The shape of profiles exhibiting blow up in forward and backward time

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Abstract

This work is concerned with blow up behaviour of solutions of the semilinear heat equation

$$u_t = u_{xx} + |u|^{p-1}u, \qquad x \in \mathbb{R}, u \in \mathbb{R}.$$

If $1 < p < 1 + \frac{2}{m}$ we find initial conditions such that the associated solution $u(t, \cdot)$ exists for negative times $-1 \leq t \leq 0$ and $\|u(t, \cdot)\|_{L^\infty}$ becomes unbounded as $t \searrow -1$. Moreover, $u(t, \cdot)$ blows up in forward time at some finite time $T > 0$ and the blow up profile $u(T, \cdot)$ possesses $m + 1$ intervals of strict monotonicity with prescribed critical values $u^1, \dots, u^m$.

After transforming the solution to similarity variables, the rescaled solution approaches a homogenous steady state κ asymptotically. We apply our results to show the existence of solutions in the strong stable manifold of the steady state κ. More precisely, we show the existence of solutions which approach κ with any prescribed exponential rate. In the original variables these solutions correspond to blow up solutions where a certain number of maxima coalesce at blow up time $t = T$.

1 Introduction

We are concerned with blowing up solutions of the heat equation

$$u_t = u_{xx} + |u|^{p-1}u, \qquad p > 1 \tag{1}$$

for $u \in \mathbb{R}$ and $x \in \mathbb{R}$. A lot of work has been done concerning the blow up of solutions of (1) and related equations such as $u_t = u_{xx} + e^u$, see [2, 3, 9, 21] and [10, 11].

Let us denote by $u(t, \cdot)$ a solution of (1) subject to the initial value u_0. We say that $u(t, \cdot)$ blows up, if there is a finite time $T > 0$ and a point $a \in \mathbb{R}$, such that

$$|u(t, x)| \to \infty, \qquad \text{as } x \to a, \quad t \nearrow T.$$

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Let $B = B(u_0)$ denote the set of all blow up points of $u(t,\cdot)$. Then the blow up set $B(u_0)$ consists of isolated points, see Matano [4]. Moreover, it is well known (see theorem 6) that there exists a function $\tilde{u}(x) =: u(T,x) \in L^\infty(\mathbb{R}\setminus B)$, such that $u(t,\cdot) \to \tilde{u}(\cdot)$ in L^∞_{loc} as $t \nearrow T$.
As has been proved by Merle [23], there exist initial conditions such that the associated solution $u(t,\cdot)$ blows up in finite time at finitely many prescribed locations. This result has been extended recently by Fiedler and Matano [6]. They generalized this result to solutions which additionally exist for all negative times $t \leqslant 0$ and approach zero as $t \to -\infty$. In this work we will focus on solutions $u(t,\cdot)$ of (1) on the unbounded domain $\mathbb{R}$ instead, which exist on some finite time interval $-1 < t < T$ and blow up in forward *and* backward time $t \searrow -1$. Moreover, we want to show that the values of the (bounded or unbounded) extremal values of the solution $u(t,\cdot)$ at blow up time $t = T$ can be prescribed arbitrarily, as long as there are only finitely many. In order to state these statements more precisely, let us focus on the derivation of our main results in more detail.

Presentation of the results
Instead of studying blow up solutions of (1) directly we can alternatively work with a different equation after passing to self similarity variables. To be more precise, let us consider the following change of variables

$$\begin{aligned} v(s,y) &= (t+1)^{\frac{1}{(p-1)}} u(t,x), \\ x &= (t+1)^{1/2} y, \quad t = e^s - 1. \end{aligned} \tag{2}$$

Then $v(s,y)$ solves

$$\begin{aligned} v_s &= v_{yy} + \frac{y}{2} v_y + \frac{1}{p-1} v + |v|^{p-1} v, \\ v(0,\cdot) &= u_0(\cdot) \end{aligned} \tag{3}$$

for $s > 0$. We will show in section 2 that the linear part L of (3) is selfadjoint and possesses only simple real eigenvalues $\lambda_0 > \lambda_1 > \ldots$, if we consider L as an operator on some wheighted space H^1_ρ (see (17) for a definition of $H^1_\rho \subset H^1_{loc}$). More explicitly, the eigenvalues of L are given by

$$\lambda_j = \frac{1}{p-1} - \frac{j+1}{2}, \qquad j = 0, 1, 2, \ldots.$$

In particular, exactly for the values $p_k = 1 + \frac{2}{k+1}$, $k \in \mathbb{N}$, there exists a simple eigenvalue zero. Fixing any $p < p_k$ we can count the unstable eigenvalues; these are characterized by the fact that their real part is strictly positive. Let us denote by E^m_u for any $0 < m \leqslant k$ the unstable eigenspace associated to the eigenvalues $\lambda_0, \ldots \lambda_{m-1}$. Since each eigenvalue λ_j of L has geometric and algebraic multiplicity one (see lemma 2), we have

$$E^m_u = \operatorname{span} \langle \phi_0, \ldots, \phi_{m-1} \rangle,$$

where ϕ_j denotes the eigenvector to the eigenvalue λ_j. Lemma 1 in section 2 characterizes these eigenvectors: each ϕ_j possesses exactly j zeros and $j+1$

local extrema. Note that the dimension of the (p-dependent) eigenspace E_u^m increases and becomes arbitrarily large as p approaches 1 from above.
The existence of the isolated unstable eigenvalues of L now strongly suggestes the existence of a m-dimensional, strong unstable manifold W_m in the unstable manifold W^u of zero. Let us remind that the unstable manifold is characterized by the fact that solutions starting in W^u converge to zero in backward time $s \to -\infty$ with exponential rate. $W_m \subset W^u$ has the additional property that solutions decay at least with exponential rate $(\lambda_{m-1} - \delta)$, where $\delta > 0$ is some small number, and at most with rate $\lambda_0 \cdot s$ as $s \to -\infty$ (see theorem 5 in section 3).
Let us choose an odd number m now, pick $v_*(\cdot) \in W_m$ and consider the associated solution $v = v(s, \cdot)$ with $v(0, \cdot) = v_*(\cdot)$. Then we expect that generically

$$c \cdot e^{(\lambda_{m-1}+\delta)s} \leqslant \|v(s,\cdot)\|_{H^1_\rho} \leqslant C \cdot e^{(\lambda_{m-1}-\delta)s} \tag{4}$$

for $s \to -\infty$ and some constants $c, C > 0$. In other words, we expect $v(s, \cdot)$ to approach zero along the eigenvector $\phi_{m-1}(\cdot)$ which corresponds to the least unstable eigenvalue of L. This suggests that $v(s, \cdot)$ possesses m local extrema, at least for sufficiently small $s << 0$ and $v(s, 0) \neq 0$, since $v(s, \cdot)$ approaches zero along the eigenvector $\phi_{m-1}(\cdot)$ and we show in section 3 that this is indeed true. We can therefore try to "follow" the extremal values of the spatial profile $v(s, \cdot)$ on the maximal time interval $(-\infty, \tilde{T})$, as long as they are well defined. As we show in section 3 the maximal existence interval is indeed finite; hence, $0 < \tilde{T} < \infty$ and $v(s, \cdot)$ blows up in finite time. Focusing on the amount of local extrema of $v(s, \cdot)$ we observe that these cannot increase in time (see section 2.2), though cancellations are possible. However, if we exclude the latter case the extrema of the spatial profile $v(s, \cdot)$ will exist up to the blow up time $s = \tilde{T} > 0$. There, at least one extremal value has to become unbounded. Of course we can ask, whether it is possible to prescribe the extremal values of the solution $v(s, \cdot)$ at blow up time $s = \tilde{T} > 0$. As we will show in section 3 this is indeed possible.
What are the consequences for the associated solution $u(t, \cdot)$ of the original equation defined via (15)? Note first that $u(t, \cdot)$ exists on the time interval $-1 < t < T$ for some finite $T > 0$ which can be explicitly expressed in terms of the former blow up time $\tilde{T}$ of $v(s, \cdot)$. Moreover, $u(t, \cdot)$ possesses m local extrema up to the blow up time. A prescription of the extremal values of $v(\tilde{T}, \cdot)$ now leads to a prescription of extremal values of the blow up profile $u(T, \cdot)$ via the transformation (15).
Moreover, we can also specify the shape of the solution profile $u(t, \cdot)$ near $t \approx -1$. Indeed, using the transformation (2) and taking into account (4), we obtain:

$$\begin{aligned} u(t,x) &\approx (1+t)^{\lambda_{m-1}} \cdot \phi_{m-1}\left(\frac{x}{\sqrt{t+1}}\right) \cdot (1+t)^{\frac{1}{1-p}} \\ &= (1+t)^{-\frac{m}{2}} \phi_{m-1}\left(\frac{x}{\sqrt{t+1}}\right) \end{aligned} \tag{5}$$

for $t \approx -1$. In the case $\psi_{m-1}(0) \neq 0$ (which is satisfied if m is odd) we observe

that all the m extremal locations of $u(t,\cdot)$ coalesce at $x=0$ in backward time $t \searrow -1$ and cause a blow up of the value $u(t,0)$.
In this way we have obtained a complete picture concerning the *global* dynamics of all solutions $u(t,\cdot)$ of (1) on their domain of definition and which are induced by solutions $v(s,\cdot)$ in the unstable manifold W^u. Let us summarize our results.

Theorem 1
Select an integer $k \geqslant 1$ and consider the equation

$$\dot{u} = u_{xx} + |u|^{p-1}u,$$

for $u, x \in \mathbb{R}$ and $1 < p < 1 + \frac{2}{k}$. Then for any integer $0 < m \leqslant k$ choose values $\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^m \in \mathbb{R} \cup \{\pm\infty\}$ such that $\tilde{u}^i \in \{\pm\infty\}$ for at least one $i \in \{1, \ldots, m\}$ and define $\tilde{u}^0 = \tilde{u}^{m+1} = 0$. Moreover, let

$$\kappa(-1)^j(\tilde{u}^{j+1} - \tilde{u}^j) \geq 0 \tag{6}$$

for some fixed sign $\kappa \in \{\pm 1\}$, $j = 0, 1 \ldots, m$, and fix a positive real number $T > 0$. Then there exists an initial condition u^0 with associated solution $u(t,\cdot)$ on $-1 < t < T < \infty$. Additionally, $u(t,\cdot)$ blows up in forward time at T with piecewise monotone blow up profile $u(T,\cdot)$. If the inequalities in (6) are strict then $u(T,\cdot)$ is piecewise strictly monotone with $m+1$ intervals of strict monotonicity. Furthermore, $u(T,\cdot)$ possesses m critical values $u^1, \ldots, u^m$ with

$$u^j = \tilde{u}^j \cdot (1+T)^{1/1-p}, \qquad j = 1, \ldots, m$$

and $\mu(u^{u^0}(t,\cdot)) = m$ for all $-\tau < t < T$; where μ is an integer-valued function (also called lap-number) which counts the amount of local extrema of $u^{u^0}(t,\cdot)$ (see section 2.2 for a precise definition).

Remark
Let us point out that we can always associate critical values to the blow up profile if $u_*(\cdot) = u(T(u^0),\cdot)$ is piecewise strictly monotone (which is the case if (6) is satisfied strictly). If $u(T(u^0),\cdot)$ is constant on some interval, however, we call its value on this interval also critical.

Remark
If m is an *odd* integer, the solutions $u(t,\cdot)$ in theorem 1 actually blow up as $t \searrow -1$ as we have argued before. More precisely, $|u(t,0)|$ becomes unbounded for $t \searrow -1$. The reason for that is that the corresponding solution $v(s,\cdot)$ approaches zero along the eigenvector ϕ_{m-1}, which satisfies $\phi_{m-1}(0) \neq 0$ if m is odd. This not true for m even. Nevertheless, we can also construct solutions with m prescribed extremal values at blow up time, such that the solutions blow up in backward time $t \searrow -1$, if m is *even*. However, we then have to choose p close to some p_{k-1} with $k \geqslant m$, see section 5 for details.

Theorem 1 states that we can choose values $\tilde{u}^1, \ldots, \tilde{u}^m \in \mathbb{R} \cup \{\pm\infty\}$ and then find an initial value u_0, such that the corresponding solution $u(t,\cdot)$ of (1) blows up at some blow up time T with these critical values up to a scalar multiple.

Note, however, that the amount of critical values, here m, poses severe restrictions on the exponent p. In particular, we can make statements only in the regime $1 < p < 1 + \frac{2}{m}$. We discuss a possible extension of theorem 1 to the case of arbitrary $p > 1$ in the section 5.

Flatter behaviour
Of particular interest is the exceptional case that we choose the extremal values in theorem 1 to be $\tilde{u}_1 = \tilde{u}_2 = \ldots = \tilde{u}_m = \infty$ for some odd integer m. Then there exists a solution $u^*(t,\cdot)$ which possesses exactly the critical values $\tilde{u}_1, \ldots, \tilde{u}_m$ at blow up time $T > 0$. In general, we cannot say *where* the extremal values $\tilde{u}_j$ of the blow up profile $u^*(t,\cdot)$ are attained. But here the case is different. Indeed, since the solution $u^*(t,\cdot)$ can only blow up at isolated points, $m/2$ maxima have to coalesce at some point a at the blow up time $t = T > 0$ (see corollary 1 in section 4). While approaching the blow up point a these maximal values become unbounded as $t \nearrow T$.
In order to understand this situation better, we consider the following change of both dependent and independent variables defined by

$$\begin{aligned} w^*(\tau, z) &= (T-t)^{\frac{1}{p-1}} u^*(t,x) \\ z &= (x-a)/\sqrt{T-t}, \qquad \tau = -\ln(T-t). \end{aligned} \tag{7}$$

Now w^* solves

$$w_\tau = \frac{1}{\tilde{\rho}} \partial_z \left(\tilde{\rho} \partial_z w \right) - \frac{1}{p-1} w + |w|^{p-1} w, \tag{8}$$

where $\tilde{\rho} = \tilde{\rho}(z) := e^{\frac{-z^2}{4}}$. Studying the behaviour of $u^*(t,\cdot)$ near blow up time $t = T$ is equivalent to studying the large-time behaviour of $w^*(\tau,\cdot)$. It follows from the results in [10, 11] that

$$w^*(\tau, z) \to \kappa := (p-1)^{\frac{1}{1-p}} \quad \text{as} \quad \tau \to \infty.$$

The natural question is *how* w^* approaches κ. Concerning general blow up solutions u of (1), this question has been adressed in [10, 11, 15, 16, 17, 21, 22]. More precisely, one of the two cases occurs:

$$\begin{aligned} &i) \quad w^*(\tau,\cdot) \sim \kappa + \frac{\kappa}{2p\tau} \left(1 - \frac{1}{2} z^2 \right), \qquad \text{or} \\ &ii) \quad w^*(\tau,\cdot) \sim \kappa + C e^{(1-\frac{j}{2})\tau} H_j(\cdot) + o(e^{(1-\frac{j}{2})\tau}), \qquad \text{as } \tau \to \infty, \end{aligned} \tag{9}$$

for some even integer j, where the convergence takes place in L^∞_{loc}, see [10, 15, 17]. Here, the function H_j is given by

$$H_j(z) = \sum_{n=0}^{j/2} \frac{j!}{n!(j-2n)!} (-1)^n z^{j-2n}. \tag{10}$$

Note that H_j possesses exactly $j/2$ maxima on $\mathbb{R}$. Using the transformation (7) we observe that solutions w behaving as in ii) correspond to solutions u of (1), where at least $j/2$ maxima coalesce at some point a at blow up time

$t = T > 0$. Of course, more extrema of $u(t, \cdot)$ could accumulate at a as $t \to T$ with a rate slower than $\sqrt{T-t}$. These might then not be "visible" in the L^∞_{loc}-norm. However, this is not the case (as we show in section 4) and we can show that $w^*(\tau, \cdot)$ satisfies ii) exactly for $j = m+1$ (see section 4). The precise result is the following:

Theorem 2 (Flatter behaviour)
Choose some even number $j \in \mathbb{N}$. Then for any $1 < p < 1 + \frac{2}{j}$ there exists an initial value $u^{0,*} \in H^1_\rho$, such that the associated solution $u^*(t, \cdot)$ blows up at some finite time $t = T > 0$, and exists for all $-1 < t < T$ and $\|u^*(t, \cdot)\|_{L^\infty}$ becomes unbounded as $t \searrow -1$. Moreover, if we denote by $w^*(\tau, z)$ the function defined as in (7) then

$$w^*(\tau, z) = (p-1)^{-\frac{1}{p-1}} + Ce^{(1-\frac{j}{2})\tau} H_j(z) + o(e^{(1-\frac{j}{2})\tau}) \tag{11}$$

as $\tau \to \infty$, where convergence takes place in H^1_{loc}. Alternatively, in terms of the solution $u^*(t, x)$

$$\lim_{t \nearrow T} \left((T-t)^{\frac{1}{p-1}} u^*(t, a + \xi (T-t)^{1/j}) \right) = (p-1)^{\frac{-1}{p-1}} (1 + C\xi^j)^{\frac{-1}{p-1}} \tag{12}$$

holds uniformly on compact sets of ξ, where C is some constant.

At least in the special case $j = 4$ the existence of such solutions $u^*(t, \cdot)$ on the positive time-interval $0 \leqslant t < T$ has been obtained by Velazquéz [16]; even in the case of arbitrary $p > 1$. The existence of solutions $u(t, \cdot)$ of (1) behaving as in (12) for *any* even integer j was conjectured in [16], but not proved until now. Let us point out that generically we expect solutions $w(\tau, \cdot)$ to satisfy property i) of (9) as has been proved in [17]. More precisely, it has been shown there that if a solution $w(\tau, \cdot)$ approaches κ with exponential rate as $\tau \to \infty$ (as in (9), ii)), then in every neighborhood of its initial value with respect to the sup-norm there exists a point $\tilde{w}_0$, such that the solution corresponding to $\tilde{w}_0$ converges to κ for $\tau \to \infty$ with algebraic rate. Furthermore, the behaviour i) in (9) is stable with respect to small perturbations of the initial data, see theorem 2 in [18].

The characterization of blow up profiles $u(T, \cdot)$ as in theorem 1 has first been provided by Fiedler and Matano [6] who considered the equation

$$u_t = u_{xx} + |u|^{p-1} u + m \cdot u, \qquad 0 < x < 2\pi \tag{13}$$

for some $m \in \mathbb{N}$ on the *bounded* interval $[0, \pi]$ with Dirichlet boundary conditions. Here, compared to the equation (1), one can prove the existence of an unstable manifold of zero directly and its dimension depends on the linear contribution $m \cdot u$ in (13). As shown in [6], solutions starting in the unstable manifold blow up in finite time and their blow up profile at blow up time can be prescribed analogous to the statement of our theorem 1. The main difference to our situation is the fact that these solutions are *ancient* solutions; that is, these solutions exist for all negative times $t \leqslant 0$ and approach

zero as $t \to -\infty$. In other words, they can be regarded as heteroclinic orbits connecting the trivial steady state to a point at infinity.
From a technical point of view, our work is selfcontained. We should mention this, since the proof of the continuous dependence of the critical values of the (compactified) blow up profile upon its initial value (which is a crucial ingredient in the proof of theorem 1) has not been proved until now. This technicality has been used in [6] and proved in [7], though the latter work has not been published up to now. Moreover, since we are mainly interested in the case $p \sim 1$, we have to abandon the assumption that our nonlinearity is two times differentiable. This, however, will effect the smoothness of the unstable manifold of zero of equation (3). Problems of these type are absent in [6].

The work is divided into six sections. In section 2 we discuss properties of the equation (3), which will be used throughout the whole work. In particular, we address the issue of nodal properties of solutions of (3) in section 2.2. The proof of existence of the strong unstable manifold $W_m \subset W^u$ is addressed in section 3. By constructing a blow up map in sections 3.1 and 3.2, which encodes the critical values of the blow up profile, we can then show that solutions in W_m blow up at prescribed values. Furthermore, the main theorem 1 is proved in sections 3.1 and 3.2. It should be mentioned that the proof follows along the lines of [6], although various additional technical difficulties (which have been mentioned in the previous paragraph) show up in our setting. Finally, we prove theorem 2, which is concerned about the existence of solutions with flatter behaviour, in section 4. Finally, we discuss possible extensions of theorem 1 to the case of arbitrary $p > 1$ in section 5.

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2 Preliminaries

In this section we want to study the equation

$$u_t = u_{xx} + |u|^{p-1}u, \tag{14}$$

where $u \in \mathbb{R}$, $x \in \mathbb{R}$ and $p > 1$. We consider a solution $u(t, x)$ with initial condition $u(0, \cdot) = u_0$, then the rescaled function

$$v(s, y) = (1 + t)^{\frac{1}{(p-1)}} u(t, x), \tag{15}$$

with $x = (t + 1)^{1/2}y$, $t = e^s - 1$ satisfies

$$\begin{aligned} v_s &= v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v + |v|^{p-1}v, \\ v(0, y) &= u_0(y) \end{aligned} \tag{16}$$

for $s > 0$.

2.1 Ljapunov function

Equation (16) possesses a Ljapunov function, which is given by

$$E(v) = \int_{\mathbb{R}} \left\{ \frac{1}{2} v_y^2 - \frac{1}{2(p-1)} v^2 - \frac{1}{p+1} |v|^{p+1} \right\} \rho dy$$

for $v \in H_\rho^1$. Here, $H_\rho^1(\mathbb{R}, \mathbb{R})$ denotes the set of all H^1-functions ϕ, such that

$$\|\phi\|_\rho^2 := \int_{\mathbb{R}} |\phi(y)|^2 \rho(y) dy + \int_{\mathbb{R}} |\phi'(y)|^2 \rho(y) dy < \infty \tag{17}$$

and $\rho(y) = e^{\frac{y^2}{4}}$. In particular, functions in H_ρ^1 decay exponentially. It is now easy to check that if $v(s)$ is a solution of (16) then

$$E(v(s)) - E(v(0)) = \int_0^s |v_s(t)|_2^2 dt \leqslant 0,$$

where $|v|_2$ denotes the norm with respect to the space L_ρ^2.

2.2 The zero-number and the lap-number

A discrete Ljapunov function is given by the zero number $z(v)$, which is well defined for any $v \in H_\rho^1$ that possesses only finitely many sign changes. More precisely, if for some $v_0 \in H_\rho^1$ the supremum M of

$$\{i \in \mathbb{N} : \exists -\infty < x_1 < \ldots < x_i < \infty \text{ with } f(x_j) \cdot f(x_{j+1}) < 1\}$$

is finite, we can define $z(v_0) = M$. Then the results of Angenent in [1] imply that the zero number z along any solution $v(s)$, $v(0) = v_0$, cannot increase in time. Since this observation will turn out to be very crucial in our analysis, we collect some results of the zero number, see [1]:

Theorem 3 (Angenent)
Consider the equation

$$u_t = u_{xx} + q(x,t)u, \qquad x \in \mathbb{R}, \quad t_1 < t < t_2, \tag{18}$$

for $q \in L^\infty$. Let u be a solution of (18), subject to the initial value $u(t_1, \cdot) = u_0(\cdot)$, such that $|u(t,x)| \leqslant A \exp(Bx^2)$ for some constants $A, B > 0$. Then the following is true:

A) For each $t \in (t_1, t_2)$ the zero set of $u(t, \cdot)$ is a discrete subset of $\mathbb{R}$.

B) If u_0 has finitely many sign changes then $t \to z(u(t))$ is nonincreasing on (t_1, t_2).

C) If $u(t)$ is a nontrivial solution of (18) and $u(t_*, x_*) = u_x(t_*, x_*) = 0$ at some (t_*, x_*) then $z(u(t)) > z(u(s))$ for any $s \in (t_*, t_2)$ and $t \in (t_1, t_*)$.

D) If $u(t)$ is a nontrivial solution and $\xi(t) : (t_1, t_2) \to \mathbb{R}$ denotes a continuous curve of zero of $u(t, \cdot)$ then $\xi(t)$ does not converge to $\pm\infty$ as $t \searrow t_0$ for any $t_0 \in (t_1, t_2)$.

In order to see why analogous results are valid for the equation

$$v_s = v_{yy} + \frac{y}{2} v_y + \frac{1}{p-1} v + |v|^{p-1} v,$$

let $v(s,\cdot)$ for $s > 0$ be a solution in the space H^1_ρ, such that $v(0,\cdot) \in H^1_\rho$ has finitely many sign changes. Then $u(t,x)$, which is defined via (15) for $t > 0$, defines a solution of $u_t = u_{xx} + |u|^{p-1}u$ and the results of Angenent apply with the choice $q := |u|^{p-1}$. In particular, $u(t,x)$ and therefore also $v(y,s) = (1+t)^{\frac{1}{p-1}} u(t,x)$ have only finitely many zeros for each t and s, respectively. Differentiating the equation

$$u_t = u_{xx} + |u|^{p-1} u$$

with respect to x we observe that theorem 3 also applies to $\tilde{u}(t,x) := u_x(t,x)$. We conclude that $u_x(t,\cdot)$ possesses only finitely many zeros and so does $v_y(s,\cdot)$ for each s on account of (15). Hence, if a zero of $v_y(s,\cdot)$ persists after varying s slightly it corresponds to a local extremum of $v(s,\cdot)$ by part C) of theorem 3. Therefore, we can count the number of local extrema along a solution $v(s,\cdot)$ of (3) by the *minmax number* or *lap number* μ, which is defined by

$$\mu(v(s,\cdot)) := z(v_y(s,\cdot)).$$

Theorem 3 now implies in particular that $s \mapsto \mu(v(s,\cdot))$ cannot increase in time.

2.3 The linearization at the steady state

Let us denote by L the linear part of (16), that is

$$L : \mathcal{D}(L) \subset H^1_\rho \to H^1_\rho$$
$$L\phi = \phi_{yy} + \frac{y}{2}\phi_y + \frac{1}{p-1}\phi.$$

It is straightforward to show, that L, defined on the domain $\mathcal{D}(L) = C^\infty_0$, admits a selfadjoint extension. We will work with this extension from now on. As shown in [27] L possesses eigenvalues λ_j of finite multiplicity with

$$\lambda_0 > \lambda_1 > \ldots > \lambda_j > \ldots.$$

More explicitly, if we denote by ϕ_j the corresponding eigenvectors, we have

$$\lambda_j = \frac{1}{p-1} - \frac{j+1}{2}, \tag{19}$$
$$\phi_j(y) = \frac{d^j}{dy^j} \exp(-y^2/4) = \psi_j(y) \exp(-y^2/4), \quad j = 0, 1, 2, \ldots,$$

where ψ_j denotes the j-th Hermite polynomial. It is known that ψ_j (and therefore ϕ_j) possesses exactly j zeros, see [27] and the references therein.

Lemma 1
ϕ_j has exactly j zeros and $j+1$ local extrema.

Proof
Note that

$$\partial_y \phi_j(y) = 0 \qquad \Longleftrightarrow \qquad \phi_{j+1}(y) = 0,$$

so ϕ_j possesses exactly $j+1$ critical points, where the first derivative vanishes. Using theorem 3 one can now show that critical actually correspond to local extrema rather than saddle points. □

Moreover, every eigenvalue λ_j of L is a simple:

Lemma 2
Fix any $p > 1$ and let let λ_j be an eigenvalue of L. Then the kernel of $L - \lambda_j$ is one-dimensional and there exists no vector $\psi \in H^3_\rho$ such that $(L - \lambda_j)\psi = \phi_j$.

Proof
Using Fourier transformation it is not hard to show that the geometric multiplicity of λ_j is one for each j, see proposition 2.3 in [20]. Note that since $L - \lambda_j$ is self adjoint the corresponding eigenvector ϕ_j is contained in a complement to the range, since $\mathrm{Rg}(L - \lambda_j)^\perp = \ker((L - \lambda_j)^*) = \ker(L - \lambda_j)$. Hence, there cannot exist an vector ψ with $(L - \lambda_j)\psi = \phi_j$. This proves the claim. □

$\{\phi_j\}_{j=0,1,2,\ldots}$ is an orthonormal basis of H^1_ρ after a suitable normalization. Let us set $p_k := 1 + \frac{2}{k+1}$ and observe that

$$\begin{cases} \lambda_k > 0, & 1 < p < p_k \\ \lambda_k = 0, & p = p_k \\ \lambda_k < 0, & p_k < p. \end{cases} \tag{20}$$

Using this fact, the following result could be proved by Yanagida et al in [27]:

Theorem 4
Define $p_k = 1 + \frac{2}{k+1}$ for any integer k.

a) If $1 < p \leqslant p_k$, then any nontrivial solution of (14), such that the initial condition $u_0 \in H^1_\rho$ possesses k sign changes, blows up in finite time.

b) $p_k < p$, then there exists an initial value u_0, such that the solution exists globally in time.

3 Blow up behaviour in the unstable manifold

In this section we construct the unstable manifold of the steady state zero of equation (16) and show that solutions starting on this manifolds blow up in finite time. We encode the blow up profile by the values of its (normalized) local extrema and show that any prescribed sequence of extrema is actually realized by a specific solution at blow up time $t = T$.

Let us now take a closer look at the equation

$$v_s = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v + |v|^{p-1}v \tag{21}$$

in the space H^1_ρ. Note that the linear equation $v_s = Lv$ generates an analytic semigroup, since L is selfadjoint and $L - \lambda$ is bounded from above for $\lambda = \lambda_0 = 3$.
If we consider the sectorial operator $-L$ as a densely defined operator in the space L^2_ρ we can define the associated fractional power spaces X^α, see [14] for a definition. In particular, $X^0 = L^2_\rho$ and $X^1 = \mathcal{D}(L) = H^2_\rho$. If α is close to 1 then $X^\alpha \hookrightarrow BC^0_{\rho^{1/2}}$, where the latter space is equipped with the norm

$$\|\varphi\|_{\infty,\rho^{1/2}} = \sup_{y\in\mathbb{R}} \rho(y)^{1/2}|\varphi(y)|,$$

see proposition 3.1 in [27]. Moreover, any $\varphi \in X^\alpha$ is locally a C^1-map for any α sufficiently close to 1, since $\phi \in H^2_\rho$ implies $\phi \in C^1_{loc}$.
Alternatively, we can consider the operator L as a densely defined map on H^1_ρ. Then we can define the associated fractional power spaces Y^α for $\alpha \in [0,1]$ with $Y^0 = H^1_\rho$ and $Y^1 = \mathcal{D}(L) = H^3_\rho$.
Let us fix some $1 < p < p_{k-1}$. We can now prove the existence of a strong unstable, m-dimensional unstable manifold W_m for any $0 < m \leqslant k$:

Theorem 5 (The unstable manifold)
Let m, k be integers with $0 < m \leqslant k$ and $1 < p < p_{k-1}$. Then the equation

$$v_s = v_{yy} + \frac{y}{2}v_y + \frac{1}{p-1}v + |v|^{p-1}v$$

possesses a strong unstable manifold $W_m \subset H^2_\rho$. W_m is of Lipschitz class when considered as a submanifold of the space Y^α. It is of class C^1 when regarded as a submanifold of X^α for some fixed $\alpha \in (0,1)$. Moreover, W_m has the following properties:

i) *W_m is given as a graph over $E_m := span\{\phi_0, \phi_1, \ldots, \phi_{m-1}\}$. If we consider W_m as a C^1-manifold in the space X^α, then $E_m = T_0 W_m$*

ii) *$\lambda_{m-1} > 0$ and $v^* \in W_m$ if and only if*

$$\lim_{s\to-\infty} v(s,\cdot)e^{(\lambda_{m-1}-\eta)|s|} = 0$$

for any fixed $\eta > 0$, which is small enough.

iii) *$\lim_{s\to-\infty} v(s,\cdot)/\|v(s,\cdot)\|_{H^3_\rho} \in E_m$ is an eigenfunction for any initial value $v^* \in W_m$, with convergence in the Y^α-norm for any fixed $\alpha \in (0,1)$. In particular, the convergence is true in BC^0 and in C^2_{loc} (see the discussion before this theorem).*

iv) *$z(v^*) \leqslant m-1$ and $\mu(v^*) \leqslant m$ for any $v^* \in W_m$.*

v) The H^1_ρ-orthogonal eigenprojection

$$P : W_m \to E_m$$

is injective.

vi) For small $\rho > 0$, the local unstable manifold

$$W_m^{loc} := P^{-1}\{\phi \in E_m : \|\phi\|_{H^1_\rho} < \rho\}$$

depends continuously on $b \in \mathbb{R}$. Here, W_m^{loc} can be considered either as a submanifold in X^α or Y^α.

vii) Any solution $v(s,\cdot)$ of (21) with initial condition $v^* \in W^m$ blows up in finite time $s = T > 0$.

Proof
Let us first observe that the linear part L of equation (21) admits an analytic semigroup e^{Ls} in the state space $X^0 := L^2_\rho$ and likewise $Y^0 := H^1_\rho$, since L can be viewed as a selfadjoint operator on both spaces. We consider the space H^1_ρ first. Due to the isolated eigenvalues of L, which have multiplicity one and since L is selfadjoint there exist closed subspaces E^s, E^u of H^1_ρ such that $e^{Ls}\big|_{E^s}$ decays exponentially in forward time $s \to \infty$. $e^{Ls}\big|_{E^u}$ can be extended to a group on the finite dimensional space E^u, which decays exponentially in backward time $s \to -\infty$, see [29, 14]. Since e^{Ls} is an analytic semigroup so are $e^{Ls}\big|_{E^u}$, $e^{Ls}\big|_{E^s}$ on their domains E^u, E^s, respectively.
Next, we choose an $1 > \alpha > 0$ such that $Y^\alpha \hookrightarrow H^2_\rho$ with continuous embedding. On account of lemma 3, which is stated after this proof, we know that the superposition map

$$f : Y^\alpha \to H^1_\rho; \quad (f(v))(y) := \tilde{f}(v(y))$$

associated to $\tilde{f}(y) = |y|^{p-1}y$ satisfies

$$\|f(u) - f(v)\|_{H^1_\rho} \leqslant M_\varepsilon \|u - v\|_{Y^\alpha}$$

for all $u, v \in B_\varepsilon(0)$. Here, M_ε is a positive constant which tends to zero for $\varepsilon \to 0$ (note that $\|v\|_{H^2_\rho} \leqslant C\|v\|_{Y^\alpha}$ for any $v \in Y^\alpha$ by our choice of α). From this calculation we easily deduce that

$$\|f_{mod}(u) - f_{mod}(v)\|_{H^1_\rho} \leqslant M_\varepsilon \|u - v\|_{Y^\alpha}.$$

for any $u, v \in Y^\alpha$, where f_{mod} is defined by

$$f_{mod} : v \to \chi_\varepsilon(\|v\|_{Y^\alpha})|v|^{p-1}v \tag{22}$$

and where $\chi_\varepsilon : \mathbb{R} \to \mathbb{R}$ denotes a smooth cut-off function with compact support in $[-\varepsilon, \varepsilon]$. We can therefore set up a variation-of-constants-formula in the space $BC^{-\gamma} := BC^{-\gamma}((-\infty, 0], Y^\alpha)$ for $\gamma > 0$ small; a space which contains all continuous functions v which are bounded with respect to the norm

$$\|v\|_{-\gamma, Y^\alpha} = \sup_{s \leqslant 0} e^{\gamma|s|} \|v(s)(\cdot)\|_{Y^\alpha}.$$

The existence of a Lipschitz manifold W_m in the space Y^α for some fixed $\alpha \in (0,1)$ now follows from the results in [14, 31]. We will choose α from now on sufficiently large, such that $\phi \in Y^\alpha$ satisfies $\phi \in C^2_{loc}$. This is possible since $Y^1 = H^3_\rho$ satisfies this property.
Alternatively, we can work in the space $X^0 = L^2_\rho$. Now f_{mod}, which is defined in (22), induces a C^1-map from X^α to L^2_ρ and we can set up a variation-of-constants-formula in the space $BC^{-\gamma}((-\infty,0],X^\alpha)$ if $\gamma > 0$ is small. Again, the existence of a strong unstable manifold of class C^1, which is a submanifold of X^α, $\alpha \in (0,1)$, follows from the results in [14, 31]. In particular, this proves the claims i), ii) and vi).
Let us now verify point iii). In order to prove this claim we will consider the manifold W^{loc}_m as a submanifold of Y^α. By construction of W^{loc}_m we have

$$W^{loc}_m = \text{graph}(\Psi_m),$$

for some Lipschitz map $\Psi_m : E_m \cap U \to Y^\alpha_s$, where Y^α_s denotes the orthogonal complement to $E_m \subset Y^\alpha$ with respect to the space Y^α and where U denotes a small neighborhood of zero. By shrinking U if necessary, we can assure that the Lipschitz constant of Ψ_m becomes arbitrarily small (and we remind that $\Psi_m(0) = 0$). By invariance of W^{loc}_m we can write any solution $v(s,\cdot) \in W^{loc}_m$ locally in the form

$$v(s,\cdot) = Pv(s,\cdot) + \Psi_m(Pv(s,\cdot)) =: v^u(s,\cdot) + \Psi_m(v^u(s,\cdot)),$$

where P, as before, denotes the projection with range E_m associated to the inner direct sum $E_m \oplus Y^\alpha_s = Y^\alpha$. In particular, $v^u(s,\cdot) \in E_m$, where all norms are equivalent; we are therefore free to consider E_m with the L^2_ρ-scalar product. Now $v^u(s,\cdot)$ solves an ordinary differential equation in backward time $s \leqslant 0$, which is called the *reduced equation* on the unstable manifold. The vectorfield associated to the reduced equation is C^1. Indeed, in the $X^0 = L^2_\rho$ setting the vectorfield f is a C^1-map and so is the reduced equation which coincides with the one of our case. By the proof of lemma 4.3.2 in [12]

$$\lim_{s\to-\infty} v^u(s,\cdot)/\|v^u(s,\cdot)\|_{L^2_\rho} \in E_m.$$

In particular, we conclude convergence in the Y^α-norm; i.e.

$$\lim_{s\to-\infty} v^u(s,\cdot)/\|v^u(s,\cdot)\|_{Y^\alpha} \in E_m,$$

since this norm is well defined on E_m and all norms are equivalent on any finite dimensional space. On account of $v^u(s,\cdot) = Pv(s,\cdot)$ (and in particular $\|v^u(s,\cdot)\|_{Y^\alpha} \to 0$ if and only if $\|v(s,\cdot)\|_{Y^\alpha} \to 0$), we also get

$$\lim_{s\to-\infty} v^u(s,\cdot)/\|v(s,\cdot)\|_{Y^\alpha} \in E_m.$$

We now look at

$$\frac{\Psi_m(v^u(s,\cdot))}{\|v^u(s,\cdot)\|_{Y^\alpha}}.$$

Because $\Psi_m : E_m \cap U \to Y_s^\alpha$ is a Lipschitz map with Lipschitz-constant ε in some small neighbourhood $U = U(\varepsilon)$ and $v^u(s,\cdot) \in U(\varepsilon)$ for all $s \leqslant s_*$ and some $s_* << 0$ small enough we conclude

$$\frac{\|\Psi_m(v^u(s,\cdot))\|_{Y^\alpha}}{\|v(s,\cdot)\|_{Y^\alpha}} \leqslant \varepsilon \frac{\|v_u(s,\cdot)\|_{Y^\alpha}}{\|v(s,\cdot)\|_{Y^\alpha}} \leqslant M\varepsilon \frac{\|v(s,\cdot)\|_{Y^\alpha}}{\|v(s,\cdot)\|_{Y^\alpha}} = M\varepsilon$$

for $M = \|P\|_{L(E_m,Y^\alpha)}$ which is independent of $\varepsilon > 0$. This shows that

$$\frac{v(s,\cdot)}{\|v(s,\cdot)\|_{Y^\alpha}} = \phi + 2\varepsilon\tilde{v}$$

if $s \leqslant s_*$ is small enough, where $\tilde{v}$ denotes some suitable unit vector in Y^α and $\phi \in E_m$. This proves iii).
Let us now prove *iv*) and take any $v^* \in W_m$. Then the solution $v(s,\cdot)$ to the initial value v^* possesses at most $m-1$ sign changes, that is $z(v(s,\cdot)) \leqslant m-1$ for $s << 0$ small enough. This is true due to property iii) which states that $v(s,\cdot)$ approaches zero along an eigenvector in the BC^0-norm. Counting sign changes, this proves that $v(s,\cdot)$ can not have less than $m-1$ sign changes. Now note that any function $\varphi \in X^\alpha$ satisfies $\varphi \in C^1_{loc}$ if α is close enough to 1. Therefore, the convergence in iii) actually holds true in C^1_{loc}. Observe that any zero of any eigenfunction ϕ_k is simple; meaning that $|\partial_y \phi_k|$ has a constant sign, bounded away from zero, near any zero. Convergence of the rescaled solution $h(t) = u(t,\cdot)/\|u(t,\cdot)\|_{H^2_\rho}$ in C^1_{loc} implies that $\partial_y h$ also has a constant sign near any zero of ϕ_k. In particular, there cannot be any accumulation of zeros of h near any zero of ϕ_k. However, there may be a zero of $v(s,\cdot)$ that converges towards $\pm\infty$ as $s \to -\infty$ which is not near any zero of ϕ_k (note that v decays exponentially near $\pm\infty$). Since $v(s) = v(s,\cdot) \in W^{loc}_m$, it admits the representation

$$v(s) = \sum_{j=0}^{k} a_j\phi_j + \Gamma(\sum_{j=0}^{k} a_j\phi_j) = \sum_{j=0}^{k} a_j\phi_j + o(|a_0| + \dots |a_k|)$$

for fixed s and suitable coefficients $a_j \in \mathbb{R}$, where $W^{loc}_m = \text{graph}(\Gamma)$. Hence, there cannot be any zero of $v(s,\cdot)$ near $\pm\infty$ if we show that the element $\Gamma(\phi)$, $\phi := \sum_{j=0}^{k} a_j\phi_j$, lies in a function space where functions approach zero with a faster rate as $|y| \to \infty$ than ϕ. Now note that for any $\tilde{\phi} \in T_0W_m$

$$\Gamma(\tilde{\phi}) = \int_{-\infty}^{0} \left(e^{L(-s)}\big|_{E^u}\right) |v_*(s)|^{p-1} v_*(s) ds, \tag{23}$$

where $v_*(s)$ denotes the solution of (3) in W^{loc}_m associated to $v_*(0) = \tilde{\phi} + \Gamma(\tilde{\phi})$. Now if $w \in Y^\alpha$ the function $|w|^{p-1}w$ decays with some exponential rate $e^{y^2/4+\gamma|y|}$ for some $\gamma > 0$ small enough. As one can show by using (23), $\Gamma(\tilde{\phi})$ also lies in some function space where functions approach zero with rate at least $e^{y^2/4+\gamma|y|}$. Now note that $v_*(s)$ in (23) can be written in the form $v_*(s) = \phi(s) + \Gamma(\phi(s))$ for some function $\phi(s)$, which is in T_0W_m for each fixed s. A simple bootstrap argument now completes the proof. This finally shows

that zeros of $v(s)$ can approach $\pm\infty$ as $s \to -\infty$, which shows $z(v(s,\cdot)) \leqslant m-1$ for $s < 0$ small enough.
Moreover, since the amount of zeros cannot increase in time s the claim follows. Analogously, we can now prove that $\mu(v(s,\cdot)) \leqslant m$ if $s < 0$ is sufficiently small and $v(0,\cdot) = v^*(\cdot) \in W_m$. The argumentation proceeds analogously: On account of convergence of the solution $v(s,\cdot)$ in BC^0 and, say $z(v(s,\cdot)) = k$ for any $s < 0$ small enough, where $0 \leqslant k \leqslant m-1$, we conclude the existence of at least $k+1$ extrema; namely at least one between any two zeros of $v(s,\cdot)$. But note that $v(s,\cdot)/\|v(s,\cdot)\|_{Y^\alpha}$ actually converges to an eigenfunction with respect to the Y^α-norm which implies convergence in C^2_{loc}. Since all extrema of any eigenfunction are regular (that is, the second derivative is different from zero) there can actually be no more than m extrema. Indeed, any extremum corresponds to a simple zero of $\partial_y v(s,\cdot)$ and

$$\frac{\partial_y v(s,\cdot)}{\|v(s,\cdot)\|_{Y^\alpha}}$$

still approaches the derivative of an element in E_m with respect to the C^1_{loc}-norm in backward time $s \to -\infty$, which proves the claim.
We prove v) next and consider two different elements $v^1, v^2 \in W_m$ such that $P(v^1 - v^2) = 0$. Then

$$0 \neq v^1 - v^2 \in E_m^\perp = \text{ span}\{\phi_m, \phi_{m+1}, \ldots\}.$$

In order to show that this leads to a contradiction let $v^1 - v^2 = \sum_i a_i \phi_i$ with $a_1 \in \mathbb{R}$ and $i \geqslant m$. Without loss of generality assume $a_m \neq 0$. We consider the function

$$\tilde{v}(s) := \sum_i a_i e^{\lambda_i s} \phi_i.$$

Then $v(s)$ is defined for $s \in \mathbb{R}$ and solves the linear equation $v_s = Lv$. Since $a_m \neq 0$ and $\lambda_m > \lambda_j$ for $j > m$ we conclude that $\tilde{v}(s) \sim a_m \phi_m e^{\lambda_m s}$ for $s >> 0$. In particular $z(\tilde{v}(s)) \geqslant m$ for $s \geqslant 0$. Via the transformation (15), $\tilde{v}$ now defines a solution $\tilde{u}$ of the linear equation $u_t = u_{xx}$. Moreover, since zeros remain uneffected by the transformtion, $z(\tilde{u}(t)) \geqslant m$ for $t >> 0$ and $z(\tilde{u}(0)) \leqslant m-1$ which contradicts the fact that the zero number is nonincreasing.
Finally, if $b = 1$, then the claim *vii)* follows from theorem 4 since $1 < p < p_{m-1}$. Indeed, any $v^* \in W_m$ satisfies $z(v^*) \leqslant m-1$. Since v^* lies in the unstable manifold, the solution $v(s)$ with initial value v^* cannot converge to zero in forward time $s \to \infty$ due to the Ljapunov function. Moreover, there exist no other equilibria with less or equal than $m-1$ sign changes, see [27] for details. □

Lemma 3
Let $p > 1$. Then the map

$$\begin{aligned} f : H^2_\rho &\to H^1_\rho \\ f(\varphi)(y) &= |\varphi(y)|^{p-1}\varphi(y) \end{aligned}$$

is Lipschitz continuous on $B_\varepsilon(0)$ and the Lipschitz constant tends to zero if $\varepsilon \to 0$.

Proof
We only consider the case $p < 2$. Indeed, in the case $p \geq 2$ the map $x \mapsto |x|^{p-1}x$ is actually C^2. For $u, v \in B_\varepsilon(0) \subset H^2_\rho$ we look at

$$\begin{aligned} \|\partial_y |u(y)|^{p-1}u(y) \;\; &- \;\; \partial_y |v(y)|^{p-1}v(y)\|^2_{L^2_\rho} \\ &= \int_{-\infty}^{\infty} \rho(y) \left[p|u(y)|^{p-1}u'(y) - p|v(y)|^{p-1}v'(y)\right]^2 dy \\ &\leqslant \left(\int_{-\infty}^{\infty} |\rho(y) \left[p|u(y)|^{p-1}u'(y) - p|v(y)|^{p-1}v'(y)\right]|dy\right)^2 \end{aligned}$$

It is therefore sufficient to show that

$$\int_{-\infty}^{\infty} |\rho(y) \left[p|u(y)|^{p-1}u'(y) - p|v(y)|^{p-1}v'(y)\right]|dy \leqslant M \cdot |u - v|_{H^2_\rho}. \tag{24}$$

for some constant $M = M(\varepsilon)$ which tends to zero for $\varepsilon \to 0$. We compute

$$\begin{aligned} &\int_{-\infty}^{\infty} |\rho \left[p|u|^{p-1}u' - p|v|^{p-1}v'\right]|dy \qquad (25) \\ \leqslant &\int_{-\infty}^{\infty} \rho|\kappa|^{p-1}|u' - v'|dy + \int_{-\infty}^{\infty} \rho|\gamma| \cdot ||u|^{p-1} - |v|^{p-1}|dy \end{aligned}$$

for $\kappa(y) := u(y)$ and $\gamma(y) := v'(y)$ or $\kappa(y) := v(y)$ and $\gamma(y) := u'(y)$; e.g. in order to obtain the estimate with $\kappa(y) = u(y)$ and $\gamma(y) = v'(y)$ we would add and substract the term $|u|^{p-1}v'$ and apply the triangle inequality.
Note that only the second term of the right hand side of the last inequality (25) causes problems. Indeed, the first term is estimated readily by

$$\int_{-\infty}^{\infty} \rho|\kappa|^{p-1}|u' - v'|dy \leqslant \int_{-\infty}^{\infty} \rho dy \cdot \|\kappa\|^{p-1}_\infty \cdot \|u - v\|_{H^2_\rho}$$

and since $\|\kappa\|^{p-1}_\infty$ is near zero for $\kappa \in B_\varepsilon(0)$ everything is fine.
Without loss of generality we may assume from now on that the functions $u(\cdot), v(\cdot)$ nowhere vanish identically; that is $v(y) \neq 0, u(y) \neq 0$ except on a countable subset of $\mathbb{R}$. Moreover, we can restrict our attention to the case $|u(y)| \neq |v(y)|$ holds except on a countable set. We can then define the function

$$g(y) = \left(\begin{array}{lll} |v(y)|, & \text{if } |u(y)| > |v(y)| & \text{for } y \in \mathbb{R} \\ |u(y)|, & \text{if } |v(y)| > |u(y)| & \text{for } y \in \mathbb{R} \end{array} \right.$$

which is a well defined, Lebesgue integrable function on $\mathbb{R}$. Let us set $\hat{\gamma}(y) := |v'(y)|$ if $g(y) = |v(y)|$ and $\hat{\gamma}(y) := |u'(y)|$ if $g(y) = |u(y)|$. By the mean value theorem we have

$$\int_{-\infty}^{\infty} \hat{\gamma} \cdot ||u|^{p-1} - |v|^{p-1}|dy \leqslant \int_{-\infty}^{\infty} \hat{\gamma} \cdot (p-1)\frac{1}{\xi^{2-p}} \cdot |u - v|dy$$

for some $\xi = \xi^{u(y),v(y)} \in [u(y), v(y)]$ if $u(y) > v(y)$ or $\xi \in [v(y), u(y)]$ otherwise.
Moreover, we can estimate

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{\gamma} \cdot (p-1)\frac{1}{\xi^{2-p}} \cdot |u - v|dy \;\; &\leqslant \;\; \int_{-\infty}^{\infty} \hat{\gamma} \cdot (p-1)\frac{1}{g^{2-p}}dy \cdot |u - v|_\infty \\ &\leqslant \;\; \int_{-\infty}^{\infty} \hat{\gamma} \cdot (p-1)\frac{1}{g^{2-p}}dy \cdot \|u - v\|_{H^2_\rho}. \end{aligned}$$

Note that we can "ignore" ρ in the integrand on account of the mean value theorem for integrals. Let us now approximate $\hat{\gamma}$ in the L^1-norm by functions $z_n \in C^\infty \cap L^2_{\rho^{1/2}}$. Moreover, we choose functions $\psi_n \in C^\infty \cap L^2_{\rho^{1/2}}$ such that $\psi_n \to g$ uniformly. Note, that we can choose $\pm\psi_n' = z_n > 0$ (with the appropriate choice of sign $\pm$) except at a set of small Lebesgue measure, say $1/n$, since $\pm g' = \hat{\gamma} > 0$ up to sign change almost everywhere. Now fix some $\sigma > 0$. Choosing n large enough we conclude

$$\begin{aligned}\int_{-a}^{b} z_n \cdot (p-1)\frac{1}{\psi_n^{2-p}}dy &\leqslant \left|\pm\int_{-a}^{b} \psi_n' \cdot (p-1)\frac{1}{\psi_n^{2-p}}dy\right| + \sigma \\ &\leqslant \left|\int_{\psi_n(-a)}^{\psi_n(b)} (p-1)\frac{1}{\zeta^{2-p}}d\zeta\right| + \sigma \\ &\leqslant ||\psi_n(-a)|^{p-1} - |\psi_n(b)|^{p-1}| + \sigma \\ &\leqslant |g(-a)|^{p-1} + |g(b)|^{p-1} + \tilde{\sigma}\end{aligned}$$

if $\tilde{\sigma} > 0$ denotes some small number. Now letting $n \to \infty$ and $a, b \to \infty$ proves

$$\int_{-\infty}^{\infty} \hat{\gamma} \cdot (p-1)\frac{1}{g^{2-p}}dy \cdot |u-v|_\infty \leqslant M|u-v|_\infty \qquad (26)$$

for some small number $M = M(\tilde{\sigma}, \varepsilon)$. Since (26) holds for a dense set $u, v \in B_\varepsilon(0)$ it actually holds on $B_\varepsilon(0)$. □

Theorem 5 tells us that every solution in W_m blows up in finite time. In order to associate a blow up profile $v(T, \cdot)$ to any initial condition $v^* \in W_m$ we need the following result from [4] formulated in terms of equation (16).

Theorem 6
Let v be a solution of (16), subject to the initial value $v^* \in H^1_\rho$, with blow up time T and blow up set B. Then there exists a $\tilde{v}(\cdot) =: v(T, \cdot) \in L^\infty_{loc}(\mathbb{R} \setminus B)$ such that $v(s, \cdot) \to \tilde{v}(\cdot)$ locally uniformly in $\mathbb{R} \setminus B$ as $s \to T$.

This theorem is first of all true in the case of equation (14), see [22]. But since any solution $v(s, \cdot)$ gives rise to a solution $u(t, \cdot)$ of (14) via (15), the results carry over to equation (16). Indeed, if $u(t, \cdot) \to \tilde{u}(\cdot)$ as t approaches the blow up time $\tilde{T} > 0$, then the identity

$$v(y, s) = (e^s)^{1/(p-1)} u(e^{s/2}y, e^s - 1)$$

defines a blow up profile $\tilde{v} \in L^\infty_{loc}(\mathbb{R} \setminus B)$ for $s = T$. On account of the results in [4], the blow up set B consists of isolated points. We therefore consider $\tilde{v}(\cdot) =: v(T, \cdot)$ from now on as the blow up profile, which is a well defined object.
We now want to state a result which is concerned about the dependence of the blow up time $T = T(v^*)$ with respect to the initial data. The next lemma has been proved in [28] and we state a slightly weaker version in terms of equation (21).

Lemma 4
Let v, v_n be solutions of (21) with initial data ψ and ψ_n, respectively. Suppose that v blows up at $s = T < \infty$. If $\psi_n \to \psi$ in $H^1_{\rho/2}$ as $n \to \infty$, then for every sufficiently large n the solution v_n blows up in finite time, and its blow up time T_n satisfies $T_n \to T$ as $n \to \infty$.

3.1 The encoded blow up profile

We now want to construct an initial condition $v^* \in H^1_\rho$ in W_m, such that the associated blow up profile $v(T, \cdot)$ has m prescribed extrema

$$0 < v^1 > v^2 < v^3 >< \dots, \qquad \text{if m is even} \tag{27}$$

and

$$0 > v^1 < v^2 > v^3 <> \dots, \qquad \text{if m is odd}$$

at locations

$$-\infty < y_1 < y_2 < \dots < y_m < \infty.$$

Note that (27) is true for the eigenfunctions

$$\phi_{2k}(y) = \frac{\partial^{2k}}{\partial y^{2k}} e^{-y^2/4}, \qquad k \in \mathbb{N}.$$

We now construct a map $\Gamma : S^{m-1}_{\zeta,+} \to [-\pi/2, \pi/2]^m$, which encodes the critical values of the blow up profile associated to any initial value $v \in S^{m-1}_{\zeta,+}$. Here,

$$S^{m-1}_{\zeta,+} \subseteq W^{loc}_m, \qquad S^{m-1}_{\zeta,+} \cap W^{loc}_{m-1} = \emptyset$$

denotes a hemisphere which is constructed in the following way. Since the manifold W_{m-1} can be locally represented as a graph over the eigenspace E_{m-1}, W^{loc}_{m-1} intersects a small sphere S^{m-1}_ζ with radius ζ in $W^{loc}_m \supset W^{loc}_{m-1}$ in the equator S^{m-2}_ζ. Moreover, we have

$$\lim_{s \to -\infty} \frac{v(s, \cdot)}{\|v(s, \cdot)\|_{Y^\alpha}} \in \operatorname{span}\{\phi_{m-1}\} \tag{28}$$

in Y^α for $v^* \in S^{m-1}_\zeta \setminus W_{m-1}$, see property iii) of theorem 5. This implies

$$\lim_{s \to -\infty} \mu(v(s, \cdot)) = m.$$

$S^{m-1}_{\zeta,+}$ now denotes the hemisphere of v^*, where the limit in (28) provides the positive multiple of ϕ_{m-1}.
We now think of a situation where no extrema of the profile $v(s, \cdot)$ coalesce before the blow up time. Let us denote the locations of the m extrema of $v(s, \cdot)$ by $-\infty < \xi_1(s) < \dots < \xi_m(s) < \infty$. Moreover, we assume that

$$\xi_j := \lim_{s \nearrow T} \xi_j(s) \tag{29}$$

exists and the limits are pairwise different. Let us consider the case that m is an odd number and set $\xi_0 := -\infty$. Then the first extremum of ϕ_{m-1} is a maximum and so is the first extremum of $v(s,\cdot)$ at $y = \xi_1(s)$ for sufficiently small $s < 0$. Since we assume that no extrema of $v(s,\cdot)$ collapse for $-\infty < s < T$, we can define

$$v^j := \begin{cases} \sup_{(\xi_{j-1},\xi_j)} v(T,\cdot), & \text{if } j \in \{1,3,5,7\ldots,m\} \\ \inf_{(\xi_{j-1},\xi_j)} v(T,\cdot) & \text{if } j \in \{2,\ldots,m-1\} \end{cases}$$

for $j = 1,\ldots,m$. Note that by definition $v^j \in \{\pm\infty\}$ at least for one j. Of course v^j would coincide with the value $v(T,\xi_j)$, if this value would always be well defined. Now the blow up map $\Gamma(v^*)$ can be defined by

$$\Gamma(v^*) := (\arctan(v^1),\ldots,\arctan(v^m)).$$

However, more degenerate cases may occur. In order to define Γ in all cases appropriately we have to be more careful.

3.2 The blow up map

In this section we want to define the blow up map Γ for all $v^* \in S^{m-1}_{\zeta,+}$. Let us note that $S^{m-1}_{\zeta,+}$ is compact, which will be an important fact throughout our analysis. We start with the following definition. Let s^{v^*} for any $v^* \in S^{m-1}_{\zeta,+}$ denote the first dropping time of the minmax number μ, that is

$$s^{v^*} := \sup\{s \in (-\infty,T) : \mu(v(s,\cdot)) = m\}.$$

Here, $T = T(v^*)$ denotes the blow up time of the solution $v(s,\cdot)$ with respect to the initial value v^*. Let us denote by $\xi_1^{v^*}(s) < \xi_2^{v^*}(s) < \ldots < \xi_m^{v^*}(s)$ as before the locations of the m extrema of $v(s,\cdot)$; these locations are well defined as long as $s < s^{v^*}$.

What can we say if $s = s^{v^*} = T(v^*)$ for some initial value $v^* \in W_m$? The first thing to observe is that if some location $\xi_j(s)$ tends to $\pm\infty$ as s approaches the blow up time, then $v^j = 0$. Indeed, this follows by lemma 13 in the appendix which states that the values of $v(s,y)$ are small if s is near T and $|y|$ is large enough. But other degenerate cases can happen and we shall introduce the following quantities:

$$\begin{aligned} \underline{\xi}_j &= \liminf_{s\to s^{v^*}} \xi_j^{v^*}(s), \\ \bar{\xi}_j &= \limsup_{s\to s^{v^*}} \xi_j^{v^*}(s), \qquad j = 1,\ldots,m. \end{aligned}$$

Here, the index v^* indicates the dependence of the solution with respect to the initial value $v^* \in W_m$. For readability, we suppress the dependence of v^* in most cases, if no confusion is possible. As $s \to s^{v^*} = T$ we also have to deal with the case that

$$\bar{\xi}_j > \underline{\xi}_{j+1}.$$

Lemma 5
Assume that $s^{v^*} = T$ *and* $\bar{\xi}_j > \underline{\xi}_{j+1}$ *for some* $j = 1, \ldots, m$ *and* $v^* \in W_m$. *Then* $I = (\underline{\xi}_{j+1}, \bar{\xi}_j)$ *does not belong to the blow up set of* $v(s, \cdot)$ *and the blow up profile* $v(T, \cdot)$ *is constant on* $(\underline{\xi}_{j+1}, \bar{\xi}_j)$.

Proof
Without loss of generality we assume that $j = 1$ and that m is an odd number. We know that the blow up set B corresponding to $v^{v^*}(s, \cdot)$ consists of isolated points. Hence almost every point $y \in I$ satisfies $y \notin B$ and there exists an open neighborhood $U(y)$ such that

$$B \cap U(y) = \emptyset.$$

Theorem 6 tells us that $v(s, \cdot) \to v(T, \cdot)$ on $U(y)$ with respect to $L^\infty(U(y), \mathbb{R})$. In particular,

$$|v(s, \cdot)| < M$$

on $U(y)$ for some constant $M > 0$ and s near T. Observe that $\bar{\xi}_i \neq \underline{\xi}_i$ for $i = 1, 2$, since otherwise the dropping time s^{v^*} would be smaller than T. We can now find subsequences s_n^i such that

$$\xi_1(s_n^1) \in U(y), \qquad \xi_2(s_n^1) \notin U(y),$$
$$\xi_2(s_n^2) \in U(y), \qquad \xi_1(s_n^2) \notin U(y),$$

if $n > 0$ is large enough, where for all $s \leqslant s^{v^*}$

$$v(s, \xi_j(s)) = v^j$$

is a maximum for $j = 1$ and a minimum for $j = 2$. Due to theorem 6 we have

$$\lim_{n \to \infty} v(s_n^1, \cdot) = v(T, \cdot) = \lim_{n \to \infty} v(s_n^2, \cdot) \tag{30}$$

on $U(y)$. Note that $v(T, \cdot)$ is continuous on U, since this function is the uniform limit of continuous functions. We first consider the situation that we can choose $U(y)$ small enough such that no other extrema of $v(s, \cdot)$ for $s \approx T$ enter $U(y)$ as $s \to T$. Then all values of $v(s_n^1, \cdot)$ in $U(y)$ are smaller than v^1 (since m is an odd number and v^1 is a maximum) and all values of $v(s_n^2, \cdot)$ in $U(y)$ are larger than v^2. Together with the fact that the convergence in (30) is uniformly on $U(y)$ with respect to the L^∞-norm this shows that

$$v(T, \cdot) \equiv c$$

on some neighborhood $\tilde{U} \subset U(y)$, where c is some appropriate constant. On account of lemma 18 we know that $\partial U(y)$ does not belong to the blow up set. Hence we can choose $\tilde{y} \in \partial U(y)$ and proceed analogously. This shows that

$$B \cap I = \emptyset, \qquad \text{and} \qquad v(T, \cdot) \equiv c \text{ on } I.$$

Let us now assume that one extremum location $\xi_j(s)$ of $v(s, \cdot)$ for $j \geqslant 2$ enters $U(y)$ as $s \to T$; without loss of generality we assume

$$\liminf_{s \to T} \xi_3(s) = \underline{\xi}_2, \qquad \xi_3(s) > \xi_2(s).$$

Then the above arguments show that $v(T,\cdot)$ is constant on $\tilde{U}(y)\cap(-\infty,y)$ and proceeding as above one can then show that

$$v(T,\cdot)\equiv c, \qquad \text{on } (\underline{\xi}_1, y).$$

Since y can be choosen arbitrarily close to $\bar{\xi}_2$ this shows the claim. □

Let us now define the blow up map in the case that m is an odd number.

Definition 1
Let $1<p<p_{m-1}$, $v^*\in S^{m-1}_{\zeta,+}$.

a) *Consider $s^{v^*}<T$ and let $\xi_j\in\mathbb{R}\cup\{\pm\infty\}$ be defined as in (29) for $j=1,\dots,m$. We set $\xi_0:=-\infty$ and*

$$v^j=v^{j,v^*}=\begin{cases}\sup\{v(s^{v^*},y): y\in(\xi_{j-1},\xi_j)\}, & j\in\{1,3,5,\dots,m\}, m \text{ odd}\\ \inf\{v(s^{v^*},y): y\in(\xi_{j-1},\xi_j)\}, & j\in\{2,\dots,m-1\}, m \text{ odd}\end{cases}$$

for $j=1,\dots,m$ if m is odd. Then we define

$$\Gamma(v^*):=(\arctan(v^1),\dots,\arctan(v^m)).$$

b) *Let $s^{v^*}=T$; then define $\underline{\xi}_0:=\bar{\xi}_0:=-\infty$. For $j=1,\dots,m$ we set*

$$v^j:=\begin{cases}\sup\{v(s^{v^*},y): y\in(\bar{\xi}_{j-1},\underline{\xi}_j)\}, & j\in\{1,3,5,\dots,m\}, \bar{\xi}_{j-1}<\underline{\xi}_j\\ \inf\{v(s^{v^*},y): y\in(\bar{\xi}_{j-1},\underline{\xi}_j)\}, & j\in\{2,4,\dots,m-1\}, \bar{\xi}_{j-1}<\underline{\xi}_j\\ \sup\{v(s^{v^*},y): y\in(\underline{\xi}_j,\bar{\xi}_{j-1})\}, & j\in\{1,2,3,4,\dots,m\}, \bar{\xi}_{j-1}>\underline{\xi}_j\\ \lim_{n\nearrow\infty} v(s_n,\xi_n), & \bar{\xi}_{j-1}=\underline{\xi}_j,\end{cases}$$

where s_n is a sequence which converges to T as $n\to\infty$ and $\xi_n\to\underline{\xi}_j$, with $\xi_n\neq\underline{\xi}_j$ as $n\to\infty$. Define

$$\Gamma(v^*):=(\arctan(v^1),\dots,\arctan(v^m)).$$

For readability we have suppressed the dependence of b, which occurs in the nonlinearity of equation (21). Let us note that the definition of $\Gamma(v^*)$ coincides with the definition of the previous section, if the solution $v(s,\cdot)$ possesses m isolated extrema throughout the interval of existence $(-\infty,T)$. We make a few comments to the definitions in part b). Let us consider the case that

$$\bar{\xi}_{j-1}>\underline{\xi}_j$$

for some j. Then lemma 5 actually implies that the blow up profile $v(T,\cdot)$ is continuous on the interval $I=(\underline{\xi}_j,\bar{\xi}_{j-1})$ and we can replace the supremum in the above definition of v^j by an infimum. If $\xi:=\underline{\xi}_j=\bar{\xi}_{j-1}$ and ξ does not belong to the blow up set B then then $v(s,\cdot)\to v(T,\cdot)$ uniformly on a small neighborhood U of ξ. Since $v(T,\cdot)$ is continuous we can also choose the value $v(T,\xi)$ in the definition of Γ; in particular our definition is well defined and independent of the sequence $s_n \nearrow T$, $\xi_n \to \xi$. In the case $\xi \in B$, lemma 18 applies and for every sequence $\xi_n \to \xi$, $s_n \nearrow T$ we have

$$|v(s_n, \xi_n)| \to \infty, \qquad n \to \infty,$$

which shows that Γ is well defined. In the case $s^{v^*} = T$ we have $v^j = \pm\infty$ for at least one value j which leads to $\Gamma_j(v^*) = \arctan(v^j) = \pm\pi/2$.

We now normalize the vector $\Gamma(v^*)$ and consider the map

$$\begin{aligned} \tilde{\Gamma} &: S^{m-1}_{\zeta,+} \to \bar{D}^{m-1} \\ \tilde{\Gamma} &: v^* \mapsto \frac{\Gamma(v^*)}{|\Gamma(v^*)|_2}, \end{aligned}$$

where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^m$. Furthermore, $\bar{D}^{m-1} \subset S^{m-1} \subset \mathbb{R}^m$ denotes the closure of all unit vectors satisfying the inequalities (27). The proof of the next result is proven in appendix A.

Proposition 1
The map $\tilde{\Gamma}$ is continuous on its domain of definition.

3.2.1 Surjectivity of the blow up map

In this section we want to prove surjectivity of the blow up map $\tilde{\Gamma}$. The following extended definition of the blow up map will turn out to be very helpful. Consider

$$\begin{aligned} \tilde{\Gamma}^{ex} &: \mathcal{D} \to \bar{D}^{m-1} \\ &\mathcal{D} := \{(s, v^*);\ v^* \in S^{m-1}_{\zeta,+},\ -\infty < s \leq T(v^*)\}, \end{aligned}$$

where for $j = 1, \ldots, m$ and $\xi_0 := -\infty$ the j-th component $\tilde{\Gamma}^{ex}_j$ is defined by

$$\tilde{\Gamma}^{ex}_j(s, v^*) = \begin{cases} \max_y(v(s,y)), & s < s^{v^*},\ y \in (\xi_{j-1}(s), \xi_j(s)),\ j \in \{1, 3, 5, \ldots, m\}, \\ \min_y(v(s,y)), & s < s^{v^*},\ y \in (\xi_{j-1}(s), \xi_j(s)), j \in \{2, 4, \ldots, m-1\}, \\ \Gamma_j(v^*), & s \geq s^{v^*} \end{cases}$$

Again, we normalize the vector $\tilde{\Gamma}^{ex}$ for every $s, v^* \in \mathcal{D}$, but refrain from introducing new notation.

Lemma 6
The (normailzed) map $\tilde{\Gamma}^{ex}$ is continuous with respect to s, v^*.

Proof
Let us consider continuity in v^* first and fix some $s < s^{v^*}$, since otherwise the claim follows from proposition 1. Then it follows from lemma 15 in the appendix that the dropping time s^{v^*} depends continuously on v^*. Moreover, lemma 14 implies that the zeros of $v_y^{v_n}(s, \cdot)$ approach the corresponding zeros of $v_y^{v^*}(s, \cdot)$ for any sequence of initial data $v_n \to v^*$ in H^1_ρ as $n \to \infty$. Since the solution $v(s, \cdot)$ depends continuously on the initial data v^* in the space Y^α (that is, locally in C^2_{loc}) the claim follows. Continuity of s is trivial in the case

$s \geq s^{v^*}$ and follows from theorem 6 in the case $s = s^{v^*}$. $\square$

We now want to prove surjectivity of $\tilde{\Gamma}$ and first show that $\mathrm{Rg}(\tilde{\Gamma}) \subset D^{m-1}$. It suffices to prove

$$\deg\,(\tilde{\Gamma}^{ex}(T(\cdot),\cdot), S^{m-1}_{\zeta,\varepsilon}, \tilde{d}) \neq 0$$

for any arbitrary $\tilde{d} \in D^{m-1}$. Here, we have written $T(\cdot)$ in order to emphasize the dependence of the blow up time of its initial data. Moreover, $\varepsilon > 0$ is chosen small enough and $S^{m-1}_{\zeta,\varepsilon} \subset Y^\alpha$ denotes the closed $(m-1)$-dimensional disk of points in the hemisphere $S^{m-1}_{\zeta,+}$ staying a distance at least $\varepsilon > 0$ from the equator $\partial S^{m-1}_{\zeta,+} = S^{m-2}_{\zeta}$ of S^{m-1}_{ζ}. Since the equator lies in W_{m-1} we have $\mu(v) \leq m-1$ for all past history and $v \in \partial S^{m-1}_{\zeta,+}$.

Lemma 7
There exists $\varepsilon > 0$ small enough such that

$$\mu \leqslant m-1, \qquad \text{on } \partial S^{m-1}_{\zeta,\varepsilon}.$$

Proof
For $v^* \in S^{m-2}_{\zeta}$ we have $\mu \leqslant m-1$. Let us fix some time $-s \leqslant -1$ and consider the point $v_{-s} \in W_m$ of the backward orbit through v^* at time $-s$. Then $\mu(v_{-s}) \leqslant m-1$ and we can assume that $-s$ is not a dropping time. Thus if we consider a sufficiently small neighborhood U of v_{-s} in $W_m$with respect to the Y^α-topology then we can assume that for all $v \in U$ we have $\mu(v) \leqslant m-1$ by continuous dependence of the dropping time, see lemma 15. Since the amount of strict local maxima and minima cannot increase in forward time, we conclude the existence of a neighborhood $\tilde{U}$ of v^* in W_m, such that $\mu(v) \leqslant m-1$ for all $v \in \tilde{U}$. Moreover, since S^{m-2}_{ζ} is compact we can find finitely many of these neighborhoods which cover S^{m-2}_{ζ} in W_m. In particular, there exists a radius $\kappa > 0$, such that the m-dimensional ball $B_\kappa(v)$ is contained in one of the $\tilde{U} = \tilde{U}(v^*)$ for every $v \in S^{m-2}_{\zeta}$. Choosing $\varepsilon := \kappa/2$ proves the claim. $\square$

Lemma 7 implies that the dropping time s^{v^*} is strictly negative for any $v^* \in \partial S^{m-1}_{\zeta,\varepsilon}$. This implies that at least two extrema collapse at some (negative) time $-s = s^{v^*} < 0$. In other words

$$\tilde{\Gamma}^{ex}(s, v^*) \in \partial \bar{D}^{m-1} \qquad \text{for all } s \geqslant 0. \tag{31}$$

Lemma 8
$\tilde{\Gamma} : S^{m-1}_{\zeta,+} \to \bar{D}^{m-1}$ *is surjective.*

Proof
Let us first show that for any $\tilde{d} \in D^{m-1}$

$$\deg\,(\tilde{\Gamma}(\cdot), S^{m-1}_{\zeta,\varepsilon}, \tilde{d}) = \deg\,(\tilde{\Gamma}^{ex}(T(\cdot),\cdot), S^{m-1}_{\zeta,\varepsilon}, \tilde{d}) \neq 0$$

holds, which proves the lemma in view of the approximation lemma 10. Note that we pick $S^{m-1}_{\zeta,\varepsilon} \subset S^{m-1}_{\zeta,+}$ here. Let us first observe that $v^* \mapsto \tilde{\Gamma}^{ex}(T(v^*), v^*)$ is continuous, since the maps

$$(s, v^*) \mapsto \tilde{\Gamma}^{ex}(s, v^*), \qquad v^* \mapsto T(v^*)$$

are both continuous with respect to $v^* \in S^{m-1}_{\zeta,\varepsilon} \subset S^{m-1}_{\zeta,+}$. In particular, the degree is well defined on account of continuity of $\tilde{\Gamma}^{ex}$ and the fact that

$$\tilde{\Gamma}^{ex}(T(v^*), v^*) \in \partial D^{m-1}, \qquad \text{for } v^* \in \partial S^{m-1}_{\zeta,\varepsilon}$$

due to (31). Invoking a standard homotopy

$$h(\tau, v^*) := \tilde{\Gamma}^{ex}(\tau \cdot T(v^*), v^*)$$

for $0 \leqslant \tau \leqslant 1$, we obtain

$$\deg\,(\tilde{\Gamma}^{ex}(T(\cdot), \cdot), S^{m-1}_{\zeta,\varepsilon}, \tilde{d}) = \deg\,(\tilde{\Gamma}^{ex}(0, \cdot), S^{m-1}_{\zeta,\varepsilon}, \tilde{d})$$

by (31) and homotopy invariance of Brouwer degree. Similarly, the degree is independent of the particular choice of $\tilde{d} \in D^{m-1}$. Moreover, by standard deformation of the domain $S^{m-1}_{\zeta,\varepsilon}$ the degree does not depend on the choice of small enough ε, ζ.

The next homotopy deforms the nonlinearity $f(v) = f^0(v) = |v|^{p-1}v$ to its linearization $f^1(0) = 0$ by

$$f^\tau(v) := (1 - \tau) f(v),$$

for $0 \leqslant \tau \leqslant 1$. The strong unstable $W_m = W^{f^\tau}_m$, the hemispherical disks $S^{m-1}_{\zeta,\varepsilon} = S^{m-1,f^\tau}_{\zeta,\varepsilon}$, and the dropping times $s^{v^*} = s^{v^*,f^\tau}$ all depend continuously on f^τ. Note that we can achieve that (31) holds troughout the homotopy. Indeed, let us consider the sphere $S^{m-2,f^1} = \partial S^{m-1,f^1}_{\zeta}$ with respect to the vector field $f^1(v) = 0$. In particular, $S^{m-2}_{\zeta} \subset E_{m-1}$ and $\mu(v) \leqslant m-1$ for any $v \in S^{m-2}_{\zeta}$. Then there exists a neighborhood U of S^{m-2,f^1} in the space Y^α such $\mu(v) \leqslant m-1$ for all $v \in U$. The sphere $S^{m-1,f^\tau}_{\zeta,\varepsilon}$ can be parametrized over E_m and we can choose $\zeta, \varepsilon > 0$ small enough, such that $\partial S^{m-1,f^1}_{\zeta,\varepsilon} \subset U$. Since the unstable manifold W_m depends continuously on the vectorfield and f^1 is close to f^0 in a sufficiently small neighborhood of $v = 0$, we conclude that we can choose $\zeta, \varepsilon > 0$ small enough, such that

$$\partial S^{m-1,f^\tau}_{\zeta,\varepsilon} \subset U$$

for all $\tau \in [0,1]$, which proves $\mu(v) \leqslant m-1$ for all $v \in S^{m-1,f^\tau}_{\zeta,\varepsilon}$. Hence we can provide that the dropping times remain strictly negative for $0 \leqslant \tau \leqslant 1$. This argumentation shows that

$$\tilde{\Gamma}^{ex}(0, v^*) = \tilde{\Gamma}^{ex}(0, v^*, f^\tau) \in \partial D^{m-1}$$

for $\tau \in [0,1]$ and $v^* \in \partial S^{m-1,f^\tau}_{\zeta,\varepsilon}$. Hence, we conclude

$$\deg\,(\tilde{\Gamma}^{ex}(0, \cdot), S^{m-1,f^0}_{\zeta,\varepsilon}, \tilde{d}) = \deg\,(\tilde{\Gamma}^{ex}(0, \cdot), S^{m-1,f^1}_{\zeta,\varepsilon}, \tilde{d})$$

for any $\tilde{d} \in D^{m-1}$. Note that

$$W^{f^1}_m = E_m = \operatorname{span}\{\phi_0, \ldots, \phi_{m-1}\}.$$

Since we consider the case that m is an odd number we can define

$$\tilde{d} = \frac{(\arctan(w_1), \ldots, \arctan(w_m))}{|(\arctan(w_1), \ldots, \arctan(w_m))|_2} \in D^{m-1},$$

where $0 < w_1 > w_2 < \ldots$ denote the values of the extrema of the eigenfunction ϕ_{m-1}. An explicit computation for this specific choice in lemma 9 below shows that

$$\deg\,(\tilde{\Gamma}^{ex}(0,\cdot), S^{m-1,f^1}_{\zeta,\varepsilon}, \tilde{d}) \neq 0. \tag{32}$$

Since D^{m-1} is connected, (32) remains true for all $\tilde{d} \in D^{m-1}$, which proves $D^{m-1} \subseteq \mathrm{Rg}(\tilde{\Gamma})$. The approximation lemma 10 then completes the proof of surjectivity. □

Let us now prove that (32) is satisfied.

Lemma 9
For $f(v) = f^1(v) = 0$ and

$$\tilde{d} = \frac{(\arctan(w_1), \ldots, \arctan(w_m))}{|(\arctan(w_1), \ldots, \arctan(w_m))|_2} \in D^{m-1},$$

where m is an odd number and $0 < w_1 > w_2 < \ldots$ denote the values of the extrema of the eigenfunction ϕ_{m-1}, we have

$$deg\,(\tilde{\Gamma}^{ex}(0,\cdot), S^{m-1,f^1}_{\zeta,\varepsilon}, \tilde{d}) \neq 0. \tag{33}$$

Proof
Since $\tilde{d} \in D^{m-1}$ the value $\tilde{d}$ can only be attained by v^* with $\mu(v^*) = m$ and therefore $v^* \notin \partial S^{m-1,f^1}_{\zeta,\varepsilon}$. In particular,

$$v^* \in E_m = \mathrm{span}\{\phi_0, \ldots, \phi_{m-1}\}$$

itself possesses m extrema with (normalized) values $0 < w_1 > w_2 < w_3 > \ldots$. Certainly the value $\tilde{d}$ is attained by

$$v^* = \beta\phi_{m-1} \tag{34}$$

for some $\beta > 0$. We now determine the local degree of $\tilde{\Gamma}^{ex}(0,\cdot)$ at $v^* = \phi_{m-1}$ with respect to the normal plane $\langle v^*\rangle^{\perp} = E_{m-1}$ to v^* at v^*. This degree is given by the sign of the determinant of the linearization of $\tilde{\Gamma}^{ex}(0,\cdot)$, in the nondegenerate case. Differentiating $v^*(\xi_j) = v^j$ at a nondegenerate absolute extremum

$$(v^* + \varepsilon\eta)_y(\xi_j + \varepsilon y_j) = 0$$

with respect to ε at $\varepsilon = 0$ gives

$$Dv^j \cdot \eta = \eta(\xi_j)$$

for the derivative at v^*. It now suffices to show that the augmented matrix

$$M := \begin{pmatrix} \phi_0(\xi_1) & \phi_1(\xi_1) & \ldots & \phi_{m-2}(\xi_1) & \phi_{m-1}(\xi_1) \\ \vdots & \vdots & & \vdots & \vdots \\ \phi_0(\xi_m) & \phi_1(\xi_m) & \ldots & \phi_{m-2}(\xi_m) & \phi_{m-1}(\xi_m) \end{pmatrix} \tag{35}$$

has a nonzero determinant when evaluated at the pairwise different, extremal locations ξ_j, $j = 0, \dots, m$, of the $(m-1)$-th eigenfunction

$$\phi_{m-1}(y) = \psi_{m-1}(y) e^{-y^2/4}$$

where ψ_{m-1} denotes the $(m-1)$-th hermite polynomial. Note that only the last column in M has been augmented, which accounts for radial collapse.
Let us now show that $\ker(M) = \{0\}$. Suppose a linear combination $\phi \in E_m$ of elements in E_m vanishes at m points $\xi_1, \xi_2, \dots, \xi_m$. Since $z(\phi) \leqslant m-1$ for any nontrivial linear combination, we conclude that $\phi = 0$.
This would complete the proof if $v_* = \beta\phi_{m-1}$ would be the only preimage of $\tilde{d}$ with respect to $\tilde{\Gamma}^{ex}(0, \cdot)$. Since this is not immediately clear we let $\tilde{v}_*$ be another preimage. Then an analogous argumentation leads to the determination of the sign of a matrix $\tilde{M}$ (as in (35)). In particular, the columns of (the augmented) matrix $\tilde{M}$ are again given by suitable linear combinations of the vectors $\phi_0, \dots, \phi_{m-1}$. Arguing as above we can then show that the determinant of $\tilde{M}$ does not vanish. Moreover, the *sign* of the determinant must coincide with the sign of $\det(M)$. Indeed, otherwise there would exist pairwise distinct locations $\tilde{\xi}_1, \dots \tilde{\xi}_m$ and a nontrivial linear combination of vectors $\phi_0, \dots, \phi_{m-1}$ which would vanish when evaluated at $\tilde{\xi}_1, \dots, \tilde{\xi}_m$. This is not possible! Hence, the degree (given as a sum of positive integers) does not vanish. This proves the claim. □

We prove the approximation lemma next.

Lemma 10
Let $\vec{w} = (w^1, w^2, \dots, w^m)$ satisfy nonstrict inequalities (27) such that $w^j = \pm\infty$ for some j. Then there exists $v^* \in W_m$ such that

$$v^j = w^j,$$

where the v^j denote the local extrema of the blow up profile $v(T(v^*), \cdot) = v^{v^*}(T(v^*), \cdot)$. Moreover, $\mu(v(s, \cdot)) = m$ for all $-\infty < s < T(v^*)$.

Proof
Approximate the vector $\vec{w}$ by a sequence $\vec{w}_n = (w_n^1, \dots, w_n^{m-1})$ of vectors satisfying (27) strictly, and use compactness of $S^{m-1}_{\zeta,\varepsilon}$; see also lemma 3.2 in [6].
□

This ends the prove of theorem 1.

4 Flatter behaviour

Let us now focus on the blow up of solutions of the equation

$$\begin{aligned} u_t &= u_{xx} + |u|^{p-1}u, \\ u(0, \cdot) &= u^0 \end{aligned} \tag{36}$$

which blow up at finite time $t = T(u^0)$ and some point $a \in \mathbb{R}$. We define

$$\begin{aligned} u(t,x) &= (T-t)^{-\frac{1}{p-1}} w(z,\tau), \\ x-a &= z(T-t)^{1/2} \qquad \tau = -\log(T-t), \end{aligned} \tag{37}$$

so that in the new variables, w satisfies

$$w_\tau = w_{zz} - \frac{1}{2} z w_z w + w + r(w), \tag{38}$$

where $r(w) = |w|^{p-1}w - \frac{p}{p-1}w$. We remind that H_n has been defined in (10): Alternatively, we can write $H_n = \phi_n / \|\phi_n\|_{L^2_{\tilde{\rho}}}$, $n = 0, 1, 2, \ldots$, where ϕ_n denotes the standard n-th Hermite polynomial and so that $\|H_n\|_{L^2_{\tilde{\rho}}} = 1$ for all n. Here $L_{\tilde{\rho}}(\mathbb{R}, \mathbb{R})$ denotes the space of all Lebesgue integrable functions ψ satisfying

$$\|\psi\|^2_{L^2_{\tilde{\rho}}} = \int_{-\infty}^{\infty} e^{-y^2/4} |\psi(y)|^2 dy < \infty.$$

We can now state a result concerning possible blow up behaviours for (36), which has been proved in [15, 16].

Theorem 7 (Velázquez)
Assume that a solution $u(t,x)$ of (36) blows up at $t = T(u^0)$ and some point $x = a$. Define w via (37), then one of the following cases occurs

a)

$$w(\tau, z) = (p-1)^{-\frac{1}{p-1}}$$

for any $\tau > 0$, or

b)

$$w(\tau, z) = (p-1)^{-\frac{1}{p-1}} - \frac{(4\pi)^{1/4}(p-1)^{-\frac{1}{p-1}}}{\sqrt{2}p} \cdot \frac{H_2(z)}{\tau} + o\left(\frac{1}{\tau}\right)$$

as $\tau \to \infty$, or

c) there exists an even number $m \geqslant 4$ such that

$$w(\tau, z) = (p-1)^{-\frac{1}{p-1}} + Ce^{(1-\frac{m}{2})\tau} H_m(y) + o(e^{(1-\frac{m}{2})\tau})$$

as $\tau \to \infty$, where convergence takes place in $H^1_{\tilde{\rho}}$.

Let us now outline our strategy for proving theorem 2. In particular, we are now interested in solutions $w(\tau, z)$ of (38), which behave as in point iii) of the last theorem. As we can see from the change of variables (37), the desired solutions correspond to blow up solutions $u(t,x)$ of the original equation (36), where at least $\frac{m}{2}$ maxima coalesce at the blow-up point a as $t \to T$. However, there could be more maxima locations of $u(t,\cdot)$ approaching a as $t \to T$, which remain undetected in the (τ, z)-variables, since the norm in $L^2_{\tilde{\rho}}$ does not capture the beaviour for large $|z|$. That this is not the case will be shown in the next lemma, which is due to Velázquez (see proposition 2.1 in [17]).

Having established this fact we then use theorem 1 for showing the existence of solutions $u(t,x)$ of (36), which have the property that exactly $m/2$ maxima locations approach the blow up point as $t \to T$. Let us stress that *every* blow up solution of theorem 1 actually translates to a solution $w(t,z)$ of (38) converging to κ as $\tau \to \infty$. Indeed, this can only be possible if the L^∞-norm $u(t,x)$ can be bounded by $M \cdot (T-t)^{-\frac{1}{p-1}}$ for some $M > 0$. Solutions, which do not satisfy this property are called blow up solutions of type-II, see [25]. However, type-II blow up does not occur in the scalar equation (36), see [25, 10, 11]. Let us now state the following lemma of Velázquez (see proposition 2.1 in [17].)

Lemma 11
Let $1 < p < p_{m-1}$, where $p_{m-1} = 1 + \frac{2}{m}$. Consider an initial value $u_0 \in S^{m-1}_{\zeta,+}$, such that the solutions $u(t,x)$ of (36) blows up at $t = T$ and some point $x = a$. Then there exists a $\delta > 0$ and $R > 0$ with the following properties. Every maximum of $u(t,\cdot)$, which approaches the blow up point $x = a$ as $t \to T$, lies in the set $\mathcal{C}_a = \{x : |x-a| \leqslant R \cdot \sqrt{T-t}\}$. Moreover, if we let $w(\tau,z)$ as defined in (37), the following cases can occur.

i) If $w(\tau,z)$ behaves as in theorem 7,b), $u(t,\cdot)$ has a single maximum in $\mathcal{C}_a$ for any $t \in [T(u^0) - \delta, T(u^0))$.

ii) If $w(\tau,z)$ behaves as in theorem 7,c), $u(t,\cdot)$ has exactly $\frac{m}{2}$ maxima in $\mathcal{C}_a$ for any $t \in [T(u^0) - \delta, T(u^0))$.

Proof
We only indicate the main line of the proof and refer to [17], proposition 2.1, for further details. Let us assume that $a = 0$. We argue that there exists an $R > 0$ such that every maximum location of $u(t,\cdot)$, which approaches $a = 0$ for $t \to T$, lies in the interval

$$\mathcal{C} = \{x : |x| \leqslant R \cdot \sqrt{T-t}\}.$$

Assume that this is wrong. On account of $u_0 \in S^{m-1}_{\zeta,+} \subset W_m$ we first of all know that the blow up set of $u(t,\cdot)$ is finite. Now there exists a curve of maxima $\Sigma(t)$ of $u(t,\cdot)$, such that $\lim_{t\to T} \Sigma(t) = 0$, and a sequence $t_n \to T$, such that

$$|\Sigma(t_n)| \geqslant n\sqrt{T-t_n}. \tag{39}$$

Note that the curve $\Sigma(t)$ depends smoothly on t. Furthermore,

$$u(\Sigma(t_n), t_n) < ((p-1)(T-t_n))^{-\frac{1}{p-1}},$$

which is proved in lemma 2.5 and 2.2 of [17]; see also the discussion after equation (2.26) in [17]. Let us now define $M(t) := u(\Sigma(t), t)$, and observe that

$$\dot{M}(t) - M(t)^p \leqslant 0,$$

since we can assume without loss of generality that $u(\Sigma(t), t) > 0$. Then

$$M(t) \leqslant (M(t_n)^{-(p-1)} - (p-1)(t-t_n))^{-\frac{1}{p-1}}, \qquad \text{for } t > t_n. \tag{40}$$

Since $M(t_n) \leqslant (\delta_1 + (p-1)(T-t_n))^{-\frac{1}{p-1}}$ for some $\delta_1 > 0$, (40) shows that $M(t)$ stays uniformly bounded as $t \nearrow T$, which is a contradiction. The rest of the claims of the theorem now follow by observing that $w(\tau, \cdot)$ has exactly one maximum on the compact interval $[-R, R]$ if $\tau > 0$ is large enough and if alternative b) of theorem 7 applies. Similarly, one can treat the case that $w(\tau, z)$ behaves as in theorem 7, c). □

Now, theorems 1, 7 and lemma 11 imply the following corollary.

Corollary 1 (Flatter behaviour)
Choose some even number $m_* \in \mathbb{N}$. Then for any $1 < p < 1 + \frac{2}{m_*}$ there exists an initial value $u^0 \in H^1_\rho$, such that the associated solution $u(t, \cdot)$ blows up at some finite time $t = T$, and exists for all $-1 < t < T(u^0)$. Moreover, if we denote by $w(\tau, z)$ the function defined as in (37) then alternative c) of theorem 7 is satisfied; that is

$$\lim_{t \nearrow T} \left((T-t)^{\frac{1}{p-1}} u(t, a + \xi (T-t)^{1/m_*}) \right) = (p-1)^{\frac{-1}{p-1}} (1 + C\xi^{m_*})^{\frac{-1}{p-1}} \tag{41}$$

holds uniformly on compact sets of ξ, where $C > 0$ is some constant.

Proof
For any given even number m_* there exists an initial value $u^0 \in H^1_\rho$ such that the corresponding solution $u(t, \cdot)$ blows up at finite time $t = T(u_0)$ and possesses exactly $m_*/2$ maxima for every $t < T$, which coalesce at $t = T(u^0)$ at some point $x = a$. Indeed, we may apply theorem 1 to the case $m := m_* - 1$, $\kappa = 1$, with prescribed critical values

$$u^1 = u^2 = \ldots = u^m = +\infty.$$

As we already argued in the introduction this corresponds exactly to the desired case, since the blow up profile cannot remain constant between two different blow up locations. Now note that alternative b) or c) of theorem 7 applies to the solution w which is associated to u via (37). But case b) is ruled out by lemma 11, i). Therefore, case c) of theorem 7 holds exactly for $m = m_*$, since all other cases are ruled out by the alternative ii) of lemma 11. Finally, statement (41) is true on account of [17]. □

5 Discussion

In this section we want to discuss possible generalizations of theorem 1 to the case of arbitrary $p > 1$.
For the existence of solutions u of (1) with k prescribed extrema at blow up time we have focused on solutions in the strong unstable manifold of the trivial steady state of (3) in the regime $1 < p < p_{k-1} = 1 + 2/k$. The reason for this is the fact that the eigenvector ϕ_{k-1} corresponds to an *unstable* eigenvalue $\lambda_{k-1} > 0$ for $p < p_{k-1}$. As p is increased through the critical value p_{k-1} a Pitchfork bifurcation occurs near zero. This means that two additional,

nontrivial steady states of (21) are born near zero for $p \approx p_k$ (see [5, 13] for details on Pitchfork bifurcations). Since we know by the results of Yanagida et al [27] that there do not exist equilibria with $k-1$ sign changes if $p < p_{k-1}$, the branch of steady states has to be directed to the *right* (see also figure 1). Every point on the bifurcation branch corresponds to a nontrivial equilibria of (21).

As we increase p through p_{k-1} the trivial steady state becomes stable. Hence, by exchange of stability, the additional steady state of one branch, which we call $\psi(p)$ from now on, is born unstable. $\psi(p)$ possesses exactly $k-1$ sign changes for $p \approx p_{k-1}$. Moreover, the unstable eigenvalues and eigenfunctions of the linarization $L(p)$ of (21) at $\psi(p)$ are close to the corresponding ones for L. In particular, in the case $p > p_{k-1}$ there exists an eigenvalue $\tilde{\lambda}_{k-1}$ of $L(p)$ close to λ_{k-1}, which is unstable. The corresponding eigenvector again possesses exactly k local extrema. We can therefore proceed completely analogous to characterize all solutions $v(s,\cdot)$ of (21) in the unstable manifold of $\psi(p)$ for $p > p_{k-1}$ as in theorem 1. Of course, one basic ingredient in the proof is again the fact that all solutions in the unstable manifold of $\psi(p)$ blow up in finite time. This is still true if p is sufficiently close to p_{k-1}.

To summarize, these remarks show why in (a slightly modified version of) theorem 1 also provides the existence of solutions $u(t,\cdot)$ on $-1 < t < T$ which blow up as $t \searrow -1$ if m is even. In order to extend theorem 1 to *arbitrary* $p > 1$, however, one has to understand the *global* nature of the bifurcation branches that bifurcate from the p_{k-1}, $k \geq 1$. We have depicted three possibilities in figure 1. In any case a branch bifurcating from p_{k-1} can never enter the region $p < p_{k-1}$ again. Indeed, the amount of zeros of the steady state $\psi(p)$ can at most decrease along the branch. We actually expect that the amount of zeros is constant. However, zeros of $\psi(p)$ could "vanish" through infinity as we move along the branch.

The desired situation is depicted in figure 1, a): The branch is globally parametrized over the p-axis and no (secondary) bifurcations occur. However, saddle node bifurcations (as in b)) may appear and the branch may even be confined only to a bounded region $p_{k-1} \leqslant p < M$ for some $M > 0$ (see figure 1, c)). But even in case a) one has to show that solutions in the unstable manifold of $\psi(b)$ blow up in finite time also for large $p > p_{k-1}$.

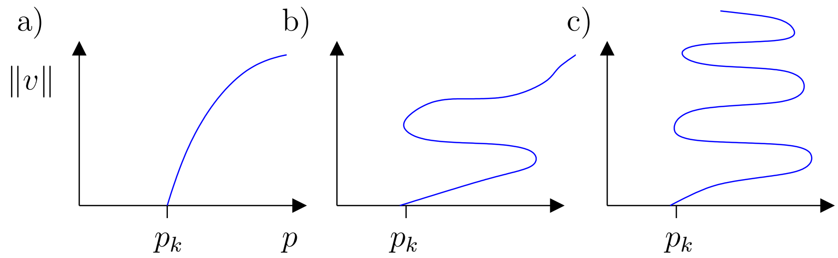

Figure 1: Three possible bifurcation branches.

6 Appendix: Continuity of the blow up map

In this section we want to show that the blow up map Γ depends continuously on the initial data $u_0 \in S^{m-1}_{\zeta,+}$. We closely follow the proof in [28]. Let us start by considering a solution u of (14) with blow up time T and $a \in \mathbb{R}$. We define $E_a(u)$ by

$$\begin{aligned} E_a(u) &= T^{2/(p-1)+1/2} \int_{\mathbb{R}} \left\{ \frac{1}{2} u_x^2 - \frac{1}{p+1} |u|^{p+1} \right\} \exp\left(-\frac{(x-a)^2}{4T} \right) dx \\ &+ \frac{1}{2(p-1)} T^{2/(p-1)-1/2} \int_{\mathbb{R}} u^2 \exp\left(-\frac{(x-a)^2}{4T} \right) dx \end{aligned}$$

and state the following result, see theorem 2.1 and 3.5 in [22]:

Lemma 12
Let u be a solution of (14) with initial value u_0 and blow up time T.

a) *For any $\eta > 0$, there exists $\sigma > 0$ such that if $E_a(u_0) < \sigma$, then a is not a blow up point of u and*

$$|u(t,a)| \leqslant \eta (T-t)^{-1/(p-1)}, \qquad \text{in } [T/2, T).$$

b) *Let $0 < c_1 < c_2$ be fixed and $c_1 \leqslant T \leqslant c_2$. Then there exists $\delta > 0$, such that if*

$$|u(t,a)| \leqslant \delta (T-t)^{-1/(p-1)}, \qquad \text{in } [a-r, a+r] \times [T/2, T)$$

for some $r > 0$, then

$$|u(t,a)| \leqslant M, \qquad \text{in } [a-r/2, a+r/2] \times [T/2, T),$$

where $M = M(c_1, c_2, \delta) > 0$.

With the help of this lemma, we can prove that blow up occurs only on a bounded interval, which is uniform with respect to the initial value $v^* \in S^{m-1}_{\zeta,+}$.

Lemma 13
Let $v(s,\cdot) = v(s,\cdot,v^*)$ the solution solution of (16) subject to the initial condition $v^* \in S^{m-1}_{\zeta,+}$ with blow up time $T = T(v^*)$. Then for any $\gamma > 0$ there exists an $\alpha > 0$, such that

$$|v(s,y)| < \gamma, \qquad \text{in } [(-\infty, -\alpha] \cup [\alpha, \infty)] \times [0, T(v^*))$$

for any $v^* \in S^{m-1}_{\zeta,+}$.

Proof
In order to apply lemma 12, we look at the function

$$u(t,x) := (t+1)^{-1/(p-1)} v(s,y),$$

where $y = x(t+1)^{-1/2}$ and $s = \ln(t+1)$. Since $v^* \in S^{m-1}_{\zeta,+} \subset H^2_\rho$, we conclude $u(0,\cdot) = v^* \in H^2_\rho$, u solves

$$u_t = u_{xx} + |u|^{p-1}u$$

and blows up at some time $t = \tilde{T}$. We consider $E_a(v^*)$, which satisfies

$$\begin{aligned} E_a(v^*) &= \tilde{T}^{2/(p-1)+1/2} \int_{\mathbb{R}} \left\{ \frac{1}{2}(v^*)_x^2 - \frac{1}{p+1}|v^*|^{p+1} \right\} \exp\left(-\frac{(x-a)^2}{4T} \right) dx \\ &+ \frac{1}{2(p-1)} \tilde{T}^{2/(p-1)-1/2} \int_{\mathbb{R}} (v^*)^2 \exp\left(-\frac{(x-a)^2}{4T} \right) dx \\ &\leqslant K_1 \cdot \exp\left(-\frac{(M-a)^2}{4\tilde{T}} \right) \end{aligned}$$

for some $K_1 > 0$ and $M > 0$ large enough. Moreover, M can be chosen independent of $v^* \in S^{m-1}_{\zeta,+}$ and $a \in \mathbb{R}$. Indeed, on account of $v^* \in H^2_\rho \cap S^{m-1}_{\zeta,+}$ or even $v^* \in H^3_\rho$ the functions $v^*, (v^*)_x$ decay exponentially for $y \to \pm\infty$; uniformly with respect to $v^* \in S^{m-1}_{\zeta,+}$, since $S^{m-1}_{\zeta,+}$ is compact. Note also the embedding theorems for the weighted Sobolev spaces H^k_ρ, see lemma 2.1 of [19] or [20], which state that $H^1_\rho(\mathbb{R},\mathbb{R}) \hookrightarrow L^{\infty,\rho^{1/2}}(\mathbb{R},\mathbb{R})$, where $\|\psi\|_{\infty,\rho} =$ ess $\sup_y \rho(y)^{1/2}|\psi(y)|$.
We conclude that

$$\lim_{a\to\infty} E_a(v^*) = 0$$

uniformly in $v^* \in S^{m-1}_{\zeta,+}$. An analogous statement holds for $a \to -\infty$. Without loss of generality we restrict to the case $a > 0$. By Lemma 12 there are positive constants b, M_1 with

$$|u(t,x)| \leqslant M_1, \qquad \text{in } [b,\infty) \times [T/2, T)$$

uniformly with respect to $v^* \in S^{m-1}_{\zeta,+}$. From parabolic regularity theory there is $M_2 > 0$ such that

$$|u_t(t,x)| \leqslant M_2, \qquad \text{in } [2b,\infty) \times [T/2, T) \tag{42}$$

for all v^*. Let us now choose a positve constant r with $2rM_2 < \varepsilon/4$. For any $v^* \in S^{m-1}_{\zeta,+}$ there are constants $C = C(r, v^*), K = K(r)$ and a neighborhood $U(v^*) \subset S^{m-1}_{\zeta,+}$ of v^* such that

$$|T(v^*) - T(v)| < r$$

and

$$|u^v(t,x)| \leqslant C\exp(-Kx^2), \qquad \text{in } \mathbb{R} \times [0, T(v^*) - r)$$

for any $v \in U$. Taking a constant $d = d(r, v^*) > 2b$ with

$$C\exp(-Kx^2) < \varepsilon/4 \qquad \text{for } x \geqslant d$$

we get

$$|u^v(t,x)| \leqslant \varepsilon/2, \qquad \text{in } [d,\infty) \times [0, T(v))$$

for any $v \in U$ by (42). Since $S^{m-1}_{\zeta,+}$ is compact, we can find $\alpha > 0$ such that

$$|u^v(t,x)| \leqslant \varepsilon/2, \qquad \text{in } [\alpha,\infty) \times [0,T(v))$$

for all $v \in S^{m-1}_{\zeta,+}$. This proves the lemma. □

With this preparations we can now proceed to prove the continuity of Γ. We first consider the case $b = 1$ in equation (21) and study the dependence of $\Gamma(v^*) = \Gamma(v^*, 1)$ on $v^* \in S^{m-1}_{\zeta,+}$. For readability, we will suppress the dependence of b in the following sections.

6.1 Dropping time less than the blow up time

We first consider the case that $s^{v^*} < T(v^*)$. That is, some extrema of the solution $v(s,\cdot)$ collapse at some point before the blow up time $T = T(v^*)$. Let us start with a result that implies the continuous dependence of the location $\xi_j(s)$ of the j-th extremum of $v(s,\cdot)$ with respect to the initial value.

Remark
We want to point out that throughout this section we make use of the fact that the nonlinearity $f(v) = |v|^{p-1}v$ defines a local Lipschitz-map when regarded as a map from Y^α to H^1_ρ. Hence, there exists a flow in the space Y^α which can be proved by setting up a variation of constants formula in the space Y^α, see [14, 29]. In particular, solutions depend continuously on initial conditions in the space Y^α.

Lemma 14
Consider $v^* \in S^{m-1}_{\zeta,+}$ and suppose that $s^{v^*} < T(v^*)$. Let $\{v_n\}$ be a sequence in $S^{m-1}_{\zeta,+}$ satisfying $v_n \to v^*$ as $n \to \infty$. Then for each $s \in (0, s^{v^*})$, $v^{v_n}_y(s,\cdot)$ has exactly m zeros for every large $n >> 0$ and the j-th zero $\xi^{v_n}_j(s)$ of $v^{v_n}_y(s,\cdot)$ satisfies $\xi^{v_n}_j(s) \to \xi^{v^*}_j(s)$ for $n \to \infty$.

Proof
Let $s \in (0, s^{v^*})$ be fixed. Then $v^{v^*}_{yy}(s, \xi^{v^*}_j(s)) \neq 0$ by theorem 3. Indeed, theorem 18 of Angenent holds for $u(t,x)$ and some suitable function $q \in BC^1$ (note that our nonlinearity f is C^1 when considered as map from $\mathbb{R}$ to $\mathbb{R}$), where $u(t,x)$ is defined by the transformation (15). By differentiating we see that theorem 18 is also applicable to $\tilde{u}(t,x) := u_x(t,x)$. Now observe that $v_y(y,s) = 0$ if and only if $u_x(t,x) = 0$, where the variables y, s are defined after (15). Theorem 18, C) therefore shows that $u_{xx}(t,x) \neq 0$ (since $s < s^{v^*}$), if (t,x) corresponds to $(s, \xi^{v^*}_j(s))$ via the transformation after (15). Hence, $v_{yy}(s, \xi^{v^*}_j(s)) \neq 0$. Thus

$$v^{v^*}_y(s, \xi^{v^*}_j(s) - \gamma) \cdot v^{v^*}_y(s, \xi^{v^*}_j(s) + \gamma) < 0, \qquad j = 1, 2, \ldots, m$$

for small $\gamma > 0$. By continuity of $v^{v^*}_y(s,\cdot)$ with respect to $v^*_y \in S^{m-1}_{\zeta,+}$ in the space $Y^\alpha \hookrightarrow H^2_\rho$ we have

$$v^{v_n}_y(s, \xi^{v^*}_j(s) - \gamma) \cdot v^{v_n}_y(s, \xi^{v^*}_j(s) + \gamma) < 0, \qquad j = 1, 2, \ldots, m$$

for large $n > 0$. Therefore, every function $v_y^{v_n}(s,\cdot)$ has at least one zero in $(\xi_j^{v^*}(s)-\gamma, \xi_j^{v^*}(s)+\gamma)$ for every j. Since $z(v_y^{v_n}(s,\cdot)) = \mu(v^{v_n}(s,\cdot)) \leqslant m$ we have proved the lemma, since $\gamma > 0$ was arbitrary. □

Lemma 15
Suppose that $s^{v^*} < T(v^*)$. If $v_n \to v^*$ in $S_{\zeta,+}^{m-1}$ as $n \to \infty$, then $s^{v_n} < T(v_n)$ for sufficiently large n and $s^{v_n} \to s^{v^*}$ for $n \to \infty$.

On account of lemma 14 the set of zeros of $v_y^{v_n}(s^{v_n}-\delta,\cdot)$ consists of m points for every large $n >> 0$. This shows that $s^{v_n} \geqslant s^{v^*} - \delta$ which implies

$$\liminf_{n\to\infty} s^{v_n} \geqslant s^{v^*}$$

and we have to show that

$$\limsup_{n\to\infty} s^{v_n} \leqslant s^{v^*}. \tag{43}$$

Assume first that exactly k zeros of $v_y^{v^*}(s,\cdot)$ reduce to one point at $(s,y) = (s^{v_*}, y_*)$ and no zero escapes to infinity, i.e.,

$$\lim_{s\to s^{v^*}} \xi_j^{v^*}(s) = y_*, \qquad j = i,\dots,i+k-1$$

for some $k \geqslant 2$ and some $1 \leqslant i < m-1$. Then two different cases may occur

I) k is even or

II) k is odd.

Let us only prove I), since II) can be proven analogously. If k is even, we can assume that $v_y^{v^*}(s^{v^*},y) > 0$ for y near y_* and $y \neq y_*$. Then for every $\delta_1 > 0$ there exists $\gamma > 0$, such that

$$v_y^{v^*}(s,y) > 0, \qquad y \in \mathcal{S},$$

and

$$\xi_i^{v^*}(s),\dots,\xi_{i+k-1}^{v^*}(s) \in (y_* - \gamma_1, y_* + \gamma_1)$$

for $s \in [s^{v^*}-\delta_1, s^{v^*})$, where we have set

$$\mathcal{S} = (\{y_* \pm \gamma_1\} \times [s^{v^*}-\delta_1, s^{v^*}+\delta_1]) \cup ([y_*-\gamma_1, y_*+\gamma_1] \times \{s^{v^*}+\delta_1\}).$$

By continuity of the solution v with respect to the initial value v^* in the space Y^α, we have

$$v_y^{v_n}(s,y) > 0, \qquad y \in \mathcal{S},$$

if n is large so that $v_y^{v_n}(s^{v^*}+\delta_1, y)$ has no zero in $[y_*-\gamma_1, y_*+\gamma_1]$. By lemma 14, $v_y^{v_n}(s^{v^*}-\delta_1, y)$ has exactly k zeros in $(y_*-\gamma_1, y_*+\gamma_1)$ if n is large enough. Therefore

$$z(v_y^{v_n}(s^{v^*}-\delta_1, y)) > z(v_y^{v_n}(s^{v^*}+\delta_1, y)),$$

so that $s^{v_n} \leqslant s^{v^*} + \delta_1$. This proves (43), since $\delta_1 > 0$ can be made arbitrarily small.

Let us now consider the case that one extremum location $\xi_j^{v^*}(s)$ of $v^{v^*}(s,\cdot)$ converges to infinity as $s \nearrow s^{v^*}$ for one $j = 1, \dots, m$. Then for any $N > 0$ there exists a $\delta_2 > 0$ such that

$$\xi_j^{v^*}(s) > N, \qquad s \geqslant s^{v^*} - \delta_2. \tag{44}$$

Indeed, since $s^{v^*} < T(v^*)$ the zero set of $v_y^{v^*}(s,\cdot)$ for $s \leqslant s^{v^*}$ near s^{v^*} contains at most m points which implies that the extremum location $\xi_j(s)$ cannot accumulate at some point $y_* \in \mathbb{R}$ (and therefore the limes inferior and superior coincide). On account of lemma 14 we conclude that

$$\xi_j^{v_n}(s) > N, \qquad s > s^{v^*} - \delta_2 \tag{45}$$

if $n >> 0$ is large enough. Now choose a sequence $\{s_N\}_{N\in\mathbb{N}}$ with $s_N \nearrow s^{v^*}$, such that (44) holds for $s = s_N$. For each N we then choose $n = n(N)$ large enough such that $\xi_j^{v_n}(s_N) > N$. Note that $n(N) \to \infty$ as $n \to \infty$. This implies that

$$\xi_j^{v_n}(s_N) \to \infty, \qquad \text{as } N \to \infty,$$

which shows that $\limsup s^{v_n} \leqslant s^{v^*}$ and everything is proved. $\square$

Let us now prove continuity of Γ in the case $s^{v^*} < T(v^*)$.

Lemma 16
Suppose that $s^{v^*} < T(v^*)$ and that $v_n \to v^*$ in $S^{m-1}_{\zeta,+}$ as $n \to \infty$. Then $\Gamma(v_n) \to \Gamma(v^*)$.

Proof
By the previous lemma we can assume that $s^{v_n} < T(v_n)$ for all n and that $s^{v_n} \to s^{v^*}$ as $n \to \infty$. We now show that the j-th component $\Gamma_j(v^*)$ is continuous. We first claim that for every $y \in \mathbb{R}$ and $\varepsilon > 0$ there is a $M = M(y,\varepsilon) > 0$, such that

$$|v^{v_n}(s^{v_n}, y) - v^{v_n}(s^{v_n}, x)| \leqslant M|y-x|, \qquad x \in B_\varepsilon(y), \tag{46}$$

if $n > 0$ is large enough. In order to prove this claim, let us note that $v^{v_n}(s,\cdot) \in Y^\alpha$ depends continuously on $s \approx s^{v_*}$, $s < T(v_n)$ and the initial condition $v_n \in S^{m-1}_{\zeta,+}$. This proves $v^{v_n}(s,\cdot) \in C^1_{loc}$ and

$$|v^{v_n}(s,y) - v^{v_n}(s,x)| \leqslant \max_{z,\tilde{s}} |v_y^{v_n}(\tilde{s}, z)| \cdot |y-x|, \tag{47}$$

where the maximum is taken with respect to $\tilde{s} \in B_\varepsilon(s^{v^*}), z \in B_\varepsilon(y)$. In particular, the maximum is well-defined and (46) is proved.
Now consider the j-th zeros $\xi_j^{v^*}(s)$, $\xi_j^{v_n}(s)$ of $v_y^{v^*}(s,\cdot)$ and $v_y^{v_n}(s,\cdot)$, respectively. Assume first that k zeros of $v_y^{v^*}(s,\cdot)$, say

$$\xi_i^{v^*}(s), \dots, \xi_{i+k-1}^{v^*}(s)$$

accumulate at $(s,y) = (s^{v^*}, y_*)$. Then we have

$$\Gamma_j(v^*) = c, \qquad j = i+1, \dots, i+k-1$$

for some appropriate $c \in \mathbb{R}$ which coincides with the value of the extrema of $v^{v^*}(s,\cdot)$ that coalesce at y^* for $s \nearrow s^{v^*}$. For any $\gamma > 0$ there exists a $\delta > 0$ with

$$|v_y^{v^*}(s, y_* \pm \gamma)| > 0, \qquad s \in [s^{v^*} - \delta, s^{v^*} + \delta]$$

and

$$\xi_i^{v^*}(s), \ldots, \xi_{i+k-1}^{v^*}(s) \in (y_* - \gamma, y_* + \gamma)$$

for $s \in [s^{v^*} - \delta, s^{v^*})$. By lemma 14 we conclude

$$\xi_i^{v_n}(s), \ldots, \xi_{i+k-1}^{v^n}(s) \in (y_* - \gamma, y_* + \gamma)$$

for $s \in [s^{v^*} - \delta, s^{v_n})$ if $n > 0$ is large enough. Using (46), we can therefore conclude for $j = i+1, \ldots, i+k-1$ that

$$\begin{aligned} |\Gamma_j(v_n) - c| &\leqslant \max_{z \in (y_* - \gamma, y_* + \gamma)} |v^{v_n}(s^{v_n}, z) - c| \\ &\leqslant \max_z |v_y^{v_n}(s^{v_n}, z)| \cdot |z - \xi_{j-1}| \\ &\leqslant 2M \cdot \gamma \end{aligned}$$

if $n > 0$ is large. Since M only depends on y, ε and $\gamma > 0$ can be chosen arbitrarily small we have shown that $\Gamma_j(v_n) \to c$ for $j = i+1, \ldots, i+k-1$. Let us now consider the case that $\Gamma_j(v^*) = c$ for one $j = 1, \ldots, m$, but $\xi_{j-1}^{v^*}(s) \to \infty$ as $s \nearrow s^{v^*}$. From this we conclude that $c = 0$, see lemma 13. Then $\xi_j^{v_n}(s) \to \infty$ for $s \nearrow s^{v_n}$, $n \to \infty$, on account of $s^{v_n} \to s^{v^*}$ and lemma 14. Since $v^{v_n}(s,\cdot) \in Y^\alpha$ depends continuously on the initial data v_n and $s \approx s^{v^*}$ we conclude that $v^{v_n}(s,\cdot)$ is small on the interval $[\kappa, \infty)$ for some $\kappa >> 0$; uniformly with respect to $v_n \approx v^*$ and $s \approx s^{v_*}$. Choose $\varepsilon > 0$; then for s close to s^{v^*} and n large enough we have

$$|\Gamma_j(v_n)| \leqslant \max_{(\xi_j(s), \infty)} |v^{v_n}(s,\cdot)| \leqslant \varepsilon$$

which again proves $\Gamma_j(v_n) \to c = 0$ as $n \to \infty$. □

6.2 Dropping time equal to the blow up time

We now consider the case $s^{v^*} = T(v^*)$ and start by proving the following lemma.

Lemma 17
Let $v^* \in S_{\zeta,+}^{m-1}$ and suppose that $s^{v^*} = T(v^*)$. Let $\{v_n\}$ be a sequence in $S_{\zeta,+}^{m-1}$ satisfying $v_n \to v^*$ as $n \to \infty$. Then $s^{v_n} \to s^{v^*} = T(v^*)$ for $n \to \infty$.

Proof
For $\delta > 0$ small enough, the set of zeros of $v_y^{v_n}(T(v^*) - \delta, \cdot)$ consists of m points for large n by lemma 14, which shows $s^{v_n} \geqslant T(v^*) - \delta$. We conclude that

$$\liminf_{n \to \infty} s^{v_n} \geqslant T(v^*).$$

Since $s^{v_n} \leqslant T(v_n)$ for all n and $T(v_n) \to T(v^*)$ as $n \to \infty$ we obtain $\limsup_{n\to\infty} s^{v_n} \leqslant T(v^*)$, which proves the claim. □

The next two lemmas have been proved in [28] and we state them in terms of equation (21).

Lemma 18
Let v be a solution of (21) with blow up time T. If $v(s,a) \to \infty$ (resp. $-\infty$) as $s \to T$, then for any $M > 0$ there exists a $\delta > 0$ such that

$$v(s,y) \geqslant M(\text{ resp. } v(s,y) \leqslant -M), \qquad \text{in } [a-\delta, a+\delta] \times [T-\delta, T).$$

In particular $v(y,T) \to \infty$ (resp. $-\infty$) as $y \to a$.

The next result concerns the continuity of the blow up profile with respect to the initial data.

Lemma 19
Let v_n and v be solutions of (21), with initial data $v_{0,n}$ and v_0 and blow up times T_n and T, respectively. Denote by B the blow up set of v. Suppose that $v_{0,n} \to v_0$ in H^1_ρ as $n \to \infty$. Then

a) $v_n(T_n, \cdot) \to v(T, \cdot)$ locally uniformly in $\mathbb{R} \setminus B$ as $n \to \infty$.

b) Let $\{s_n\}$ be a sequence satisfying $s_n < T_n$ for all n and $s_n \to T$ as $n \to \infty$. Then $v_n(s_n, \cdot) \to v(T, \cdot)$ locally uniformly in $\mathbb{R} \setminus B$ as $n \to \infty$.

c) If a is a blow up point of v to $+\infty$ (resp. $-\infty$), then there is a sequence $\{y_n\}$ such that $y_n \to a$ and $v_n(T_n, y_n) \to \infty$ (resp. $-\infty$) as $n \to \infty$.

d) Suppose a is a blow up point of v to $+\infty$ (resp. $-\infty$). If $s_n < T_n$ for all n and $s_n \to T$ as $n \to \infty$, then for all small $\varepsilon > 0$,

$$\begin{aligned} &\max \quad \{v_n(s_n, y) : y \in [a-\varepsilon, a+\varepsilon]\} \to +\infty \\ (resp.\ &\min \quad \{v_n(s_n, y) : y \in [a-\varepsilon, a+\varepsilon]\} \to +\infty). \end{aligned}$$

Remark
This lemma is proved analogously to the case of bounded domains; see [23]. Let us note that the statements of the lemma are actually still true if we replace the convergence in H^1_ρ in the assumption by a convergence in $L^\infty \cap H^1$, see proposition 2 in [24].

Finally, we can prove continuity of $\Gamma(v^*)$ in the case $s^{v^*} = T(v^*)$.

Lemma 20
Suppose that $s^{v^*} = T(v^*)$. If $v_n \to v^*$ as $n \to \infty$, then $\Gamma(v_n) \to \Gamma(v^*)$ as $n \to \infty$.

Proof
We consider four different cases.
Case I)

We first consider the case that no limes superior or inferior of the extremum locations $\xi_j^{v^*}(s)$ tends to $\pm\infty$ as $s \nearrow s^{v^*} = T(v^*)$. Moreover, we suppose that $\underline{\xi}_j < \bar{\xi}_{j-1}$ for some j. Let $[d,e] \subset \mathbb{R}$ be a maximal interval such that

$$v^{v^*}(T(v^*), y) = c \qquad \text{in } [d,e]$$

for some constant $c \in \mathbb{R}$ and $[\underline{\xi}_j^{v^*}, \bar{\xi}_{j-1}^{v^*}] \subseteq [d,e]$. Indeed, such an interval exists and is well defined on account of lemma 5. Since $[d,e]$ is bounded, for every small $\varepsilon > 0$ there exists $K_1 > 0$ such that

$$v^{v^*}(T(v^*), d-\varepsilon) < c - K_1, \quad \text{or} \quad v^{v^*}(T(v^*), d-\varepsilon) > c + K_1$$

and

$$v^{v^*}(T(v^*), e+\varepsilon) < c - K_1 \quad \text{or} \quad v^{v^*}(T(v^*), e+\varepsilon) > c + K_1$$

Let B denote the blow up set of the solution v associated to the initial value v^*. Then $B \cap [d,e] = \emptyset$ by lemma 18 and we conclude that $B \cap [d-\varepsilon, e+\varepsilon] = \emptyset$ for sufficiently small ε. It follows from lemma 19 (a) that

$$v^{v_n}(T(v_n), d-\varepsilon) < c - \frac{K_1}{2} \quad \text{or} \quad v^{v_n}(T(v_n), d-\varepsilon) > c + \frac{K_1}{2}$$

and

$$v^{v_n}(T(v_n), e+\varepsilon) < c - \frac{K_1}{2} \quad \text{or} \quad v^{v_n}(T(v_n), e+\varepsilon) > c + \frac{K_1}{2}$$

if n is large enough. Since on account of lemma 19 the values of $v^{v_n}(s,\cdot)$ are arbitrarily close to c on $[d,e]$ if $n > 0$ is large enough, it follows by the mean value theorem that for every fixed $s < T$, which is close to T, there exists an $n_0 > 0$ and at least one point $y_n^s \in [d-\varepsilon, d]$ and one point $z_n^s \in [e, e+\varepsilon]$, such that

$$|v_y^{v_n}(s, y_n^s)| \neq 0, \qquad |v_y^{v_n}(s, z_n^s)| \neq 0$$

for every $n > n_0$. Note that we have used that for fixed $s < T(v^*)$ we have $s < T(v_n)$ if n is large enough. Hence by lemma 14 we conclude

$$\xi_{j-1}^{v_n}(s), \xi_j^{v_n}(s) \in (d-\varepsilon, e+\varepsilon) \qquad \text{in } [T(v^*)-\delta, s^{v_n})$$

when n is large. Finally this shows

$$\xi_{j-1}^{v_n}, \xi_j^{v_n} \in (c-\varepsilon, d+\varepsilon)$$

in the case $s^{v_n} < T(v_n)$ and

$$[\underline{\xi}_j^{v_n}, \bar{\xi}_{j-1}^{v_n}] \subset (c-\varepsilon, d+\varepsilon)$$

if $s^{v_n} = T(v_n)$. Now it follows from lemma 19 that

$$\Gamma_j(v_n) \to \Gamma_j(v^*)$$

as $n \to \infty$.

Case II)
If in the same case $\bar{\xi}_{j-1}^{v^*} > \underline{\xi}_j^{v^*}$ at least one of the locations $\xi_i^{v^*}(s)$, $i = j-1, j$, tends to $\pm\infty$ for $s \nearrow T(v^*)$ we can proceed exactly the same way. Note that if $\xi_{j-1}^{v^*}(s) \to \infty$ as $s \to T(v^*)$ we actually conclude that $\Gamma_j(v^*) = 0$, no matter whether $\xi_j(s)$ stays bounded as $s \to T(v^*)$ or not. Indeed, the blow up profile $v(T(v^*), \cdot)$ is constant on $[\underline{\xi}_j^{v^*}, \bar{\xi}_{j-1}^{v^*}]$ by lemma 5 and lemma 13 actually implies that the constant is zero.

Case III)
If $\underline{\xi}_j^{v^*} = \bar{\xi}_{j-1}^{v^*} =: \xi$ two things can happen: Either ξ belongs to the blow up set B of $v^{v^*}(s, \cdot)$ or not. If ξ does belong to B an application of lemma 19, d) implies that

$$\Gamma_j(v_n) \to \Gamma_j(v^*), \qquad \text{as } n \to \infty.$$

If on the other hand $\xi \notin B$, the same argumentation as in case I) applies and proves continuity of Γ_j.

Case IV)
The case $\bar{\xi}_{j-1}^{v^*} < \underline{\xi}_j^{v^*}$ is now straightforward and we omit it. □

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