ and C^1 for all $p \in P_N D(A)$. Furthermore, $q(t) = \mathbf{\Psi}(p(t))$ is uniformly bounded in C^1, regardless of the choice of $p(t)$. $\square$

LEMMA 6.6. *If the nonlinearity $g(u) \in \mathcal{G}$ in Equation (126) is Lipschitz continuous from $D(A^{\beta})$ into $D(A^{\varepsilon})$ with $0 < \varepsilon < \beta \leq 1$ and $\beta - \varepsilon < \frac{1}{2}$, and bounded in $D(A^{\beta})$, then the inertial manifold $\mathcal{M}$ is Lipschitz in $D(A) = H^2 \cap \{Neumann\ Boundary\ Conditions\}$.*

We note that such conditions are easily achieved if we restrict the scalar nonlinearity $g(u) : \mathbb{R} \to \mathbb{R}$ to have a bounded first derivative. Then, as a result of the Krasnoselskii theorem [**38**], the evaluation operator g is globally Lipschitz from $D(A)$ into $D(A)$. Let $0 < \varepsilon \leq 1/2$. It follows that, for all $u, v \in D(A) = H^2 \cap \{Neumann\ Boundary\ Conditions\}$, we have $\| g(u) - g(v) \|_1 \geq \| g(u) - g(v) \|_{1/2+\varepsilon}$ and

$$\| g(u) - g(v) \|_1 \leq \widetilde{C} \| u - v \|_1$$

combined imply that

$$\| g(u) - g(v) \|_{1/2+\varepsilon} \leq \widetilde{C} \| u - v \|_1 .$$

In other words, the evaluation operator g is Lipschitz continuous from $D(A)$ into $D(A^{1/2-\varepsilon})$.

PROOF. There are two alternate methods to prove this lemma. The first method is to use bootstrapping [**24**]. Due to the Lipschitz behavior of the nonlinearity g as put forth in this lemma, and the fact that A is a sectorial operator, it follows that a certain degree of smoothing occurs. Thus, for any initial condition $u_0 \in \mathcal{M} \subset X^{\alpha}$, for example any initial condition on a heteroclinic orbit, the corresponding solution $u(t, \cdot) \subset \mathcal{M} \subset D(A)$ for $t > t_0$ [**18**]. Thus, if $\mathcal{M}$ is Lipschitz with values in C^1 at time t_0, it is Lipschitz with values in $D(A)$ at time $t > t_0$. Since $\mathcal{M}$ is positively invariant, we may simply study $\mathcal{S}(1)\mathcal{M}$ to ensure higher regularity.

The second method is to return to the calculations presented throughout this chapter. Let us set $\alpha = 1-\varepsilon$. The evaluation operator g being Lipschitz from $D(A^\beta)$ to $D(A^\varepsilon)$ is equivalent to

$$\| g(u(t,\cdot)) - g(v(t,\cdot)) \|_\varepsilon \leq \widetilde{L} \| u(t,\cdot) - v(t,\cdot) \|_\beta$$

for some constant $\widetilde{L}$ dependent on g. Then we are able to repeat the calculations presented earlier in the chapter, performing those in Sections 6.1 and 6.2 in the β-norm and β-inner product. This transforms the spectral gap condition (92) into

$$\begin{gathered} \mu_{N+1}^{1-\beta+\varepsilon} > 2\widetilde{L}, \quad \mu_N > b \\ \mu_{n+1} - \mu_n \geq 2\widetilde{L}(\mu_{n+1}^{\beta-\varepsilon} + \mu_n^{\beta-\varepsilon}), \end{gathered} \tag{128}$$

which can be fulfilled for $\widetilde{A} = -\frac{d^2}{dx^2}$ so long as $\beta - \varepsilon < \frac{1}{2}$. Finally, in the proof of Lemma 6.4 we may evaluate $\| \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \|_1$ by separating the A_1 operator provided by the norm into an A_1^α term in front, as before, and an A_1^ε term multiplying the nonlinearity. This allows us to repeat the calculations and achieve

$$\| \mathbf{\Psi}(p_1(t)) - \mathbf{\Psi}(p_2(t)) \|_1 \leq \widetilde{C}_2 \| p_1(t) - p_2(t) \|_1 .$$

Furthermore, allowing the β-norm of $g(u)$ to be bounded ensures that $\| \mathbf{\Psi}(p(t)) \|_1 \leq \widetilde{C}_3$. $\square$

CHAPTER 7

Asymptotics of Grow-Up Solutions

We now consider the behavior of grow-up solutions in the inertial manifold $\mathcal{M}$. For $\mathcal{M} = Graph[\mathbf{\Psi}]$, with $\mathbf{\Psi}$ the fixed point of the Lyapunov-Perron Operator $\mathcal{J}$, the dynamics on $\mathcal{M}$ are completely determined by the differential equation

$$\begin{aligned}\frac{dp}{dt} + Ap = P_N g(p + \mathbf{\Psi}(p)), \quad p = \sum_{i=0}^{N} \widehat{p}_i(t)\Phi_i(x) &= \sum_{i=0}^{N} \langle p(t,\cdot), \Phi_i(\cdot)\rangle_0 \, \Phi_i(x) \\ &= \sum_{i=0}^{N} \langle u(t,\cdot), \Phi_i(\cdot)\rangle_0 \, \Phi_i(x),\end{aligned} \tag{129}$$

which is an $(N+1)$-dimensional ordinary differential equation. For such an ordinary differential equation, we may provide the solution as follows for a clearer view on the asymptotic behaviors. Let us denote the unstable portion of p via $p^u(t)$ and the stable portion via $p^s(t)$. We decompose the projection operator into two parts, i.e. $P_N = P_N^u + P_N^s$, where P_N^u projects onto the modes $0, \ldots, \lfloor\sqrt{b}\rfloor$ and P_N^s projects onto the modes $\lfloor\sqrt{b}\rfloor + 1, \ldots, N$ if $\lfloor\sqrt{b}\rfloor < N$, otherwise $P_N = P_N^u$. In other words, $p^u(t) = \sum_{i=0}^{\lfloor\sqrt{b}\rfloor} \widehat{p}_i(t)\Phi_i(x)$ and $p^s(t) = \sum_{i=\lfloor\sqrt{b}\rfloor+1}^{N} \widehat{p}_i(t)\Phi_i(x)$. We remind the reader that in order to fulfill the spectral gap condition, N may be chosen such that some stable modes are included in p. Then

$$\begin{aligned} p^u(t,\cdot) = p^u(t) &= e^{-AP_N^u t} p_0^{h,u} + \int_{\infty}^{t} e^{-AP_N^u(t-s)} P_N^u g(u(s)) ds \\ p^s(t,\cdot) = p^s(t) &= e^{-AP_N^s t} p_0^s + \int_0^t e^{-AP_N^s(t-s)} P_N^s g(u(s)) ds. \end{aligned} \tag{130}$$

We note that $p^s(t)$ remains bounded while only the second term in $p^u(t)$ remains bounded. It is the $e^{-A^u t} p_0^{h,u}$ term in $p^u(t)$ which determines the asymptotic behavior of $p(t,\cdot)$ and $u(t,\cdot)$ when $u(t,\cdot)$ is a grow-up solution.

Henceforth, we will return to our chosen operator $A = -\frac{d^2}{dx^2} - bI$. Because these grow-up solutions are contained in $\mathcal{M}$, it follows that their behavior is determined by (129). It was our perceived inability to evaluate the limiting behavior of grow-up solutions in higher norms that left open the possibility of the zero number dropping at $t = \infty$. Studying the behavior on an inertial manifold in C^1 provides a great deal of clarification in this respect.

Since $\mathcal{M}$ is a Lipschitz manifold in C^1, it follows that any solution $u(t, \cdot)$ contained in $\mathcal{M}$ which limits to some object at infinity must limit to that object in the C^1-norm. It is possible for a function $\frac{u(t,\cdot)}{||u(t,\cdot)||_{L^2}}$ to be L^2-close to any $\Phi_i(\cdot)$ with the appropriate sign while having zero number greater than i, because the L^2-norm only measures the difference between the two solutions, and not the difference between their first or second derivatives. Recall that both $\Phi_i(x)$ and $\frac{u(t,x)}{\|u(t,x)\|_{L^2}}$ have only simple zeros for all but finitely many times $t_k \in \mathbb{R}^+$. In order for the zero number to be greater than i while the two functions are L^2-close, it follows that at some point where $\Phi_i(x)$ crosses the x-axis with either strictly positive or strictly negative slope, the function $\frac{u(t,x)}{||u(t,x)||_{L^2}}$ must cross the x-axis multiple times, with slope that is positive, negative, and zero. Thus, if any given $\Phi_i(\cdot)$ does not have the same zero number as $\frac{u(t,\cdot)}{||u(t,\cdot)||_{L^2}}$, then $\lim\limits_{t\to\infty} ||\frac{u(t,\cdot)}{||u(t,\cdot)||_{L^2}} - \frac{\Phi_i(\cdot)}{\|\Phi_i(\cdot)\|_{L^2}}||_{C^1} \neq 0$.

Lemma 7.1. *Let v be a hyperbolic stationary solution to*

$$\begin{aligned} & u_t = u_{xx} + bu + g(u), \quad x \in [0, \pi] \\ (131) \qquad & u_x(t, 0) = u_x(t, \pi) = 0 \\ & b > 0, \quad g(u) \in \mathcal{G} \end{aligned}$$

such that $i(v) = n + 1$ and $l(v) \in \{n + 1, n\}$. Fix $k \leq n$, $\sigma \in \{1, -1\}$ such that there does not exist any stationary solution $w \notin EJ_v$ with $sign(w(0) - v(0)) = \sigma$ and $l(w) = k$, and if $k = l(v) = n$, there additionally does not exist any stationary solution $w \in EJ_v$ such that $sign(w(0) - v(0)) = \sigma$ and $l(w) \leq l(v)$. Let $u^\sigma_k(t, \cdot)$ be the solution to (131) guaranteed by Lemma 5.4, i.e. $z(u^\sigma_k(t, \cdot) - v(\cdot)) = k$ for all $0 \leq t < \infty$, $sign(u^\sigma_k(t, 0) - v(0)) = \sigma$, $\lim\limits_{t\to\infty} \| u^\sigma_k(t, \cdot) \|_{L^2} = \infty$, and $\lim\limits_{t\to-\infty} u^\sigma_k(t, \cdot) = v$.

Then

$$\lim_{t\to\infty} \| \frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} - \frac{\Phi_k^\sigma(\cdot)}{\| \Phi_k^\sigma(\cdot) \|_{L^2}} \|_{C^1} = 0, \tag{132}$$

or in other words, the zero numbers of $u_k^\sigma(t,\cdot)$ *and* $u_k^\sigma(t,\cdot) - v(\cdot)$ *do not drop at infinity.*

PROOF. Since $g(u) \in \mathcal{G}$, it follows that there exists a completed inertial manifold for the dynamical system defined by Equation (131). As described in Chapter 6, all solutions to Equation (131) including u_k^σ can be decomposed into a finite portion living in $P_N L^2$ and an infinite portion which may be described by the mapping $\mathbf{\Psi} : P_N L^2 \to Q_N L^2$ over the finite portion which is Lipschitz with values in C^1. As $P_N L^2$ is a finite-dimensional orthogonal subspace of L^2, it follows that $P_N L^2 = P_N D(A) = P_N H^2 \cap \{\textit{Neumann Boundary Conditions}\}$. Furthermore, due to $\mathbf{\Psi}$ being derived from the fixed point of a Lyapunov-Perron operator, it follows that if $u(t,\cdot) \in H^2$ is a solution to Equation (131), then $\mathbf{\Psi}(u(t,\cdot)) \in Q_N D(A) = Q_N H^2 \cap \{\textit{Neumann Boundary Conditions}\}$. Thus, we may decompose $u_k^\sigma(t,\cdot)$ as follows:

(133)

$$u_k^\sigma(t,\cdot) = p(t,\cdot) + q(t,\cdot)$$

$$p(t,\cdot) = \sum_{i=0}^{N} \langle u_k^\sigma(t,\cdot), \Phi_i(\cdot)\rangle_0 \, \Phi_i(\cdot) \in P_N H^2 \cap \{\textit{Neumann Boundary Conditions}\}$$

$$q(t,\cdot) = \mathbf{\Psi}(p(t,\cdot)) \in Q_N H^2 \cap \{\textit{Neumann Boundary Conditions}\}.$$

Thus

$$\frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} = \frac{p(t,\cdot) + q(t,\cdot)}{\| p(t,\cdot) + q(t,\cdot) \|_{L^2}} \tag{134}$$

and

$$\frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{C^1}} = \frac{p(t,\cdot) + q(t,\cdot)}{\| p(t,\cdot) + q(t,\cdot) \|_{C^1}} \tag{135}$$

Recall that all invariant sets, including grow-up solutions in the unstable manifold of a bounded stationary solution, live on the finite-dimensional inertial manifold $\mathcal{M}$. Additionally, recall that $q(t,\cdot)$ is uniformly bounded in the C^1-norm and thus all weaker norms as well.

Since $q(t,\cdot)$ is bounded while $p(t,\cdot)$ grows to infinity along with $u_k^\sigma(t,\cdot)$ in any applicable norm, we have that

$$\lim_{t\to\infty} \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} = \lim_{t\to\infty} \frac{p(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} = \lim_{t\to\infty} \frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}}$$

and furthermore

$$\lim_{t\to\infty} \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} = \lim_{t\to\infty} \frac{p(t,\cdot)}{\| u_k^\sigma(t,\cdot) - v(\cdot) \|_{L^2}} = \lim_{t\to\infty} \frac{u_k^\sigma(t,\cdot) - v(\cdot)}{\| u_k^\sigma(t,\cdot) - v(\cdot) \|_{L^2}}.$$

. We recall from Chapter 5 that for some $i \leq k$,

$$\lim_{t\to\infty} \left\| \frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{L^2} = 0 \tag{136}$$

and thus

$$\lim_{t\to\infty} \left\| \frac{u_k^\sigma(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{L^2} = 0. \tag{137}$$

Since $u_k^\sigma(t,\cdot)$ lives on the inertial manifold $\mathcal{M} \subset C^1_{Lip}$, which is finite dimensional, it follows that if $u_k^\sigma(t,\cdot)$ limits in $\mathcal{M}$ to some object in the L^2-norm, it must limit to the same object in the C^1-norm, due to norm equivalence in finite dimensions. It is the finite-dimensionality of our C^1 inertial manifold which is key to our proving C^1-convergence.

Thus

$$\begin{aligned} \lim_{t\to\infty} \left\| \frac{u_k^\sigma(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{C^1} &= \lim_{t\to\infty} \left\| \frac{p(t,\cdot)+q(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{C^1} \\ &\leq \lim_{t\to\infty} \left\| \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{C^1} + \lim_{t\to\infty} \left\| \frac{q(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} \right\|_{C^1} \\ &= \lim_{t\to\infty} \left\| \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{C^1} + \lim_{t\to\infty} \frac{\| \mathbf{\Psi}(p(t,\cdot)) \|_{C^1}}{\| p(t,\cdot) \|_{L^2}} \end{aligned}$$

Since $\mathbf{\Psi}(p(t,\cdot))$ is uniformly bounded in C^1 while $p(t,\cdot)$ grows to infinity in the L^2-norm, it follows that

$$\lim_{t\to\infty} \left\| \frac{u_k^\sigma(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{C^1} \leq \lim_{t\to\infty} \left\| \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \right\|_{C^1}.$$

Since $p(t,\cdot)$ and $\Phi_i^\sigma(\cdot)$ are both in the finite-dimensional subspace $P_N D(A)$ it follows that

$$\lim_{t\to\infty} \| \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \|_{C^1} \leq \lim_{t\to\infty} \widetilde{C} \| \frac{p(t,\cdot)}{\| p(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \|_{L^2} = 0$$

and therefore

$$\lim_{t\to\infty} \| \frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \|_{L^2} = 0 \Rightarrow \lim_{t\to\infty} \| \frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\| \Phi_i^\sigma(\cdot) \|_{L^2}} \|_{C^1} = 0.$$

But $\frac{u_k^\sigma(t,\cdot)}{\|u_k^\sigma(t,\cdot)\|_{L^2}}$ can only limit to a particular Φ_i^σ in the C^1-norm if it has the same zero number as that object for all $t \geq t^*$, t^* some finite time past which the zero numbers of the grow-up solution and the shifted grow-up solution coincide. Therefore, the object to which the grow-up solution $u_k^\sigma(t,\cdot)$ limits in C^1 must be the eigenfunction projected to infinity with lap number and zero number k. In other words, $i = k$ and thus

$$\lim_{t\to\infty} \| \frac{u_k^\sigma(t,\cdot)}{\| u_k^\sigma(t,\cdot) \|_{L^2}} - \frac{\Phi_k^\sigma(\cdot)}{\| \Phi_k^\sigma(\cdot) \|_{L^2}} \|_{C^1} = 0 \tag{138}$$

for all $u_k^\sigma(t,\cdot)$ as previously defined. Therefore, the zero number and shifted zero number of $u_k^\sigma(t,\cdot)$ does not drop at $t = \infty$. We note here that this also holds for a function $u(t,\cdot)$ on any heteroclinic which grows to infinity. The results from Chapter 5 which are used herein as well as the calculations in this proof may be performed for any grow-up solution in the unstable manifold of a stationary solution. We must simply determine the largest finite dropping time t_k such that $z(u(t,\cdot) - v(\cdot)) = k$ for all $t \in [t_k, \infty)$. It then follows that for such solutions the zero number does not drop at $t = \infty$, and thus the C^1 limit proven above holds in these cases as well. □

REMARK 7.2. *We choose the splitting of* $u(t,\cdot)$ *into* p *and* q *terms rather than* u^u *and* u^s *terms to aid in the later generalization of the argument to non-hyperbolic equilibria* v*. The choice of* N*, which determines the location of the splitting between* p *and* q*, may be made such that any center eigenspace is included in the finite-dimensional subspace* $P_N D(A)$*.*

REMARK 7.3. *If we have higher regularity of our nonlinearity $g(u)$ sufficient to induce smoothing, then $\mathcal{M}$ is a Lipschitz manifold with values in H^2, and thus the analysis obtained in Lemma 7.1 may be obtained for the H^2-norm. We note that convergence in norms stronger than C^1 is not actually needed to prevent the dropping of the zero number at $t = \infty$.*

1. Implications for Convergence of Transfinite Heteroclinics

We now have all the tools we need to determine the asymptotic behavior of grow-up solutions in the unstable manifolds of bounded equilibria, and thus enumerate the possible asymptotic behaviors of any arbitrary grow-up solution.

THEOREM 7.4. *Let $g(u) \in \mathcal{G}$, $b > 0$, and let v be a hyperbolic stationary solution of*

$$\begin{aligned} u_t &= u_{xx} + bu + g(u), \quad x \in [0, \pi] \\ u_x(t, 0) &= u_x(t, \pi) = 0 \end{aligned} \tag{139}$$

such that $i(v) = n + 1$ and $l(v) \in \{n + 1, n\}$. For every $\sigma \in \{1, -1\}$ and $0 \le k \le n$ such that v is not blocked from connecting to infinity by Lemma 5.3, v connects via heteroclinic orbit to an equilibrium at infinity $\mathbf{\Phi}_k^\sigma$ with $l(\mathbf{\Phi}_k^\sigma) = z(\mathbf{\Phi}_k^\sigma) = k$ and $sign(\mathbf{\Phi}_k^\sigma(0)) = \sigma$. In other words, there exists a trajectory $u_k^\sigma(t, \cdot) \in W^u(v)$ such that

$$\lim_{t \to -\infty} u_k^\sigma(t, \cdot) = v$$

$$sign(u_k^\sigma(t, 0) - v(0)) = \sigma$$

$$\lim_{t \to \infty} z(u_k^\sigma(t, \cdot) - v(\cdot)) = k$$

$$\lim_{t \to \infty} \| u_k^\sigma(t, \cdot) \|_{L^2} = \infty$$

$$\lim_{t \to \infty} \left\| \frac{u_k^\sigma(t, \cdot)}{\| u_k^\sigma(t, \cdot) \|_{L^2}} - \frac{\Phi_k^\sigma(\cdot)}{\| \Phi_k^\sigma(\cdot) \|_{L^2}} \right\|_{C^1} = 0$$

where $\Phi_k^\sigma(x) = \sigma \cos(kx)$ and $\frac{\mathbf{\Phi}_k^\sigma(\cdot)}{\|\mathbf{\Phi}_k^\sigma(\cdot)\|_{L^2}} = \Phi_k^\sigma(\cdot)$.

PROOF. Let us fix a k and σ such that the assumptions of the theorem hold. As discussed in Chapter 5, we may not limit to any Φ_l^σ where $l^2 > b$ as the corresponding modes of the grow-up solutions are bounded, while lower modes grow to infinity. Note that if $(l(v))^2 > b$, there will always exist blocking solutions $w_j^{\pm}$ for every j such that $\lfloor\sqrt{b}\rfloor < j \le l(v)$, since each n-branch in the bifurcation diagram limits to the line $b = n^2$.

Lemma 5.4 implies that for any such k and σ there exists an initial condition $u_0 = u_0(k, \sigma) \in W^u(v)$ and a corresponding solution $u(t, \cdot)$ to (139) such that

$$z(u(t,\cdot) - v(\cdot)) = k \ for \ all \ 0 \le t < \infty$$

$$sign(u(t, 0) - v(0)) = \sigma$$

$$\lim_{t\to-\infty} u(t,\cdot) = v$$

$$\lim_{t\to\infty} ||u(t,\cdot)||_{L^2} = \infty.$$

We denote such a solution for a fixed k and σ by $u(t,\cdot) = u_k^{\sigma}(t,\cdot)$. Then it follows that

$$\lim_{t\to-\infty} u_k^{\sigma}(t,\cdot) = v$$

and

$$sign(u_k^{\sigma}(t, 0) - v(0)) = \sigma.$$

Since the L^2-norm of any object is less than or equal to its H^2-norm or C^1-norm, Lemma 5.4 further implies that not only does $\lim_{t\to\infty} \parallel u_k^{\sigma}(t,\cdot) \parallel_{L^2} = \infty$, but

$$\lim_{t\to\infty} \parallel u_k^{\sigma}(t,\cdot) \parallel_{H^2} = \infty$$

and

$$\lim_{t\to\infty} \parallel u_k^{\sigma}(t,\cdot) \parallel_{C^1} = \infty.$$

We choose N and corresponding projections P_N and Q_N as previously defined, such that $N \ge n$ and N fulfills the spectral gap condition (92). Because we have chosen $g(u) \in \mathcal{G}$ and therefore C^2 and uniformly Lipschitz with values in L^2, this is quite simple. By Lemmas 6.4 and 6.5 there then exists a completed inertial manifold for (139) which contains all invariant sets. Since $W^u(v)$ is an invariant set by definition, it follows that $W^u(v) \subset \mathcal{M}$ and thus $u_k^\sigma(t,\cdot) \subset \mathcal{M}$. Therefore, the dynamics of $u_k^\sigma(t,\cdot)$ are determined by the $(N+1)$-dimensional ODE

$$
\begin{aligned}
&\frac{dp}{dt} = -Ap + P_N g(p + \boldsymbol{\Psi}(p)) \\
&p = [p_0, \ldots, p_N], \quad p_i(t,\cdot) = \widehat{p}_i(t)\Phi_i(\cdot) = \langle u_k^\sigma(t,\cdot), \Phi_i(\cdot)\rangle_0 \, \Phi_i(\cdot) \\
&\frac{d}{dx}p_i(t,0) = \frac{d}{dx}p_i(t,\pi) = 0,
\end{aligned}
\tag{140}
$$

where Φ_i is the ith eigenfunction of the operator $A = -\frac{d^2}{dx^2} - bI$ with Neumann boundary conditions. Since $\lim_{t\to\infty} \parallel u_k^\sigma(t,\cdot) \parallel_{L^2} = \infty$ and $||u_k^\sigma(t,\cdot)|| \leq ||\widetilde{u}_k^\sigma(t,\cdot)|| + ||v||$ with v a stationary solution, it follows that $\lim_{t\to\infty} \parallel \widetilde{u}_k^\sigma(t,\cdot) \parallel_{L^2} = \infty$.

Lemma 5.4 implies that $z(u_k^\sigma(t,\cdot) - v(\cdot)) = k$ for $0 \leq t < \infty$. Recalling Lemma 4.2, the set of times when $u_k^\sigma(t,\cdot) - v(\cdot)$ has simple zeros is open and dense in $\mathbb{R}^+$. Further, the times when $u_k^\sigma(t,\cdot) - v(\cdot)$ does not have simple zeros coincide with the dropping times. Since $u_k^\sigma(t,\cdot)$ is defined such that $u_k^\sigma(t,\cdot) - v(\cdot)$ has no finite positive dropping times, it follows that $u_k^\sigma(t,\cdot) - v(\cdot)$ has only simple zeros for all $t \in (0,\infty)$. Further, the $\boldsymbol{\Phi}_i^\sigma$ are defined to have only simple zeros as well.

We know that $\lim_{t\to\infty} \parallel \frac{u_k^\sigma(t,\cdot)}{\|u_k^\sigma(t,\cdot)\|_{L^2}} - \frac{\Phi_i^\sigma(\cdot)}{\|\Phi_i^\sigma(\cdot)\|_{L^2}} \parallel_{L^2} = 0$ for some $\boldsymbol{\Phi}_i^\sigma$ at infinity as time goes to infinity (see Section 5.2 for the detailed calculations). We know that for $\frac{u_k^\sigma(t,\cdot)}{||u_k^\sigma(t,\cdot)||_{L^2}}$ to limit to $\frac{\Phi_i^\sigma(\cdot)}{\|\Phi_i^\sigma(\cdot)\|_{L^2}}$ in C^1, the functions $u_k^\sigma(t,\cdot)$ and $\Phi_i^\sigma(\cdot)$ must have the same zero number. Since both $\frac{u_k^\sigma(t,\cdot)}{||u_k^\sigma(t,\cdot)||_{L^2}}$ and $\frac{\Phi_i^\sigma(\cdot)}{\|\Phi_i^\sigma(\cdot)\|_{L^2}}$ have only simple zeros for $t \in (0,\infty)$, they cannot be C^1-close and have differing zero numbers. By Lemma 5.4, the zero number of $\widetilde{u}_k^\sigma(t,\cdot)$ cannot drop at any positive finite time. Since the norm of $\widetilde{u}_k^\sigma(t,\cdot)$ grows to infinity along with $u_k^\sigma(t,\cdot)$, it follows that whenever $\widetilde{u}_k^\sigma(t,\cdot)$ is sufficiently large, $z(\widetilde{u}_k^\sigma(t,\cdot)) = z(u_k^\sigma(t,\cdot))$. Thus $z(u_k^\sigma(t,\cdot))$ must be constant in some neighborhood of $t = \infty$. It follows that neither the shifted nor unshifted zero number may drop at $t = \infty$ and $\lim_{t\to\infty} z(\widetilde{u}_k^\sigma(t,\cdot)) = \lim_{t\to\infty} z(u_k^\sigma(t,\cdot)) = k$, in other words $i = k$. Thus, v connects to $\boldsymbol{\Phi}_k^\sigma$ along some trajectory $u_k^\sigma(t,\cdot)$. $\square$

We recall that an n-branch is always to the left of an $n+1$-branch and a pitchfork branch with higher lap number will always be nested within a pitchfork branch with lower lap number. It then follows that the bounded stationary solution on either side of v in the bifurcation diagram which maximizes $|w(0) - v(0)|$, blocks a given v from connecting to infinity, and is not itself blocked will have the smallest lap number of any solution on that side of v to which v may connect. Recalling that $l(w) = z(w - \beta(w)) = z(w - v)$ in such cases, it further follows that this furthest solution w will have the smallest shifted zero number of any stationary solution to which v may connect. Therefore, there exist two minimal values of K^- and K^+ wherein, for $k < K^+$, there do not exist any stationary solutions w with $sign(w(0) - v(0)) > 0$ and $l(w) = k$ to which v connects, and for $k < K^-$, there do not exist any stationary solutions w with $sign(w(0) - v(0)) < 0$ and $l(w) = k$ to which v connects. Then v connects to $K^+ + K^-$ "equilibria" at infinity $\mathbf{\Phi}_j^\sigma$, or more specifically, v has heteroclinics connecting to $\mathbf{\Phi}_0^+, \ldots, \mathbf{\Phi}_{K^+}^+, \mathbf{\Phi}_0^-, \ldots, \mathbf{\Phi}_{K^-}^-$. We illustrate this via Figure 7, which depicts the bifurcation diagram of stationary solutions to

$$u_t = u_{xx} + bu + 25\sin(u), \quad x \in [0, \pi]$$
$$u_x(t, 0) = u_x(t, \pi) = 0. \tag{141}$$

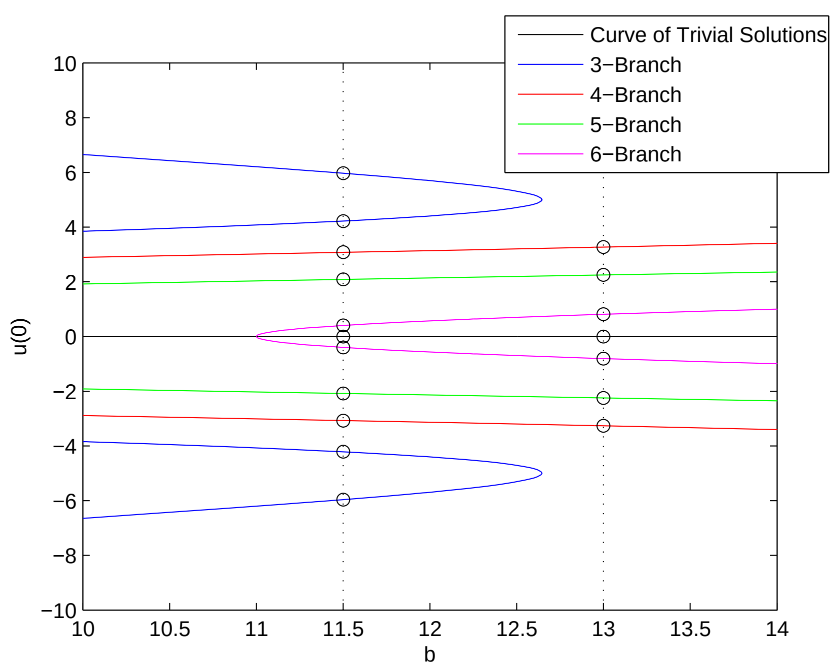

FIGURE 7. Bifurcation diagram for $g(u) = 25\ sin(u)$ with stationary solutions depicted for $b = 11.5$ and $b = 13$

For $b = 11.5$, there exist eleven stationary solutions to Equation (141), all of which are hyperbolic. For this choice of b, it is clear that $K^+ = K^- = 3$ for all but the outermost stationary solutions. For the topmost stationary solution, that depicted by the point at $(11.5, 5.96657)$, $K^+ = 4$ while $K^- = 3$. By Theorem 7.4, this stationary solution has a heteroclinic connection to $\mathbf{\Phi}_3^+$, but does not connect to $\mathbf{\Phi}_3^-$. The opposite is true for the stationary solution at $(11.5, -5.96657)$. In this case, $K^+ = 3$, while $K^- = 4$, and thus the stationary solution connects to $\mathbf{\Phi}_3^-$ but not $\mathbf{\Phi}_3^+$. Since $K^+ \geq 3$ and $K^- \geq 3$ for all stationary solutions of Equation (141) with $b = 11.5$, it follows that every bounded stationary solution connects to every element of $\{\mathbf{\Phi}_k^+,\ \mathbf{\Phi}_k^- \mid 0 \leq k < 3\}$. For $b = 13$, the situation is simplified. For every stationary solution, $K^+ = K^- = 4$. Thus, every stationary solution connects to every element of $\{\mathbf{\Phi}_k^+,\ \mathbf{\Phi}_k^- \mid 0 \leq k < 4\}$.

Any grow-up solution will, by definition, not limit to a bounded equilibrium. We recall that the y-map maps a region $\Sigma \in W^u \backslash \{v\}$ to S^n when $i(v) = n + 1$. Lemma 4.6 and Corollary 4.7 imply that there exists an initial condition u_0 corresponding to every choice of dropping times and sign. For simplicity, we first consider only positive sign, i.e. the situation wherein $u(0,0) - v(0) > 0$, i.e. $\sigma = 1$. Given K^+ as defined above, it follows that every initial condition corresponding to dropping times wherein $t_{K^+-1} = \infty$ must correspond to a grow-up solution. Such a grow-up solution will limit to $\mathbf{\Phi}_i^+$, where i corresponds to the smallest i for which the dropping time $t_i < \infty$ for $i < K^+$. Thus, there is a K^+-dimensional subset of W^u which limits to $\mathbf{\Phi}_0^+$, a $(K^+ - 1)$-dimensional subset limiting to $\mathbf{\Phi}_1^+$, up to a possibly solitary one-dimensional heteroclinic in $W^u(v)$ limiting to $\mathbf{\Phi}_{K^+-1}^+$. For the case wherein $u(0,0) - v(0) < 0$, i.e. $\sigma = -1$, the same holds for K^- and $\mathbf{\Phi}_i^-$.

2. Heteroclinic Connections Among Bounded Equilibria

Now that we have determined the behavior of the grow-up heteroclinic trajectories, we may return to the study of connections between bounded equilibria.

Lemma 7.5. *A given hyperbolic stationary solution $v \in E$ connects to all $w \in E$ not excluded by Lemmas 5.1, 5.6, and 5.7. These are the stationary solutions w with $i(w) < i(v)$ for which there is no stationary solution $\overline{w}$ such that $\overline{w}(0)$ lies between $v(0)$ and $w(0)$ and satisfies $z(v - \overline{w}) \leq z(w - \overline{w})$.*

In order to prove Lemma 7.5, we instead prove a lemma which lays out explicitly which bounded stationary solutions a given v connects to and show that this lemma implies Lemma 7.5. For this lemma we require definitions for terms we shall use frequently. We define a number of frequently referred to sets as follows:

$$L_n := \{v \in E \mid l(v) = n \text{ or } v \equiv \eta^*\},$$

$$\Omega_E(v) := \{w \in E \mid \text{v connects to } w \neq v\},$$

$$W_k(v) := \{w \in L_k \mid \quad w(0) \in EJ_v \backslash EJ_{\underline{v}},\ i(w) < i(v)\},$$

and for $0 \leq k < i(v)$ we define

$\overline{v}_k$ is the bounded stationary solution $\widehat{v}$ with $l(\widehat{v}) = k$ such that

$$\widehat{v} \notin EJ_v, \quad \widehat{v}(0) > v(0) \text{ is minimal,}$$

$\underline{v}_k$ is the bounded stationary solution $\widehat{v}$ with $l(\widehat{v}) = k$ such that

$$\widehat{v} \notin EJ_v, \quad \widehat{v}(0) < v(0) \text{ is maximal,}$$

$\underline{v}$ is the stationary solution $w \in EJ_v \cap L_{l(v)}$ with maximal $w(0)$,
$\overline{v}$ is the stationary solution $w \in EJ_v \cap L_{l(v)}$ with minimal $w(0)$.

We introduce here a number of ancillary lemmas which help to determine all connections between bounded equilibria. Lemma 7.6 was proven in [**19**] and is of great use in both the dissipative and slowly non-dissipative case. Lemma 7.7 was proved in [**7**] and carries over with modifications to Neumann boundary conditions, as the proof does not rely on (139) being dissipative.

Lemma 7.6. *Let v_1, v_2, and v_3 be stationary solutions of (139), with v_2 hyperbolic.*

(1) *If $v_j \in E$ connects to v_k then $i(v_j) > i(v_k)$ and $W^u(v_j) \cap W^s(v_k)$ is a C^2 imbedded submanifold of the Hilbert space of dimension $i(v_j) - i(v_k) \geq 1$.*

(2) *If v_1 connects to v_2, and v_2 connects to v_3, then v_1 connects to v_3. This is known as the Cascading Principle.*

(3) *Let $v_j \in E$ connect to $v_k \in E$. Then $cl(W^u(v_j) \cap W^s(v_k)$ consists of v_j, v_k and all $w \in E$ such that v_j connects to w which connects to v_k, as well as their connections.*

Lemma 7.7. *Let v connect to $w \in E$, and let $i(v) = l(v) + 1$, $i(w) < l(v)$, and $w \in EJ_v$. Then there exists a $\underline{w}$ such that $\underline{w} \in EJ_v$ and v connects to $\underline{w}$ which connects to w.*

Lemma 7.8. *Let v be a hyperbolic stationary solution of (139). Then v connects to the following stationary solutions:*

(1) *If $v = \eta^*$, or if $v \neq \eta^*$ and $i(v) = l(v)$, then*

$$\Omega_E(v) = \{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}.$$

(2) *If $v(0) > \eta^*$ and $i(v) = l(v) + 1$, then*

$$\Omega_E(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

(a) $\Omega_1 = \{\overline{v}_k \mid K^+ \leq k < i(v)\}$,

(b) $\Omega_2 = \{\underline{v}_k \mid K^- \leq k < i(v) - 1\}$, *and if* $K^- < i(v)$, *either*

(c) $\Omega_3 = \{\underline{v}_k \mid k = i(v) - 1\}$ *if* $EJ_v = \emptyset$, *or else* $\Omega_3 = \underline{v} \cup \bigcup_{k<l(v)} W_k$.

(3) *If $v(0) < \eta^*$ and $i(v) = l(v) + 1$, then*

$$\Omega_E(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where

(a) $\Omega_1 = \{\underline{v}_k \mid K^- \leq k < i(v)\}$,

(b) $\Omega_2 = \{\overline{v}_k \mid K^+ \leq k < i(v) - 1\}$, *and if* $K^+ < i(v)$, *either*

(c) $\Omega_3 = \{\overline{v}_k \mid k = i(v) - 1\}$ *if* $EJ_v = \emptyset$, *or else* $\Omega_3 = \overline{v} \cup \bigcup_{k<l(v)} W_k$.

(4) *If* v *is a spatially homogeneous stationary solution* $v(x) \equiv \eta \neq \eta^*$ *such that* $i(v) = l(v) + j$ *where* $j > 1$, *then*

$$\Omega_E(v) = \{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}.$$

Proof. Lemma 4.6 and Corollary 4.7 imply that for any $0 \leq k < i(v)$ and $\sigma \in \{1, -1\}$, there exists an initial condition $u_0 \in H^2$ such that $z(u(t, \cdot) - v(\cdot)) = k$ for $0 \leq t < \infty$ and $sign(u(t, 0) - v(0)) = \sigma$. Let us fix values for k and σ, and assume that there exists a stationary solution $w \notin EJ_v$ with Morse index $i(w) = k < i(v)$, and therefore $l(w) \in \{k, k-1\}$, such that $sign(w(0) - v(0)) = \sigma$. This is equivalent to $K^+ \leq k$. If there exists a stationary solution w with $l(w) = k - 1 = i(w) - 1$, then there must be a stationary solution $\widehat{w}$ such that $l(\widehat{w}) = k = i(\widehat{w})$ and $\widehat{w}(0)$ lies between $w(0)$ and $v(0)$, since both n-branches and pitchfork branches do not cross, and n-branches are nested with increasing lap number left to right, while pitchfork branches are nested with increasing lap number inwards, as proven in Chapter 3 and Lemma 5.8.

For ease of notation, let us refer to the stationary solution $\widehat{w}$ simply as w. Then $i(w) = k = l(w)$, where $k < i(v)$, and $sign(w(0) - v(0)) = \sigma$. If there exists at least one such stationary solution then there exists a minimal (respectively maximal) solution for $\sigma = 1$ (respectively $\sigma = -1$). By definition, this stationary solution is $\overline{v}_k$ (respectively $\underline{v}_k$). Due to the nesting of branches, if v is on a nonzero n-branch, then $\overline{v}_k$ (respectively $\underline{v}_k$) is on the k-branch, while if v is on a set of pitchfork branches $\overline{v}_k$ (respectively $\underline{v}_k$) is in the same set of pitchfork branches for $k > 0$.

If a given $\overline{v}_k$ or $\underline{v}_k$ exists, then by Lemma 4.8 and Corollary 4.9 it follows that $z(v - \overline{v}_k) = z(\overline{v}_k - v) = l(\overline{v}_k) = k$ and $z(v - \underline{v}_k) = z(\underline{v}_k - v) = l(\underline{v}_k) = k$. Then by the Infinite Blocking Lemma, v does not connect to any objects at infinity with zero number greater than or equal to k and appropriate sign at their left intercept. Most especially, it does not connect to $\{\mathbf{\Phi}_k^\sigma,\ \mathbf{\Phi}_{k+1}^\sigma, \ldots\}$. Furthermore, by Lemma 7.1 the shifted zero number does not drop at infinity, thus v cannot connect via $u_k^\sigma(t,\cdot)$ to any $\{\mathbf{\Phi}_0^\sigma,\ \ldots,\ \mathbf{\Phi}_{k-1}^\sigma\}$. Thus, it follows that the solution $u_k^\sigma(t,\cdot)$ corresponding to the choice of k and σ must remain bounded, and therefore must limit to some bounded equilibrium.

We shall now study each case in turn:

Case 1: If $v = \eta^*$ we may apply Lemma 4.6, and if $v \neq \eta^*$ we may apply Corollary 4.7, since it follows that for any spatially homogeneous stationary solution which is not the trivial solution, if $i(v) = l(v)$ then the stationary solution is asymptotically stable. Let us assume that a given $\overline{v}_k$ exists, i.e. that there exists at least one stationary solution w such that $w(0) > v(0)$, $w \notin EJ_v$, and $l(w) = k < i(v)$. Then $z(v-\overline{v}_k) = z(\overline{v}_k - v) = l(\overline{v}_k) = k < i(v)$ and $i(\overline{v}_k) < i(v)$. By Lemma 4.6 or Corollary 4.7, there exists a solution $u(t,\cdot)$ such that $\lim\limits_{t\to-\infty} u(t,\cdot) = v(\cdot)$, $z(u(t,\cdot) - v(\cdot)) = k$ for all $t \in [0,\infty)$, and $(u(t,0) - v(0)) > 0$ for all time. Since $z(\overline{v}_k - v) = k$, the Infinite Blocking Lemma comes into play, and $u(t,\cdot)$ is blocked from connecting to infinity, i.e. it must remain bounded and limit to some bounded equilibrium w with $z(w-v) = k$. Since $\overline{v}_k$ is the minimal solution above EJ_v, there do not exist any stationary solutions $w \notin EJ_v$ between v and $\overline{v}_k$ to invoke the Finite Blocking Lemma.

Furthermore, v does not connect to any stationary solution $w_j \in EJ_v$ such that $i(w) = j$. To see why, recall that

$$z(v^1 - v^2) = \begin{cases} l(v^1) \geq 1 & if\ range(v^2) \subset range(v^1) \\ 0 & if\ range(v^2) \cap range(v^1) = \emptyset \end{cases} \tag{142}$$

for any two solutions of (139). It follows that $z(v - w_j) = l(v) = i(v)$, since $w_j \in EJ_v$ implies that $range(w_j) \subset range(v)$. But Lemma 5.7 implies that v cannot connect to any such w_j. Further, if $j > i(v)$ the connection is blocked by Lemma 5.6 as well. Therefore, for every v in Case 1, it follows that v does not connect to any stationary solutions in EJ_v.

Thus, for a given k such that $\overline{v}_k$ exists, v is blocked from connecting to infinity or any finite stationary solution with lap number k except $\overline{v}_k$ by Lemmas 5.1-5.7.

But by the LaSalle Invariance Principle, the solution corresponding to the associated $u_0 \in W^u(v)$ must limit to some bounded equilibrium. Therefore $\lim_{t\to\infty} u_k^+(t,\cdot) = \overline{v}_k$.

The same arguments hold for $\underline{v}_k$ by the replacement of $w(0) > v(0)$ with $w(0) < v(0)$, $\sigma = +1$ with $\sigma = -1$, $u(t,0) - v(0) < 0$ for all time, minimal replaced by maximal, and $\lim_{t\to\infty} u_k^-(t,\cdot) = \underline{v}_k$.

Finally, for any stationary solution $w_j \notin EJ_v$ such that $i(w_j) \geq i(v)$, v is blocked from connecting to w_j by Lemma 5.6. Therefore, for $v = \eta^*$ or if $v \neq \eta^*$ and $i(v) = l(v)$, it follows that

$$\Omega_E(v) = \{\underline{v}_k | K^- \leq k < i(v)\} \cup \{\overline{v}_k | K^+ \leq k < i(v)\}.$$

Case 2: We assume that $v(0) > \eta^*$ and $i(v) = l(v)+1$. Therefore v must be on an n-branch, but not encased in any pitchforks, as the spatially homogeneous stationary solution inside any set of pitchfork branches has $i(v) \geq 2 = l(v) + 2$, and the Morse index on each pitchfork branch is equal to the lap number. Since $v(0) > \eta^*$, it follows that either $EJ_v = v$ for v a spatially homogeneous stationary solution, and therefore all other stationary solutions are not in EJ_v, or else $v(0) > v(\frac{\pi}{n})$ for $l(v) = n = i(v) - 1$.

If $\sigma = +1$, then for $k \geq K^+$ there exists at least one stationary solution w_k such that $w_k(0) > v(0)$, w_k is on the k-branch, and $l(w) = i(w) = k$. Since $v(0) > \eta^*$, it follows that for any stationary solution w such that $w(0) > v(0)$, $w \notin EJ_v$. We may now apply Lemma 4.6 (for spatially homogeneous v) or Corollary 4.7 (for all other v in this case), which imply that for all $0 \leq k < i(v)$, and thus especially for $K^+ \leq k < i(v)$, there exists an initial condition u_0 such that $z(u(t,\cdot) - v(\cdot)) = k$ and $sign(u(t,0) - v(0)) = +1$ for all $t \in [0, \infty)$. $Range(v) \subset Range(w)$ since $w(0) > v(0) > \eta^*$, and therefore $z(v - w) = z(w - v) = l(w) = i(w) = k \leq l(v)$. Thus, the Infinite Blocking Lemma takes effect, and $u(t,\cdot)$ is blocked from connecting to infinity and must remain bounded and limit to a bounded equilibrium with shifted zero number k.

By the Finite Blocking Lemma, all stationary solutions $w \neq \overline{v}_k$ such that $l(w) = k$ and $w(0) > v(0)$ have connections blocked by $\overline{v}_k$ if they are on the k-branch, and are excluded if they are on a pitchfork branch. But, by the LaSalle Invariance Principle, $u(t,\cdot)$ must limit to some bounded equilibrium with $z(w-v) = k = l(\overline{v}_k)$, and therefore $\lim\limits_{t\to\infty} u_k^+(t,\cdot) = \overline{v}_k$ for all $K^+ \leq k < i(v)$.

Now let us take $\sigma = -1$. Lemma 4.6 (for spatially homogeneous v) or Corollary 4.7 (for all other v) imply that for all $0 \leq k < i(v)$, and thus especially for $K^- \leq k < i(v)$, there exists an initial condition $u_0 \in W^u(v)$ such that $z(u(t,\cdot) - v(\cdot)) = k$ and $sign(u(t,0) - v(0)) = -1$ for all $t \in [0,\infty)$.

Let us first consider the case where $EJ_v = \emptyset$. Then for any non-pitchfork w such that $w(0) < v(0)$, $Range(v) \subset Range(w)$, therefore $z(v-w) = l(w)$. For any w on a pitchfork branch, $z(v-w) = 0$, and thus $u(t,\cdot)$ does not limit to this equilibrium. For $K^- \leq k$, it follows that $z(v - \underline{v}_k) = l(\underline{v}_k) = k \leq l(v)$. Thus, the Infinite Blocking Lemma comes into effect, and $u(t,\cdot)$ is blocked from connecting to infinity and must limit to a bounded equilibrium. By definition of $\underline{v}_k$, no blocking stationary solution exists between v and $\underline{v}_k$, and $\underline{v}_k$ blocks connections to all "lower" stationary solutions with lap number equal to k. Therefore, by the LaSalle Invariance Principle $\lim\limits_{t\to\infty} u_k^-(t,\cdot) = \underline{v}_k$ for all $K^- \leq k \leq i(v) - 1$.

Now we consider the case where $EJ_v \neq \emptyset$. We take each individual choice of $k < i(v)$ separately. We must break J_v into three regions, recalling that $l(v) = n$: $(v(\frac{\pi}{n}), \underline{v}(\frac{\pi}{n}))$, $[\underline{v}(\frac{\pi}{n}), \underline{v}(0)]$, and $(\underline{v}(0), v(0))$.

We first consider $k = i(v) - 1$, or $k = n$ to use the notation we have assigned. Lemma 4.6 and Corollary 4.7 imply that there exists one initial condition in $W^u(v)$ such that $z(u(t,\cdot) - v(\cdot)) = n$ and $sign(u(t,0) - v(0)) = -1$ for all $t \in [0,\infty)$. If there exist multiple non-pitchfork stationary solutions $w \notin EJ_v$ such that $l(w) = n$ and $w(0) - v(0) < 0$, then $z(v - \underline{v}_n) = n$ and $z(w - \underline{v}_n) = n$, thus the Finite Blocking Lemma implies that $\underline{v}_n$ blocks all of these connections. If there exist any stationary solutions $w \in EJ_v$ with $l(w) = n$, recall that $\underline{v}$ is the maximal of these. Due to the nesting of all stationary solutions in EJ_v within v in the phase portrait, it follows that $z(v - \underline{v}) = l(v) = n$. Further, $\underline{v}$ being the stationary solution with lap number n

and maximal left intercept implies that all other stationary solutions $w \in EJ_v$ with $l(w) = n$ are nested within $\underline{v}$ in the phase portrait, and therefore $z(w - \underline{v}) = l(\underline{v}) = n$.

Thus, by the Finite Blocking Lemma $\underline{v}$ blocks connections from v to any other stationary solution with lap number n in EJ_v. Additionally, for any non-pitchfork stationary solution $w \notin EJ_v$ with $l(w) = n$ and $w(0) - v(0) < 0$, including $\underline{v}_n$, $z(w - \underline{v}) = l(w) = n$ and therefore $\underline{v}$ blocks all of these connections as well. Furthermore, by the Infinite Blocking Lemma, if $\underline{v}$ exists, v does not connect to any objects below it at infinity with lap number greater than or equal to n. Thus, if there exists a $\underline{v}$, then v does not connect to $\underline{v}_n$ and only connects to $\underline{v}$ among all objects with lap number n. If there does not exist any $\underline{v}$ we understand $\{\underline{v}\} = \emptyset$.

Now let us consider $k < n$. Corollary 4.7 implies the existence of infinitely many initial conditions in $W^u(v)$ such that $z(u(t, \cdot) - v(\cdot)) = k$ and $sign(u(t, 0) - v(0)) = -1$ for all $t \in [t_k, \infty)$, $t_k \geq 0$. Let us consider one particular t_k, that of $t_k = 0$. The corollary implies the existence of an initial condition $u_0^* \in W^u(v)$ such that $z(u^*(t, \cdot) - v(\cdot)) = k$ and $sign(u^*(t, 0) - v(0)) = -1$ for $t \in [0, \infty)$. Let us consider two sets of bounded stationary solutions below v in the bifurcation diagram, $w^- \in EJ_v$ where $l(w^-) = k \neq n$ and $w^+ \notin EJ_v$ where $l(w^+) = k < n$. $Range(w^-) \subset range(v)$ while $range(w^+) \supseteq range(v)$, therefore $z(w^- - v) = l(v) = n$ while $z(w^+ - v) = l(w^+) = k < n$. Therefore no solution $w^- \in EJ_v$ is a candidate for the limit of $u^*(t, \cdot)$. Also, as before, for any $w^+ \neq \underline{v}_k$ it follows by definition of $\underline{v}_k$ that $z(v - \underline{v}_k) = l(\underline{v}_k) = k$ and $z(w^+ - \underline{v}_k) = l(w^+) = k$, thus the Finite Blocking Lemma implies that v does not connect to any $w^+ \neq \underline{v}_k$, and the existence of $\underline{v}_k$ results in the Infinite Blocking Lemma blocking connections to objects at infinity with lap number greater than or equal to k. The existence of $\underline{v}_k$ implies that $u^*(t, \cdot)$ must limit to some bounded equilibrium, and $\underline{v}_k$ is the only equilibrium which fulfills the conditions put forth by Corollary 4.7 which is not blocked. Therefore v connects to all $\underline{v}_k$ such that $K^- \leq k < n$, i.e. all $\underline{v}_k$ which exist except $\underline{v}_n$.

Now let us see if v connects to any stationary solutions in EJ_v with lap number k. Any solutions $w \in EJ_v$ with lap number greater than n are blocked by Lemma

5.6. For all stationary solutions $w \in E[\underline{v}(\frac{\pi}{n}), \underline{v}(0)]$, excluding $\underline{v}$, $range(w) \subseteq range(\underline{v})$ and therefore $z(v - \underline{v}) = l(v) = n = l(\underline{v}) = z(w - \underline{v})$. Thus v is blocked by $\underline{v}$ from connecting to any $w \in E[\underline{v}(\frac{\pi}{n}), \underline{v}(0))$ by the Finite Blocking Lemma. This leaves us only to consider those stationary solutions $w \in (v(\frac{\pi}{n}), \underline{v}(\frac{\pi}{n})) \cup (\underline{v}(0), v(0))$ for which connections exist and which are blocked. We follow the method by which Brunovský and Fiedler proved such connections in [**7**].

The method is laid out as follows. In reference to the notation used in [**7**], we denote $\mathbb{L} := \underline{v} \cup \bigcup_{k<l(v)} W_k(v)$ and $\mathbb{K} := \mathbb{L} \cap (L_{n-1} \cup L_n)$. We have already shown that v does not connect to any $w \notin EJ_v$ such that $w(0) < v(0)$ and $l(w) = z(v - w) = n$, neither does it connect to any $w \in [\underline{v}(\frac{\pi}{n}), \underline{v}(0))$, therefore $\Omega_3 \subset \mathbb{L}$. We proceed by induction. In Step 1 we use the induction hypothesis to prove that v connects to all elements of $\mathbb{L}$, provided v connects to all elements of $\mathbb{K}$. In Step 2 we prove that v connects to some elements of $\mathbb{K}$. We conclude this portion of the proof by referencing a result of Brunovský and Fiedler showing that if v connects to some solution in $\mathbb{K}$, it connects to its neighbors in $\mathbb{K}$, and by extension all $w \in \mathbb{K}$.

Let us consider the case $l(v) = n = 0$, and assume as usual that there exists at least one spatially homogeneous stationary solution $w \in E(-\infty, v(0))$. We construct the proof in a more general form than necessary so that it will be applicable to a higher choice of n or the transference to Dirichlet boundary conditions. By the Finite Blocking Lemma, v must connect to the maximal such w. If $EJ_v = \emptyset$, then $w = \underline{v}_0$ by definition. Any solution in EJ_v blocks connections to $E(-\infty, v(\frac{\pi}{n})]$, thus if $EJ_v \neq \emptyset$ then v connects to the maximal $w \in EJ_v$, which is by definition $\underline{v}$. Thus we have proven that v connects to $\mathbb{L}$ in the case where $n = 0$. We assume this continues to hold for $n - 1 \geq 0$ and prove it holds for n.

Step 1. If v connects to all elements of $\mathbb{K}$, then v connect to all elements of $\mathbb{L}$.

Let $w \in \mathbb{L} \backslash \mathbb{K}$. Then $l(w) < n - 1$. Since n-branches may not intersect and originate in increasing order as b increases, it follows that there must be elements of L_{n-1} in $\mathbb{L}$. If $w(0) > \underline{v}(0)$, we denote the minimal such solution between w and v by w_1, if $w(0) < \underline{v}(\frac{\pi}{n})$, we denote the maximal such solution below w by w_1. Therefore

v connects to w_1 by assumption, as $w_1 \in \mathbb{K}$. By Lemma 3.14 and the choice of w_1 being minimal or maximal, it follows that $i(w_1) = l(w_1) + 1 = n$. By our choice of w_1, it follows that there does not exist any other stationary solution in L_{n-1} between w and w_1, thus $w \in \mathbb{L}(w_1)$. Thus, by the induction hypothesis w_1 connects to w and therefore v connects to w by Lemma 7.6. Since w was chosen arbitrarily, we have now completed Step 1.

Step 2. The stationary solution v connects to some element of $\mathbb{K}$.

By Corollary 4.7, v must connect to some solution w which is below v in the bifurcation diagram, wherein $z(w - v) = n$. Since EJ_v is nonempty by assumption, it follows that $w(0) > v(\frac{\pi}{n})$, since any such w would block connections to contenders below EJ_v (recall that for any $w \in EJ_v$, $z(w-v) = z(v-w) = l(v) = n$). If $i(w) = n$, then $w \in L_{n-1} \cup L_n$ by Lemma 3.11. We have already shown that v is blocked from connecting to $w \in EJ_v$ if $w \notin \mathbb{L}$. Therefore, $w \in \mathbb{K}$ and v connects to w.

Now let us suppose $i(w) < n$. Then by Lemma 7.7, v connects to some stationary solution $\underline{w}$ such that $i(\underline{w}) = n$ with $\underline{w} \in EJ_v$. As before, $\underline{w} \in L_{n-1} \cup L_n$. Again it follows that $\underline{w} \in \mathbb{K}$ and thus v connects to $\underline{w}$. Thus, Step 2 is completed.

Step 3. If v connects to some $w \in \mathbb{K}$, then it connects to its neighbors in $\mathbb{K}$, provided they exist [**7**].

We do not here reproduce the method used to prove this result, as the changes are either notational or related to the change of boundary conditions and require simple substitutions for updating. Steps 1 and 2 are largely unchanged from those in [**7**] and the interested reader will find all necessary changes in the translations of these steps to this context.

Thus, the result proved in [**7**] carries over and v connects to all elements of $\mathbb{L}$. Therefore, for Case 2

$$\Omega_E = \{\overline{v}_k \mid K^+ \leq k < i(v)\} \cup \{\underline{v}_k \mid K^- \leq k < i(v)-1\} \cup \Omega_3.$$

$$If\ K^- \neq i(v),\ either$$

$$\Omega_3 = \{\underline{v}_k \mid k = i(v)-1\}\ if\ EJ_v = \emptyset,$$

$$or\ else\ \Omega_3 = \underline{v} \cup \bigcup_{k<l(v)} W_k.$$

Case 3: The case $v(0) < \eta^*$, $i(v) = l(v)+1$ is symmetric to Case 2. As before, v must be on an n-branch, but not encased in pitchforks. Since $v(0) < \eta^*$ it follows that either $EJ_v = v$ for v a spatially homogeneous stationary solution, and therefore all other stationary solutions are outside EJ_v, or else $v(0) < v(\frac{\pi}{n})$ for $l(v) = n = i(v)-1$.

If $\sigma = -1$, then for $k \geq K^-$ there exists at least one stationary solution w_k such that $w_k(0) < v(0)$, w_k is on the k-branch, and $l(w) = i(w) = k$. Since $v(0) < \eta^*$, it follows that for any stationary solution w such that $w(0) < v(0)$, $w \notin EJ_v$. We may now apply Lemma 4.6 or Corollary 4.7, which imply that for $K^- \leq k < i(v)$ there exists an initial condition u_0 such that $z(u(t,\cdot)-v(\cdot)) = k$ and $sign(u(t,0)-v(0)) = -1$ for all $t \in [0,\infty)$. $Range(v) \subset Range(w)$ since $w(0) < v(0) < \eta^*$, and therefore $z(v-w) = z(w-v) = l(w) = i(w) = k \leq l(v)$. Thus, the Infinite Blocking Lemma takes effect and $u(t,\cdot)$ is blocked from connecting to infinity, and must remain bounded and limit to a bounded equilibrium with shifted zero number k.

By the Finite Blocking Lemma, all stationary solutions $w \neq \underline{v}_k$ such that $l(w) = k$ and $w(0) < v(0)$ have connections blocked by $\underline{v}_k$ if they are on the k-branch, and are excluded if they are on a pitchfork branch. But, by the LaSalle Invariance Principle, $u(t,\cdot)$ must limit to some bounded equilibrium with $z(w-v) = k = l(\underline{v}_k)$, and therefore $\lim_{t\to\infty} u_k^-(t,\cdot) = \underline{v}_k$ for all $K^- \leq k < i(v)$.

Now let us take $\sigma = +1$. Lemma 4.6 or Corollary 4.7 imply that for $K^+ \leq k < i(v)$ there exists an initial condition $u_0 \in W^u(v)$ such that $z(u(t,\cdot)-v(\cdot)) = k$ and $sign(u(t,0)-v(0)) = +1$ for all $t \in [0,\infty)$. Let us first consider the case where $EJ_v = \emptyset$. Then for any non-pitchfork w such that $w(0) < v(0)$, $Range(v) \subset Range(w)$, therefore $z(v-w) = l(w)$. For any w on a pitchfork branch, $z(v-w) = 0$, and thus $u(t,\cdot)$ does not limit to this equilibrium. For $K^+ \leq k$, it follows that $z(v-\overline{v}_k) = l(\overline{v}_k) = k \leq l(v)$. Thus the Infinite Blocking Lemma comes into effect, and $u(t,\cdot)$ is blocked from connecting to infinity and must limit to a bounded equilibrium. By definition of $\overline{v}_k$, no blocking stationary solution exists between v and $\overline{v}_k$, and $\overline{v}_k$ blocks connections to all "higher" stationary solutions with lap number equal to k. Therefore, by the LaSalle Invariance Principle, $\lim\limits_{t\to\infty} u_k^+(t,\cdot) = \overline{v}_k$ for all $K^+ \leq k \leq i(v)-1$.

Now we consider the case where $EJ_v \neq \emptyset$. Again we take each individual choice of $k < i(v)$ separately. We now break J_v into three regions, recalling that $l(v) = n$: $(v(0), \overline{v}(0))$, $[\overline{v}(0), \overline{v}(\frac{\pi}{n})]$, and $(\overline{v}(\frac{\pi}{n}), v(\frac{\pi}{n}))$.

We first consider $k = n = i(v)-1$. Lemma 4.6 and Corollary 4.7 imply that there exists one initial condition in $W^u(v)$ such that $z(u(t,\cdot)-v(\cdot)) = n$ and $sign(u(t,0)-v(0)) = +1$ for all $t \in [0,\infty)$. If there exist multiple non-pitchfork stationary solutions $w \notin EJ_v$ such that $l(w) = n$ and $w(0)-v(0) > 0$, then $z(v-\overline{v}_n) = n$ and $z(w-\overline{v}_n) = n$, thus the Finite Blocking Lemma implies that $\overline{v}_n$ blocks all of these connections. If there exist any stationary solutions $w \in EJ_v$ with $l(w) = n$, recall that $\overline{v}$ is the minimal of these. Due to the nesting of all stationary solutions in EJ_v within v in the phase portrait, it follows that $z(v-\overline{v}) = l(v) = n$. Further, $\overline{v}$ being the stationary solution with lap number n and minimal left intercept in EJ_v implies that all other stationary solutions $w \in EJ_v$ with $l(w) = n$ are nested within $\overline{v}$ in the phase portrait, and therefore $z(w-\overline{v}) = l(\overline{v}) = n$.

Thus, by the Finite Blocking Lemma $\overline{v}$ blocks connections from v to any other stationary solution with lap number n in EJ_v. Additionally, for any non-pitchfork stationary solution $w \notin EJ_v$ with $l(w) = n$ and $w(0)-v(0) > 0$, including $\overline{v}_n$, $z(w-\overline{v}) = l(w) = n$ and therefore $\overline{v}$ blocks all of these connections as well. Furthermore, by the Infinite Blocking Lemma, if $\overline{v}$ exists, v does not connect to any objects above it at infinity with lap number greater than or equal to n. Thus, if there exists a $\overline{v}$, then v does not connect to $\overline{v}_n$ and only connects to $\overline{v}$ among all objects with lap number n. If there does not exist any $\overline{v}$ we understand $\{\overline{v}\} = \emptyset$.

Now let us consider $k < n$. Corollary 4.7 implies the existence of infinitely many initial conditions in $W^u(v)$ such that $z(u(t,\cdot) - v(\cdot)) = k$ and $sign(u(t,0) - v(0)) = +1$ for all $t \in [t_k, \infty)$, $t_k \geq 0$. Let us consider one particular t_k, that of $t_k = 0$. The corollary implies the existence of an initial condition $u_0^* \in W^u(v)$ such that $z(u^*(t,\cdot) - v(\cdot)) = k$ and $sign(u^*(t,0) - v(0)) = +1$ for $t \in [0, \infty)$. Let us consider two sets of bounded stationary solutions above v in the bifurcation diagram, $w^- \in EJ_v$ where $l(w^-) = k \neq n$ and $w^+ \notin EJ_v$ where $l(w^+) = k < n$. $Range(w^-) \subset range(v)$, while $range(w^+) \supseteq range(v)$, therefore $z(w^- - v) = l(v) = n$ while $z(w^+ - v) = l(w^+) = k < n$. Therefore no solution $w^- \in EJ_v$ is a candidate for the limit of $u^*(t, \cdot)$. Also, as before, for any $w^+ \neq \overline{v}_k$ it follows by definition of $\overline{v}_k$ that $z(v - \overline{v}_k) = l(\overline{v}_k) = k$ and $z(w^+ - \overline{v}_k) = l(w^+) = k$, thus the Finite Blocking Lemma implies that v does not connect to any $w^+ \neq \overline{v}_k$, and the existence of $\overline{v}_k$ results in the Infinite Blocking Lemma blocking connections to objects at infinity with lap number greater than or equal to k. The existence of $\overline{v}_k$ implies that $u^*(t, \cdot)$ must limit to some bounded equilibrium, and $\overline{v}_k$ is the only equilibrium which fulfills the conditions put forth by Corollary 4.7 which is not blocked. Therefore v connects to all $\overline{v}_k$ such that $K^+ \leq k < n$, i.e. all $\overline{v}_k$ which exist except $\overline{v}_n$.

Now we show that v connects to $\mathbb{L}$. For all stationary solutions $w \in E[\overline{v}(0), \overline{v}(\frac{\pi}{n})]$, excluding $\overline{v}$, $range(w) \subseteq range(\overline{v})$ and therefore $z(v - \overline{v}) = l(v) = n = l(\overline{v}) = z(w - \overline{v})$. Thus v is blocked by $\overline{v}$ from connecting to any $w \in E(\overline{v}(0), \overline{v}(\frac{\pi}{n})]$ by the Finite Blocking Lemma. This leaves us only to consider those stationary solutions $w \in (v(0), \overline{v}(0)) \cup (\overline{v}(\frac{\pi}{n}), v(\frac{\pi}{n}))$ to determine which connections exist and which are blocked. Here we denote $\mathbb{L} := \overline{v} \cup \bigcup\limits_{k<l(v)} W_k$, $\mathbb{K}$ is defined as before. We have already shown that $\Omega_3 \subset \mathbb{L}$.

We consider the case $l(v) = n = 0$ and assume, as usual, that there exists at least one spatially homogeneous stationary solution $w \in E(v(0), \infty)$. By the Finite Blocking Lemma, v must connect to the minimal such w. We construct the proof in a more general form than necessary so that it will be applicable to a higher choice of n or the transference to Dirichlet boundary conditions. If $EJ_v = \emptyset$, then $w = \overline{v}_0$ by definition. Any solution in EJ_v blocks connections to $E(v(\frac{\pi}{n}), \infty]$, thus if $EJ_v \neq \emptyset$ then v connects to the minimal $w \in EJ_v$, which is by definition $\overline{v}$. Thus we have proven v connects to $\mathbb{L}$ in the case $n = 0$. We assume this continues to hold for $n - 1 \geq 0$ and show it holds for n.

Step 1. If v connects to all elements of $\mathbb{K}$, then v connect to all elements of $\mathbb{L}$.

Let $w \in \mathbb{L} \backslash \mathbb{K}$. Then $l(w) < n - 1$. Since n-branches may not intersect and originate in increasing order as b increases, it follows that there must be elements of L_{n-1} in $\mathbb{L}$. If $w(0) < \overline{v}(0)$, we denote the maximal such solution between w and v by w_1, if $w(0) > \overline{v}(\frac{\pi}{n})$, we denote the minimal such solution above w by w_1. Therefore v connects to w_1 by assumption, as $w_1 \in \mathbb{K}$. By Lemma 3.14 and the choice of w_1 being minimal or maximal, it follows that $i(w_1) = l(w_1) + 1 = n$. By our choice of w_1, it follows that there does not exist any other stationary solution in L_{n-1} between w and w_1, thus $w \in \mathbb{L}(w_1)$. Thus, by the induction hypothesis, w_1 connects to w and therefore v connects to w by Lemma 7.6. Since w was chosen arbitrarily, we have now completed Step 1.

Step 2. The stationary solution v connects to some element of $\mathbb{K}$.

By Corollary 4.7, v must connect to some solution w which is above v in the bifurcation diagram, wherein $z(w - v) = n$. Since EJ_v is nonempty by assumption, it follows that $w(0) < v(\frac{\pi}{n})$, since any such w would block connections to contenders above EJ_v (recall that for any $w \in EJ_v$, $z(w-v) = z(v-w) = l(v) = n$). If $i(w) = n$, then $w \in L_{n-1} \cup L_n$ by Lemma 3.11. We have already shown that v is blocked from connecting to $w \in EJ_v$ if $w \notin \mathbb{L}$. Therefore, $w \in \mathbb{K}$ and v connects to w.

Now let us suppose $i(w) < n$. Then by Lemma 7.7, v connects to some stationary solution $\overline{w}$ such that $i(\overline{w}) = n$ with $\overline{w} \in EJ_v$. As before, $\overline{w} \in L_{n-1} \cup L_n$. Again it follows that $\overline{w} \in \mathbb{K}$ and thus v connects to $\overline{w}$. Thus, Step 2 is completed.

Step 3. If v connects to some $w \in \mathbb{K}$, then it connects to its neighbors in $\mathbb{K}$ provided they exist [**7**].

Again, the proof of Step 3 carries over completely from [**7**]. Thus, the result proved in [**7**] carries over and v connects to all elements of $\mathbb{L}$. Therefore, for Case 3

$$\Omega_E = \{\underline{v}_k | K^- \leq k < i(v)\} \cup \{\overline{v}_k | K^+ \leq k < i(v) - 1\} \cup \Omega_3.$$

$$If\ K^+ \neq i(v),\ either$$

$$\Omega_3 = \{\overline{v}_k | k = i(v) - 1\}\ if\ EJ_v = \emptyset,$$

$$or\ else\ \Omega_3 = \overline{v} \cup \bigcup_{k<l(v)} W_k.$$

Case 4: The final case addresses the situation of spatially homogeneous stationary solutions inside pitchfork branches, as these are the only solutions not addressed by Cases 1, 2, and 3. Let us recall from Chapter 3 that for every set of pitchforks, the outermost pitchfork branches have lap number and Morse index equal to 1. Therefore $K^+ \leq 1$ and $K^- \leq 1$. We may apply Lemma 4.6, which implies that for all $K^+ \leq k < i(v)$ there exist solutions $u_k^+(t, \cdot)$ in the unstable manifold of v for which $z(u_k^+(t, \cdot) - v(\cdot)) = k$ and $sign(u(t, 0) - v(0)) = +1$ for all $t \in [0, \infty)$. Additionally, for all $K^- \leq k < i(v)$ there exist solutions $u_k^-(t, \cdot)$ in the unstable manifold of v for which $z(u_k^-(t, \cdot) - v(\cdot)) = k$ and $sign(u(t, 0) - v(0)) = -1$ for all $t \in [0, \infty)$. Let us fix k.

As v is a spatially homogeneous stationary solution, $EJ_v = \emptyset$. Therefore $z(v - \overline{v}_k) = l(\overline{v}_k) = k$, and for any stationary solution w with $l(w) = k$ and $w(0) > \overline{v}_k(0)$, $z(w - \overline{v}_k) = l(w) = k$ or $z(w - \overline{v}_k) = 0$ for w on a different set of pitchforks. Thus either w is excluded outright, or $\overline{v}_k$ blocks connections to all such stationary solutions above v by the Finite Blocking Lemma, as well as objects at infinity with lap number greater than or equal to k by the Infinite Blocking Lemma. Since $u_k^+(t, \cdot)$ must therefore limit to some bounded equilibrium, $\lim_{t\to\infty} u_k^+(t, \cdot) = \overline{v}_k$. Additionally, $z(v - \underline{v}_k) = l(\underline{v}_k) = k$, and for any stationary solution w with $l(w) = k$ and $w(0) < \underline{v}_k(0)$, $z(w - \underline{v}_k) = l(w) = k$ or $z(w - \underline{v}_k) = 0$ if w is on a different set of pitchforks. Thus either w is excluded outright, or $\underline{v}_k$ blocks connections to all such stationary solutions below v by the Finite Blocking Lemma, as well as the objects at infinity with negative left intercept and lap number greater than or equal to k by the Infinite Blocking Lemma. Since $u_k^-(t, \cdot)$ must also limit to some bounded equilibrium, $\lim_{t\to\infty} u_k^-(t, \cdot) = \underline{v}_k$. Therefore, for any spatially homogeneous stationary solution v such that $i(v) = l(v) + j$ with

$j > 1$,

$$\Omega_E(v) = \{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}.$$

This completes the proof. □

REMARK 7.9. *We have only excluded connections by invoking the Finite Blocking Lemma, the Infinite Blocking Lemma, and Lemmas 5.6 and 5.7, which have been referred to as Morse Blocking and Zero Number Blocking in other works. All stationary solutions in E, the set of bounded equilibria, which were not blocked by any of these lemmas were proven to have connections from v. Thus, the proof of Lemma 7.5 is contained within the proof of Lemma 7.8.*

CHAPTER 8

Equilibria at Infinity

We have previously described the limiting behavior on the unbounded portions of unstable manifolds of bounded equilibria. In this chapter we shall take these assertions to the next level by discerning exactly what we mean by "equilibria at infinity" and introducing recently developed techniques for the depiction of behavior at infinity.

1. The Non-compact Global Attractor

In Chapter 2 we determined that any solution to

$$\begin{gathered} u_t = u_{xx} + bu + g(u), \quad x \in [0,\pi] \\ u_x(t,0) = u_x(t,\pi) = 0 \\ g(u) \in \mathcal{G} \end{gathered} \tag{143}$$

which does not limit to a bounded equilibrium cannot remain bounded, and that for $b > 0$, the corresponding compact semigroup $\mathcal{S}$ is non-dissipative. Thus, for $b > 0$, the global attractor $\mathcal{A}$ must be a non-compact set. Therefore, we cannot use the classical definition of global attractor, as all definitions relying on compactness of the attractor no longer apply.

We define our attractor $\mathcal{A}$ as follows: a non-compact global attractor is the minimal set $\mathcal{A} \subset H^2 \cap \{Neumann\ Boundary\ Conditions\} = X$ such that $\mathcal{A}$ is positively invariant and attracts all bounded sets in X, i.e. its basis of attraction is X. Thus, $\mathcal{A}$ is a functional invariant set for the semigroup $\mathcal{S}$ such that for all $t \geq 0$, $\mathcal{S}(t)\mathcal{A}$=$\mathcal{A}$. Normally, the term global attractor is used to refer to the maximal compact invariant set, a set which additionally fulfills the above definition for a dissipative dynamical system, but in our case, non-compactness forms the starting point for our investigation, and thus cannot be excluded.

As proven by Temam in [**39**], for a continuous semigroup of operators on a Banach space X possessing a Lyapunov functional V which is defined and continuous in X, and a compact global attractor $\mathcal{A}$, the global attractor $\mathcal{A}$ is the set of unstable manifolds of E, the set of bounded equilibria, i.e.

$$\mathcal{A} = \bigcup_{v \in E} W^u(v). \tag{144}$$

Additionally, if E is discrete, $\mathcal{A}$ is the union of E and the heteroclinics connecting one equilibrium of E to another.

Unfortunately, Temam's proof uses the compactness of the global attractor. Thus, we must take a different approach. If we consider the set of solutions which remain bounded in some sufficiently large ball in H^2 for $t \geq T(u_0)$, with $T(u_0)$ some fixed time dependent on each forward bounded solution, then we may apply Temam's proof to this subset. Therefore, we may discern that the bounded portion of the global attractor may be defined as previously. This leaves the unbounded portion to be studied, i.e. the grow-up heteroclinics. If one can construct a structure of "equilibria at infinity" which, along with their connecting orbits, attract all grow-up solutions, then we may retain (144) as a valid definition of the construction of a non-compact global attractor. As we shall describe in this chapter, that is exactly what we do.

2. The Poincaré Compactification

In order to properly study and visualize behavior at and within infinity, we need an extra tool. To this end, we introduce the Poincaré compactification. This construction, originally introduced by Poincaré in [**25**] for two dimensions, was modified by Hell [**17**] to provide a "compactification" of an infinite-dimensional Hilbert space in such a way that infinity is mapped onto a whole sphere. This "compactification" is not, in fact, compact, as the "compactification" transforms our infinite-dimensional space into an infinite-dimensional manifold whose boundary is the sphere at infinity. Its use allows a great deal of work on infinity which is not possible in H^2 alone. In addition, we shall apply the recent results of Hell [**17**] on the application of the Conley index at infinity and the structure of the attractor within infinity, in order to illustrate the behavior of the dynamical system within this region. Furthermore, these results will provide more information on the objects to which our grow-up solutions limit.

We herein produce the formulation discussed in [**17**] as applied to our choice of Hilbert space and class of reaction-diffusion equations.

We consider the Hilbert Space $H^2 \cap \{Neumann\ Boundary\ Conditions\} = X$ with scalar product $\langle u, v\rangle = \int_0^\pi uv + u_x v_x + u_{xx} v_{xx} dx$. In addition, we add a vertical direction onto our space and identify our original space X with the affine hyperplane $X \times \{+1\}$, which is tangent to the unit sphere at its north pole. We project the hyperplane centrally onto the upper hemisphere $\mathcal{H} = \{(\chi, z) \in X \times \mathbb{R} | \langle \chi, \chi\rangle + z^2 = 1, z \geq 0\}$. Given any point M on the hyperplane $X \times \{+1\}$, the straight line through M and the center of the unit sphere intersects the sphere at two antipodal points, one on the upper hemisphere and one on the lower hemisphere. We define the projection $\mathcal{P}(M)$ as the intersection point on the upper hemisphere, thus $\mathcal{P}(M) = (\chi, z)$ with $z \geq 0$. This projection is illustrated in Figure 8. As the point M is allowed to go to infinity, its image under the Poincaré projection moves to the equator of the infinite-dimensional sphere, $\mathcal{H}_e := \{(\chi, 0) \in X \times \mathbb{R} \mid \langle \chi, \chi\rangle = 1\}$, which is a sphere in its own right, and is called the "sphere at infinity". This projection provides the ability to distinguish directions at infinity and obtain precision in the study of dynamics at infinity.

Because the origin of $X \times \mathbb{R}$, the point M, and its image under projection $\mathcal{P}(M)$ are all colinear, the coordinates (χ, z) of $\mathcal{P}(M)$ may be computed as follows:

$$
\begin{aligned}
\chi &= \frac{u}{(1 + \langle u, u\rangle)^{1/2}} \\
z &= \frac{1}{(1 + \langle u, u\rangle)^{1/2}}.
\end{aligned}
\tag{145}
$$

It is easier to study invariant sets on a plane than on a sphere. Thus, we project again onto several tangent vertical hyperplanes of $X \times \mathbb{R}$ and study the linearization at equilibria there. This construction is illustrated in Figure 9.

We fix a vector e in the unit sphere of X such that $(e, 0)$ lies on the equator of the unit sphere of $X \times \mathbb{R}$. We project a point $(\chi, z) = \mathcal{P}((u, 1))$ of the upper hemisphere

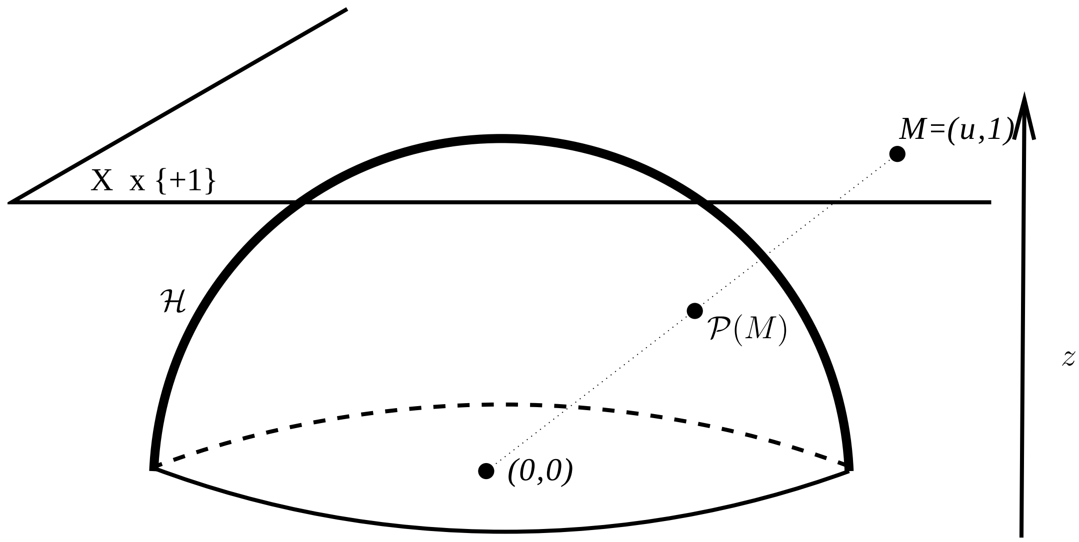

FIGURE 8. The Poincaré compactification and projection

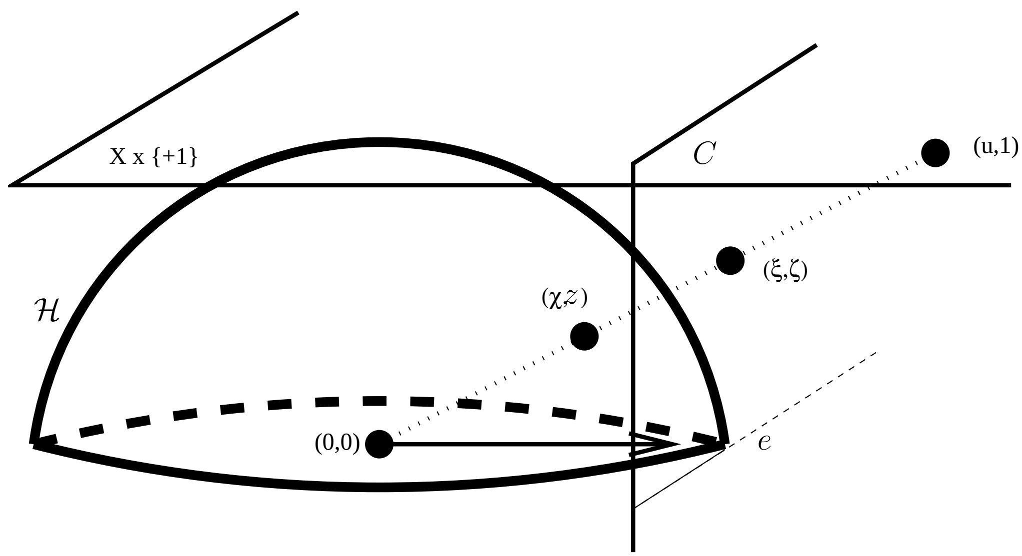

FIGURE 9. A chart of the sphere at infinity and its surroundings

$\mathcal{H}$ onto the vertical hyperplane C which is tangent to the equator at the point $(e, 0)$, constructing the projection such that this projected point lies on the line defined by $(u, 1)$ and $(0, 0)$. This is well-defined if the line through $(u, 1)$, (χ, z), and $(0, 0)$ has an intersection with the affine half-hyperplane C which is orthogonal to $(e, 0)$ and tangent to the Poincaré hemisphere. Equivalently, this requires that $\langle \chi, e\rangle$ or $\langle u, e\rangle$ is strictly nonnegative.

Using the colinearity, we define the projected point $M' = (\xi, \zeta) \in C$ by the following formulae:

$$(\xi, \zeta) = \frac{1}{\langle u, e\rangle}(u, 1) = \frac{1}{\langle \chi, e\rangle}(\chi, z). \tag{146}$$

We now recall that the original space is a Hilbert space, thus we have a countable orthonormal basis of X in which coordinates $(\widehat{u}_n)_{n\in\mathbb{N}_0}$ of any function $u \in X$ may be defined. We choose for the vector e the same basis vectors as were discussed in previous chapters, specifically the eigenfunctions of the operator A, so as to obtain equations in our coordinate system. For Φ_i, the i-th basis vector, or its negative $-\Phi_i$, we project onto the affine hyperplanes $\{\xi_i = \pm 1\}$ of $X \times \mathbb{R}$, and thus (146) is equivalent to

$$\begin{aligned} \xi_n &= \pm\frac{\widehat{u}_n}{\widehat{u}_i} \quad \textit{for all } n \in \mathbb{N} \\ \zeta &= \pm\frac{1}{\widehat{u}_i}, \end{aligned} \tag{147}$$

which holds for all $u \in X$ with ith mode nonzero. The projections onto the hyperplanes $\{\xi_i = \pm 1\}_{i\in\mathbb{N}_0}$ build an atlas of $\mathcal{H} \setminus \{(0,1)\}$. To be precise, each chart defined by (147) is a bijection between $\{(\chi, z) \in \mathcal{H} \mid \langle \chi, \Phi_i\rangle > 0\}$ and the half-hyperplane $C_i^+ := \{\xi_i = 1, \zeta \geq 0\}$ or $\{(\chi, z) \in \mathcal{H} \mid \langle \chi, \Phi_i\rangle < 0\}$ and the half-hyperplane $C_i^- := \{\xi_i = -1, \zeta \geq 0\}$.

Now that the geometric aspects of the Poincaré Compactification have been illustrated, we focus on how the differential equations themselves are transformed. We consider a differential equation of the form

$$u_t = \mathcal{L}(u) = u_{xx} + bu + g(u) = -Au + g(u) \tag{148}$$

on the Hilbert space X. Taking the derivative of (145) with respect to time, we obtain the following equations for $z \neq 0$ and $\mathcal{L}_z$, the homothety of $\mathcal{L}$ with factor z given by $\mathcal{L}_z := z\mathcal{L}(z^{-1}) = \frac{d^2}{dx^2} + bI + zg(z^{-1})$:

$$\begin{aligned} \chi_t &= \langle \chi, \mathcal{L}_z(\chi)\rangle\, \chi \mathcal{L}_z(\chi) - \langle \chi, \mathcal{L}_z(\chi)\rangle\, \chi \\ z_t &= -\langle \mathcal{L}_z(\chi), \chi\rangle\, z. \end{aligned} \tag{149}$$

Taking the derivative of (146) with respect to time, we obtain the following equations for $\zeta \neq 0$:

$$\begin{aligned} \xi_t &= -\langle \mathcal{L}_\zeta(\xi), e\rangle \xi + \mathcal{L}_\zeta(\xi) \\ \zeta_t &= -\langle \mathcal{L}_\zeta(\xi), e\rangle \zeta. \end{aligned} \tag{150}$$

Finally, we can view these equations in their coordinates for a given choice of basis vector $\pm\Phi_i$. For a fixed i, (150) becomes:

$$\begin{aligned} (\xi_n)_t &= \mp\mathcal{L}^i_\zeta(\xi)\xi_n + \mathcal{L}^n_\zeta(\xi) \qquad for\ all\ n \in \mathbb{N} \\ \zeta_t &= \mp\mathcal{L}^i_\zeta(\xi)\zeta \end{aligned} \tag{151}$$

where the $(\xi_n)_{n\in\mathbb{N}_0}$ are the coordinates of ξ in the basis $(\Phi_n)_{n\in\mathbb{N}_0}$, and $\mathcal{L}^n_\zeta(\xi) := \langle \mathcal{L}_\zeta(\xi), \Phi_n\rangle$ is the nth component of $\mathcal{L}_\zeta(\xi)$ with respect to the basis $(\Phi_n)_{n\in\mathbb{N}_0}$.

We recall that μ_i is the ith eigenvalue of the operator A from previous chapters, where $\mu_i = i^2 - b$. For our particular choice of operator $\mathcal{L}$, these equations become

$$\begin{aligned} (\xi_n)_t &= (\mu_i - \mu_n)\xi_n + (\langle g_\zeta(\xi), \Phi_i\rangle \xi_n + \langle g_\zeta(\xi), \Phi_n\rangle) \\ &= (i^2 - n^2)\xi_n + (\langle g_\zeta(\xi), \Phi_i\rangle \xi_n + \langle g_\zeta(\xi), \Phi_n\rangle) \\ \zeta_t &= -\langle g_\zeta(\xi), \Phi_i\rangle \zeta \end{aligned} \tag{152}$$

in the half-hyperplane $\{\xi_i = \pm 1,\ \zeta \geq 0\}$. The Poincaré compactified equation does not need to be normalized, since our nonlinearity is well-behaved and sublinear. The zeta equation limits to $\zeta_t = 0$ as zeta approaches zero, confirming that the equator is invariant. Because the nonlinearity g is sublinear, the terms $\langle g_\zeta(\xi), \Phi_k\rangle$ in Equation (152) are zero for $\zeta = 0$ by definition of g_ζ. Additionally g_ζ limits to zero as ζ limits to zero. Thus, on the equator Equation (152) becomes simply

$$(\xi_n)_t = (\mu_i - \mu_n)\xi_n = (i^2 - n^2)\xi_n \ for\ all\ i \neq n. \tag{153}$$

For each fixed $i \in \mathbb{N}_0$ the half-hyperplane $C^{\pm} = \{\xi_i = \pm 1,\ \zeta \geq 0\}$ contains exactly one equilibrium of Equation (152):

$$\mathbf{\Phi}_i^{\pm} : \begin{cases} \xi_i = \pm 1 \\ \xi_n = 0 \qquad for\ n \neq i \\ \zeta = 0. \end{cases} \tag{154}$$

Thus, on the sphere at infinity there exist a countable infinity of equilibria $\mathbf{\Phi}_i^{\pm}$ of (152) with coordinates in the Poincaré hemisphere $\mathcal{H}$ given by

$$\mathbf{\Phi}_i^{\pm} : \begin{cases} \chi_i = \pm 1 \\ \chi_n = 0 \qquad for\ n \neq i \\ z = 0. \end{cases} \tag{155}$$

The stability of these equilibria is determined through the study of Equation (153). For $i = 0$, the quantity $\mu_0 - \mu_n$ is always negative and the two equilibria $\mathbf{\Phi}_0^{\pm}$ are stable. For $i \geq 1$, $\mu_i - \mu_n$ is positive for $0 \leq n \leq i-1$ and negative for $n \geq i+1$, thus the equilibria $\mathbf{\Phi}_i^{\pm}$ have i unstable directions and infinitely many stable directions. In [**17**], Hell proved that for fixed $i \in \mathbb{N}_0$, $\sigma \in \{+1, -1\}$, and $n \neq i$, the ξ_n-axis is invariant and consists of heteroclinics from

$$\mathbf{\Phi}_i^{\sigma}\ to\ \mathbf{\Phi}_n^{\pm}\ if\ \mu_i - \mu_n > 0,\ i.e.\ n \leq i-1$$

$$\mathbf{\Phi}_n^{\pm}\ to\ \mathbf{\Phi}_i^{\sigma}\ if\ \mu_i - \mu_n < 0,\ i.e.\ n \geq i+1.$$

Additionally, a generic initial condition in the i-dimensional unstable manifold of $\mathbf{\Phi}_i^{\pm}$ converges to $\mathbf{\Phi}_0^{\pm}$.

We direct the reader to note that the equilibria $\mathbf{\Phi}_i^{\pm}$ in the sphere at infinity are the same objects that transfinite heteroclinics limit to, as we have proven in previous chapters.

3. The Global Attractor Decomposed

Based on the gradient-structure of the flows to (143), the Lyapunov functional, and the compactness of the semigroup, it follows that each orbit which remains bounded tends to an equilibrium. In Theorem 7.4 we proved that a grow-up heteroclinic tends towards a $\mathbf{\Phi}_i^{\pm}$ at infinity, which the work of Hell [**17**] proved were equilibria at infinity, and whose connections within infinity were determined. Thus, all solutions to (143) limit to some equilibrium, and the non-compact attractor $\mathcal{A}$ shares a second definition with those of traditional global attractors: it is the set of equilibria and their heteroclinics. We now introduce a theorem which ties together all the previous results on these heteroclinics to provide an explicit deconstruction of the attractor. We remind the reader that a complete decomposition for the attractor requires all bounded equilibria to by hyperbolic, and that the set of nonlinearities such that this holds is open and dense set in the set of nonlinearities in $\mathcal{G}$ depending only on u. The results introduced in the following theorem only provide the explicit connection structure for one equilibrium at a time and thus, Theorem 8.1 holds for all nonlinearities $g(u) \in \mathcal{G}$. Let us now define $\Omega(v)$ to be the set of equilibria distinct from v to which v connects via heteroclinic trajectory. Thus, $\Omega_E(v) \subset \Omega(v)$, as $\Omega(v)$ may include equilibria at infinity, rather than exclusively those equilibria in E.

THEOREM 8.1. *Let $g(u) \in \mathcal{G}$, $b > 0$, and let $v \in E$ be a bounded hyperbolic stationary solution of (143). Then v connects to all bounded equilibria $w \in E$ and equilibria at infinity $\mathbf{\Phi}_i^{\pm}$ which are not blocked by Lemmas 5.1 - 5.7.*

Equivalently, v connects to bounded equilibria and equilibria at infinity as follows:

(1) *If $v = \eta^*$, or if $v \neq \eta^*$ and $i(v) = l(v)$, then*
$\Omega(v) = \{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}$
$\cup\ \{\mathbf{\Phi}_k^+ \mid 0 \leq k < K^+\} \cup \{\mathbf{\Phi}_k^- \mid 0 \leq k < K^-\}.$

(2) *If $v(0) > \eta^*$ and $i(v) = l(v) + 1$, then*
$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$, *where*
$\Omega_1 = \{\overline{v}_k \mid K^+ \leq k < i(v)\}$,
$\Omega_2 = \{\underline{v}_k \mid K^- \leq k < i(v) - 1\}$,
if $K^- < i(v)$, either $\Omega_3 = \{\underline{v}_{i(v)-1}\}$ if $EJ_v = \emptyset$, or else $\Omega_3 = \underline{v} \cup \bigcup_{k<l(v)} W_k$,
$\Omega_4 = \{\mathbf{\Phi}_k^+ \mid 0 \leq k < K^+\}$, *and*
$\Omega_5 = \{\mathbf{\Phi}_k^- \mid 0 \leq k < K^-\}$.

(3) *If* $v(0) < \eta^*$ *and* $i(v) = l(v) + 1$*, then*

$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$*, where*

$\Omega_1 = \{\underline{v}_k \mid K^- \leq k < i(v)\}$,

$\Omega_2 = \{\overline{v}_k \mid K^+ \leq k < i(v) - 1\}$,

if $K^+ < i(v)$*, either* $\Omega_3 = \{\overline{v}_{i(v)-1}\}$ *if* $EJ_v = \emptyset$*, or else* $\Omega_3 = \overline{v} \cup \bigcup_{k<l(v)} W_k$,

$\Omega_4 = \{\mathbf{\Phi}_k^+ \mid 0 \leq k < K^+\}$*, and*

$\Omega_5 = \{\mathbf{\Phi}_k^- \mid 0 \leq k < K^-\}$.

(4) *If* v *is a spatially homogeneous stationary solution* $v(x) = \eta \neq \eta^*$ *such that* $i(v) = l(v) + j$ *with* $j > 1$*, then*

$\Omega(v) = \{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}$

$\cup \{\mathbf{\Phi}_k^+ \mid 0 \leq k < K^+\} \cup \{\mathbf{\Phi}_k^- \mid 0 \leq k < K^-\}$.

Proof. **Case 1:** By Lemma 7.8, v connects to $\{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}$ when $K^+, K^- < i(v)$ (otherwise these sets are empty sets). By definition, for $k < K^+$ and $k < K^-$, there do not exist any stationary solutions respectively above or below v which block connections to infinity. Theorem 7.4 thus proved that for every $\sigma = +1$ and $k < K^+$ or $\sigma = -1$ and $k < K^-$, there exist trajectories $u_k^\sigma(t, \cdot)$ connecting v to $\mathbf{\Phi}_k^\sigma$ in forward time. Thus, Case 1 is proved through combining the previous results of Lemma 7.8 and Theorem 7.4.

Case 2: By Lemma 7.8, v connects to $\{\overline{v}_k \mid K^+ \leq k < i(v)\} \cup \{\underline{v}_k \mid K^- \leq k < i(v) - 1\}$ when $K^+ < i(v)$ and $K^- < i(v) - 1$ (otherwise these sets are empty sets). Additionally, the lemma proved that if $EJ_v = \emptyset$, then v also connects to $\underline{v}_{i(v)-1}$, and if $EJ_v \neq \emptyset$, then v connects to all elements of the set $\underline{v} \cup \bigcup_{k<l(v)} W_k$. By definition, for $k < K^+$ and $k < K^-$, there do not exist any stationary solutions which block connections to infinity. Theorem 7.4 thus proved that for every $\sigma = +1$ and $k < K^+$ or $\sigma = -1$ and $k < K^-$, there exist trajectories $u_k^\sigma(t, \cdot)$ connecting v to $\mathbf{\Phi}_k^\sigma$ in forward time. Thus, Case 2 is proved through combining the previous results of Lemma 7.8 and Theorem 7.4.

Case 3: By Lemma 7.8, v connects to $\{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)-1\}$ when $K^- < i(v)$ and $K^+ < i(v)-1$ (otherwise these sets are empty sets). Additionally, the lemma proved that if $EJ_v = \emptyset$, then v also connects to $\overline{v}_{i(v)-1}$, and if $EJ_v \neq \emptyset$, then v connects to all elements of the set $\overline{v} \cup \bigcup\limits_{k<l(v)} W_k$. By definition, for $k < K^+$ and $k < K^-$, there do not exist any stationary solutions which block connections to infinity. Theorem 7.4 thus proved that for every $\sigma = +1$ and $k < K^+$ or $\sigma = -1$ and $k < K^-$, there exist trajectories $u_k^\sigma(t, \cdot)$ connecting v to $\mathbf{\Phi}_k^\sigma$ in forward time. Thus, Case 3 is proved through combining the previous results of Lemma 7.8 and Theorem 7.4.

Case 4: By Lemma 7.8, v connects to $\{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}$ when $K^+, K^- < i(v)$ (otherwise these sets are empty sets). By definition, for $k < K^+$ and $k < K^-$, there do not exist any stationary solutions which block connections to infinity. Theorem 7.4 thus proved that for every $\sigma = +1$ and $k < K^+$ or $\sigma = -1$ and $k < K^-$ there exist trajectories $u_k^\sigma(t, \cdot)$ connecting v to $\mathbf{\Phi}_k^\sigma$ in forward time. Thus, Case 4 is proved through combining the previous results of Lemma 7.8 and Theorem 7.4. $\square$

Let us denote by $\mathcal{G}_h$ the class of nonlinearities $g(u) \in \mathcal{G}$ wherein all stationary solutions and equilibria at infinity for Equation (143) are hyperbolic and g is dependent only on u. We remind the reader that $\mathcal{G}_h$ is open and dense in the set of nonlinearities in $\mathcal{G}$ depending only on u.

Corollary 8.2. *For any fixed $b > 0$ and $g(u) \in \mathcal{G}_h$, the non-compact global attractor $\mathcal{A}$ is comprised of the bounded equilibria $v \in E$, their heteroclinic connections as defined by Theorem 8.1, the equilibria at infinity $\mathbf{\Phi}_k^\pm$, $k \leq \lfloor\sqrt{b}\rfloor$, and the connections of each $\mathbf{\Phi}_k^\pm$ to the set $\{\mathbf{\Phi}_i^+,\ \mathbf{\Phi}_i^- \mid i < k\}$.*

Proof. The dynamical system defined by Equation (143) with $b > 0$ and $g(u) \in \mathcal{G}_h$ is a gradient system with a strict Lyapunov functional possessing a compact semigroup, as discussed in Chapter 2. Therefore all trajectories which remain bounded must limit to some bounded equilibrium. Furthermore, the existence of an inertial manifold $\mathcal{M}$ which contains all invariant sets implies that all bounded regions in $H^2 \cap \{Neumann\ Boundary\ Conditions\}$ are exponentially attracted to $\mathcal{M}$. Additionally, all solutions not in $\mathcal{M}$ are tracked by solutions in $\mathcal{M}$, which themselves are attracted to the set of stationary solutions and their heteroclinics. Finally, in [**17**] Hell proved that each $\mathbf{\Phi}_k^{\pm}$ connects to all those $\mathbf{\Phi}_i^{+}$, $\mathbf{\Phi}_i^{-}$ for which $i < k$. Thus, for any initial condition $u_0 \in H^2 \cap \{Neumann\ Boundary\ Conditions\}$ such that $u_0 \notin \mathcal{A}$, the corresponding solution $u(t, \cdot)$ to Equation (143) may be decomposed into the Fourier series

$$u(t,\cdot) = \sum_{j=0}^{\infty} \widehat{u}_j(t)\Phi_j(x), \quad \widehat{u}_j(t) = \langle u(t,x), \Phi_j(x)\rangle_0\,,$$

where each mode is determined by the equation

$$\widehat{u}_j(t) = e^{(b-j^2)t}\widehat{u}_j(0) + \int_0^t e^{(b-j^2)(t-s)} g_j(s)ds. \tag{156}$$

Therefore, for $j \geq k = \lfloor \sqrt{b} \rfloor$, the quantity $b - j^2$ is negative, and the jth mode must remain bounded. If the solution $u(t, \cdot)$ remains bounded, it must limit to a stationary solution $v \in E$, and therefore will limit to $\mathcal{A}$. If $u(t, \cdot)$ is a grow-up solution, then the jth modes will die out relative to the norm of u, and therefore u may only limit to one of the $\mathbf{\Phi}_j^{\pm}$ where $j \leq k$. Since $\mathcal{A}$ contains the only objects at infinity to which any initial conditions may limit, it therefore contains the only objects at infinity to which any sets of initial conditions may limit. Thus $\mathcal{A}$, which is comprised of the set of bounded equilibria, their heteroclinics, and the attracting equilibria at infinity with their heteroclinics, forms the minimal invariant set which attracts all bounded sets in $H^2 \cap \{Neumann\ Boundary\ Conditions\}$. $\square$

CHAPTER 9

Case Study: $f(u) = bu + asin(u)$

Now that we have laid out the results in their most general forms, let us look at an interesting example to see exactly how these results may be applied and what information may be obtained. To this end, we shall focus on a simple choice of nonlinearity $g(u)$ which produces a wealth of interesting results.

Let us consider the nonlinearity

$$g(u) = a\sin(u), \quad a \in \mathbb{R}. \tag{157}$$

It is clear that such a nonlinearity easily fulfills our requirements; it is not only C^2 but C^∞, is bounded with upper bound $\overline{\gamma} = |a|$ and lower bound $\underline{\gamma} = -|a|$, thus $\Gamma = |a|$. It is not only Lipschitz continuous in L^2 but C^1 in H^2 with bound $\| Dg \| \leq |a|$, thus the constants c and C_1 in Chapter 6 are $c = \frac{\sqrt{\pi}|a|}{\mu_{N+1}} = \frac{\sqrt{\pi}|a|}{(N+1)^2-b}$ and $C_1 = |a|$. Our evolutionary equation is now

$$\begin{aligned} u_t &= u_{xx} + bu + a\sin(u), \quad x \in [0,\pi] \\ u_x(t,0) &= u_x(t,\pi) = 0. \end{aligned} \tag{158}$$

As shown in Lemma 2.3 in Chapter 2, the system generated by Equation (158) is clearly non-dissipative for $b > 0$. In addition, for $b = 0$ the system is also non-dissipative, although its behavior is noticeably different than that occurring for $b > 0$, as it is not a slowly non-dissipative system. For $a \neq 0$, the partial differential equation

$$u_t = u_{xx} + a\sin(u), \quad x \in [0,\pi]$$

$$u_x(t,0) = u_x(t,\pi) = 0$$

has infinitely many spatially homogeneous equilibria $u(x) = n\pi,\ n \in \mathbb{Z}$. In addition, one can quickly derive that half of these equilibria, those for which $u(x) = 2n\pi$, have Morse index $i(u) = 1$, and that their unstable manifolds connect them to the two adjacent equilibria. As we cannot bound the set of equilibria, the system is clearly non-dissipative, despite not having asymptotically linear growth.

1. The Time Map and Bifurcation Diagrams

The function $g(u) = a\ sin(u)$ being odd provides for a number of simplifications regarding the time map and bifurcation diagram. First and foremost is that $\eta^* = 0$. Additionally, for any choice of η not contained on or inside a homoclinic orbit in the phase plane, $\alpha(\eta) = -\eta$. Due to $sin(u)$ being an odd function, the phase portrait is symmetric across both the vertical and horizontal axes. Therefore, for $v(0) = \eta$ contained in a homoclinic orbit such that v is a stationary solution of Equation (158), $-v$ is also a stationary solution of (158), with $\alpha(-v(0)) = -\alpha(v(0))$, $i(v) = i(-v)$ and $l(v) = l(-v)$. It thus follows that the bifurcation diagram is symmetric across the horizontal axis.

As $g(u) = a\ sin(u)$ has infinitely many intersections with the u-axis, it follows that the n-branches will intersect the lines $b = n^2$ infinitely many times, as long as $a \neq 0$. For $a = 0$, the lines $b = n^2$ are exactly the n-branches. Figures 10, 11, and 12 illustrate three examples of such bifurcation diagrams, dependent on the choice of a. As dictated by Lemmas 3.4 and 3.9, each n-branch intersects the horizontal axis at $b = n^2 - a$ and asymptotically approaches the line $b = n^2$ as $|u(0)|$ increases. For $a \neq 0$ we can see how the nonlinearity deforms the bifurcation diagram of the linear equation, with the linear growth term overpowering the perturbation as $|u(0)|$ becomes large. For any given b, as we increase $|a|$ away from zero, the perturbations of the n-branches from the straight lines in Figure 11 become larger, as can be seen by comparing Figure 10 to Figure 12. Thus, for any given b, the number of n-branches intersecting a line in the bifurcation diagram at that value of b will increase as $\mid a \mid$ increases.

We can also see how the choice of nonlinearity affects the pitchfork branches bifurcating off the 0-branch. In Figure 10, only the pitchfork branches on the portion of the curve nearest 0 are clearly visible, for higher segments the regions in which the pitchfork branches exist become miniscule, and the deviations from the 0-branch correspondingly difficult to discern. For the second sets of pitchforks, those in the

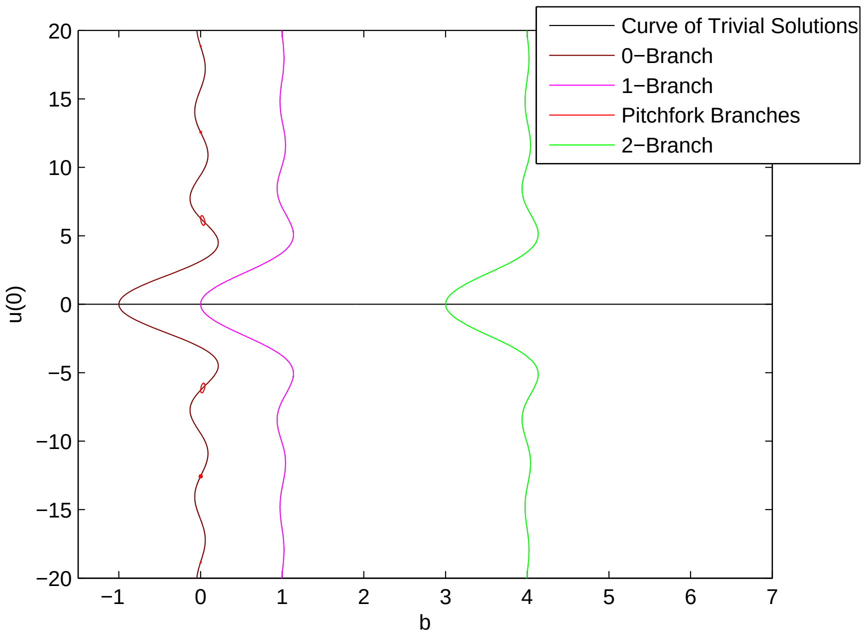

FIGURE 10. Bifurcation diagram for $a = 1$ and $g(u) = a\ sin(u)$

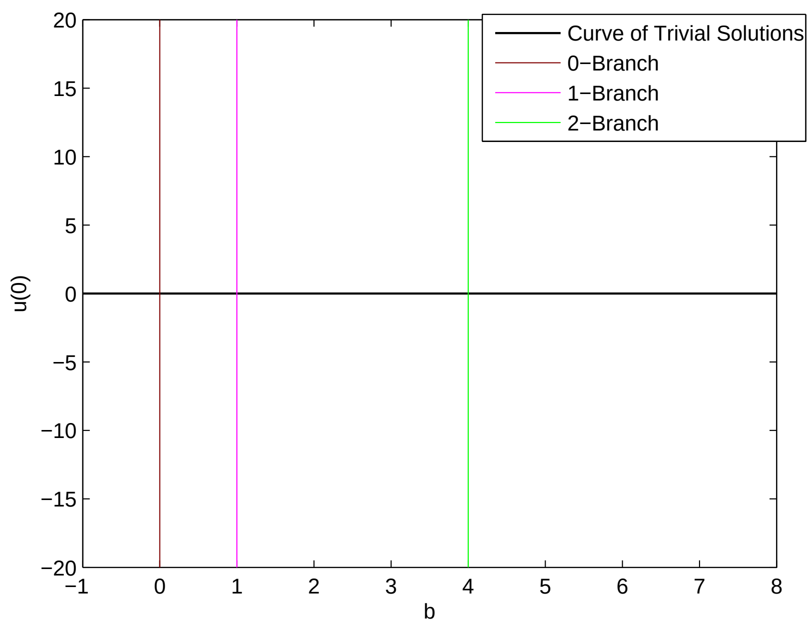

FIGURE 11. Bifurcation diagram for $g(u) = 0$

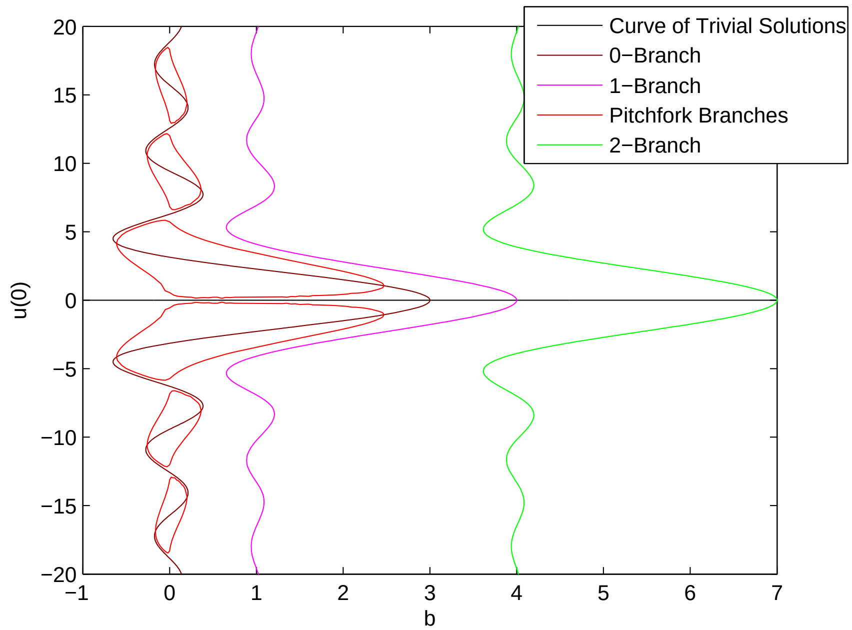

FIGURE 12. Bifurcation diagram for $a = -3$ and $g(u) = a\ sin(u)$

region of $(0, 4\pi)$ and $(0, -4\pi)$, the length of the pitchfork is less than .0001 and is only visible in the figure as a visual artifact.

On the other hand, in Figure 12, all six sets of pitchforks in the depicted region are clearly visible to the naked eye, and their deviations from the 0-branch have become large enough to show distinct behavior. Comparing Figures 10 and 12 illustrates the way the value of Γ, i.e. the size of the nonlinearity, relative to b can make all deviations from the linear equation more pronounced. Both the wiggling nature of the bifurcations and the distance of the pitchfork branches from the 0-branch become noticeably more pronounced for $\Gamma = 3$, as shown in Figure 12, as opposed to $\Gamma = 1$, as shown in Figure 10.

We now fix two different choices of a and b and investigate the results for these two systems. For the first choice, we let $a = 16$ and $b = 10$. For this choice of constants, Equation (158) becomes

$$\begin{aligned} u_t &= u_{xx} + 10u + 16\sin(u), \quad x \in [0, \pi] \\ u_x(t, 0) &= u_x(t, \pi) = 0. \end{aligned} \tag{159}$$

We may plot the time map, as shown in Figure 13, and locate all non-spatially homogeneous bounded stationary solutions to (159) via the intersections of the time map with the various lines at heights $\frac{\pi}{n}$.

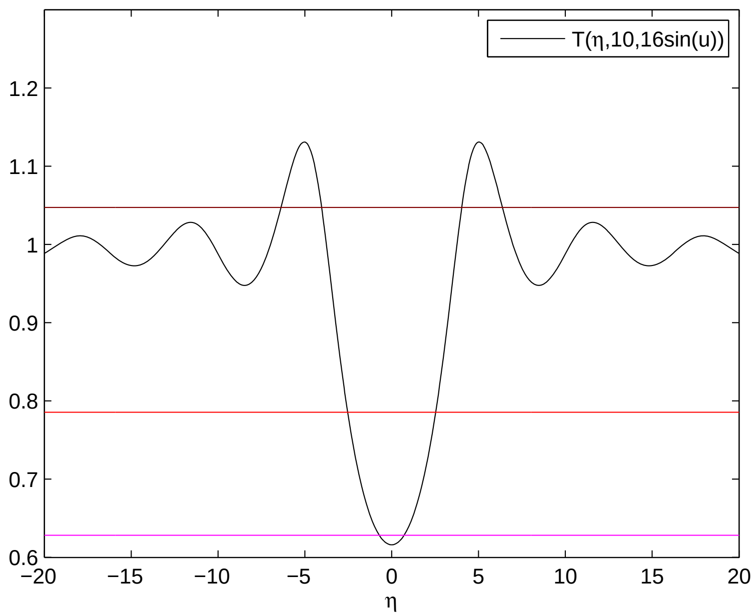

FIGURE 13. Plot of the time map for Equation (159)

It is clear from Figure 13 that in addition to the trivial solution $u(x) \equiv 0$, there exist two bounded stationary solutions with lap number 5, two with lap number 4, and four with lap number 3. As the time map does not cross the lines at π, $\frac{\pi}{2}$, or $\frac{\pi}{n}$ for $n > 5$, it follows that there do not exist any stationary solutions with lap number 1 or 2 or lap number greater than 5. Since $10u + 16sin(u) = 0$ only at $u = 0$, it follows that there do not exist any nontrivial spatially homogeneous stationary solutions. Figure 14 depicts the relevant region of the bifurcation diagram, with all the stationary solutions for $b = 10$ emphasized, while Figure 15 depicts the phase portrait for

$$0 = u_{xx} + 10u + 16\sin(u) \tag{160}$$

with the stationary solutions of Equation (159) depicted in colors corresponding to their lap numbers and matching the colors chosen in Figure 14. Each curve in the

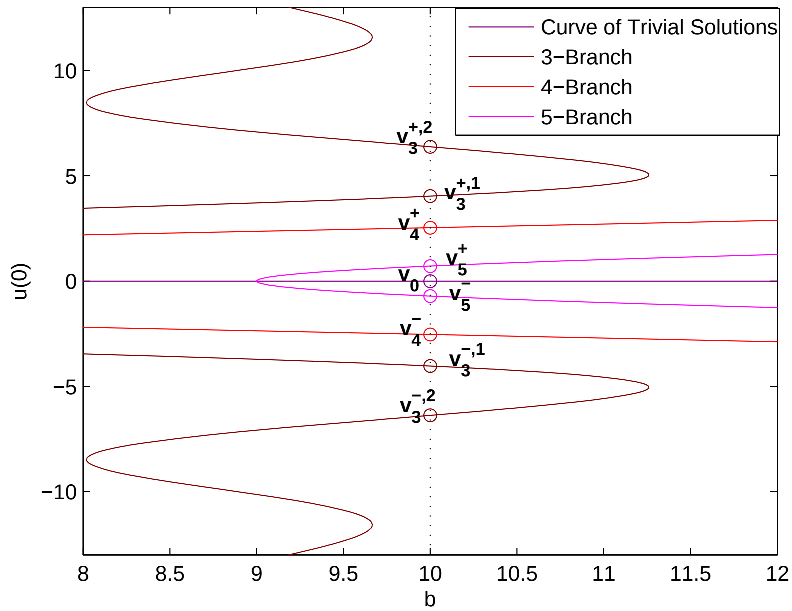

FIGURE 14. Bifurcation diagram for $g(u) = 16\ sin(u)$ depicting all bounded stationary solutions for $b = 10$

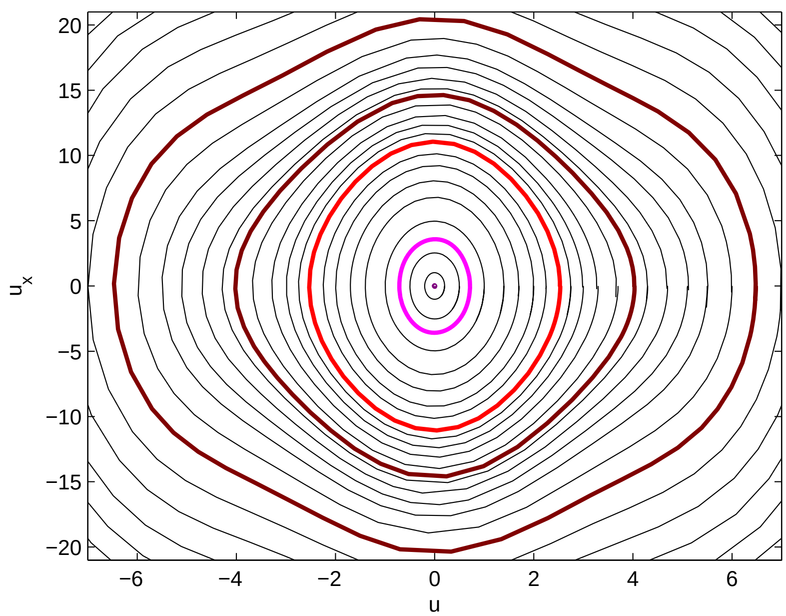

FIGURE 15. Phase portrait for $0 = u_{xx} + 10u + 16\ sin(u)$ depicting all bounded stationary solutions of Equation (159)

phase portrait corresponds to two stationary solutions of Equation (159), representing its two intersection points with the u-axis, while the fixed point in the phase portrait corresponds to one unique stationary solution, the spatially homogeneous solution $u(x) \equiv 0$. We provide identifying notation in Figure 14 for each bounded equilibrium for use in decomposing the heteroclinic structure of the attractor.

For our second choice of constants, we let $a = -3$ and $b = 0.7$, focusing in on a particular choice of b from the bifurcation diagram presented in Figure 12. This allows us to depict the differing behavior of the time map, phase portrait, and connections in the attractor for a choice of b wherein pitchfork branches exist. For this choice of constants, Equation (158) becomes

$$\begin{gathered} u_t = u_{xx} + 0.7u - 3\sin(u), \quad x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{gathered} \tag{161}$$

We may plot the time map, as shown in Figure 16, and locate all non-spatially homogeneous bounded stationary solutions to (161) via the intersections of the time map with the lines at heights $\frac{\pi}{n}$.

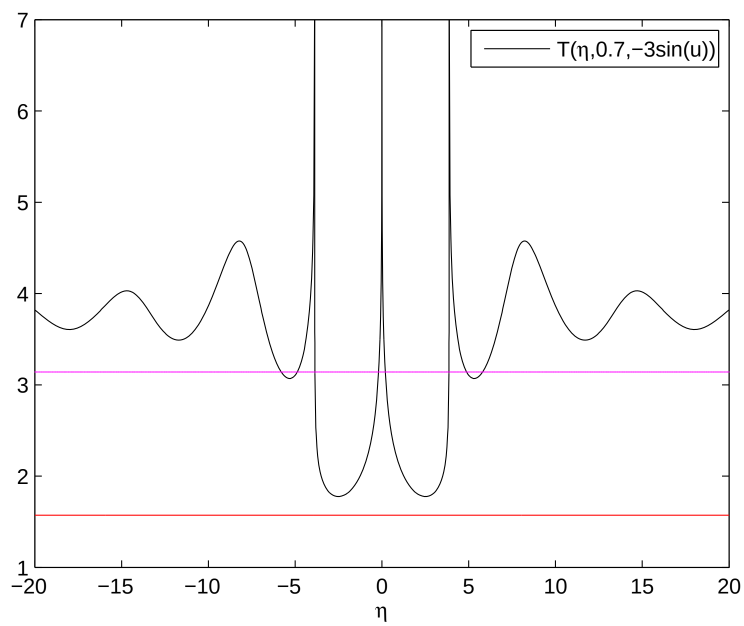

FIGURE 16. Plot of the time map for Equation (159)

From Figure 16 it is clear that in addition to the trivial solution $u(x) \equiv 0$, there exist two more spatially homogeneous solutions, and all other stationary solutions have lap number 1. We have plotted the line at height $\frac{\pi}{2}$ to illustrate the fact that there do not exist any stationary solutions of Equation (161) with lap number greater than 1, as the time map does not cross the line at $\frac{\pi}{2}$. Further, since $\eta = 0$ corresponds to a fixed point and $T(0, 0.7, -3sin(u)) = \infty$, it follows that $u(x) \equiv 0$ must be a saddle point in the phase portrait. Thus, the other singularities in the time map must be at its δ-points, the intersection of the homoclinic orbit with the u-axis. The minima between these singularities correspond to the β-points of these homoclinics, which are stationary solutions of Equation (161).

Figure 17 depicts the relevant region of the bifurcation diagram, with all the stationary solutions for $b = 0.7$ emphasized, while Figure 18 depicts the phase portrait for

$$0 = u_{xx} + 0.7u - 3sin(u) \tag{162}$$

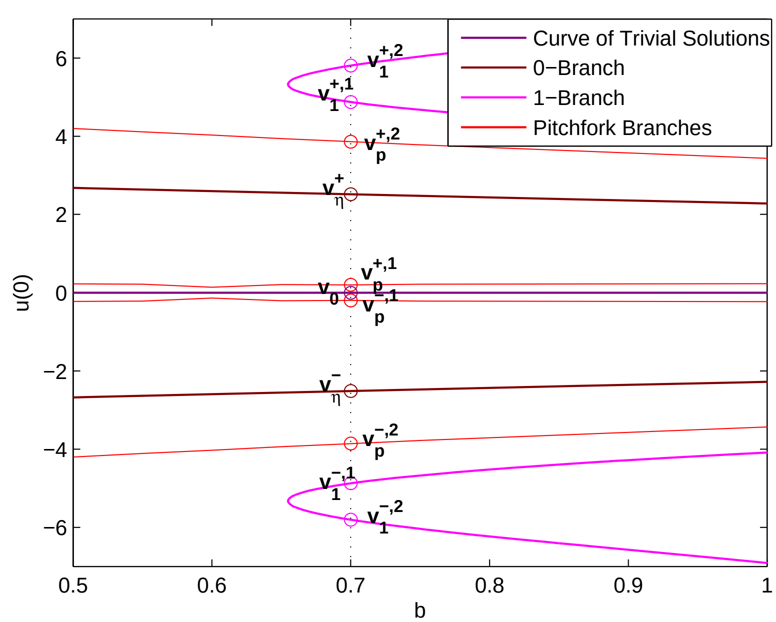

FIGURE 17. Bifurcation diagram for $g(u) = -3\ sin(u)$ depicting all bounded stationary solutions for $b = 0.7$

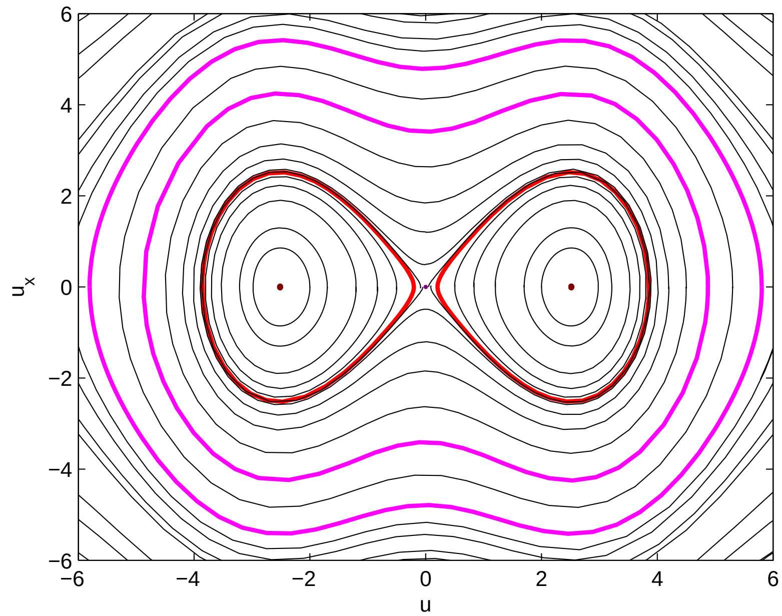

FIGURE 18. Phase portrait for $0 = u_{xx} + 0.7u - 3\ sin(u)$ depicting all bounded stationary solutions of Equation (161)

with the stationary solutions of Equation (161) depicted in colors corresponding to their lap numbers and matching the colors chosen in Figure 17. Each curve in the phase portrait corresponds to two stationary solutions of Equation (161), representing its two intersection points with the u-axis, while fixed points in the phase portrait correspond to the three spatially homogeneous stationary solutions $v_0(x) \equiv 0$, $v_\eta^+(x) \equiv 2.5146$, and $v_\eta^-(x) \equiv -2.5146$.

It is clear from a comparison between Figures 16 and 18 that those regions in the time map diagram which lie between singularities do indeed correspond to the regions in the phase portrait bounded by homoclinic orbits.

2. The Non-compact Global Attractor for $g(u) = a\ sin(u)$

The nonlinearity $g(u) = a\ sin(u)$ is globally Lipschitz from L^2 into L^2, thus it is sufficiently well-behaved to ensure the existence of a completed inertial manifold. Thus, Theorems 7.4 and 8.1 apply and we are able to explicitly determine all heteroclinic connections for a generic form of Equation (158), whether bounded, transfinite, or intra-infinite. Thus, for our two choices of sinusoidal nonlinearities presented in the previous section, we can explicitly determine all connections which exist. We present these connections in two ways, first as a table and then in figure form.

In the study of Equation (159), the heteroclinic connections between equilibria are listed in Table 1, and depicted in Figure 19. We carry over the color scheme from Figure 14 and depict equilibria at infinity with differing lap numbers using differing colors as well.

Table 1. Global attractor decomposition for Equation (159)

	Equilibria to Which Heteroclinic Connections Exist in Forward Time
v_0	$v_5^+,\ v_5^-,\ v_4^+,\ v_4^-,\ v_3^{+,1},\ v_3^{-,1},\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
v_5^+	$v_4^+,\ v_4^-,\ v_3^{+,1},\ v_3^{-,1},\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
v_5^-	$v_4^+,\ v_4^-,\ v_3^{+,1},\ v_3^{-,1},\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
v_4^+	$v_3^{+,1},\ v_3^{-,1},\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
v_4^-	$v_3^{+,1},\ v_3^{-,1},\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_3^{+,1}$	$\mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_3^{-,1}$	$\mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_3^{+,2}$	$v_3^{+,1},\ \mathbf{\Phi}_3^+,\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_3^{-,2}$	$v_3^{-,1},\ \mathbf{\Phi}_3^-,\ \mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_3^+$	$\mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_3^-$	$\mathbf{\Phi}_2^+, \mathbf{\Phi}_2^-,\ \mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_2^+$	$\mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_2^-$	$\mathbf{\Phi}_1^+,\ \mathbf{\Phi}_1^-,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_1^+$	$\mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_1^-$	$\mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_0^+$	Stable Equilibrium
$\mathbf{\Phi}_0^-$	Stable Equilibrium

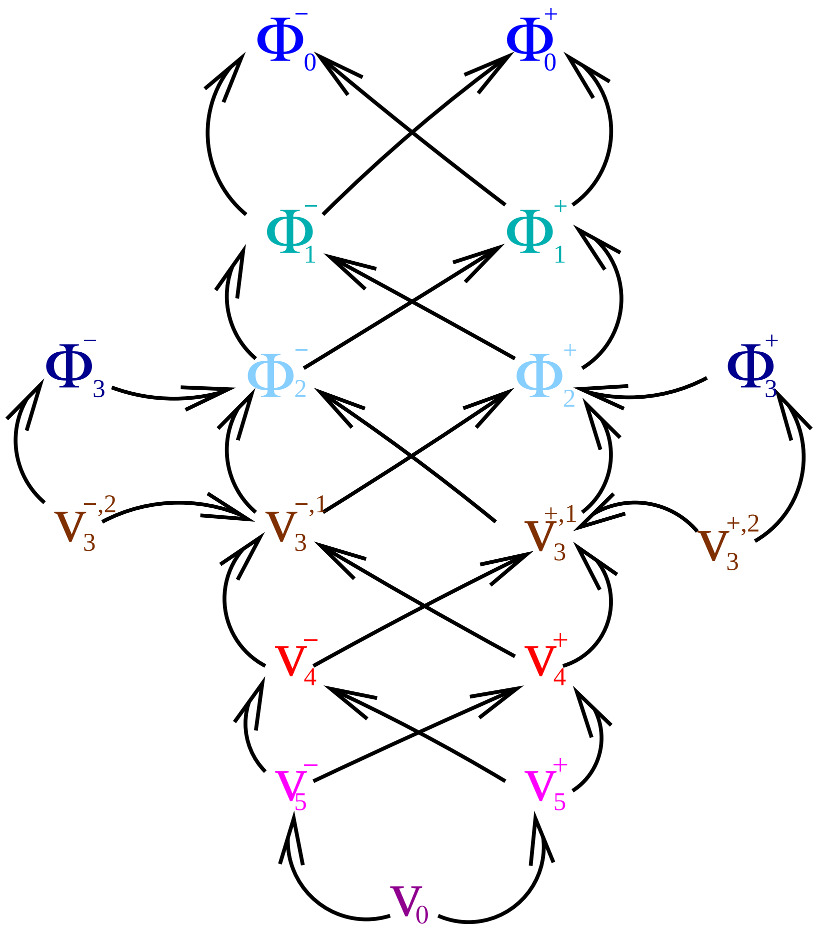

FIGURE 19. Graphical depiction of the non-compact global attractor for Equation (159)

Due to the number of total equilibria, including those equilibria at infinity, it becomes increasingly difficult to gain insight from a visual depiction if all heteroclinic connections are depicted. Thus, we adopt a standard practice in the depiction of attractors with Chafee-Infante structure, which takes advantage of the Cascading Principle. We presume that any time a connection is depicted from v^1 to v^2 and another is depicted from v^2 to v^3, then there exists a direct connection from v^1 to v^3.

Additionally, we remind the reader that any time the Morse indices of two equilibria with a heteroclinic connection differ by more than one, there in fact exist infinitely many such connections corresponding to the infinite number of possible dropping times of the lap number, rather than just the singular connection drawn.

The only time there is truly only one heteroclinic between two equilibria is when the Morse indices of these equilibria differ by one.

We remind the reader that the various $\mathbf{\Phi}$ solutions are equilibria at infinity. As shown in both Figure 19 and Table 1, all equilibria of Equation (159) experience cascading connections, i.e. A connecting to B which connects to C implies that A connects to C via a separate trajectory.

TABLE 2. Global attractor decomposition for Equation (161)

	Equilibria to Which Heteroclinic Connections Exist in Forward Time
v_0	Stable Equilibrium
$v_p^{+,1}$	$v_0,\ \mathbf{\Phi}_0^+$
$v_p^{-,1}$	$v_0,\ \mathbf{\Phi}_0^-$
v_η^+	$v_p^{+,1},\ v_p^{+,2},\ \mathbf{\Phi}_0^+,\ v_0$
v_η^-	$v_p^{-,1},\ v_p^{-,2},\ v_0,\ \mathbf{\Phi}_0^-$
$v_p^{+,2}$	$v_0,\ \mathbf{\Phi}_0^+$
$v_p^{-,2}$	$v_0,\ \mathbf{\Phi}_0^-$
$v_1^{+,1}$	$v_1^{+,2},\ v_p^{+,2},\ v_0,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_1^{-,1}$	$v_1^{-,2},\ v_p^{-,2},\ v_0,\ \mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_1^{+,2}$	$\mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$v_1^{-,2}$	$\mathbf{\Phi}_0^+,\ \mathbf{\Phi}_0^-$
$\mathbf{\Phi}_0^+$	Stable Equilibrium
$\mathbf{\Phi}_0^-$	Stable Equilibrium

For Equation (161), the situation is rather more complicated, due to the existence of pitchfork bifurcations and solutions of lower lap number inhabiting $EJ_{v_1^{\pm,1}}$. We list the heteroclinic connections between equilibria in Table 2, and depict the more complicated attractor in Figure 20.

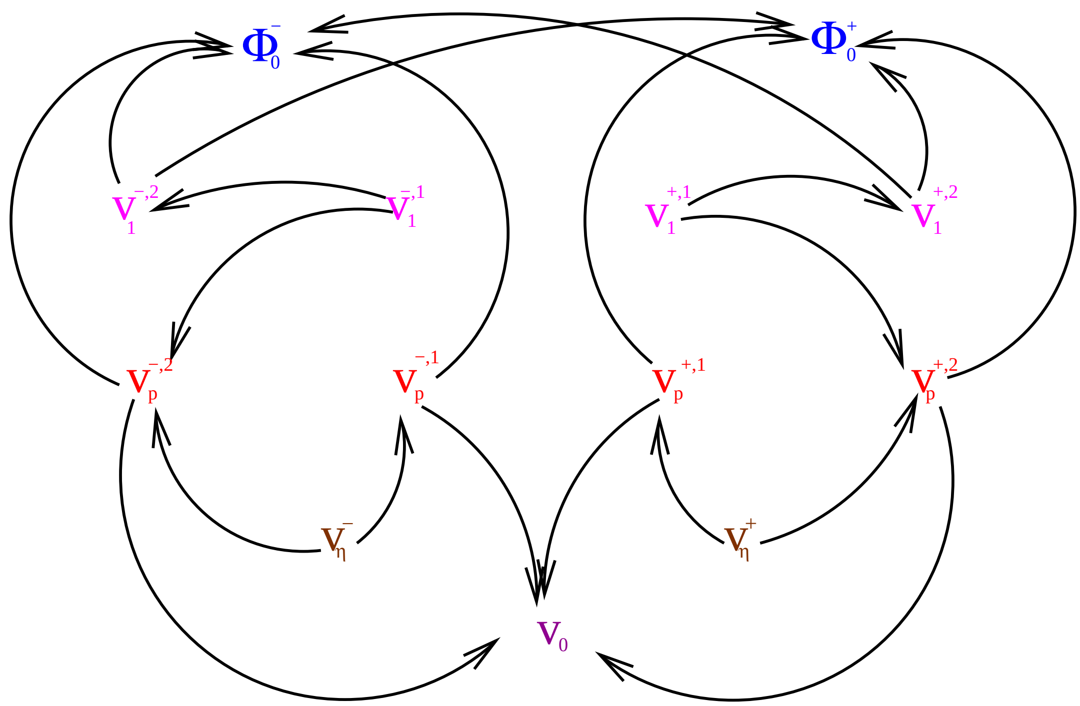

Figure 20. Graphical depiction of the non-compact global attractor for Equation (161)

It becomes clear that despite the simplicity of the nonlinearity presented by $g(u) = a\sin(u)$, Equation (158) produces a variety of interesting phenomena that we may study through the use of the results presented in previous chapters.

CHAPTER 10

The Dirichlet Case

Up to this point, we have only studied the asymptotic behavior of solutions to

$$u_t = u_{xx} + bu + g(u), \quad x \in [0, \pi], \quad g(u) \in \mathcal{G} \tag{163}$$

for Neumann boundary conditions. But we may extend this work to Dirichlet boundary conditions with a reasonable amount of effort. In this chapter we address what adjustments need to be made and in what ways the global picture has changed. Henceforth we study the equation

$$\begin{aligned} u_t &= u_{xx} + bu + g(u), \quad x \in [0, \pi] \\ u(t, 0) &= u(t, \pi) = 0, \quad g(u) \in \mathcal{G}. \end{aligned} \tag{164}$$

1. Zero Number

Recall the simple relation between the lap number and the zero number of a given C^1 function $u(t, x)$,

$$l(u) = \left\{ \begin{array}{ll} z(u_x) + 1, & \text{if } u_x \neq 0 \\ 0, & \text{if } u_x \equiv 0 \end{array} \right\}. \tag{165}$$

In [**4, 23, 40**] it was proven that the zero number of a solution $u(t, x)$ to

$$u_t = u_{xx} + f(u)$$

with Dirichlet boundary conditions is nonincreasing in forward time. Further, in [**4**] it was proven that the zero number of any solution to

$$u_t = u_{xx} + f(x, u)$$

is nonincreasing in forward time provided that $f(x, 0) = 0$ for all $x \in [0, \pi]$. Thus, we have the same properties on the zero number for Dirichlet boundary conditions as we used in the study of Neumann boundary conditions.

2. Non-dissipativity

Unlike the case of Neumann boundary conditions, there exists at most one spatially homogeneous stationary solution in the Dirichlet boundary condition equation (164), the trivial solution $u^*(x) \equiv 0$. In fact, if $g(u) \neq 0$, there are no spatially homogeneous stationary solutions to Equation (164). Thus, Lemma 2.2 does not carry over to the Dirichlet case.

LEMMA 10.1. *Given a scalar parabolic equation of the form (164), the corresponding semigroup $\mathcal{S}$ is not point dissipative if $b > 1$.*

PROOF. For a given $b > 1$ and $g(u) \in \mathcal{G}$, where $\underline{\gamma} \leq g(u) \leq \overline{\gamma}$, we are able to again use the time map and bifurcation diagram to determine an ordering of the bounded equilibria. We extract from this set those equilibria with zero number equal to 0. As a consequence of Lemma 10.5, for any $b > 1$ the set of stationary solutions with zero number equal to 0 is bounded, although the bound may be very large for b close to 1. Thus, the Lyapunov functional over the set of stationary solutions with zero number equal to 0 is bounded below.

Let us consider an initial condition $u_0 = a\ sin(x)$. The value of the Lyapunov functional $V(u_0)$ may be studied as follows:

$$V(u) = \int_0^\pi \frac{1}{2}u_x^2(s) - \frac{b}{2}u^2(s) - G(u(s))ds$$

$$\Rightarrow V(u_0) = V(asin(x)) = \int_0^\pi \frac{a^2}{2}cos^2(s) - \frac{ba^2}{2}sin^2(s) - G(asin(s))ds$$

$$\leq \int_0^\pi \frac{a^2}{2}cos^2(s) - \frac{ba^2}{2}sin^2(s) - \underline{\gamma}asin(s)ds$$

$$= \int_0^\pi \frac{a^2}{4}(1 + cos(2s)) - \frac{ba^2}{4}(1 - cos(2s)) - \underline{\gamma}asin(s)ds$$

$$= \frac{a^2\pi}{4} + 0 - \frac{ba^2\pi}{4} + 0 - 2a\underline{\gamma} = \frac{(1-b)\pi}{4}a^2 - 2a\underline{\gamma}.$$

Since $1 - b < 0$, by a sufficiently large choice of a we may always find an initial condition u_0 at which the Lyapunov functional has lower value than at any bounded stationary solution with zero number equal to 0. Since $z(u_0) = z(asin(x)) = 0$, it follows that the $z(u(t,\cdot)) = 0$ for all forward time for the solution $u(t,\cdot)$ where $u(0,\cdot) = u_0 = a\sin(x)$. The value of the Lyapunov functional along the orbit given by $u(t,\cdot)$ must decrease as time moves forward, but due to the nonincrease of the zero number [**4, 23, 40**], this orbit cannot contain any solutions with zero number greater than 0 for t positive. Since the Lyapunov functional at all bounded equilibria w with $z(w) = 0$ has higher value than at u_0, it follows that these bounded equilibria cannot be in the omega limit set of u_0. By Lemma 2.1 we know that an orbit not limiting to any bounded equilibrium cannot remain bounded for all time. Thus, we have determined that the trajectory $u(t,\cdot)$ corresponding to the above described initial condition u_0 does not remain in any bounded set for all time. As we have now discovered at least one trajectory which does not remain bounded for all forward time, it follows that the semigroup $\mathcal{S}$ is not point dissipative, and thus is not compact dissipative.

Thus, for arbitrary nonlinearity $g(u) \in \mathcal{G}$ and $b > 1$, the semigroup $\mathcal{S}$ is non-dissipative. $\square$

REMARK 10.2. *If the domain of x is changed from $[0,\pi]$ to $[0,L]$, the semigroup $\mathcal{S}$ is not point dissipative for $b > \frac{L^2}{\pi^2}$.*

3. The Time Map

Recalling the definitions given in the beginning of Chapter 3, we reiterate the definition of the time map $T(\eta, f)$ and the nth time map $T_n(\eta, f)$ for Dirichlet boundary conditions. For Dirichlet conditions, the time map $T(\eta, f)$ determines the period of a periodic solution to the second order equation

$$u_{xx} + f(u) = 0 \tag{166}$$

intersecting the point $(u(0), u_x(0)) = (0, \eta)$. It determines the "time" in x needed for a solution in the phase plane to travel from a point on the u_x-axis to its first subsequent intersection with the u_x-axis [**3, 30, 37**]. As introduced in [**35, 36, 37**], we define the time map by studying the length of time it takes for a trajectory originating at

$(0, \eta)$ in the phase plane to reach the point $(\alpha(\eta), 0)$, its first intersection with the u-axis. This is exactly half the time it takes to reach its first intersection with the u_x-axis. Thus, assuming $\eta > 0$, we define the time map for our specific system via the formulae

$$
\begin{aligned}
T(\eta, b, g) &= \sqrt{2} \int_0^{\alpha(\eta)} \frac{du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}} \\
T(-\eta, b, g) &= \sqrt{2} \int_{\alpha(-\eta)}^0 \frac{du}{\sqrt{\frac{b}{2}(\alpha(-\eta))^2 - \frac{b}{2}u^2 + G(\alpha(-\eta)) - G(u)}}.
\end{aligned} \tag{167}
$$

As before, all stationary solutions of Equation (164) must be either the zero solution, must take "time" $\frac{\pi}{n}$, with $n \in \mathbb{N}$, to travel from $(0, \eta)$ to $(0, -\eta)$, or must take time $\frac{\pi}{n}$, with $n \in \mathbb{N}$, to travel from $(0, \eta)$ back to $(0, \eta)$, in order to fulfill the boundary conditions.

The nth time map $T_n(\eta, f)$ for solutions to Equation (166) is the nth positive zero of the solution $v(x)$ to Equation (166) which satisfies

$$v(0) = 0, \quad v_x(0) = \eta,$$

whenever this zero exists [**7**]. As with Neumann boundary conditions, the 2nd time map follows from the first, although not in the same manner. In the Dirichlet case, the formula for the 2nd time map is $T_2(\eta, f) = T(\eta, f) + T(-\eta, f)$. This follows from the Hamiltonian [**6**]. By definition, $u(0) = 0 = u(T(\eta, f))$; plugging this into Equation (9) implies that $u_x^2(0) = u_x^2(T(\eta, f))$. The equality $u_x(0) = u_x(T(\eta, f))$ is impossible for a periodic orbit, as it would imply a homoclinic orbit in the phase plane, which implies $T(\eta, f) = \infty$ and $T_2(\eta, f)$ does not exist. Thus $u_x(0) = -u_x(T(\eta, f)) = \eta$, and $T_2(\eta, f) = T(\eta, f) + T(-\eta, f)$. As $F(\alpha(\eta)) = F(\alpha(-\eta))$, it follows that we may define $T_2(\eta, b, g)$ explicitly via the formulae

$$
\begin{aligned}
T_2(\eta, b, g) &= \sqrt{2} \int_{\alpha(-\eta)}^{\alpha(\eta)} \frac{du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}}, \quad \alpha(\eta) > \alpha(-\eta) \\
T_2(\eta, b, g) &= \sqrt{2} \int_{\alpha(-\eta)}^{\alpha(\eta)} \frac{-du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}}, \quad \alpha(\eta) < \alpha(-\eta).
\end{aligned} \tag{168}
$$

Further, we can explicitly define the nth time map for any given b and g:

$$\begin{aligned}
&n\ even: \quad T_n(\eta,b,g) = \frac{n}{2}T_2(\eta,b,g) = \\
&\frac{n}{\sqrt{2}}\int_{\alpha(-\eta)}^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}} \\
&n\ odd: T_n(\eta,b,g) = \frac{n-1}{2}T_2(\eta,b,g) + T(\eta,b,g) = \\
&\frac{n-1}{\sqrt{2}}\int_{\alpha(-\eta)}^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}} \\
&+\sqrt{2}\int_{0}^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}}.
\end{aligned} \tag{169}$$

It becomes clear that, unlike in the case of Neumann boundary conditions, we cannot simply evaluate one time map and derive all our information therein. For every choice of b and g we must evaluate every successive time map until we determine the value of n for which $T_n(\eta, b, g) > \pi$ for all η, only then will we have found all bounded stationary solutions.

Recalling Figures 1 and 2, one can see that there are a number of regions in the phase portrait containing solutions to the Neumann form of Equation (166) which do not contain any solutions to the Dirichlet form. Any solutions contained within a homoclinic loop which does not intersect the u_x-axis are invisible to the Dirichlet problem. Thus, the locations of fixed points in the phase portraits, although they cannot be stationary solutions of the Dirichlet problem, do give us information on the locations of stationary solutions. With the introduction of explicit formulae for each successive time map, we may now prove analogs of the lemmas in Chapter 3.

LEMMA 10.3. *As $u_x(0) = \eta$ approaches $\pm\infty$, the time map for any choice of $g(u) \in \mathcal{G}$ and $b > 0$ fulfilling Dirichlet boundary conditions will approach the value $\frac{\pi}{\sqrt{b}}$, i.e.*

$$\lim_{\eta\to\pm\infty} T(\eta, b, g) = \frac{\pi}{\sqrt{b}}.$$

PROOF. We choose $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$, where $\overline{\eta}_+$ is defined as the value $\eta > 0$ such that $T(\eta, b, g) < \infty$ for all $\overline{\eta}_+ < \eta < \infty$, and $\overline{\eta}_-$ is defined as the value $\eta < 0$

such that $T(\eta, b, g) < \infty$ for all $-\infty < \eta < \overline{\eta}_-$. Essentially, we choose to start with η outside of all separatrices in the phase plane, as these contribute the only points where the time map is infinite [**3**]. The dominance of the linear part of $f(u) = bu + g(u)$ ensures that the region of discontinuities of T is bounded. As we have chosen η outside of the separatrices, it is ensured that $\alpha(\eta)$ is outside of the separatrices as well, as $\alpha(\eta)$ is determined by the solution to the equation

$$\frac{1}{2}\eta^2 = \frac{b}{2}\alpha^2(\eta) + G(\alpha(\eta)), \tag{170}$$

where $(\alpha(\eta), 0)$ is the unique point on the phase plane trajectory which contains the point $(u(0), u_x(0)) = (0, \eta)$ such that $sign(\alpha(\eta)) = sign(\eta)$. By definition of $\overline{\eta}_\pm$, it is clear that $b\alpha(\eta) + g(\alpha(\eta)) \neq 0$ and $b\alpha(-\eta) + g(\alpha(-\eta)) \neq 0$ for all $\alpha(\eta)$ corresponding to $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$. As $\frac{d\alpha(\eta)}{d\eta} = \frac{\eta}{b\alpha(\eta)+g(\alpha(\eta))}$, it is clear that $\frac{d\alpha(\eta)}{d\eta}$ is defined everywhere in the regions $|\eta| > \max\left\{|\overline{\eta}_+|, |\overline{\eta}_-|\right\}$.

Studying the Hamiltonian (170), and recalling that η and $\alpha(\eta)$ refer to points in the phase plane where $u = 0$ and $u_x = 0$ respectively, it is clear that solutions to $0 = u_{xx} + bu + g(u)$ which are outside of all separatrices must be nested. Further, recalling that $sign(\alpha(\pm\eta)) = sign(\pm\eta)$, it follows that $\frac{d\alpha(\eta)}{d\eta} > 0$ and $\lim\limits_{\eta\to\pm\infty} \alpha(\eta) = \pm\infty$

Thus, $\lim\limits_{\eta\to\pm\infty} T(\eta, b, g) = \lim\limits_{\eta\to\pm\infty} 2\int_0^{\alpha(\eta)} \frac{sign(\alpha(\eta))du}{\sqrt{b(\alpha(\eta))^2 - bu^2 + 2G(\alpha(\eta)) - 2G(u)}}$. Recalling the fact that $\underline{\gamma}\alpha(\eta) \leq G(\alpha(\eta)) \leq \overline{\gamma}\alpha(\eta)$ when $\underline{\gamma} \leq g(u) \leq \overline{\gamma}$, and applying a change of variables $x\alpha(\eta) = u$ enables the following calculations:

$$\begin{aligned}
&\lim_{\eta\to\infty} \sqrt{2}\int_0^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}} \\
&= \lim_{\alpha(\eta)\to\infty} 2\int_0^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{b(\alpha(\eta))^2 - bu^2 + 2G(\alpha(\eta)) - 2G(u)}} \\
&= \lim_{\alpha(\eta)\to\infty} 2\int_0^{1} \frac{\mid \alpha(\eta) \mid dx}{\sqrt{b(\alpha(\eta))^2 - b(\alpha(\eta))^2x^2 + 2G(\alpha(\eta)) - 2G(x\alpha(\eta))}} \\
&= \lim_{\alpha(\eta)\to\infty} 2\int_0^{1} \frac{dx}{\sqrt{b - bx^2 + 2\frac{G(\eta)}{(\alpha(\eta))^2} - 2\frac{G(x\alpha(\eta))}{(\alpha(\eta))^2}}}
\end{aligned}$$

$$= 2\int_0^1 \frac{dx}{\sqrt{b-bx^2}} = \frac{2}{\sqrt{b}}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{\sqrt{b}}$$

$$\lim_{\eta\to-\infty} \sqrt{2}\int_0^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{\frac{b}{2}(\alpha(\eta))^2 - \frac{b}{2}u^2 + G(\alpha(\eta)) - G(u)}}$$

$$\lim_{\alpha(\eta)\to-\infty} 2\int_0^{\alpha(\eta)} \frac{sign(\eta)du}{\sqrt{b(\alpha(\eta))^2 - bu^2 + 2G(\alpha(\eta)) - 2G(u)}}$$

$$= \lim_{\alpha(\eta)\to-\infty} 2\int_0^1 \frac{\mid \alpha(\eta) \mid dx}{\sqrt{b(\alpha(\eta))^2 - b(\alpha(\eta))^2x^2 + 2G(\alpha(\eta)) - 2G(x\alpha(\eta))}}$$

$$= \lim_{\alpha(\eta)\to-\infty} 2\int_0^1 \frac{dx}{\sqrt{b - bx^2 + 2\frac{G(\alpha(\eta))}{(\alpha(\eta))^2} - 2\frac{G(x\alpha(\eta))}{(\alpha(\eta))^2}}}$$

$$= 2\int_0^1 \frac{dx}{\sqrt{b-bx^2}} = \frac{2}{\sqrt{b}}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{\sqrt{b}}.$$

It is clear that as long as $g(u)$ is bounded and C^2, the limit of the time map as $u_x(0)$ approaches infinity remains the same. It is entirely dependent on the asymptotic linearity, rather than the specific nonlinearity. $\square$

Corollary 10.4. *As $u_x(0) = \eta$ approaches $\pm\infty$, the nth time map T_n for any choice of $g(u) \in \mathcal{G}$ and $b > 0$ fulfilling Dirichlet boundary conditions will approach the limit $\frac{n\pi}{\sqrt{b}}$, i.e.*

$$\lim_{\eta\to\pm\infty} T_n(\eta, b, g) = \frac{n\pi}{\sqrt{b}}.$$

Proof. This result follows directly from Equations (168) and (169) and Lemma 10.3. For n even,

$$\lim_{\eta\to\pm\infty} T_n(\eta,b,g) = \lim_{\eta\to\pm\infty} \frac{n}{2}T_2(\eta,b,g) = \lim_{\eta\to\pm\infty} (\frac{n}{2}T(\eta,b,g) + \frac{n}{2}T(-\eta,b,g))$$

$$= \frac{n}{2}(\lim_{\eta\to\pm\infty} T(\eta,b,g) + \lim_{\eta\to\pm\infty} T(-\eta,b,g)) = \frac{n}{2}(\frac{\pi}{\sqrt{b}} + \frac{\pi}{\sqrt{b}}) = \frac{n}{2}(\frac{2\pi}{\sqrt{b}}) = \frac{n\pi}{\sqrt{b}}$$

and for n odd,

$$\lim_{\eta\to\pm\infty} T_n(\eta,b,g) = \lim_{\eta\to\pm\infty} (\frac{n-1}{2}T_2(\eta,b,g) + T(\eta,b,g))$$

$$= \lim_{\eta\to\pm\infty} (\frac{n+1}{2}T(\eta,b,g) + \frac{n-1}{2}T(-\eta,b,g))$$

$$= \frac{n+1}{2}(\lim_{\eta\to\pm\infty} T(\eta,b,g)) + \frac{n-1}{2}(\lim_{\eta\to\pm\infty} T(-\eta,b,g))$$

$$= \frac{n+1}{2}\frac{\pi}{\sqrt{b}} + \frac{n-1}{2}\frac{\pi}{\sqrt{b}} = \frac{n\pi}{\sqrt{b}}.$$

□

Furthermore, all stationary solutions to Equation (164) are nested about each other, as they must all either oscillate around the origin or intersect the origin, and may not intersect each other.

4. The Bifurcation Diagram

As in the Neumann case, studying the time map provides us with the global bifurcation diagram of stationary solutions as well as information about it, although not as readily as in the Neumann case. In the case of Neumann boundary conditions, we only needed to plot the first time map $T(\eta, b, g)$; its intersections with lines at heights π, $\frac{\pi}{2}, \ldots,$ $\frac{\pi}{n}$ determined where there were non-spatially homogeneous stationary solutions in the bifurcation diagram and which lap numbers they possessed. For Dirichlet boundary conditions, we must plot all time maps $T_1, \ \ldots, \ T_n$ and locate their intersections with a line at height π. As $T_n > T_{n-1} > \ldots > T_1$, it follows that we are finished once we have reached a jth time map wherein $T_j(\eta, b, g) > \pi$ for all η. For a given value of η, the corresponding stationary solution has zero number j if the $j+1$st time map is equal to π at that point. Since the various time maps cannot intersect, this identifies uniquely the stationary solution in the bifurcation diagram. The natural ordering of values $\eta = u_x(0)$ leads to a logical ordering of solutions, and thus forms the vertical axis of our bifurcation diagram. By plotting various time maps and increasing b, we may construct the global bifurcation diagram of stationary solutions.

Similarly to the Neumann case, we designate a curve in the bifurcation diagram as an n-branch when said curve is parametrized over η and is comprised of all solutions with zero number equal to $n-1$. For any given j, if $T_j(\eta, b, g) = \pi$, it follows that the stationary solution with left intercept $u(0) = 0$, $u_x(0) = \eta$ must cross the u_x-axis $j-1$ times, i.e. it has zero number equal to $j-1$. Therefore, the n-branch corresponds to those values of b and $u_x(0) = \eta$ for which $T_n(\eta, b, g) = \pi$. The only other possible curve in the bifurcation diagram is the line of trivial solutions at $\eta = 0$, although this does not necessarily exist if $g(0) \neq 0$. We note that for nonlinearities $g(u)$ which are odd functions of u, the nonzero n-branches are unchanged by the transition from Neumann to Dirichlet boundary conditions, although the solutions which are represented by the points on these branches are altered.

LEMMA 10.5. *An n-branch will asymptotically approach the line $b = n^2$ in the bifurcation diagram.*

PROOF. As proven in Lemmas 10.3 and 10.4, $\lim\limits_{\eta\to\pm\infty} T_n(\eta, b, g) = \frac{n\pi}{\sqrt{b}}$ for fixed b. As we have defined an n-branch for Dirichlet boundary conditions, all solutions on an n-branch are stationary solutions of (164) with $u_x(0) = \eta$ whose nth time map $T_n(\eta, b, g) = \pi$. For a fixed $b \neq n^2$, as η limits to $\pm\infty$, $\lim\limits_{\eta\to\pm\infty} T_n(\eta, b, g) = \frac{n\pi}{\sqrt{b}} \neq \pi$. As the time map is continuous for η sufficiently large, it follows that there exists an η_b such that for $| \eta |>| \eta_b |$, the nth time map $T_n(\eta, b, g) \neq \pi$, and thus, the n-branch does not intersect the line at this fixed b. Equivalently, there are no equilibrium solutions for $| \eta |>| \eta_b |$. Only for $b = n^2$ will this property not hold. The n-branch is continuous, defined by the intersections of the nth time map T_n with the line at π for variable b. Thus, as η limits to positive or negative infinity, the n-branch must approach the line $b = n^2$. $\square$

Since there is no 0-branch or branch of spatially homogeneous stationary solutions in the Dirichlet case, it follows that there cannot be any pitchfork bifurcations in the Dirichlet bifurcation diagram. As in the Neumann case, the n-branches bifurcate from the horizontal axis (which is the curve of trivial solutions if there exists a trivial solution) at the value b fulfilling $b + g'(0) = n^2$. Again, the sign of $g'(0)$ determines whether these bifurcations open to the left or right: for $g'(0) > 0$ the bifurcation will be annihilated as b decreases, while for $g'(0) < 0$ the bifurcation will be annihilated as b increases.

The relationship between the zero number and Morse index is much the same as the relationship between the lap number and Morse index.

Lemma 10.6. *The Morse index $i(v)$ and zero number $z(v)$ of a hyperbolic stationary solution to Equation (164) are related by*

$$i(v) \in \{z(v), z(v)+1\}.$$

This has already been proven in a number of sources [**2, 6**] as a consequence of the Sturm-Picone Comparison Theorem. The proof follows nearly identically to the proof of Lemma 3.11, as well as the proof presented in [**6**], and as such will not be replicated here.

Lemma 10.7. *For v a hyperbolic stationary solution to (164) with $v_x(0) = \eta_0 \neq 0$,*

$$\begin{aligned} \eta_0 \cdot \frac{db}{d\eta}(\eta_0) > 0 &\Rightarrow i(v) = z(v) \\ \eta_0 \cdot \frac{db}{d\eta}(\eta_0) < 0 &\Rightarrow i(v) = z(v) + 1. \end{aligned} \tag{171}$$

Proof. The first portion of this proof follows from [**7**]; although the proof in that paper was for a more restrictive class of inequalities, the same proof holds for $f(u) = bu + g(u),\ g \in \mathcal{G}$. Thus, for v a hyperbolic stationary solution of Equation (164) with $\eta_0 = v_x(0) \neq 0$ and $z(v) = n$

$$\begin{aligned} \eta_0 T'_{n+1}(\eta_0) > 0 &\Rightarrow i(v) = n \\ \eta_0 T'_{n+1}(\eta_0) < 0 &\Rightarrow i(v) = n+1. \end{aligned} \tag{172}$$

As in Neumann boundary conditions, $\frac{dT_{n+1}}{db}(\eta) < 0$ for fixed η. For a fixed $\eta_0 > 0$ such that $\eta_0 T'_{n+1}(\eta_0) > 0$, as b is increased, the value of η at which $T_{n+1}(\eta) = \pi$ increases, while for $\eta_0 < 0$ such that $\eta_0 T'_{n+1}(\eta_0) > 0$, this intersection point decreases. Such changes occur in the time map for increasing b if and only if $\eta_0 \frac{db}{d\eta}(\eta_0) > 0$.

For a fixed $\eta_0 > 0$ such that $\eta_0 T'_{n+1}(\eta_0) < 0$, as b is increased, the value of η at which $T_{n+1}(\eta) = \pi$ decreases, while for $\eta_0 < 0$ such that $\eta_0 T'_{n+1}(\eta_0) < 0$, this intersection point increases. Such changes occur in the time map for increasing b if and only if $\eta_0 \frac{db}{d\eta}(\eta_0) < 0$. Thus

$$\begin{aligned} \eta_0 \cdot \frac{db}{d\eta}(\eta_0) > 0 &\Rightarrow i(v) = z(v) \\ \eta_0 \cdot \frac{db}{d\eta}(\eta_0) < 0 &\Rightarrow i(v) = z(v) + 1. \end{aligned} \tag{173}$$

□

Thus, as in the case of Neumann boundary conditions, we may produce a global bifurcation diagram of stationary solutions and determine the degeneracy of a given stationary solution, as well as its zero number and its Morse index, just from the solution's location in the bifurcation diagram.

5. The Y-Map

For Dirichlet boundary conditions we may prove the same results as in Chapter 4 for the equation

$$\begin{aligned} u_t = u_{xx} + \underbrace{bu + g(x,u)}_{f(x,u)}, \quad x \in [0, \pi] \\ u(t,0) = u(t,\pi) = 0. \end{aligned} \tag{174}$$

We retain the definitions of $\mathcal{G}$, $\mathcal{G}_0$, $\mathcal{F}$ and $\mathcal{F}_0$, and construct the y-map in an analogous manner. As we did not construct the y-map to study the lap number over time but rather to study a shifted zero number, the Dirichlet case is more straightforward than the Neumann in terms of the y-map. We provide an analogue to Proposition 4.1 for the Dirichlet case here:

Proposition 10.8. *Any equation of the form (174) where $g(x,u) = g(u)$, i.e. where g and f are only dependent on u, may be rewritten into an equivalent equation in $\widetilde{u}$ where $\widetilde{g} \in \mathcal{G}_0$ and $\widetilde{f} \in \mathcal{F}_0$.*

Proof. We introduce the change of variables $\widetilde{u} = u - v$, where v is any bounded hyperbolic stationary solution of Equation (174). Applying this change of variables transforms Equation (174) into

$$\widetilde{u}_t = \widetilde{u}_{xx} + b\widetilde{u} + g(\widetilde{u}+v) - g(v)$$

(175)
$$\widetilde{u}(t,0) = \widetilde{u}(t,\pi) = 0$$

$$\widetilde{g}(x,\widetilde{u}) = g(\widetilde{u}+v(x)) - g(v(x)), \quad \widetilde{f}(x,\widetilde{u}) = b\widetilde{u} + \widetilde{g}(x,\widetilde{u}).$$

It is clear that $g(\widetilde{u}+v(x)) - g(v(x))$ is a function of both x and $\widetilde{u}$, and additionally that $\widetilde{g}(x,0) = 0$ and thus $\widetilde{f}(x,0) = 0$. Thus, $\widetilde{f} \in \mathcal{F}_0$ and $\widetilde{g} \in \mathcal{G}_0$, and therefore Equation (175) is a scalar parabolic partial differential equation of the form (174). Furthermore, it is possible to rewrite Equation (175) to see that it is linear in $\widetilde{u}$. As nonlinearities in $\mathcal{G}$ are twice continuously differentiable, we may rewrite $g(\widetilde{u}+v)-g(v)$ as $\widetilde{g}(x,\widetilde{u}) = \int_0^1 g'(v+\theta\widetilde{u})d\theta \cdot \widetilde{u}$. $\square$

As in Chapter 4, we assume that $g(u) \in \mathcal{G}_0$; if not, we may transform Equation (174) into Equation (175) and $\widetilde{g}(x,\widetilde{u}) \in \mathcal{G}_0$.

We define the dropping times t_k as before, but must redefine σ_k as follows:

(176)
$$\sigma_k := \begin{cases} sign\ u_x(t,0)\ for\ some\ t \in (t_k, t_{k-1}), & if\ t_k < t_{k-1} \\ 0, & otherwise. \end{cases}$$

Each σ_k is well-defined since $u_x(t,0) \neq 0$ for $t_k < t < t_{k-1}$ via Lemma 10.9.

Lemma 10.9. *Given $f = f(x,u) \in \mathcal{F}_0$ and zero number $z(u(0,\cdot)) < \infty$, define the dropping times of $u(t,\cdot)$ as in (30), and assume $t_k < t_{k-1}$. Then*

$$u_x(t,0) \neq 0 \qquad for\ all\ t \in (t_k, t_{k-1})$$

and thus $sign(u_x(t,0))$ does not depend on $t \in (t_k, t_{k-1})$.

Proof. This proof follows identically to the equivalent proof in [**6**], but applied to a broader range of nonlinearities. A result of Fiedler and Brunovský from [**5**], which corresponds to the equivalent result for Neumann boundary conditions discussed in Lemma 4.2, implies that for any $t \in (t_k, t_{k-1})$, there exists an $\varepsilon > 0$ such that $u(t',x')$ has one sign only for $0 < x' < \varepsilon$ and $|t-t'| < \varepsilon$. This implies that $u_x(t',0) \neq 0$, for $|t - t'| < \varepsilon$, via the strong maximum principle [**34**], and thus $sign(u_x(t, 0))$ is independent of $t \in (t_k, t_{k-1})$. $\square$

As before, it is clear that y maps into S^n and that $y(u_0) = \sigma e_k$ implies that $0 = t_n = \ldots = t_k$ and $t_{k-1} = \infty$, or in other words, $z(u_0) = k$ and $z(u(t, \cdot)) = k$ for all finite forward time. From Lemma 10.9 it follows that the sign of $u_x(t, 0)$ cannot change for $t \in (t_k, t_{k-1})$ and thus $\sigma \cdot u_x(t, 0) > 0$ for all non-dropping times.

Lemma 4.3 holds identically for Dirichlet boundary conditions. The only changes we must introduce into the proof are noting that $u(t, x)$ is a map into $H^2(\mathbb{R}, [0, \pi]) \cap H_0^1(\mathbb{R}, [0, \pi])$ rather than $H^2(\mathbb{R}, [0, \pi]) \cap \{Neumann\ Boundary\ Conditions\}$, replacing $u(t, 0; f, u_0)$ with $u_x(t, 0; f, u_0)$, and finally replacing $u(t, 0; f, u_0) \neq 0$ by implication of Lemma 4.2 with $u_x(t, 0; f, u_0) \neq 0$ by implication of Lemma 10.9.

Recall that the Sturm-Liouville eigenfunctions of

$$\begin{aligned} & u_{xx} + b(x)u = \lambda u \\ & u(0) = u(\pi) = 0 \end{aligned} \tag{177}$$

fulfill Dirichlet boundary conditions rather than Neumann, and thus sign conventions are chosen based on $(\varphi_i(0))_x > 0$. We may proceed identically to Chapter 4 in order to prove that $y : \Sigma^n \to S^n$ is an essential mapping for the Dirichlet system. Thus, we may proceed to prove Dirichlet forms of Lemma 4.6 and Corollary 4.7.

LEMMA 10.10. *Let $v \equiv 0$ be a hyperbolic stationary solution of (174) with unstable manifold W^u of dimension $i(v) > 0$. Let $\Sigma \subset W^u \backslash \{v\}$ be homotopic in $W^u \backslash \{v\}$ to a small sphere centered at W^u of dimension $n = i(v) - 1$. For any finite sequence*

$$\begin{aligned} & 0 = \delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_0 \leq \infty \\ & s_k \in \{1, -1\}, \qquad 0 \leq k \leq n, \end{aligned} \tag{178}$$

there exists a point $u_0 \in \Sigma$ corresponding to an initial condition $u(0, \cdot) \in X$ such that the graph $t \to z(u(t, \cdot))$ is determined by (δ_k). In other words, for any $0 \leq t < \infty$,

$$\begin{aligned} & t \geq \delta_k \Leftrightarrow z(u(t, \cdot)) \leq k \\ & \delta_k < t < \delta_{k-1} \Rightarrow sign(u_x(t, 0)) = s_k. \end{aligned} \tag{179}$$

Proof. The assumption that $v \equiv 0$ is a stationary solution of Equation (174) implies that $f(x,0) = 0$, thus $f \in \mathcal{F}_0$. If we wish to focus on a nonlinearity $f(u)$ where $f(0) \neq 0$, then we assume that the nonlinearity $f(x,u) = \widetilde{f}(x,\widetilde{u})$ and drop the tildes for convenience of notation.

We first assume that the restricted y-map, $y : \Sigma \to S^n$ is essential. Therefore, y is surjective. We now define the vector ς exactly as y was defined in (30, 32, 176), replacing t_k with δ_k and σ_k with s_k. By the surjectivity of y, there exists an initial datum $u_0 \in \Sigma$ such that $y(u_0) = \varsigma$. But as we noted earlier, knowing $y(u_0)$ uniquely determines the dropping times t_k and signs σ_k of the solution $u(t,\cdot)$ corresponding to u_0. Thus, it is determined that $t_k = \delta_k$ and $\sigma_k = s_k$ whenever $\delta_k < \delta_{k-1}$.

In order to prove that y is essential, we must homotopically deform our nonlinearity f from the corresponding linear form. We define

$$f_\vartheta(x,u) := \vartheta f(x,u) + (1-\vartheta) f_u(x,0) \cdot u \tag{180}$$

or, recalling the types of nonlinearities f over which we are interested,

$$\begin{aligned} f_\vartheta(x,u) = bu + g_\vartheta(x,u) &:= bu + \vartheta g(x,u) + (1-\vartheta) g_u(x,0) \cdot u \\ \Rightarrow g_\vartheta(x,u) &:= \vartheta g(x,u) + (1-\vartheta) g_u(x,0) \cdot u \end{aligned} \tag{181}$$

with the homotopy parameter $0 \leq \vartheta \leq 1$. As we deform f, the unstable manifold of the stationary solution $v \equiv 0$ of (174) with a specific nonlinearity f_ϑ is simultaneously deformed. The linearization at $v \equiv 0$ in the homotopically deformed system

$$0 = u_{xx} + bu + \vartheta g_u(x,0)u + (1-\vartheta) g_u(x,0)u = u_{xx} + bu + g_u(x,0)u \tag{182}$$

is entirely unchanged. Additionally, $f_\vartheta \in \mathcal{F}_0$ depends continuously on ϑ as $\mathcal{F}_0$ supports the weak Whitney topology.

We denote the cut-off tangent space of $W^u(f_\vartheta)$ at $v \equiv 0$ for $\vartheta = 0$ by

$$W^u_{loc}(f_0) := span\left\{\varphi_0, \ldots, \varphi_n\right\} \cap \left\{u_0 \in X \middle| |u_0| < 2\varepsilon\right\}. \tag{183}$$

The local unstable manifolds with respect to an f_ϑ are then parametrized by diffeomorphisms

$$\rho_\vartheta : W^u_{loc}(f_0) \to W^u_{loc}(f_\vartheta) \tag{184}$$

where ρ_ϑ^{-1} is induced by the orthogonal projection onto $span\,\{\varphi_0, \dots, \varphi_n\}$. We observe that ρ_ϑ depends continuously on ϑ in the uniform C^0 topology.

Now we fix a sphere

$$\Sigma^n := \{u \in W^u_{loc}(f_0) |\ |u| < \varepsilon\} \tag{185}$$

in the cut-off unstable manifold of $v \equiv 0$ and let y^ϑ denote the restriction to $\rho_\vartheta(\Sigma^n)$ of the y-map associated to f_ϑ. After a homotopy we may assume that $\Sigma = \rho_1(\Sigma^n)$. Finally, we define

$$\overline{y}_\vartheta := y^\vartheta \cdot \rho_\vartheta : \Sigma^n \to S^n. \tag{186}$$

This mapping is well-defined, as $z(u) \leq n$ on $W^u(f_\vartheta)$. The mapping is continuous, and depends continuously on ϑ thanks to the Dirichlet form of Lemma 4.3. The Dirichlet form of Lemma 4.5 implies that $\overline{y}^0 = y_0 \cdot \rho_0 = y_0 : \Sigma^n \to S^n$ is essential. By the homotopy invariance of this property, $\overline{y}^1 = y_1 \cdot \rho_1 = y \cdot \rho_1$ is essential, and therefore y is essential.

Thus, choosing a sequence of δ_k and s_k we have shown that for any $0 \leq t < \infty$

$$t \geq \delta_k \Leftrightarrow z(u(t,\cdot)) = z(u(t,\cdot) - v) \leq k$$

$$\delta_k < t < \delta_{k-1} \Rightarrow sign(u_x(t,0)) = s_k.$$

$\square$

Corollary 10.11. *Let v be a non-trivial hyperbolic stationary solution of*

$$\begin{aligned} u_t &= u_{xx} + bu + g(u), \quad x \in [0,\pi] \\ u(t,0) &= u(t,\pi) = 0, \quad g(u) \in \mathcal{G} \end{aligned} \tag{187}$$

with Morse index $i(v) = n + 1 > 0$. Let $\Sigma \subset W^u \backslash \{v\}$ be homotopic in $W^u \backslash \{v\}$ to a small sphere centered at W^u of dimension n. For any finite sequence

$$
\begin{aligned}
&0 = \delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_0 \leq \infty \\
&s_k \in \{1, -1\}, \qquad 0 \leq k \leq n,
\end{aligned}
\tag{188}
$$

there exists a point $u_0 \in \Sigma$ corresponding to an initial condition $u(0, \cdot) \in X$ such that the graph $t \to z(u(t, \cdot) - v(\cdot))$ is determined by (δ_k). In other words, for any $0 \leq t < \infty$,

$$
\begin{aligned}
&t \geq \delta_k \Leftrightarrow z(u(t, \cdot) - v(\cdot)) \leq k \\
&\delta_k < t < \delta_{k-1} \Rightarrow sign(u_x(t, 0) - v_x(0)) = s_k.
\end{aligned}
\tag{189}
$$

Proof. We now extend the results of Lemma 10.10 to any hyperbolic stationary solution v of (187). Let u be a solution of (187) with $g(u) \in \mathcal{G}$. Then $\widetilde{u} := u - v$ satisfies

$$
\begin{aligned}
&\widetilde{u}_t = \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}(x, \widetilde{u}) \\
&\widetilde{g}(x, \widetilde{u}) := g(\widetilde{u} + v(x)) - g(v(x)),
\end{aligned}
\tag{190}
$$

noting that $\widetilde{g}(x, 0) = 0$. The eigenvalue problem of (190) at a hyperbolic stationary solution $\widetilde{w} = w - v$ is

$$
\lambda u = u_{xx} + bu + \widetilde{g}_u(x, \widetilde{w})u = u_{xx} + bu + g_u(w)u. \tag{191}
$$

If we assume that the stationary solution w in Equation (187) has Morse index j and therefore $\lambda_0 > \lambda_1 > \ldots > \lambda_{j-1} > 0 > \lambda_j > \ldots$, with corresponding eigenfunctions $\varphi^0, \ldots, \varphi^{j-1}, \ldots$, it is clear that $\widetilde{w}$ must have the same eigenfunctions and eigenvalues, and therefore the same Morse index, since the eigenvalue problem for a stationary solution w of (187) is the same as the eigenvalue problem for the corresponding stationary solution $\widetilde{w}$ of (190).

The arguments within the proof of Lemma 10.10 hold in the shifted system, but regarding initial datum $\widetilde{u}_0 = u_0 - v$ and corresponding solutions $\widetilde{u} = u - v$. This is due to the fact that the zero number of the difference of two solutions to Equation

(187) is nonincreasing in time. Thus, Lemma 10.10 asserts that there exists an initial datum $\widetilde{u}_0$ such that for $0 \leq t < \infty$

$$t \geq \delta_k \Leftrightarrow z(\widetilde{u}(t,\cdot)) \leq k$$

$$\delta_k < t < \delta_{k-1} \Rightarrow sign(\widetilde{u}_x(t,0)) = s_k$$

for any choice of sequences

$$0 = \delta_n \leq \delta_{n-1} \leq \ldots \leq \delta_0 \leq \infty$$

$$s_j \in \{1,-1\}\,, \quad 0 \leq j \leq n.$$

□

Finally, in place of Lemma 4.8 we introduce Lemma 4.2 of [**6**]. We redefine the set J_v to be $(-v_x(0), v_x(0))$, recalling that the symmetry in the phase portrait ensures that $v_x(\frac{\pi}{n}) = -v_x(0)$ for a stationary solution with lap number n.

Lemma 10.12. *Consider an equation*

$$u_t = u_{xx} + f(u)$$

$$x \in [0,\pi],\ t \geq 0,\ f \in C^2 \tag{192}$$

$$u(t,0) = u(t,\pi) = 0.$$

Let v^1 and v^2 be two distinct stationary solutions. Then $\mid v_x^1(0) \mid \geq \mid v_x^2(0) \mid$ implies

$$z(v^1 - v^2) = z(v^1). \tag{193}$$

We do not require an analogue to Corollary 4.9, as there do not exist any pitchfork bifurcations in the bifurcation diagram for Dirichlet conditions, except those of the n-branches from the b-axis.

6. Heteroclinic Connections and the Asymptotic Behavior of Grow-Up Solutions

Now that we have proved the existence of an analogous y-map for Dirichlet boundary conditions, we may proceed to prove analogous blocking lemmas and an analogous grow-up lemma.

LEMMA 10.13. (Finite Blocking Lemma for Dirichlet Boundary Conditions) *Let v and w be two distinct stationary solutions of Equation (187), v hyperbolic, and let $\overline{w}(0,\cdot)$ be a function which solves Equation (187) such that $w_x(0)$ lies strictly between $v_x(0)$ and $\overline{w}_x(0,0)$. Then*

$$z(v-w) \leq z(\overline{w}-w)$$

implies that v does not connect to $\overline{w}$.

We remind readers that v connects to a function $\overline{u}(0,\cdot)$ solving Equation (187) if there exists some solution $u(t,\cdot)$ to (187) such that $u(T,\cdot) = \overline{u}(0,\cdot)$ for some $T \geq 0$ and $\lim\limits_{t\to-\infty} u(t,\cdot) = v$.

PROOF. We proceed by contradiction. Assume that v connects to $\overline{w}$ via a trajectory $u(t,\cdot)$, $t \in (-\infty, T]$. Then $\widetilde{u} = u - w$ satisfies an equation of the form

$$\begin{aligned} \widetilde{u}_t &= \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}(x,\widetilde{u}) \\ \widetilde{g}(x,\widetilde{u}) &:= g(\widetilde{u} + w(x)) - g(w(x)), \quad \widetilde{g}(x,0) = 0. \end{aligned} \tag{194}$$

Via the results of Lemma 10.10 and Corollary 10.11, we may assume that $w = 0$ without loss of generality by working in the shifted system (194). Thus, either $\widetilde{v}_x(0) < 0 < \widetilde{\overline{w}}_x(0,0)$ or $\widetilde{\overline{w}}_x(0,0) < 0 < \widetilde{v}_x(0)$. The nonincrease of the zero number $z(u(t,\cdot))$ on the trajectory connecting v to $\overline{w}$ and the concomitant nonincrease of $z(\widetilde{u}(t,\cdot))$ on the trajectory connecting $\widetilde{v}$ to $\widetilde{\overline{w}}$ imply that

$$z(v-w) = z(\widetilde{v}) \geq z(\widetilde{\overline{w}}) = z(\overline{w}-w).$$

Recalling Lemma 10.9, we know that $z(\widetilde{v}) \neq z(\widetilde{\overline{w}})$ as $\widetilde{v}_x(0)$ and $\widetilde{\overline{w}}_x(0,0)$ have opposite signs. Therefore $z(v-w) > z(\overline{w}-w)$ if v connects to $\overline{w}$ and the lemma is proved by contraposition. $\square$

REMARK 10.14. *If we let $\overline{w}(0,\cdot)$ be a stationary solution of Equation (187), then the Dirichlet Finite Blocking Lemma 10.13 still holds with $T = \infty$.*

Lemma 10.15. (Infinite Blocking Lemma for Dirichlet Boundary Conditions) *Let v and w be two distinct stationary solutions to (187), v hyperbolic. Let $\sigma = sign(w_x(0) - v_x(0))$. If*

$$z(v - w) \leq k,$$

then all trajectories $u(t,\cdot)$ in the unstable manifold of v wherein $sign(u_x(t,0) - v_x(0)) = \sigma$ and $z(u(t,\cdot) - v(\cdot)) = j \geq k$ for all $t \in (t_j, \infty]$ with $t_j < \infty$ remain bounded. In other words, v does not contain any heteroclinic connections to objects at infinity with zero number greater than or equal to k.

Proof. We proceed by contradiction. Assume that v connects to an object at infinity Φ^∞ via a given trajectory $u(t,\cdot)$, $t \in \mathbb{R}$. We require that the renormalized Φ^∞ have only simple zeros, which can be shown to hold for all relevant objects at infinity. Thus, the zero number of Φ^∞ is well defined; let $z(\Phi^\infty) = z(\Phi^\infty - v) = j$ and $sign(\Phi^\infty_x(0)) = \sigma$. This implies that $u(t,\cdot)$ limits to infinity in the C^1-norm (and thus all lower norms) as t goes to infinity, i.e. $u(t,\cdot)$ is a grow-up solution. Let $\overline{w}(0,\cdot)$ be any function such that $w_x(0)$ lies strictly between $v_x(0)$ and $\overline{w}_x(0,0)$ and $z(\overline{w} - w) = j$. By Lemma 10.13, v does not connect to $\overline{w}$, since $j \geq k$.

If v connects to Φ^∞ where $z(\Phi^\infty) = j$ via a trajectory $u(t,\cdot)$, then in the shifted equation, v must connect to $\Phi^\infty - w = \Phi^\infty$ via a trajectory $\widetilde{u}(t,\cdot)$. It then follows that $z(u(t,\cdot) - v(\cdot)) \geq j$ for all time $t \in \mathbb{R}$. This implies that at some time $T < \infty$, the value of $w_x(0)$ must lie between $u_x(T,0) = \overline{w}_x(0,0)$ and $v_x(0)$, since it lies between $\Phi^\infty_x(0)$ and $v_x(0)$. But this leads to a contradiction, as the Dirichlet Finite Blocking Lemma 10.13 prevents $u_x(t,\cdot)$ from ever crossing w_x in its left intercept. Therefore, v cannot connect to any object at infinity with zero number greater than or equal to k. Thus, any trajectories in the unstable manifold of v where the shifted zero number never drops below k must remain bounded. $\square$

Lemma 10.16. *Let v be a hyperbolic stationary solution to (187) such that $i(v) = n+1$. Fix k such that $0 \leq k \leq n$ and $\sigma \in \{1, -1\}$. If v is not blocked from connecting to infinity in the sense of Lemma 10.15, i.e. there does not exist a stationary solution*

w to (187) such that $z(w) = k$, $w \notin EJ_v$, and $sign(w_x(0) - v_x(0)) = \sigma$, and when $k = n = z(v)$, there additionally does not exist any $w \in EJ_v$ such that $z(w) \leq n$, then there exists an initial condition $u_0 \in W^u(v)$ and a corresponding solution $u(t, \cdot)$ to (187) such that the following hold:

$$
\begin{aligned}
&z(u(t,\cdot) - v(\cdot)) = k \text{ for all } 0 \leq t < \infty \\
&sign(u_x(t,0) - v_x(0)) = \sigma \\
&\lim_{t \to -\infty} u(t,\cdot) = v \\
&\lim_{t \to \infty} \| u(t,\cdot) \|_{L^2} = \infty.
\end{aligned}
\tag{195}
$$

Proof. We may apply Lemma 10.10 for $v \equiv 0$ or Corollary 10.11 for v nontrivial to the set of solutions of (187), choosing

$$
\delta_j := \begin{cases} 0 & for \quad j \geq k \\ \infty & for \quad j < k \end{cases}
$$

$$
s_k := \sigma.
$$

By Lemma 10.10 or Corollary 10.11 there exists an initial condition $u_0 \in W^u(v)$ corresponding to our choice of k and σ, and the lemma or corollary asserts that for the solution $u(t, \cdot)$ corresponding to $u(0, \cdot) = u_0$, the properties $z(u(t, \cdot) - v) = k$ and $sign(u_x(t, 0) - v_x(0)) = \sigma$ hold for all $0 \leq t < \infty$. Since u_0 is in the unstable manifold of v, this implies that $\lim_{t \to -\infty} u(t, \cdot) = v$.

Further, $u(t, \cdot)$ cannot connect to some bounded equilibrium $\overline{w} \in EJ_v$ with zero number less than k. For $u(t, \cdot)$ to connect to $\overline{w}$, the zero number in the shifted system $z(\widetilde{u}(t, \cdot)) = z(u(t, \cdot) - v)$ must drop at $t = \infty$. To show why this is not possible, let us assume that $\lim_{t \to \infty} u(t, \cdot) = \overline{w}$ with $z(\overline{w}) < k$. Then $\lim_{t \to \infty} \widetilde{u}(t, \cdot) = \overline{w} - v = \widetilde{\overline{w}}$. Because $\widetilde{\overline{w}}(t, \cdot) \neq 0$, it follows that $\widetilde{\overline{w}}$ must have only simple zeros, as it solves the ordinary differential equation $0 = \widetilde{u}_{xx} + b\widetilde{u} + \widetilde{g}(x, \widetilde{w})$. Any solution in a small neighborhood of $\widetilde{\overline{w}}$ must also have simple zeros. Thus, the shifted zero number is constant over some small neighborhood of $\widetilde{\overline{w}}$, and therefore in this neighborhood $z(\widetilde{u}) = z(\widetilde{\overline{w}}) = z(\overline{w} - v) < k$.

But if this is the case, then $z(u(t,\cdot)-v(\cdot))$ would have to drop at some finite time as $u(t,\cdot)$ approaches $\overline{w}$. Therefore, if $u(t,\cdot)$ connects to a bounded equilibrium, this equilibrium must fulfill $z(w-v)=k$, which occurs when $z(w)=k$ for $w \notin EJ_v$, or when $k=z(v)$ for $w \in EJ_v$. Since $k=z(v)$ implies that $k=n$, and $u(t,\cdot)$ will only limit to stationary solutions $w \in EJ_v$ if $i(w)<i(v)$ and therefore $z(w) \leq n$ by Lemma 10.17, it follows that the only bounded stationary solutions to which $u(t,\cdot)$ may connect are those listed in the lemma. Additionally, Lemma 10.10 and Corollary 10.11 imply that the sign of $(u_x(t,0)-v_x(0))$ remains always positive or always negative, therefore, if $u(t,\cdot)$ were to limit to any bounded equilibrium w, it would have to be one such that $w_x(0)-v_x(0)=\sigma$.

Since there does not exist any bounded stationary solutions w fulfilling these conditions, we may conclude that $u(t,\cdot)$ cannot limit to any bounded stationary solution. But as Lemma 2.1 states, $u(t,\cdot)$ can then not be bounded in any ball, no matter how large, and therefore $\lim\limits_{t\to\infty} \parallel u(t,\cdot) \parallel = \infty$ in any appropriate norm in H^2. Since the L^2-norm of $u(t,\cdot)$ must be less than or equal to the H^1, H^2 and C^1 norms, we choose it for the formulation of our lemma, to guarantee the infiniteness of the other norms. Finally, knowing that $u(t,\cdot)$ grows to be infinitely large, we may conclude that there exist infinitely many times $t^l>0$ at which $z(u(t^l,\cdot)-v)=z(u(t^l,\cdot))$. $\square$

Remark 5.5 holds for Dirichlet boundary conditions with the substitution of zero number for lap number and $w_i^{\pm}(0)$ for $(w_i^{\pm}(0))_x$. We have thus determined when trajectories in the unstable manifold of a stationary solution for the Dirichlet problem are grow-up solutions.

Lemma 10.17. *If $v,\ w \in E$ satisfy $i(w) \geq i(v)$, then v does not connect to w.*

Proof. In [**19**], Henry proved that for v and w stationary solutions to (187), not necessarily hyperbolic, the stable and unstable manifolds of v and w intersect transversely if they intersect at all. Since $dim\ W^u(v)=i(v)\leq i(w)=\ codim\ W^s(w)$, it follows that $dim\ W^u(v) \cap W^s(w) \leq 0$. Since $v \neq w$, it follows that v cannot connect to w. $\square$

Lemma 10.18. *Let $v \in E$ be a hyperbolic stationary solution of (187) and let $w \in E$ be a second stationary solution, $w \neq v$, such that $z(v-w) \geq i(v)$. Then v does not connect to w.*

Proof. Let us assume that v connects to w, i.e. that there exists an initial condition $u_0 \in W^u(v)$ such that $\lim_{t\to-\infty} u(t,\cdot) = v$ and $\lim_{t\to\infty} u(t,\cdot) = w$ for the corresponding solution $u(t,\cdot)$. Then Lemma 10.17 implies that $i(w) < i(v)$. Further, Fiedler and Brunovský proved in [**4**] that $z(u-v) \geq i(v)$ for $u \in W^s(v)\backslash\{v\}$ and $z(u-v) < i(v)$ for $u \in W^u(v)\backslash\{v\}$.

By Lemma 10.10 or Corollary 10.11, any stationary solution to which v connects must satisfy $z(w-v) = k$ and $sign(w_x(0)-v_x(0)) = \sigma$ for some choice of $0 \leq k < i(v)$ and $\sigma \in \{1,-1\}$, due to the property that $z(u_0 - v) < i(v)$ for all initial datum u_0 in the unstable manifold of v. Thus $k = z(v-w) < i(v)$, and the lemma is proved by contraposition. $\square$

Finally, we provide a Dirichlet form of Lemma 5.8.

Lemma 10.19. *Given any non-trivial stationary solution v where $i(v) = z(v)+1$, let w be the nearest stationary solution such that $w \notin EJ_v$ and $z(w) \leq z(v)$. If w exists, then $z(w) = i(w)$.*

Proof. Due to the spacing of n-branch origination points, and the fact that they cannot intersect, it follows that the $n-1$-branch, if it exists, is between the n-branch and all lower branches, and the $n+1$-branch is between the n-branch and all higher branches. By Lemma 10.7, we know that for $v_x(0) = \eta_0$, $(\eta_0) \cdot \frac{db}{d\eta} < 0$. It follows that the nearest branch, both above and below EJ_v, must be either the $n+1$-branch or the n-branch if any stationary solutions exist either above or below EJ_v. Let us assume there exists a stationary solution $w \notin EJ_v$ such that $z(w) \leq z(v)$, and let us fix $z(w) = k$.

For $i(w) = k+1$, we would need $w_x(0) \cdot \frac{db}{d\eta} \mid_{w_x(0)} < 0$. But since lower branches always exist to the left of higher branches, it follows that we must first have a region where $\eta_0 \cdot \frac{db}{d\eta} > 0$ on the $n+1$-branch if it crosses the line at b, then on the n-branch, and so forth for all intermediary branches, or else the k-branch could not cross the line at b. But the stationary solutions in these regions have $i(w_{n+1}) = z(w_{n+1})$, $i(w_n) = z(w_n)$, ... Thus, there exists a stationary solution w_j between v and all solutions such that $z(w) = k < j \leq n = z(v)$. Furthermore, the continuity of the k-branch to its origination point implies that the k-branch itself must first cross the line at b with $(\eta - \eta^*) \cdot \frac{db}{d\eta} > 0$. Therefore, excluding solutions in EJ_v and those with $z(w) > z(v)$, it follows that the nearest stationary solutions both above and below v such that $z(w) = k \leq n$ must fulfill $z(w) = i(w) = k$ if they exist. If they do not exist, then there are no stationary solutions with $z(w) \leq k$, as all lower lap number stationary solutions are to the left of the k-branch. $\square$

We define K^+ and K^- as before, but with respect to zero number rather than lap number. Remark 5.5 then ensures that there do not exist any bounded equilibria not in EJ_v with left boundary slope greater than $v_x(0)$ and zero number less than K^+, and that there do not exist any bounded equilibria not in EJ_v with left boundary slope less than $v_x(0)$ and zero number less than K^-. Then Lemma 10.16 on infinite liberalism states that there exist heteroclinics in $W^u(v)$ which grow to infinity with $K^+ + K^-$ distinct behaviors denoted by sign and asymptotic zero number. In the vast majority of cases, there will be infinitely many heteroclinics of each type, in a small minority of situations (such as when the difference between the Conley index of v and its limiting object differs by only 1), there will be only one heteroclinic of a given type.

We consider a specific grow-up solution. Let $u_k^{\pm}(t, \cdot)$ denote the solution with $z(u_k^{\pm}(t, \cdot) - v(\cdot)) = k$ for $0 \leq t < \infty$ with $u_x(t, 0) > v_x(0)$ for u_k^+ and $u_x(t, 0) < v_x(0)$ for u_k^-. For u_k^+ it is clear that $k < K^+$. For u_k^- it is clear that $k < K^-$. Our analysis in Section 5.2 holds for Dirichlet boundary conditions with only the replacement of the eigenfunctions $\Phi_j(x) = c_j \cos(jx)$ with $\Phi_j(x) = c_j \sin((j+1)x)$, i.e. for the Dirichlet case the 0th eigenfunction is $c_0 \sin(x)$. The analysis proceeds as before, implying the limit of $\frac{u(t,\cdot)}{\|u(t,\cdot)\|_{L^2}}$ to a single eigenfunction $\Phi_k^{\pm}$ with $k \leq K^+$ or $k \leq K^-$ for the appropriate sign, but again we reach the inability to determine C^1-closeness via a straightforward evaluation of the strong limit.

The construction of the inertial manifold in the case of Dirichlet boundary conditions is essentially unchanged, we must simply replace the Neumann eigenfunctions with the Dirichlet and $H^2 \cap \{Neumann\ Boundary\ Conditions\}$ with $H^2 \cap H_0^1$. Thus, all results follow for our equation with Dirichlet boundary conditions just as in Neumann boundary conditions.

The existence of an inertial manifold which is Lipschitz with values in C^1 provides the ability to determine C^1-closeness as well as L^2-closeness of solutions, and thus prevents the dropping of the zero number at infinity for unbounded solutions. Combining all these elements yields Dirichlet versions of Theorem 7.4 and Theorem 8.1.

THEOREM 10.20. *Let $g(u) \in \mathcal{G}$, $b > 1$, and let v be a hyperbolic stationary solution of*

$$
\begin{aligned}
u_t &= u_{xx} + bu + g(u), \quad x \in [0, \pi] \\
u(t, 0) &= u(t, \pi) = 0
\end{aligned}
\tag{196}
$$

such that $i(v) = n + 1$ and $z(v) \in \{n + 1, n\}$. For every $\sigma \in \{1, -1\}$ and $0 \leq k \leq n$ such that v is not blocked from reaching infinity by Lemma 10.15 , v connects to an equilibrium at infinity $\mathbf{\Phi}_k^\sigma$ with $z(\mathbf{\Phi}_k^\sigma) = k$ and $sign(\mathbf{\Phi}_k^\sigma(0))_x = \sigma$. In other words there exists a trajectory $u_k^\sigma(t, \cdot) \in W^u(v)$ such that

$$\lim_{t \to -\infty} u_k^\sigma(t, \cdot) = v$$

$$sign((u_k^\sigma)_x(t, 0) - v_x(0)) = \sigma$$

$$\lim_{t \to \infty} z(u_k^\sigma(t, \cdot) - v(\cdot)) = k$$

$$\lim_{t \to \infty} \| u_k^\sigma(t, \cdot) \|_{L^2} = \infty$$

$$\lim_{t \to \infty} \| \frac{u_k^\sigma(t, \cdot)}{\| u_k^\sigma(t, \cdot) \|_{L^2}} - \frac{\Phi_k^\sigma(\cdot)}{\| \Phi_k^\sigma(\cdot) \|_{L^2}} \|_{C^1} = 0$$

where $\Phi_k^\sigma(x) = \sigma sin((k + 1)x)$ and $\frac{\mathbf{\Phi}_k^\sigma(\cdot)}{\|\mathbf{\Phi}_k^\sigma(\cdot)\|_{L^2}} = \Phi_k^\sigma(\cdot)$.

PROOF. The proof of Theorem 10.20 follows exactly from the proof of Theorem 7.4, as it is the culmination of the results derived in previous chapters. We only need to replace in the Dirichlet-appropriate notation, zero number $z(v)$ for lap number $l(v)$, Lemma 10.16 for Lemma 5.4, and $u_x(t,0) - v_x(0)$ for $u(t,0) - v(0)$. Thus the dynamics of $u_k^\sigma(t,\cdot)$ are determined by the $(N+1)$-dimensional ODE

$$\begin{aligned} \frac{dp}{dt} &= -Ap + P_N g(p + \mathbf{\Psi}(p)) \\ p = [p_0, \ldots, p_N], \quad p_i(t,\cdot) &= \widehat{p_i}(t)\Phi_i(\cdot) = \langle u_k^\sigma(t,\cdot), \Phi_i(\cdot)\rangle_0 \, \Phi_i(\cdot) \\ p_i(t,0) &= p_i(t,\pi) = 0, \end{aligned} \tag{197}$$

with Φ_i the ith eigenfunction of the operator $A = -\frac{d^2}{dx^2} - bI$ with Dirichlet boundary conditions. $\square$

For the Dirichlet problem we define our frequently referred to sets as follows:

$$Z_n := \{v \in E | z(v) = n \text{ or } v \equiv 0\},$$

$$\Omega_E(v) := \{w \in E | \text{v connects to } w \neq v\},$$

$$W_k(v) := \{w \in Z_k | \;\; w_x(0) \in EJ_v \backslash EJ_{\underline{v}}, \; i(w) < i(v)\},$$

and for $0 \leq k < i(v)$ we define

$\overline{v}_k$ is the bounded stationary solution $\widehat{v}$ with $z(\widehat{v}) = k$ such that

$$\widehat{v} \notin EJ_v, \quad \widehat{v}_x(0) > v_x(0) \text{ is minimal},$$

$\underline{v}_k$ is the bounded stationary solution $\widehat{v}$ with $z(\widehat{v}) = k$ such that

$$\widehat{v} \notin EJ_v, \quad \widehat{v}_x(0) < v_x(0) \text{ is maximal},$$

$\underline{v}$ is the stationary solution $w \in EJ_v \cap Z_{z(v)}$ with maximal $w_x(0)$,

$\overline{v}$ is the stationary solution $w \in EJ_v \cap Z_{z(v)}$ with minimal $w_x(0)$,

and

$$\mathbb{L} := \underline{v} \cup \bigcup_{k<z(v)} W_k(v) \text{ or } \mathbb{L} := \overline{v} \cup \bigcup_{k<z(v)} W_k(v)$$

as appropriate, and

$$\mathbb{K} := \mathbb{L} \cap (Z_{n-1} \cup Z_n).$$

Lemmas 7.6 and 7.7 were independent of the choice of boundary conditions and simply require the replacement of $l(v)$ with $z(v)$. Lemmas 7.5 and 7.8 on the other hand, require minor changes for their application to Dirichlet boundary conditions. We must replace Neumann-specific notation, i.e. the lap number, definition of σ, the definition of sets and special solutions, and references to lemmas in previous chapters with the Dirichlet-specific definitions, references and zero number. Finally, we use the result proved in [**7**] which showed that for Dirichlet boundary conditions, if v connects to any element of $\mathbb{L}$, then it connects to all elements of $\mathbb{L}$, which carries over to the slowly non-dissipative equation.

THEOREM 10.21. *Let $g(u) \in \mathcal{G}$, $b > 1$ and let $v \in E$ be a bounded hyperbolic stationary solution of (196). Then v connects to all bounded equilibria $w \in E$ and equilibria at infinity $\mathbf{\Phi}_i^{\pm}$ which are not blocked by Lemmas 10.13 - 10.18.*

Equivalently, v connects to bounded equilibria and equilibria at infinity as follows:

(1) *If $v \equiv 0$, or if $v \neq 0$ and $i(v) = z(v)$, then*

$\Omega(v) = \{\underline{v}_k \mid K^- \leq k < i(v)\} \cup \{\overline{v}_k \mid K^+ \leq k < i(v)\}$

$\cup \; \{\mathbf{\Phi}_i^+ \mid 0 \leq k < K^+\} \cup \{\mathbf{\Phi}_i^- \mid 0 \leq k < K^-\}$.

(2) *If $v_x(0) > 0$ and $i(v) = z(v) + 1$, then*

$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$, *where*

$\Omega_1 = \{\overline{v}_k \mid K^+ \leq k < i(v)\}$,

$\Omega_2 = \{\underline{v}_k \mid K^- \leq k < i(v) - 1\}$,

if $K^- < i(v)$, either $\Omega_3 = \{\underline{v}_{i(v)-1}\}$ if $EJ_v = \emptyset$, or else $\Omega_3 = \underline{v} \cup \bigcup_{k<z(v)} W_k$,

$\Omega_4 = \{\mathbf{\Phi}_k^+ \mid 0 \leq k < K^+\}$, *and*

$\Omega_5 = \{\mathbf{\Phi}_k^- \mid 0 \leq k < K^-\}$.

(3) *If $v_x(0) < 0$ and $i(v) = z(v) + 1$, then*

$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$, *where*

$\Omega_1 = \{\underline{v}_k \mid K^- \leq k < i(v)\}$,

$\Omega_2 = \{\overline{v}_k \mid K^+ \leq k < i(v) - 1\}$, *if* $K^+ < i(v)$, *either* $\Omega_3 = \{\overline{v}_{i(v)-1}\}$ *if* $EJ_v = \emptyset$, *or else* $\Omega_3 = \overline{v} \cup \bigcup_{k<z(v)} W_k$,

$\Omega_4 = \{\mathbf{\Phi}_k^+ \mid 0 \leq k < K^+\}$, *and*

$\Omega_5 = \{\mathbf{\Phi}_k^- \mid 0 \leq k < K^-\}$.

Proof. Again, the proof of Theorem 8.1 carries over to the Dirichlet case with a few simple changes of definition and the use of the Dirichlet lemmas and theorem provided in this chapter rather than their Neumann counterparts, as in the proof of Theorem 10.20. □

The non-compact global attractor is defined equivalently for the Dirichlet equation, with the caveat that the semigroup $\mathcal{S}$ is non-dissipative for $b > 1$ rather $b > 0$.

Thus, it is straightforward to see that all results presented in this thesis for the Neumann form of the slowly non-dissipative dynamical system carry over to the Dirichlet form with a little extra work. Although characteristics of each individual dynamical system may change, i.e. the lack of spatially homogenous stationary solutions and pitchfork branches, the greater results on the asymptotics of solutions and the non-compact attractor do not.

CHAPTER 11

Conclusion

1. Summary of Results

In this thesis we have solved the so-called "connection problem" and "asymptotics problem" in the realm of slowly non-dissipative systems, and in so doing, provided a full decomposition of the non-compact global attractor for a generic class of such systems and described the long-time behavior of solutions therein. We have extended a number of useful techniques to the realm of slowly non-dissipative systems, among these the y-map for the study of asymptotic nodal properties. We have shown the existence of characteristic behaviors for all scalar slowly non-dissipative PDEs, including the behavior of global bifurcation diagrams, as well as the existence and asymptotic behaviors of grow-up solutions.

We have proven the existence of a completed inertial manifold for a general class of grow-up systems, taking the classical inertial manifold for dissipative systems and constructing a corresponding object for slowly non-dissipative systems. We have further proven that this manifold is Lipschitz with values in C^1. This completed inertial manifold provides the same advantages as its predecessor without requiring the mollification of solutions far from the origin. We have used the existence of such a finite-dimensional attracting manifold and its smoothness properties to shed light on the behavior of solutions limiting to infinity, and determine the behavior on all transfinite heteroclinic orbits in the Hilbert space $H^2 \cap \{Boundary\ Conditions\}$, both for Dirichlet and Neumann boundary conditions.

We have defined the concept of a non-compact global attractor, and used the existence of so-called equilibria at infinity to provide an analogy between the non-compact attractor for a slowly non-dissipative system and the classical global attractor for a dissipative system. Thus doing, we have proven theorems which explicitly detail all the heteroclinic connections, bounded, transfinite, and intra-infinite, which, when combined with bounded equilibria and equilibria at infinity, provide the full structure of the non-compact attractor. Thus doing, we have introduced full solutions to the much studied connection problem for a new realm of scalar parabolic PDEs, and as a result of such work, expanded existing techniques in the theory of infinite-dimensional dynamical systems to realms previously unaddressed.

2. Future Extensions

There are three primary directions in which one may extend the work presented in this thesis. The first is to address the connection between slowly non-dissipative systems and related dissipative and fast non-dissipative systems. For a given nonlinear evolutionary equation of the form

$$u_t = u_{xx} + bu + g(u), \quad x \in [0, \pi], \tag{198}$$

we may study the correlations between this equation and a pair of fast non-dissipative and dissipative equations which limit to Equation (198), for example

$$u_t = u_{xx} + bu + g(u) + \varepsilon h(u), \quad x \in [0, \pi] \tag{199}$$

and

$$u_t = u_{xx} + bu + g(u) - \varepsilon h(u), \quad x \in [0, \pi] \tag{200}$$

where $h(u) = u^3$. Equation (200) is a dissipative nonlinear evolutionary equation with a Chafee-Infante structure. There has been much study on such equations, including full solutions of the connection problem [**6, 7**], detailed studies of the asymptotics of solutions [**22, 41**], and proofs over the existence and smoothness of inertial manifolds [**10, 11, 21**]. Additionally, much work has been produced on equations of the form (199), including studies on the connection problem for solutions which remain bounded [**31**]. By studying the behavior of the dynamical systems induced by Equations (199) and (200) as ε limits to zero, we may learn a great deal about the connection between dissipative, slowly non-dissipative, and fast non-dissipative systems. Additionally, the behaviors which carry over as ε limits to zero should be dependent not on the choice of function $h(u)$, but only on ε. Thus, we should be able to prove that alternate choices of $h(u)$ which guarantee dissipativity, such as $h(u) = u^{2n+1}$, yield the same behaviors as ε limits to zero.

Another open problem is to extend the results of this thesis to higher dimensions. Recently, convergence results for higher-dimensional dissipative parabolic PDEs were proven. Additionally, inertial manifold results exist for some higher-dimensional dissipative PDEs as well. For $n = 2$ inertial manifolds have been proven to exist for certain specific domains, particularly square and rectangular domains, and for $n = 3$ they have been proven to exist for cubic domains. But for $n > 3$ there in fact exist counterexamples to the existence of inertial manifolds for hypercubic domains, so the choices of dimension and domain of definition play a very large part in determining the viability of extending these results to higher dimensions. Despite these restrictions, a logical next step could be to determine if similar results to those proved in this thesis could be proven for certain n-dimensional forms of Equation (198). For a higher-dimensional evolutionary equation, the critical growth term would not be linear but rather would likely be the critical Sobolev exponent. In other words, an n-dimensional form of (198) might be

$$u_t = \Delta u + bu^p + g(u), \quad x \in \Omega \subset \mathbb{R}^n \tag{201}$$

where $p \geq \frac{n+2}{n-2}$ for $n \geq 3$ and $g(u)$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$.

The final direction we might take is to study a variation on Equation (198) wherein the linear growth term bu is replaced by a jumping nonlinearity, i.e.

$$u_t = u_{xx} + b^+u^+ - b^-u^- + g(u), \quad x \in [0, \pi] \tag{202}$$

wherein $u^+(t, x) = \max\{u(t, x), 0\}$ and $u^-(t, x) = \max\{-u(t, x), 0\}$. There has been a great deal of study in the past few decades on equations with jumping nonlinearities. It has been proven by Svatopluk Fučik in [**13**] that equations of the form

$$u_{xx} + b^+u^+ - b^-u^- = g(u) + f(x), \tag{203}$$

where $g(u)$ is continuous and bounded and $f \in L^1(0,\pi)$ are solvable when

$$u_{xx} + b^+u^+ - b^-u^- = 0 \tag{204}$$

is solvable. Furthermore, the solvability of (204) is determined by the location of (b^+, b^-) in a Cartesian graph called the Fučik Spectrum. Additionally, all solutions of (204) are classical solutions and can in fact be formed by "gluing" together sinusoidal functions of the form $c_1 sin(\sqrt{b^+}x - \xi_1)$ and $c_1 sin(\sqrt{b^-}x - \xi_2)$ for the Dirichlet problem or $c_1 cos(\sqrt{b^+}x - \xi_1)$ and $c_1 cos(\sqrt{b^-}x - \xi_2)$ for the Neumann problem.

While there has been much study on existence and uniqueness for evolutionary equations with jumping nonlinearities and solvability in higher dimensions, there has not yet been an attempt to study the asymptotics and connection problems answered in this thesis for an equation of the form (202). To expand on the work presented herein in such a direction would open a wealth of challenges unbroached under an evolutionary equation with the simpler linear term bu. In fact, even taking the case $g(u) = 0$ would introduce a number of fascinating new difficulties.

Bibliography

[1] S. Angenent. The zero set of a solution of a parabolic equation. *Crelle's journal für die reine und angewandte Mathematik*, 390:79–96, 1988.

[2] G. Birkhoff and G.-C. Rota. *Ordinary Differential Equations.* John Wiley and Sons, 1978.

[3] P. Brunovský and S.-N. Chow. Generic properties of stationary state solutions of reaction-diffusion equations. *Journal of Differential Equations*, 53:1–23, 1984.

[4] P. Brunovský and B. Fiedler. Numbers of zeros on invariant manifolds in reaction-diffusion equations. *Nonlinear Analysis, Theory, Methods and Applications*, 10(2):179–193, 1986.

[5] P. Brunovský and B. Fiedler. Simplicity of zeros in scalar parabolic equations. *Journal of Differential Equations*, 62:237–241, 1986.

[6] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics Reported*, 1:57–91, 1988.

[7] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. ii: The complete solution. *Journal of Differential Equations*, 81(1):106–135, 1989.

[8] P. Constantin, C. Foias, and R. Temam. Attractors representing turbulent flows. *Memoirs of the American Mathematical Society*, 314, 1985.

[9] A. Dold. *Lectures on Algebraic Topology.* Springer-Verlag, 1972.

[10] C. Foias, G. R. Sell, and R. Temam. Variétés inertielles des équations différentielles dissipatives. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 301:139–142, 1985.

[11] C. Foias, G. R. Sell, and R. Temam. Inertial manifolds for nonlinear evolutionary equations. *Journal of Differential Equations*, 73:309–353, 1988.

[12] C. Foias and R. Temam. Some analytic and geometric properties of the solutions of the navier-stokes equations. *Journal de Mathématiques Pures et Appliequées*, 58:339–368, 1979.

[13] S. Fučik. *Solvability of Nonlinear Equations and Boundary Value Problems.* Reidel, 1980.

[14] H. Haken. *Synergetics.* Springer-Verlag, 1978.

[15] J. K. Hale. *Ordinary Differential Equations.* Wiley-Interscience, 1969.

[16] J. K. Hale, L. T. Magalhaes, and W. M. Oliva. *An Introduction to Infinite Dimensional Dynamical Systems - Geometric Theory.* Springer-Verlag, 1984.

[17] J. Hell. *Conley Index at Infinity.* PhD thesis, Freie Universität Berlin, Berlin, Germany, 2009.

[18] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Springer-Verlag, 1981.

[19] D. Henry. Some infinite-dimensional morse-smale systems defined by parabolic partial differential equations. *Journal of Differential Equations*, 59:165–205, 1985.

[20] M. W. Hirsch. *Differential Topology*. Springer-Verlag, 1976.

[21] K. Lu, S.-N. Chow, and G. R. Sell. Smoothness of inertial manifolds. *Journal of Mathematical Analysis and Applications*, 169(1):283–312, 1992.

[22] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *Journal of the Faculty of Science: University of Tokyo: Section IA*, 29:401–441, 1982.

[23] K. Nickel. Gestaltaussagen über lösungen parabolischer differentialgleichungen . *Crelle's journal für die reine und angewandte Mathematik*, 211:78–94, 1962.

[24] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, 1983.

[25] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. *Ouevres*, 1881.

[26] P. Poláčik. Generic hyperbolicity in one-dimensional reaction-diffusion equations with general boundary conditions. *Nonlinear Analysis*, 11:593–597, 1987.

[27] J. C. Robinson. *Infinite-Dimensional Dynamical Systems*. Cambridge Texts in Applied Mathematics, 2001.

[28] C. Rocha. Generic properties of equilibria of reaction-diffusion equations with variable diffusion. *Proceedings of the Royal Society of Edinburgh*, 101A:45–55, 1985.

[29] C. Rocha. Realization of period maps of planar hamiltonian systems. *Journal of Dynamics and Differential Equations*, 19(3):571–591, 2007.

[30] R. Schaaf. *Global Solution Branches of Two Point Boundary Value Problems*. Springer-Verlag, 1980.

[31] S. Schulz. *Superlinear Dynamics of a Scalar Parabolic Equation*. PhD thesis, Justus-Liebig-Universitaet Giessen, 2007.

[32] G. R. Sell and J. Mallet-Paret. Inertial manifolds for reaction diffusion equations in higher space dimensions. *Journal of the American Mathematical Society*, 1(4):805–866, 1988.

[33] G. R. Sell and Y. You. *Dynamics of Evolutionary Equations*. Springer-Verlag, 2002.

[34] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*. Springer-Verlag, 1983.

[35] J. Smoller, A. Tromba, and A. Wasserman. Non-degenerate solutions of boundary-value problems. *Journal of Nonlinear Analysis*, 4:207–216, 1980.

[36] J. Smoller and A. Wasserman. Global bifurcation of steady-state solutions. *Journal of Differential Equations*, 39:269–290, 1981.

[37] J. Smoller and A. Wasserman. Generic bifurcation of steady-state solutions. *Journal of Differential Equations*, 52:432–438, 1984.

[38] M. Struwe. *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer-Verlag, 1990.

[39] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Springer-Verlag, 1988.

[40] W. Walter. *Differential and Integral Inequalities*. Springer-Verlag, 1970.

[41] T. Zelenjak. Stabilization of solutions of boundary value problems for a second-order parabolic equation with one space variable. *Differencial'nye Uravnenija*, 4:34–45, 1968.