MASTER THESIS

Time-Delayed Feedback Control of Rotationally Symmetric Systems

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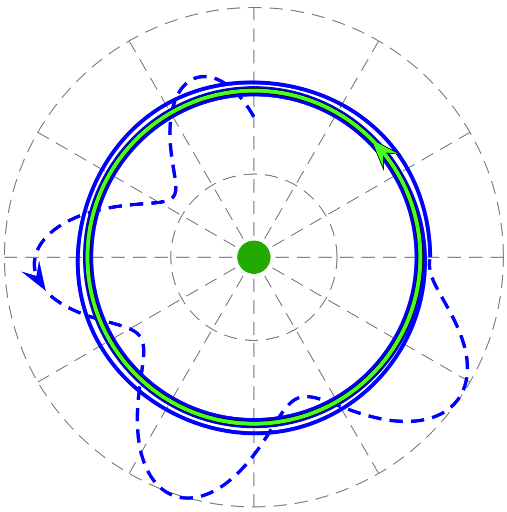

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Abstract

Time-delayed feedback control has proven to be useful to stabilize periodic orbits. We extend the concept of Pyragas control with a linear control-matrix to nonlinear non-invasive control-schemes. For a two-dimensional rotationally symmetric system we show that equivariant Pyragas control can *always* be used for stabilization if the *delay-time is small enough*. We present a method to find such a stabilizing control-scheme. The results are *not* limited to systems near a Hopf bifurcation. We give a *strict* threshold on the unstable Floquet multiplier and the delay-time at which control becomes impossible – showing the limitations of classical Pyragas control. Finally we apply the control to a system in Hopf normal form and a coupled oscillator system.

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1 Introduction

The aim of this thesis is to establish for which periodic orbits in a rotationally symmetric system stabilization by a certain time-delayed is possible. The control-method we are considering belongs to the class of time-delayed feedback control. While experimental results from physics and mathematical results near Hopf bifurcation (introduced in section 2.4) exist – see e.g. [13], [17], [20], [22] and references therein – stabilization far from the bifurcation point is still relatively unexplored [21].

1.1 Previous developments in the field

The idea of using only small perturbations to change the asymptotic behavior near a periodic orbit was famously considered by Ott, Grebogi, and Yorke in 1990 [16]. They proposed a method where control is applied at discrete time-steps. For this method they proved stabilization of a periodic orbit in chaotic systems. However this method requires quite intricate knowledge of the system's behavior near the periodic orbit, which might be hard to attain experimentally.

In 1992, Pyragas suggested a continuously applied time-delayed feedback control

$$K\left[\mathbf{x}(t) - \mathbf{x}(t - mT)\right], \tag{1.1}$$

with control-matrices K and delays that are a multiple of the period T [19]. By adding such a term, an unstable periodic orbit is to be stabilized. We can see that for T-periodic solutions the control-term (1.1) vanishes. Whenever the periodic orbit of the uncontrolled system persists, we call a control *non-invasive*.

This method of control is easily implemented experimentally. It does not require intricate knowledge of the periodic orbit one aims to stabilize, but only its period. Thus, since its introduction a large number of publications have applied it to real- and model-systems – see the references in [22]. This so called *Pyragas control* is also the basis for the control-method presented in this thesis.

The obvious question regarding the method was whether it can achieve stabilization. In 1997, Hiroyuki Nakajima proved the so called *odd-number limitation* [14] for a non-autonomous system, stating that an odd number of unstable Floquet-multipliers (see section 2.2) prevents stabilization of hyperbolic periodic orbits via Pyragas control.

This strong limitation was *assumed* (although not claimed explicitly in the paper) to also exist for autonomous systems. In 2006, this assumed 'autonomous odd-number limitation' was refuted by Fiedler et. al. using a counter-example in Hopf normal form [7].

After the odd-number limitation turned out to be non-existent for *autonomous* systems there were several results for stabilization near Hopf bifurcation via Pyragas control. Two coupled oscillators were considered in [8], extending the results to a four-dimensional example with a few additional restrictions. Based on this paper, Konstantin Bubolz and I developed modifications to the control-method in our bachelor theses ([3] and [2] respectively). There, most of the additional restrictions were lifted.

An extension to the stabilization of three coupled oscillators was developed by Isabelle Schneider [24]. Periodic orbits with a specific symmetry ("ponies on a merry-go-round") were stabilized. Formalizing such control-methods in her master thesis, she coined the term *equivariant Pyragas control* [25]. Stabilization results for quite general systems of ordinary differential equations were obtained. The equivariant Pyragas control is non-invasive only on periodic orbits with a certain *spatio-temporal symmetry*

$$\mathbf{x}_*(t) = h\,\mathbf{x}_*(t - \theta T) \tag{1.2}$$

where h is a linear map corresponding to the normalized time-delay θ – for a group-theoretical definition see the introduction of [25]. The equivariance was used to stabilize periodic orbits that could not be stabilized before – again near Hopf bifurcation.

Some earlier stabilization-schemes also comply with the definition of equivariant Pyragas control. For example the half-period delay considered in [15] requires the periodic orbit to be what the authors call 'self-symmetric', i.e. $\mathbf{x}(t) = -\mathbf{x}\left(t - \frac{1}{2}T\right)$ where T is again the period.

The stabilization approach of this thesis also is an equivariant one. We use the rotational symmetry (more precisely the S^1-*equivariance*)

$$x_*(t) = \exp\left(2\pi \mathrm{i}\,\theta T\right)\, x_*(t - \theta T) \tag{1.3}$$

with $x \in \mathbb{C}$ and $\theta > 0$ to construct a control-method. This symmetry was considered by Choe et al. in [4] as well as in my bachelor thesis [2] where it turned out to be very useful.

In a recent joint recent joint publication [23], Isabelle Schneider and I applied equivariant Pyragas control to a system of n oscillators coupled diffusively in a bidirectional ring. The aim there was to remove restrictions that exist without the use of an equivariance. The equivariance also allows one to select one of multiple periodic orbits with the same period. The result from my bachelor thesis, i.e. the use of the symmetry (1.3) is extended to this system to construct larger regions of stabilizing parameters. An application of *this thesis‘* results to such a dynamical system is presented in chapter 6.

We see that stabilization is proven *near Hopf bifurcation* in these references. An interesting result for orbits far from a bifurcation point was obtained by Hooton and Amann in 2012 [12]. They proved a necessary (but not sufficient) condition which control-matrices must satisfy to be stabilizing. This was done for an *arbitrary* autonomous system with Pyragas control of the form (1.1). In [21] this condition was extended and stabilization of a periodic orbit in the Lorentz attractor, which is chaotic and has unstable dimension one, was achieved numerically. Recently, numerical results regarding successful stabilization far from the bifurcation-point were also obtained in [18]. As their model system they chose Hopf normal form – whose stabilization will be considered in thesis, too.

1.2 Approach and structure of this thesis

Motivated by the successful application of equivariant Pyragas control near Hopf-bifurcation, the *approach of this thesis* is to apply the promising strategy of equivariant Pyragas control to orbits *far from a bifurcation point.* Our aim is to prove stabilization in a constructive manner.

Based on the existing result by Bernold Fiedler for rotationally symmetric systems [9] and the possibility to use the S^1-symmetry for an equivariant approach [23], we combine the two concepts. We will find a (main) theorem that tells us whether or not stabilization of a periodic orbit is possible, based on the unstable Floquet multiplier and delay-time alone. A second theorem will give us a means to find suitable control-schemes.

To achieve the aim, the thesis is structured as follows:

In the following chapter, we take a look at theoretical concepts that are essential for this thesis. Delayed differential equations and the stability considerations for such equations are explained. The Hopf normal form, as the canonical example of a rotationally symmetric two-dimensional systems, is introduced.

Having revisited the theory, we are now able to state our main results – the Main Theorem and Theorem 2. The proofs are carried out in the following chapter 4.

In chapter 5 an example of such a stabilizing control-method is given for Hopf normal form. The main results are successfully tested for this model system.

To show that our results can also be useful in higher-dimensional systems, in chapter 6, we adapt them to periodic orbits in a system of n coupled oscillators. We prove stabilization of periodic orbits in the considered system which might have arbitrary unstable dimension.

Finally, in chapter 7, we discuss our findings and look at potential starting points for further investigation.

2 Theoretical preliminaries

This chapter introduces the basic concepts of delay differential equations, characteristic equations and the Hopf normal form, which are important for this thesis. At least an intuitive understanding of bifurcations would also be useful to understand the proof of the main results. Bifurcation theory is not part of these preliminaries though, but we refer to [10] for an introduction.

Readers familiar with the subject can easily skip sections 2.1 and 2.2 and move directly to section 2.3, where the characteristic equation for a general two dimensional system with non-invasive delay terms is explicitly calculated.

2.1 Delay-differential equations

The control-method presented in this thesis uses a delayed term. While without control we have a system of autonomous ordinary differential equations, with control the prehistory of the orbit comes into play. Therefore the considered behavior is given by delay differential equations (for a general introduction see [5]). Why solutions exist in this case and in what sense the systems are infinite dimensional will briefly be explained in this section. A basic knowledge of ordinary differential equations is assumed (see [1] for an introduction).

Let us start by considering a delay equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-\tau_1), \mathbf{x}(t-\tau_2), \dots, \mathbf{x}(t-\tau_n)) \tag{2.1}$$

which has only discrete delays $0 < \tau_1 < \tau_2 < \cdots < \tau_n$.

For such a system initial conditions cannot consist of only one point $\mathbf{x}(t=0) \in \mathbb{R}^n$ as the delayed terms $\mathbf{x}(t-\tau_j)$ would not be defined at zero. Instead we have to use a *prehistory-function* $\mathbf{x}_0(\cdot) \in \mathrm{C}^0([-\tau_n, 0])$ as initial condition, giving us the necessary values for $-\tau_n \leq t \leq 0$.

Contrary to most partial differential equations the existence and uniqueness of a solution to this problem is not more difficult than for ordinary differential equations – and it is proven in the same way.

Consider the problem only for times $t \in [0, \tau_1]$. Then the delayed terms $\mathbf{x}(t - \tau_j)$ can be replaced by the values of the prehistory $\mathbf{x}_0(t - \tau_j)$ for all $j = 1, \ldots, n$. Those are defined as long as $t \leq \tau_1$. Only the non-delayed term $\mathbf{x}(t)$ remains and (2.1) has become an *ordinary* differential equation. Thus the Picard-Lindelöf theorem (see [1]) applies, giving us existence and uniqueness of solutions.

To obtain the trajectories for $t > \tau_1$ the method of steps is used: For $t \in [\tau_1, 2\tau_1]$ we reapply the process with our new found solution as prehistory. This way we can find the solution in a stepwise manner as long as it exists – so, if no blow-up appears, forever.

In this thesis we will only deal with autonomous equations without a t-dependence in $\mathbf{f}$ and only a single delay. In the discussion we will come back to the multiple delay case though.

2.2 Determining stability

The problem considered in this thesis is one of asymptotic stability of periodic orbits in delay-equations. Therefore in this section we will briefly look at how a stability analysis in this setting generally is performed.

Let us first look at ordinary differential equations (without delay) to see what can be carried over to the delay differential equation case and where difficulties may arise. For periodic orbits the classical approach is to find the *Poincaré map* for a *transverse intersection* with the orbit (see [1]). Let our orbit be given by $\mathbf{x}_*(t) \in \mathbb{R}^n$. Then we need to define a $(n-1)$-dimensional differentiable sub-manifold $\mathcal{S}$ – e.g. a hyperplane. The periodic orbit must intersect this so called Poincaré section transversely, i.e.

$$\mathbf{x}_*(0) \in \mathcal{S} \text{ and} \tag{2.2}$$

$$\dot{\mathbf{x}}_*(0) \notin \operatorname{span}\left(\mathrm{D}\mathcal{S}(\mathbf{x}_*(0))\right). \tag{2.3}$$

The Poincaré map

$$\begin{aligned} \mathbf{P}\colon \quad \mathcal{S} &\to \mathcal{S} \\ \mathbf{x}(0) &\mapsto \mathbf{x}(t_R) \end{aligned} \tag{2.4}$$

assigns to every initial condition in the Poincaré section the point at which the orbit first crosses $\mathcal{S}$ again. The return-time $t_R = t_R(\mathbf{x}) > 0$ will in general be different from the period of the periodic orbit, but in a neighborhood to the periodic orbit it will exist, by continuity of the flow.

The intuition behind those definitions is that we 'turn' the periodic orbit into a fixed point of a map that is differentiable. The local stability can then be determined by looking at the derivative of the Poincaré map near the starting point of the periodic orbit $x_*(0)$. The eigenvalues μ of $\mathrm{D}\mathbf{P}|_{\mathbf{x}=\mathbf{x}_*(0)}$ are called *Floquet multipliers* of the periodic orbit. The number of Floquet multipliers outside the unit circle is called the *unstable dimension.* In this thesis we will only use the *Floquet exponent* $\tilde{a} \in \mathbb{C}$ in our formulations. It is defined as $\mu =: \exp(\tilde{a})$ and consequently unstable Floquet exponents are those with positive real part.

In the rotationally symmetric systems considered in this thesis we will be able to transform the periodic orbit to a ring of equilibria. This means we will only have to consider the stability of an equilibrium, which is considerably easier than to calculate the return map $\mathbf{P}$ and determine its eigenvalues.

So let us recollect how the stability of an equilibrium is determined in an ordinary differential equation. For a system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{2.5}$$

we can linearize at the equilibrium $\mathbf{x}_*(0)$:

$$\dot{\delta\mathbf{x}}(t) = \big(\mathrm{D}\mathbf{f}\big)\,\delta\mathbf{x}(t) \tag{2.6}$$

The stability is given by the eigenvalues, which are easily obtained from the characteristic polynomial, where the solutions η of

$$0 = \det\big(\mathrm{D}\mathbf{f} - \eta\,\mathrm{Id}\big). \tag{2.7}$$

are the eigenvalues.

This can be generalized to delay equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau)), \tag{2.8}$$

leading to a linearization

$$\dot{\delta\mathbf{x}}(t) = \big(\mathrm{D}_1\mathbf{f}\big)\,\delta\mathbf{x}(t) + \big(\mathrm{D}_2\mathbf{f}\big)\,\delta\mathbf{x}(t-\tau). \tag{2.9}$$

To understand the stability properties of this equation, let us look at the transformation from (2.6) to (2.7) in a different way. Consider an eigenfunction $\mathbf{e}_\eta$ to the eigenvalue η. Its derivative is $\dot{\mathbf{e}}_\eta = \eta\mathbf{e}_\eta$, so it solves

$$\eta\mathbf{e}_\eta(t) = \big(\mathrm{D}\mathbf{f}\big)\,\mathbf{e}_\eta(t), \tag{2.10}$$

which can be rewritten as matrix equation (2.7) by canceling out $\mathbf{e}_\eta$ on both sides. For our linearized delayed equation (2.9) we can use an exponential ansatz for the eigenfunctions, giving us

$$\mathbf{e}_\eta(t-\tau) = \exp(-\eta\tau)\,\mathbf{e}_\eta(t). \tag{2.11}$$

This allows us to rewrite (2.9) for eigenfunctions as

$$0 = \big(\mathrm{D}_1\mathbf{f}\big)\,\mathbf{e}_\eta(t) + \big(\mathrm{D}_2\mathbf{f}\big)\,\exp(-\eta\tau)\,\mathbf{e}_\eta(t) - \eta\mathbf{e}_\eta(t). \tag{2.12}$$

Canceling out the eigenfunction produces a determinant equation

$$0 = \det\left(\mathrm{D}_1\mathbf{f} + \exp(-\eta\tau)\,\mathrm{D}_2\mathbf{f} - \eta\,\mathrm{Id}\right) \tag{2.13}$$

with a right hand side that, due to the exponential term, no longer is a polynomial in η.

Remark. From (2.13) we can see why we have infinitely many eigenvalues for delay-equations. For simplicity consider the following characteristic equation for one dimension:

$$0 = \exp(-\eta\tau) - \eta \tag{2.14}$$

Let us split this equation up in real and imaginary part for $\eta = \rho + \mathrm{i}\omega$:

$$\begin{aligned} \rho &= \exp(-\rho\tau)\cos\omega\tau \\ \omega &= \exp(-\rho\tau)\sin\omega\tau \end{aligned} \tag{2.15}$$

We see that for every $N \in \mathbb{N}$ there exist $\rho < -N$ and ω (with $\cos\omega\tau \to 0$ and $\sin\omega\tau \to 1$ for $N \to \infty$) solving (2.15). Therefore infinitely many eigenvalues exist which is typical for delay-equations.

2.3 Characteristic equation for a two-dimensional system

The systems that appear in the thesis all either have dimension two or can be split into two-dimensional subsystems. Our stabilization proofs (in chapters 4, 5 and 6) require characteristic equations. We will find this equation in $\mathbb{R}^2$ with one delay explicitly. More precisely we look at linear systems (resulting from linearization)

$$\dot{\mathbf{x}}(t) = M_0\mathbf{x}(t) + M_\tau\mathbf{x}(t-\tau) \tag{2.16}$$

with $\mathbf{x} \in \mathbb{R}^2$ and arbitrary matrices $M_0, M_\tau \in \mathbb{R}^{2\times 2}$.

In this thesis, we use complex notation for the two-dimensional case – i.e. for $x_1 + \mathrm{i}x_2 =: z \in \mathbb{C}$ we consider the complex equation

$$\dot{z} = \alpha z + \beta \bar{z} + \varphi[z - z(t - \theta T)] + \psi[\bar{z} - \bar{z}(t - \theta T)] \tag{2.17}$$

which is equivalent to (2.16). The parameters $\alpha, \beta, \varphi, \psi \in \mathbb{C}$ represent all possible choices of M_0 and M_τ. We formulate the following lemma

Lemma 1. *For a linear complex one-dimensional delay differential equation* (2.17) *the characteristic equation is of the form*

$$0 = \eta - a + b_0 u + b_1 u\eta + b_2 u^2 \eta + c/\eta \tag{2.18}$$

with rescaled eigenvalues $\eta = \tilde{\eta}\theta T \in \mathbb{C}$*,* $u = u(\eta) = [1 - \exp(-\eta)]/\eta$ *and the parameters defined as*

$$\begin{aligned} a &= 2\,\mathrm{Re}\,\alpha\,\theta T \\ b_0 &= 2\,\mathrm{Re}\left(\alpha\bar{\varphi} - \beta\bar{\psi}\right)(\theta T)^2 \\ b_1 &= -2\,\mathrm{Re}\,\varphi\,\theta T \\ b_2 &= \left(|\varphi|^2 - |\psi|^2\right)(\theta T)^2 \\ c &= \left(|\alpha|^2 - |\beta|^2\right)(\theta T)^2. \end{aligned} \tag{2.19}$$

The proof of this lemma is straightforward but rather tedious. We will write (2.17) in a more convenient matrix-notation, calculate a determinant and read off the coefficients.

With an exponential ansatz we get the equation

$$\tilde{\eta}\,\mathrm{Id} = A + B + \left[1 - \mathrm{e}^{-\tilde{\eta}\theta T}\right]\Phi + \left[1 - \mathrm{e}^{-\tilde{\eta}\theta T}\right]\Psi \tag{2.20}$$

with complex eigenvalue $\tilde{\eta}$,

$$A = \begin{pmatrix} \mathrm{Re}\,\alpha & -\mathrm{Im}\,\alpha \\ \mathrm{Im}\,\alpha & \mathrm{Re}\,\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \mathrm{Re}\,\beta & \mathrm{Im}\,\beta \\ \mathrm{Im}\,\beta & -\mathrm{Re}\,\beta \end{pmatrix} \tag{2.21}$$

and Φ and Ψ defined analogously for φ and ψ respectively. To shorten the notation let us abbreviate $\alpha_\mathrm{R} := \mathrm{Re}\,\alpha$, $\alpha_\mathrm{I} := \mathrm{Im}\,\alpha$ and define this analogously for β, φ and ψ. Also we use the shorthand $\tilde{u} := [1 - \exp(-\tilde{\eta}\theta T)]$. Taking the determinant gives us the characteristic equation:

$$\begin{aligned} 0 &= \det\left(A + B + \tilde{u}\Phi + \tilde{u}\Psi - \tilde{\eta}\,\mathrm{Id}\right) \\ &= \det\begin{pmatrix} \alpha_\mathrm{R} + \beta_\mathrm{R} + \tilde{u}\varphi_\mathrm{R} + \tilde{u}\psi_\mathrm{R} - \tilde{\eta} & -\alpha_\mathrm{I} + \beta_\mathrm{I} - \tilde{u}\varphi_\mathrm{I} + \tilde{u}\psi_\mathrm{I} \\ \alpha_\mathrm{I} + \beta_\mathrm{I} + \tilde{u}\varphi_\mathrm{I} + \tilde{u}\psi_\mathrm{I} & \alpha_\mathrm{R} - \beta_\mathrm{R} + \tilde{u}\varphi_\mathrm{R} - \tilde{u}\psi_\mathrm{R} - \tilde{\eta} \end{pmatrix} \\ &= (\alpha_\mathrm{R} + \beta_\mathrm{R} + \tilde{u}\varphi_\mathrm{R} + \tilde{u}\psi_\mathrm{R} - \tilde{\eta})(\alpha_\mathrm{R} - \beta_\mathrm{R} + \tilde{u}\varphi_\mathrm{R} - \tilde{u}\psi_\mathrm{R} - \tilde{\eta}) \\ &\quad - (-\alpha_\mathrm{I} + \beta_\mathrm{I} - \tilde{u}\varphi_\mathrm{I} + \tilde{u}\psi_\mathrm{I})(\alpha_\mathrm{I} + \beta_\mathrm{I} + \tilde{u}\varphi_\mathrm{I} + \tilde{u}\psi_\mathrm{I}). \end{aligned} \tag{2.22}$$

To remove the factor θT from our exponential terms, we will divide the whole equation by $\tilde{\eta}\theta T$ and introduce the rescaled eigenvalue $\eta := \tilde{\eta} \cdot \theta T$. Consequently we also introduce $u := \tilde{u}/(\tilde{\eta}\theta T) = [1 - \exp(-\eta)]/\eta$.

We want choose the parameters φ and ψ to control the system's eigenvalues. So it is suitable to determine which terms in our rescaled η and u appear. The rescaled version of equation (2.22) is of the form

$$0 = \eta - a + b_0 u + b_1 u\eta + b_2 u^2 \eta + c/\eta \tag{2.18}$$

with real parameters a, b_0, b_1, b_2 and c. All that remains to prove Lemma 1 is to read off the coefficients from (2.22):

$$\begin{aligned}
a &= 2\alpha_{\mathrm{R}}\,\theta T \\
b_0 &= 2\left(\varphi_{\mathrm{R}}\alpha_{\mathrm{R}} - \psi_{\mathrm{R}}\beta_{\mathrm{R}} + \varphi_{\mathrm{I}}\alpha_{\mathrm{I}} - \psi_{\mathrm{I}}\beta_{\mathrm{I}}\right)(\theta T)^2 \\
b_1 &= -2\varphi_{\mathrm{R}}\,\theta T \\
b_2 &= \left(\varphi_{\mathrm{R}}^2 - \psi_{\mathrm{R}}^2 + \varphi_{\mathrm{I}}^2 - \psi_{\mathrm{I}}^2\right)(\theta T)^2 \\
c &= \left(\alpha_{\mathrm{R}}^2 - \beta_{\mathrm{R}}^2 + \alpha_{\mathrm{I}}^2 - \beta_{\mathrm{I}}^2\right)(\theta T)^2,
\end{aligned} \tag{2.23}$$

which is equivalent to the shorter (2.19). □

If the characteristic equation (2.18) is derived by considering a periodic orbit, we know the uncontrolled system has a simple real eigenvalue and the eigenvalue $\tilde{\eta} = 0$ corresponding to the periodic orbit. This is *only possible* if $c = 0$ and thus the second eigenvalue is $\tilde{a} := a/(\theta T) = 2\,\mathrm{Re}\,\alpha$. If the periodic orbit is unstable the non-zero eigenvalue is positive. For a general equilibrium equation the restriction $c = 0$ does not necessarily apply. There might be two distinct real or a pair of complex conjugated eigenvalues.

2.4 Hopf normal form

If periodic orbits bifurcate from equilibria via a Hopf-bifurcation the Hopf normal-form is of particular interest. Although in this thesis we are not interested in the behavior of the system near the bifurcation point, the normal form is useful as a model system, as its periodic orbits are rotationally symmetric. They therefore fall within the scope of our Main Theorem, which means we are able to stabilize them (see chapter 5). Additionally, we will model the dynamics of a coupled oscillator system in chapter 6 by representing each oscillator as an equation in Hopf normal-form.

For the purpose of this thesis the following is just a definition of a special right hand side in an ordinary differential equation. But this *normal-form* is actually the consequence of a deeper theory, which is explained in [11] in an intuitively accessible manner.

Definition. A system in **Hopf normal form** fullfills the ordinary differential equation

$$\dot{z}(t) = F(z(t), \lambda) := \left(\lambda + \mathrm{i} + \gamma|z(t)|^2\right) z(t) \tag{2.24}$$

with $z(t) \in \mathbb{C}$, the bifurcation parameter $\lambda \in \mathbb{R}$ and the cubic term parameter $\gamma \in \mathbb{C} \setminus \{0\}$.

As we will see, the Hopf normal form undergoes a *Hopf bifurcation* at $\lambda = 0$. That means a periodic orbit appears (or disappears) near the equilibrium, when varying λ. This bifurcation behavior, was used to refute the odd-number limitation [7] and to prove the feasibility of equivariant Pyragas control [25]. While for an overview of bifurcation theory we refer to [10], please do note that the Hopf bifurcation will become relevant in section 4.2.

To see why λ is a bifurcation parameter and what role the γ plays, we will calculate the periodic solutions of this ordinary differential equation explicitly. At the beginning of chapter 5, where we stabilize a system in Hopf normal form, a *phase-portrait* with unstable periodic orbit is plotted (figure 5.1a on page 34).

The Cartesian coordinates $\operatorname{Re} z$ and $\operatorname{Im} z$ are not very well suited for this task, so we look at (2.24) in polar coordinates $z = r \exp \mathrm{i}\varphi$:

$$\dot{r} = \left(\lambda + r^2 \operatorname{Re}\gamma\right) r \tag{2.25}$$

$$\dot{\varphi} = 1 + r^2 \operatorname{Im}\gamma. \tag{2.26}$$

We see that the radial equation (2.25) is independent of the angle φ. There might be non-zero constant choices $r = r_*$ producing $\dot{r} = 0$. Those will generally lead to rotationally symmetric periodic orbits

$$z^*(t) = r_* \exp 2\pi \mathrm{i} t/T \tag{2.27}$$

as $\dot{\varphi}$ stays constant for all times. An exception is the degenerated case at $r_*^2 \operatorname{Im}\gamma = 1$, where a circle-line of equilibria at distance r_* from the origin is produced instead. The periodic orbits have the period

$$T = \frac{2\pi}{1 + r_*^2 \operatorname{Im}\gamma}. \tag{2.28}$$

Let us look for zeros in the radial component of the Hopf normal-form. Depending on $\operatorname{Re}\gamma$ we have three possible cases:

$\operatorname{Re}\gamma = 0$ is a *degenerated* case. We have $\dot{r} = 0$, if and only if $r = 0$. Solutions spiral inwards for $\lambda < 0$, outwards for $\lambda > 0$ and are periodic (with period $T = 2\pi$) for $\lambda = 0$. This case is not further considered as it does not produce a Hopf bifurcation

$\operatorname{Re}\gamma < 0$ is the *supercritical* case. For $\lambda > 0$ there exists an additional solution for $\dot{r} = 0$ with $r^2 = r_*^2 = -\lambda/\operatorname{Re}\gamma$. This corresponds to a periodic solution z^* of the full system. From the sign of $\dot{r}$ we also see the asymptotic behavior. For initial values with $r(0) \neq r_*$, solutions spiral away from periodic orbit $-$ for $r(0) < r_*$ this means towards the equilibrium $z = 0$.

$\operatorname{Re}\gamma > 0$ is the *subcritical* case. Here the solution $r^2 = r_*^2 = -\lambda/\operatorname{Re}\gamma$ exists for $\lambda < 0$. Again the asymptotic behavior is given by the sign of $\dot{r}$. Initial values with $r(0) \neq r_*$ spiral towards the periodic orbit, making it stable.

Remark. For $\operatorname{Im}\gamma = 0$ the angular equation (2.26) is $\dot{\varphi} = 1$. So all solutions rotate around the origin with the same speed. In particular all periodic orbits will have period $T = 2\pi$, which often produces problems when considering stabilization at the bifurcation point. The stabilization results for the Hopf normal form in this thesis (chapter 5) need no special treatment of this case.

To summarize: For λ, γ such that $\lambda \operatorname{Re}\gamma < 0$, there exists a rotationally symmetric periodic orbit of (2.24). It is unstable for $\operatorname{Re}\gamma > 0$ and stable for $\operatorname{Re}\gamma < 0$.

3 Main results

After introducing some basic concepts in the previous chapter, in this chapter we state theorems containing the main results developed in this master thesis. The proofs are contained in the following chapter 4.

The system considered in the theorems is a two-dimensional autonomous ordinary differential equation

$$\dot{x}(t) = f(x(t), g=0) \tag{3.1}$$

with $x(t) \in \mathbb{C} \cong \mathbb{R}^2$. The right hand side $f\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ has to be at least in C^1, to allow claims about the asymptotic behavior near periodic orbits. We also have to demand that f changes to the first order when applying a control, i.e.

$$\partial_g\big|_{g=0} f \neq 0. \tag{3.2}$$

But most importantly we want the system to be *rotationally symmetric*, i.e. f commutes with rotations around the origin. Such rotations $h \in SO(2, \mathbb{R}) \cong S^1$ have the form

$$h = h(\theta) := \exp\left(2\pi \mathrm{i}\, \theta T\right), \tag{3.3}$$

with $\theta \geq 0$ determining the angle of the rotation.

In order to have something to stabilize, we need the system to possess an unstable periodic orbit $x_*(\cdot)$ with known minimal period T. As our system is rotationally symmetric, so is the periodic orbit.

Assume a *control-scheme* $g\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is applied in such a way that (3.1) becomes

$$\dot{x}(t) = f\big(x(t),\, g[x(t), hx(t - \theta T)]\big). \tag{3.4}$$

We need the control to be non-invasive on the periodic orbit, so it does not get destroyed by it. For this we only want to use the *spatio-temporal symmetry*

$$x_*(t) = hx_*(t - \theta T) \tag{3.5 $\cong$ 1.3}$$

of the orbit, so we demand that g vanishes if the symmetry is satisfied:

$$g(x, x) = 0 \quad \text{for all} \quad x \in \mathbb{C} \tag{3.6}$$

Our first theorem – which is an extension of the result in [9] to equivariant control-schemes – applies to *arbitrary* such non-invasive control-schemes.

Main Theorem. *Consider a rotationally symmetric system of ordinary differential equations in two dimensions. Let there exist an unstable periodic orbit with minimal period T and Floquet exponent $\tilde{a}$ in the uncontrolled system. Fix a delay-time θT with $\theta > 0$ and consider time-delayed control-schemes being noninvasive on S^1-equivariant T-periodic orbits. Let $\partial_g|_{g=0} f \neq 0$ and also let the control-scheme preserve the rotational symmetry of the system – i.e. consider* (3.4)*.*

Then there exists a control-scheme stabilizing the periodic orbit, ***if and only if*** *the unstable Floquet exponent satisfies*

$$\tilde{a} < 9/(\theta T). \tag{3.7}$$

So, in contrast to Pyragas control without equivariance [9] – where stabilization was only possible for $\tilde{a} < 9/(1 \cdot T)$ or i.e $\mu < \exp(9/T)$ – stabilization by *equivariant* Pyragas control is possible *for all* periodic orbits. As long as the rotational symmetry can be used completely and the delay-time θT can be made small enough, we only need to choose

$$\theta < 9/(\tilde{a}T) \tag{3.8}$$

to assure the existence of a stabilizing scheme.

The method of proof is partially constructive, giving us a way to find stabilizing control schemes. These will, in some sense, be the most-stabilizing, a concept we formalize in the following definition:

Definition. Fix a delay time θT and consider control schemes g as given in (3.4). Let $\tilde{\eta}_{\max}(g)$ be the most unstable Floquet exponent for the controlled system – i.e. the one with largest real part.
We call a (not necessarily unique) $g^\star$ **most-stabilizing control-scheme** if for all other schemes g

$$\operatorname{Re}\tilde{\eta}_{\max}(g) \geq \operatorname{Re}\tilde{\eta}_{\max}(g^\star), \tag{3.9}$$

i.e. the system controlled with $g^\star$ produces unstable eigenvalues whose real parts are all lower than the maximum real part of all other control-schemes.

Note: A *most-stabilizing* control-scheme is **not** necessarily *stabilizing*. But, if it is not, the system cannot be stabilized, which is an essential point in proving the backward implication of the Main Theorem.

Remark. In physics the most unstable Floquet multiplier of the system is sometimes called *master stability function*. One might think of $g^\star$ as a minimizer for the master stability function.

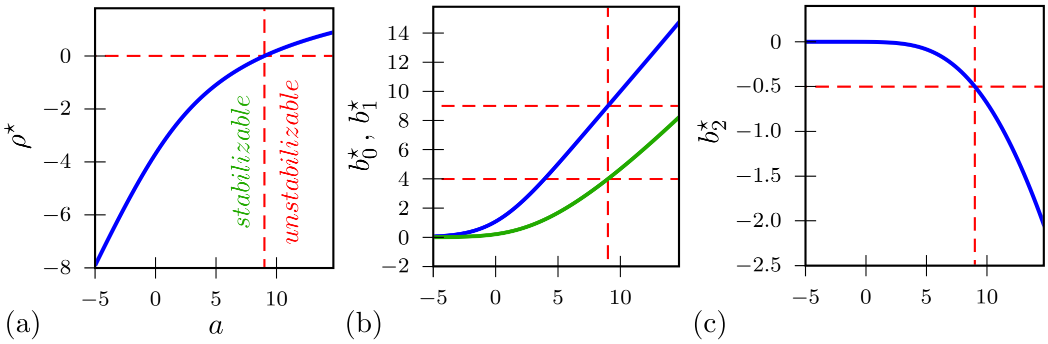


Figure 3.1: Most-stabilizing control-parameters as well as the resulting real most-unstable eigenvalue $\rho^\star$, plotted as a function of a. $\tilde{a} = a \cdot \theta T$ is the unstable Floquet exponent of the uncontrolled system. $\rho^\star$, $b_0^\star$ and $b_2^\star$ are plotted in blue, while $b_1^\star$ is green. The point $(a = b_0^\star = 9, b_1^\star = 4, b_2^\star = -\frac{1}{2})$, where stabilization becomes impossible is marked with dashed red lines. Conditions (C1), (C2) *and* (C3) *(on page 29) were used to obtain them numerically.*

The following theorem, which comes up as 'byproduct' in the proof of the main theorem, might be useful in implementing such control-schemes.

Theorem 2. *The stability properties resulting from a control scheme g as given in* (3.4) *are characterized by three real parameters b_0, b_1 and b_2. Assume non-degeneracy of the system, i.e. $\partial_g|_{g=0} f \neq 0$ and a hyperbolic periodic orbit. Then for every Floquet exponent $\tilde{a}$ in the uncontrolled system, there exists a unique choice of* most-stabilizing control-parameters *$b_0^\star$, $b_1^\star$ and $b_2^\star$ producing only eigenvalues with* $\operatorname{Re}\tilde{\eta} \leq \rho^\star \cdot \theta T \in \mathbb{R}$. *The most-stabilizing control-parameters are given implicitly by algebraic equations and characterize most-stabilizing control-schemes.*

An explanation of how the three parameters come into play follows immediately in the next chapter. The algebraic equations (C1), (C2) and (C3) are obtained in section 4.3 and can be found on page 29. Numerically, the implicit equations can easily be solved giving us suitable control-parameters for stabilization. See figure 3.1 for a plot of the most-stabilizing control-parameters.

4 Proofs of the main results

In this section the Main Theorem and Theorem 2 are proven. For this we will make use of the concepts introduced in chapter 2 and additionally use a Takens-Bogdanov bifurcation. The proof is structured into four parts:

In section 4.1 we obtain the characteristic equation for our model system by moving corotating coordinates and applying Lemma 1. In the further sections we use this equation to investigate, how the control can change the local stability of the periodic orbit.

In section 4.2 we introduce regions of parameters and calculate curves which can be used to obtain them. Properties of those curves, and thus the regions, are investigated as far as necessary for the proof.

In the third part in section 4.3 we find a fourth order Takens-Bogdanov bifurcation in our 'curve picture', giving us most-stabilizing control-parameters as defined in the previous chapter.

With those three parts we conclude the proof of Theorem 2 and our Main Theorem in section 4.4.

4.1 Calculating the characteristic equation

In this section, we will find a characteristic equation giving us information about the asymptotic behavior near a periodic orbit. The uncontrolled system we are looking at is rotationally symmetric – a symmetry which our time-delayed feedback control preserves. This allows us to transform the system to corotating coordinates. It also is non-invasive on the periodic orbit, so our control does not change the periodic orbit itself, so we can look at asymptotics near it even in the controlled case.

4.1.1 Properties of the model system

Our general model system, as introduced in the last chapter is given by

$$\dot{x} = f\big(x,\, g[x, hx(t - \theta T)]\big) \tag{3.4}$$

where x is a complex one-dimensional coordinate. The function $f(x, g)$ represents the dynamics of our system, where $f(x, 0)$ is the behavior without control. The function $g(x_1, x_2)$ is our control-scheme.

We assumed that, without control, the system has a rotationally symmetric periodic orbit with minimal period T and spatio-temporal symmetry

$$x_*(t) = hx_*(t - \theta T). \tag{3.5 $\cong$ 1.3}$$

The group action $h = h(\theta) = \exp(2\pi\mathrm{i}\theta)$ is simply a rotation. For the standard Pyragas control with integer θ – i.e. whenever the delay is a multiple of the period – this leads to $h(\theta) = \mathrm{Id}$. Then the special spatio-temporal symmetry is not used in the control-method, possibly allowing results to carry over to systems without such a symmetry.

In order to provide non-invasivity on periodic orbits with the spatio-temporal symmetry, we only consider control-terms with

$$g(x, x) = 0 \quad \text{for all} \quad x \in \mathbb{C}. \tag{3.6}$$

Let us make use of the S^1-symmetry. As f and g commute with rotations, i.e.

$$f(hx, hg) = hf(x, g) \quad \text{and} \tag{4.1}$$

$$g(hx_1, hx_2) = hg(x_1, x_2), \tag{4.2}$$

periodic orbits have the following form:

$$x_*(t) = h(t/T)\, x_*(0) = \exp(2\pi\mathrm{i}t/T)\, x_*(0) \tag{4.3}$$

Without loss of generality we can start the rotation on the real axis, i.e. $x_*(0) =: r_* \in \mathbb{R}_0^+$. For the angular velocity of the rotation we introduce the shorthand $\Omega := 2\pi/T$.

4.1.2 Transformation to corotating coordinates

The simple shape of the periodic orbit motivates a change to co-rotating coordinates

$$z(t) := h^{-1}(t/T)\, x(t) = h(-t/T)\, x(t). \tag{4.4}$$

The model system (3.4) then transforms as follows:

$$\begin{aligned}
\dot{z} &= \tfrac{d}{dt}(h(-t/T))x + h(-t/T)\dot{x} \\
&= -\mathrm{i}\Omega h(-t/T)x + h(-t/T)\, f\big(x, g[x, h(\theta)\, x(t - \theta T)]\big) \\
&= -\mathrm{i}\Omega z + f\big(z,\, g[z, h(-t/T)h(\theta)\, x(t - \theta T)]\big) \\
&= -\mathrm{i}\Omega z + f\big(z,\, g[z, h(-(t - \theta T)/T)\, x(t - \theta T)]\big) \\
&= -\mathrm{i}\Omega z + f\big(z,\, g[z, z(t - \theta T)]\big)
\end{aligned} \tag{4.5}$$

where we used the rotational symmetry of f and g to obtain the third equality. We see that the periodic orbit becomes a circle of equilibria $z_*(t) \equiv z_*(0)$ in those new coordinates. The problem of determining the stability of the periodic orbit has thus become a stability question for an equilibrium. The rotational symmetry of the system was essential in this step. Note that the group action h does not appear in the transformed equation anymore.

4.1.3 Linearization of the transformed system

In order to find the stability of the periodic orbit in the original system, we linearize at the equilibrium $z_*(0) = r_*$ in the transformed system. Inserting $z(t) = r_* + \delta z(t)$, where δz is a small change in z, leads to

$$\dot{z} = \dot{\delta z} = -\mathrm{i}\Omega(r_* + \delta z) + f\big(r_* + \delta z,\, g[r_* + \delta z, r_* + \delta z(t - \theta T)]\big). \tag{4.6}$$

To keep the convenient complex notation, we write the linearization of f as

$$f(r_* + \delta z, g) = f(r_*, g) + (\partial_z f)\,\delta z + (\partial_{\bar{z}} f)\,\delta\bar{z} + \mathcal{O}\left(|\delta z|^2\right) \tag{4.7}$$

where $\partial_z f$ and $\partial_{\bar{z}} f$ are complex-valued functions depending on g and r_*. They are not to be confused with complex derivatives, as f in general is not holomorphic.

Let us take a look at how the control influences the equation.
The non-invasivity condition $g(x, x) = 0$ will lead to the linearization

$$\begin{aligned} g(r_* + \delta z,\, r_* + \delta z(t - \theta T)) = {} & (\partial_{z_1} g)\,\delta z + (\partial_{\bar{z}_1} g)\,\delta\bar{z} \\ & + (\partial_{z_2} g)\,\delta z(t - \theta T) + (\partial_{\bar{z}_2} g)\,\delta\bar{z}(t - \theta T) \\ & + \mathcal{O}\left(|\delta z|^2\right) \end{aligned} \tag{4.8}$$

where again the $\partial_{z_1} g$, $\partial_{\bar{z}_1} g$, $\partial_{z_2} g$ and $\partial_{\bar{z}_2} g$ are complex-valued and depend on r_*. As for the case $\delta z(t) = \delta z(t - \theta T)$ the control needs to vanish, the number of terms in the first order approximation can be reduced by two via $\partial_{z_1} g = -\partial_{z_2} g$ and $\partial_{\bar{z}_1} g = -\partial_{\bar{z}_2} g$. This leaves two complex control-parameters to be chosen:

$$\begin{aligned} g(r_* + \delta z,\, r_* + \delta z(t - \theta p)) = {} & (\partial_{z_1} g)\,[\delta z - \delta z(t - \theta T)] \\ & + (\partial_{\bar{z}_1} g)\,[\delta\bar{z} - \delta\bar{z}(t - \theta T)] \;+\; \mathcal{O}\left(|\delta z|^2\right) \end{aligned} \tag{4.9}$$

For our model system we demanded that $\partial_g f \neq 0$ (at $g = 0$) since otherwise – if the system would not be changed by a control-term, or to the second order only – the linear stability could not change. The symmetry $hf(x, g) = f(x, hg)$ tells us that $\partial_g f$ can be written as complex number. To simplify the notation we, wlog, *consider* *only* $\partial_g f = 1$ – which is e.g. true if $f(x, g) = f(x, 0) + g$. Differing derivatives could be expressed by transforming g as

$$\tilde{g} = (\partial_g f)^{-1} g \tag{4.10}$$

and looking at the system for $\tilde{g}$ instead of g.

The complete linearization now reads:

$$\begin{aligned} \dot{\delta z} = &-\mathrm{i}\Omega \delta z + (\partial_z f)\, \delta z + (\partial_{\bar{z}} f)\, \delta \bar{z} \\ &+ (\partial_x g)\, [\delta z - \delta z(t - \theta p)] + (\partial_{\bar{x}} g)\, [\delta \bar{z} - \delta \bar{z}(t - \theta p)], \end{aligned} \tag{4.11}$$

4.1.4 Characteristic equation

The linearized system is autonomous (if we use an exponential ansatz for the delayed terms) and thus directly leads to a characteristic equation. Applying Lemma 1 we know the characteristic equation for such a system is

$$0 = \eta - a + b_0 u + b_1 u\eta + b_2 u^2 \eta \tag{C}$$

with the coefficients given by

$$\begin{aligned} a &= 2\,\mathrm{Re}(\partial_z f)\, \theta T \\ b_0 &= 2\,\mathrm{Re}\left((-\mathrm{i}\Omega + \partial_z f) \cdot \overline{\partial_{z_1} g} - \partial_{\bar{z}} f \cdot \overline{\partial_{\bar{z}_1} g}\right) (\theta T)^2 \\ b_1 &= -2\,\mathrm{Re}(\partial_{z_1} g)\, \theta T \\ b_2 &= \left(|\partial_{z_1} g|^2 - |\partial_{\bar{z}_1} g|^2\right) (\theta T)^2 \\ u\eta &= 1 - \exp(-\eta)\,. \end{aligned} \tag{4.12}$$

If we can choose g arbitrary, we thus can achieve every choice of control-parameters $(b_0, b_1, b_2) \in \mathbb{R}^3$.

4.2 Stabilization regions, Hopf- and saddle-curves

After having found the characteristic equation, we now need to investigate which choices of control-parameters lead to only stable eigenvalues and thus a stable controlled system. To do so, in this section we introduce regions of control-parameters which keep the real part of the eigenvalues below a certain threshold.

Definition. We call the set

$$\mathcal{B}(a,\rho) := \left\{ (b_0, b_1, b_2) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} \text{the characteristic equation only} \\ \text{admits solutions } \eta \text{ with } \operatorname{Re}\eta \le \rho \end{array} \right\} \tag{4.13}$$

a **ρ-region** of control parameters for $a, \rho \in \mathbb{R}$. Additionally, we call $\mathcal{B}(a,0)$ the **stabilizing region** of control parameters for a.

By definition ρ-regions for bigger ρ contain those for smaller ρ, i.e.

$$\rho_1 < \rho_2 \quad \text{implies} \quad \mathcal{B}(a,\rho_1) \subseteq \mathcal{B}(a,\rho_2) \tag{4.14}$$

With this definition the straightforward approach would be to try to determine the *stabilizing* region of parameters. Selecting (b_0, b_1, b_2) in the interior of $\mathcal{B}(a,0)$ will make the controlled system stable. For a numerical approach this is indeed useful as one obtains a description of the 0-regions which is easily plottable. Such a numerical investigation was performed in [9]. For a rigorous proof this might not be the best approach though, as many different cases arise – see figures 4.1 and 4.2. We therefore take a more indirect path:

We determine bifurcation curves which will form the boundary of the regions $\mathcal{B}(a,\rho)$ for arbitrary a and ρ. For very negative ρ the ρ-regions must be empty – otherwise there would need to exist control-parameters for which the characteristic equation only has solutions η with $\operatorname{Re}\eta \to -\infty$. Therefore the inclusion relation (4.14) tells us that there exists a $\rho^\star$ such that the set $\mathcal{B}(a,\rho)$ is non-empty for $\rho > \rho^\star$ but empty for $\rho < \rho^\star$.

In the next section we will then prove that the region $\mathcal{B}(a,\rho^\star) = \{(b_0^\star, b_1^\star, b_2^\star)\}$, i.e. it only contains the most-stabilizing control parameter for a (see Theorem 2).

4.2.1 Solving the characteristic equation for control-parameters

Solutions η for fixed parameters a, b_0, b_1 and b_2 are generally not algebraically accessible. So, instead of searching for solutions η, we *fix* $\operatorname{Re}\eta$ (as well as a and b_2) and solve the characteristic equation (C) for (b_0, b_1). This way we obtain two curves, one for real and one for complex eigenvalues, which together form the boundary of our the region $\mathcal{B}(a, \operatorname{Re}\eta)$. A similar method has successfully been used for characteristic equations with exponential terms – e.g. in [7] – but only purely imaginary η were considered. We will investigate this (simpler) case in our proof but will not restrict ourselves to it.

As the characteristic equation is linear in b_0 and b_1 (and now η has become just a parameter) we can easily rearrange (C) giving us (b_0, b_1):

$$b_0 + \eta b_1 = \frac{a-\eta}{u} - b_2 \cdot u\eta \tag{RC}$$

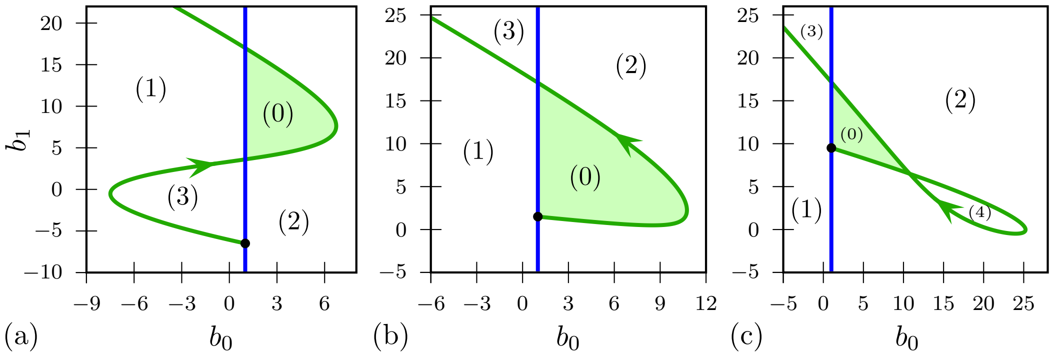


Figure 4.1: Hopf curves (green) and saddle-curves (blue) for $\rho = 0$ and different choices of b_2. The Takens-Bogdanov point TB(0) *is marked in black and the direction of rising ω is indicated by arrows. The unstable dimensions are annotated in brackets and the resulting stabilization region is shaded green. As parameters $\rho = 0$, $a = 1$ and $b_2 = 6, -2, -10$ were chosen for* (a), (b) *and* (c) *respectively.*

4.2.2 Saddle-curves

For convenience of notation let us define

$$\eta =: \rho + \mathrm{i}\omega \tag{4.15}$$

with ρ and ω real. Let us calculate the curve for real eigenvalues $\eta = \rho$, first. For those $u = (1 - \exp(-\rho))/\rho$ and the equation

$$0 = \rho - a + b_0 u + b_1 u\rho + b_2 u^2 \rho \tag{4.16}$$

are both also in the real. Thus (RC) tells us that for fixed ρ, solution parameters (b_0, b_1) lie on a straight line in $\mathbb{R}^2$:

$$b_0 + \rho b_1 = \frac{a\rho - \rho^2}{1 - \exp(-\rho)} - b_2 \cdot (1 - \exp(-\rho)) \tag{4.17}$$

The line has slope $-1/\rho$ for $\rho \neq 0$. In the special case $\rho = 0$ it is vertical and goes through $b_0 = a$. We call this straight line the *saddle-curve* of eigenvalues with real part ρ.

4.2.3 Hopf-curves

In the next step we consider eigenvalues η with imaginary part $\omega \in \mathbb{R}$ different from zero. Fortunately we can restrict our considerations to $\omega \geq 0$. To justify this, consider the complex conjugated of (C):

$$0 = \bar{\eta} - a + b_0\bar{u} + b_1\bar{u}\bar{\eta} + b_2\bar{u}^2\bar{\eta} \tag{4.18}$$

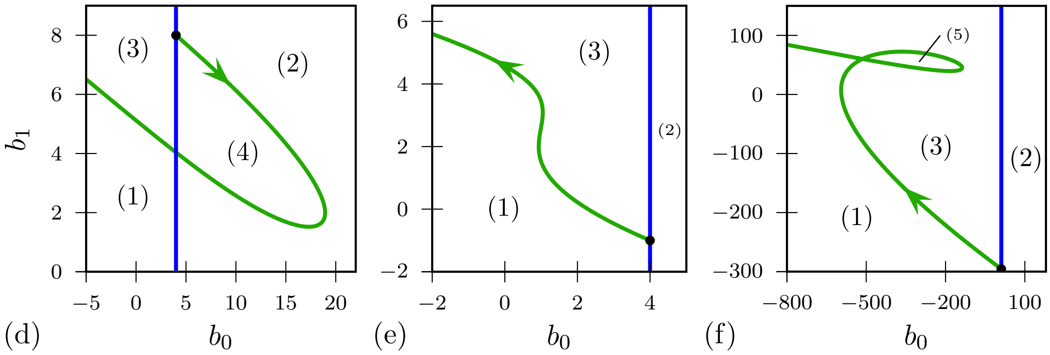


Figure 4.2: Hopf-curves (green) and saddle-curves (blue) for $\rho = 0$. The parameters (a, b_2) were chosen such that no stabilization regions exist. See also figure 4.1 for three cases where such regions exist.
The following parameters were chosen:
(d) $a = 4,\ b_2 = -7$ / (e) $a = 4,\ b_2 = 2$ / (f) $a = 11,\ b_2 = 300$

As u satisfies

$$\overline{u(\eta)} = \overline{[1 - \exp(-\eta)]/\eta} = [1 - \exp(\bar{\eta})]/\bar{\eta} = u(\bar{\eta}) \tag{4.19}$$

$\bar{\eta}$ is a solution of (C) if and only if η is also a solution. Thus we need only consider $\omega \geq 0$.

Analogously to the real eigenvalue case, we again fix a, b_2 and ρ. But this time we also consider $\omega > 0$ to be a parameter of the system. Varying ω gives us a curve of parameter values $\mathbf{b}_{\mathrm{H}}(\omega, \rho) = (b_0, b_1)$ corresponding to complex eigenvalues, which we call *Hopf-curve*:

$$b_0 + \rho b_1 + \mathrm{i}\omega b_1 = \frac{a\rho - \rho^2 - \mathrm{i}\omega\rho}{u\eta} - b_2 \cdot u\eta \tag{H}$$

where $u\eta = (1 - \mathrm{e}^{-\rho}\cos\omega + \mathrm{i}\mathrm{e}^{-\rho}\sin\omega)$.

4.2.4 Takens-Bogdanov points

At the starting point of the Hopf-curve

$$\mathrm{TB}(\rho) := \mathbf{b}_{\mathrm{H}}(\omega = 0, \rho) \tag{4.20}$$

the eigenvalue η is real and thus also lies on the saddle-curve defined by (4.17). Thus we can also parametrize it by b_1:

$$\mathbf{b}_{\mathrm{S}}(b_1, \rho) \;=\; \mathrm{TB}(\rho) + \big(\rho \cdot (b_1 - \mathrm{TB}_2(\rho))\,, b_1\big) \tag{S}$$

Remark. The point TB can be considered a Takens-Bogdanov bifurcation point for $\rho = 0$. At this point a branch of saddle-node bifurcations (the saddle-curve $\mathbf{b}_\mathrm{S}$) and a branch of of Hopf bifurcations (the Hopf-curve $\mathbf{b}_\mathrm{H}$) intersect and therefore $\eta = \rho$ is a double eigenvalue. For this thesis we will extend this nomenclature to $\rho \neq 0$ as well.

4.2.5 Plotting regions

With the expressions (H) and (S) we are able to plot both the saddle- and the Hopf-curve for any ρ and b_2 fixed. Indeed we have classified all solutions η of the characteristic equation (C), as they must lie on one of those curves for some $\rho \in \mathbb{R}$. By continuity of of the rearranged characteristic equation (RC) we know that only by crossing a curve in the (b_0, b_1)-plane, the number of eigenvalues η with $\operatorname{Re}\eta > \rho$ can change. Therefore the boundary of the ρ-region $\mathcal{B}(a, \rho)$ for each fixed b_2 is formed by those curves.

So how does one 'read the pictures' with Hopf- and saddle-curves? The parametrization via ω provides a natural orientation of the Hopf-curve. When crossing it from left to right a Hopf bifurcation increases the number of eigenvalues with $\operatorname{Re}\eta > \rho$ by two. Another change in this number happens at the saddle-curve. As its slope decreases (and $\mathrm{TB}(\rho)$ changes continuously) with ρ, saddle-curves for slightly bigger ρ lie to the left for b_1 below the Takens-Bogdanov-Point and to the right for those above. So this results in the following rule:
For $b_1 < \mathrm{TB}_2$: To the *right* of $\mathbf{b}_\mathrm{S}(\cdot, \rho)$, there exists one eigenvalue η with $\operatorname{Re}\eta > \rho$ *more* than to the left.
For $b_1 > \mathrm{TB}_2$: To the *right* of $\mathbf{b}_\mathrm{S}(\cdot, \rho)$, there exists one such eigenvalue *less* than to the left.

4.2.6 Curves for stabilizing regions

Our aim is to find the *most-stabilizing control parameters* $b_0^\star, b_1^\star, b_2^\star$ for fixed a. But, if one is only interested in a 'curve-description' of the stabilizing region $\mathcal{B}(a, 0)$, it is enough to consider only purely imaginary η. For $\rho = 0$ the $\mathbf{b}_\mathrm{H}(\omega, 0)$ on the Hopf curve are given by real and imaginary part of the right hand side of (H) in a simple explicit form:

$$b_0 = \frac{a\omega \sin\omega + \omega^2(1 - \cos\omega)}{2 - 2\cos\omega} - b_2 \cdot (1 - \cos\omega) \tag{H0}$$

$$b_1 = \frac{a(1 - \cos\omega) - \omega \sin\omega}{2 - 2\cos\omega} - (b_2/\omega)\sin\omega \tag{H1}$$

Also the Takens-Bogdanov point becomes

$$\mathrm{TB}(0) = \left(a,\ \tfrac{1}{2}a - 1 - b_2\right) \tag{4.21}$$

which one can find by taking the limit $\omega \to 0$ of (H0) and (H1).

For $\rho \neq 0$, unfortunately the expression is not that easy.

4.3 Fourth order Takens-Bogdanov bifurcation points

In this section we will prove that unique most-stabilizing control-parameters exist. Assume there exists such parameters and look at the characteristic equation for fixed $b_0 = b_0^\star$, $b_1 = b_1^\star$ and $b_2 = b_2^\star$. Then there exists (at least) one eigenvalue η with maximal real part which we call $\rho^\star$. We will to show that $\mathcal{B}(a, \rho^\star)$ consists of $(b_0^\star, b_1^\star, b_2^\star)$ only and characterize such a choice of control-parameters. Existence and uniqueness of such a point will follow from that characterization.

4.3.1 Region for the maximal real part

Let us look for a case where the ρ-region for all $\rho_1 < \rho^\star$ contains control-parameters but is empty for all $\rho_2 > \rho^\star$. In order for this to happen, the curves must admit a region for ρ_1 (and some b_2) that somehow vanishes when increasing ρ from ρ_1 through $\rho^\star$ to ρ_2.

The saddle-curve, which is a straight line, can not self-intersect. But by continuity in ρ some intersection of the two curves, or of the Hopf-curve with itself, must appear when increasing ρ above $\rho^\star$. Therefore our intersection must involve the Hopf-curve, i.e. $(b_0^\star, b_1^\star) = \mathbf{b}_{\mathrm{H}}(\omega, \rho^\star)$ for some ω and $b_2^\star$. This point must admit a region with non-empty interior for all $\rho > \rho^\star$ but changing b_2 to any value does not produce a region.

So at $\rho^\star$ we must be at a special bifurcation point, which is characterized by three conditions we will now determine.

4.3.2 Condition 1

First, let us look at the change of the Hopf-curve with changing b_2. We need to find a parameter $b_2^\star$ where an intersection of curves is insensitive to changes in b_2. Consider $b_2 = b_2 - \Delta b_2$. As the equation for the Hopf-curve (H) is linear in b_2, it becomes

$$\Delta b_0 + \rho \Delta b_1 = \Delta b_2 \cdot (1 - \exp(-\rho)\cos\omega) \tag{4.22}$$

$$\Delta b_1 = \Delta b_2 \cdot (\exp(-\rho)\sin\omega)/\omega \tag{4.23}$$

where Δb_0 and Δb_1 are the changes in the value of $\mathbf{b}_{\mathrm{H}}(\omega, \rho^\star)$. The slope of the saddle-curve does not change, but its foot-point TB does by

$$\tilde{\Delta} b_0 + \rho \tilde{\Delta} b_1 = \Delta b_2 \cdot (1 - \exp(-\rho)) \tag{4.24}$$

$$\tilde{\Delta} b_1 = \Delta b_2 \cdot \exp(-\rho). \tag{4.25}$$

If those two changes are not the same, varying b_2 also unfolds the same bifurcation as a change in ρ. So it could either produce a self-loop of $\mathbf{b}_{\mathrm{H}}$ (compare figures 4.1c and 4.2f where the loop are not stable) or an intersection of $\mathbf{b}_{\mathrm{H}}$ and $\mathbf{b}_{\mathrm{S}}$ – which both would violate our condition of being most-stabilizing. $\Delta b_1 = \tilde{\Delta} b_1$ is only satisfied at the foot-point TB, i.e for $\omega = 0$.

We thus found that the point $(b_0^\star, b_1^\star)$ must be the Takens-Bogdanov point TB$(\rho^\star)$. The region is formed by saddle- and Hopf-curve together.

4.3.3 Condition 2

Secondly *the Hopf-curve must point in the same direction as the saddle-curve.*
Assume this was not the case and the slope of $\mathbf{b}_{\mathrm{H}}(\cdot, \rho^\star)$ at zero forms an angle with $\mathbf{b}_{\mathrm{S}}$. Then slightly varying ρ from $\rho = \rho^\star$ will, by continuity in ρ, only slightly change this angle. But this means regions for $\rho < \rho^\star$ have to persist for $\rho > \rho^\star$ which is a contradiction. So this condition is necessary.

4.3.4 Condition 3

Finally *the second component of* $\mathbf{b}_H(0, \rho^\star)''$ *must also be zero.*
Assume again the contrary. Then an increase in ρ will decrease the saddle-curve's slope. But the slope of the Hopf-curve pointed in the same direction as the saddle-curve by condition 2. This would mean, there could still be no stabilization region at this point for $\rho > \rho^\star$. This is again a contradiction so this condition is necessary, too.

4.3.5 Taylor expansion of the Hopf-curve

The conditions we found essentially result in determining a fourth order Takens-Bogdanov bifurcation point. So we are interested in the behavior of the Hopf-curve near TB, where it intersects the saddle-curve. Both curves are infinitely often differentiable *(in all of the parameters)* except at poles. Looking at (H) for $\rho \neq 0$ and (H0,H1), we see that poles *only* appear for $\rho = 0$ where the divisor $(2 - 2\cos\omega)$ will become zero for $\omega = 2\pi\mathbb{Z}$. Thus we calculate the Taylor series of $\mathbf{b}_{\mathrm{H}}(\omega, \rho)$ at

$\omega = 0$. As we will see, we need it up to the third order. The left hand side of (H) has the form

$$\mathrm{lhs(H)} \;=\; b_0 + \rho\, b_1 \;+\; \mathrm{i}\omega\, b_1 \;+\; \tfrac{1}{2}\omega^2\, b_0'' + \tfrac{1}{2}\rho\omega^2\, b_1'' \;+\; \tfrac{1}{2}\mathrm{i}\omega^3\, b_1'' \;+\; \mathcal{O}\left(\omega^4\right) \tag{L}$$

where (b_0'', b_1'') is the second derivative of $\mathbf{b}_{\mathrm{H}}(\omega, \rho)$ with respect to ω at $\omega = 0$. The first derivative is zero as we will see from the Taylor expansion of rhs(H). To calculate the right hand side one might use a computer algebra system. We introduce two abbreviations

$$R := \exp(-\rho) \quad \text{and} \quad E := 1/(1 - \exp(-\rho)) \tag{4.26}$$

and obtain

$$\begin{aligned}
\mathrm{rhs(H)} = {} & (a-\rho)\rho E + b_2(R-1) \\
& + \mathrm{i}\omega \left[(\rho - a)\rho E^2 R + (a - 2\rho)E - b_2 R\right] \\
& + \omega^2 \left[(\rho - a)\rho \left(E^3 R^2 + \tfrac{1}{2}E^2 R\right) + (a - 2\rho)E^2 R + E - \tfrac{1}{2}b_2 R\right] \\
& + \mathrm{i}\omega^3 \Big[(a-\rho)\rho \left(E^4 R^3 + E^3 R^2 + \tfrac{1}{6}E^2 R\right) \\
& \qquad + (2\rho - a)\left(E^3 R^2 + \tfrac{1}{2}E^2 R\right) - E^2 R + \tfrac{1}{6} b_2 R\Big] \\
& + \mathcal{O}\left(\omega^4\right).
\end{aligned} \tag{R}$$

So, matching the coefficients on both sides of the equality we have expressions for $\mathbf{b}_{\mathrm{H}}(0, \rho) = \mathrm{TB}(\rho)$ and $\mathbf{b}_{\mathrm{H}}''(0, \rho)$.

4.3.6 Formalization of the three conditions

Using (R) and (L) we can formalize the three conditions. The constant and first order term give us $b_0^\star$ and $b_1^\star$:

$$\begin{aligned}
b_1^\star &= (\rho^\star - a)\rho^\star E^2 R + (a - 2\rho)E - b_2^\star R \\
b_0^\star &= (a - \rho^\star)\rho^\star E + b_2^\star (R-1) - \rho^\star b_1^\star
\end{aligned} \tag{C1}$$

$\mathbf{b}_{\mathrm{S}}$ is a straight line with slope $-1/\rho^\star$, so for the second condition we have to put $\mathbf{b}_{\mathrm{H}}''(0, \rho^\star) = (b_0 + \rho^\star b_1, b_1)$ – which is equivalent to the second order term of (R) being zero:

$$0 = (\rho^\star - a)\rho^\star \left(E^3 R^2 + \tfrac{1}{2}E^2 R\right) + (a - 2\rho^\star)E^2 R + E - \tfrac{1}{2}b_2^\star R. \tag{C2}$$

It only remains the condition that $\mathbf{b}_{\mathrm{H}}''(0, \rho^\star)$ is also zero:

$$\begin{aligned}
0 = {} & (a - \rho^\star)\rho^\star \left(E^4 R^3 + E^3 R^2 + \tfrac{1}{6}E^2 R\right) \\
& + (2\rho^\star - a)\left(E^3 R^2 + \tfrac{1}{2}E^2 R\right) - E^2 R + \tfrac{1}{6} b_2^\star R.
\end{aligned} \tag{C3}$$

In all three equations the stars for R and E were omitted to increase readability.

As the equations is linear in a and $b_2^\star$ the choice of $(a, b_2^\star) \in \mathbb{R}^2$ will be unique for fixed $\rho^\star$.

4.4 Conclusion of the proofs

At this point let us collect what we have proven so far:

In the first part (section 4.1) we showed that the characteristic equation for our model system (3.4) is

$$0 = \eta - a + b_0 u + b_1 u\eta + b_2 u^2 \eta \tag{C}$$

To do so, we transformed it to corotating coordinates and calculated the determinant for the linearized system using an exponential ansatz for the delayed terms.

In the second part (section 4.2) we defined ρ-regions and calculated the Hopf- and saddle-curves which form their boundaries. We also found that they intersect at the starting point $\mathbf{b}_{\mathrm{H}}(0, \rho)$ of the Hopf-curve.

In section 4.3, the third part, we proved that most-stabilizing control-parameters exist and must satisfy three conditions (C1), (C2) and (C3).

Having found those three conditions, Theorem 2 is now proved.

Numerically, it is not hard to find $\rho^\star$ and $b_2^\star$ for fixed values of $a^\star$. The curves in figure 3.1 were obtained in this way.

But as the conditions (C2) and (C3) are (affine) linear in $a^\star$ and $b_2^\star$, we can solve them for $a^\star$ and $b_2^\star$. We can easily eliminate b_2:

$$\begin{aligned} 0 = 2 \cdot (\mathrm{C2}) + 6 \cdot (\mathrm{C3}) = {} & (a - \rho^\star)\rho^\star \left(6E^4R^3 + 4E^3R^2\right) \\ & + (2\rho^\star - a)\left(6E^3R^2 + E^2R\right) - 6E^2R + 2E \end{aligned} \tag{4.27}$$

From this equation – using the monotonicity of $R = \exp(-\rho^\star)$ and $E = 1/(1 - \exp(-\rho^\star))$ in $\rho^\star$ we see that $a^\star$ *monotonically decreases with* $\rho^\star$ and we can switch between those parameters as we like (implicit function theorem).

4.4.1 Turning point of the stability

Note that the terms $E^{m+1}R^m \approx 1/\rho^m$ for small ρ and thus the conditions are not defined for the case $\rho^\star = 0$. Fortunately we can easily find the Taylor expansion for this case directly from (H0) and (H1):

$$\begin{aligned} b_0^\star + \mathrm{i}\omega b_1^\star + \tfrac{1}{2}\omega^2 b_0'' + \tfrac{1}{2}\mathrm{i}\omega^3 b_1'' = a &+ \mathrm{i}\omega\left(\tfrac{1}{2}a - 1 - b_2^\star\right) \\ &+ \omega^2\left(\tfrac{1}{12}a - \tfrac{1}{2}b_2^\star + \tfrac{1}{2}\right) \\ &+ \mathrm{i}\omega^3\left(\tfrac{1}{6}b_2^\star + \tfrac{1}{12}\right) + \mathcal{O}\left(\omega^4\right) \end{aligned} \tag{4.28}$$

Setting the second and third order term to zero, for we get $a^\star = 9$ and the most-stabilizing control parameter is

$$\left(b_0^\star, b_1^\star, b_2^\star\right) = \left(9, 4, -\tfrac{1}{2}\right). \tag{4.29}$$

This fourth order Takens-Bogdanov point was already found in [9]. It produces $\rho^\star = 0$. Therefore it marks the barrier between stabilizable and non-stabilizable systems. Only for $a < 9$ stabilization is possible – just apply a most-stabilizing control-scheme.

This proves our Main Theorem.

5 Stabilizing Hopf normal form

Having proven the Main Theorem in the last chapter, let us look a standard model system for rotationally symmetric orbits: Hopf normal form. We investigate how controls for such ordinary differential equations have to be constructed to give us free choice of the parameters b_0, b_1 and b_2 and thus allow stabilization as proven in this thesis.

Consider a complex-valued system with control of the following form

$$\dot{z} = (\lambda + \mathrm{i} + \gamma|z|^2)z \;+\; k[z - hz(t-\theta p)] + \ell z^2[\bar{z} - \bar{h}\bar{z}(t-\theta p)], \tag{CH}$$

where k and ℓ are complex parameters and $h = \exp(2\pi\mathrm{i}\theta)$. As we will show, such a control-scheme achieves stabilization whenever deemed possible by the Main Theorem.

First let us check whether the conditions of the theorem are fulfilled. The Hopf normal form has a rotational symmetry. Also the terms in k and l both commute with rotations $\exp(\mathrm{i}\varphi)$:

$$\ell(\mathrm{e}^{\mathrm{i}\varphi}z)^2 \cdot \mathrm{e}^{-\mathrm{i}\varphi}[\bar{z} - \bar{h}\bar{z}(t-\theta p)] = \mathrm{e}^{\mathrm{i}\varphi} \cdot \ell z^2[\bar{z} - \bar{h}\bar{z}(t-\theta p)] \tag{5.1}$$

Moving to co-rotating coordinates and linearizing the equation at $z = r_*$ (compare section 4.1) leads to

$$\dot{\delta z} = \gamma r_*^2(\delta z + \delta\bar{z}) + k[\delta z - \delta z(t-\theta p)] + \ell r_*^2[\delta\bar{z} - \delta\bar{z}(t-\theta p)] \tag{5.2}$$

for which Lemma 1 gives us the characteristic equation

$$\begin{aligned} 0 = \eta \;-\; 2r_*^2 \operatorname{Re}\gamma\,\theta T \;&+\; 2r_*^2 \operatorname{Re}\left(\gamma\left[\bar{k} - r_*^2\bar{\ell}\right]\right)(\theta T)^2\, u \\ &-\; 2\operatorname{Re} k\,\theta T\; u\eta \\ &+\; \left(|k|^2 - r_*^4|\ell|^2\right)(\theta T)^2\, u^2\eta \end{aligned} \tag{5.3}$$

with $u = (1 - \exp(-\eta))/\eta$.

In the characteristic equation (5.3) we can achieve all choices of control-parameters b_0, b_1 and b_2 by varying k and ℓ. As we have a method to find most-stabilizing

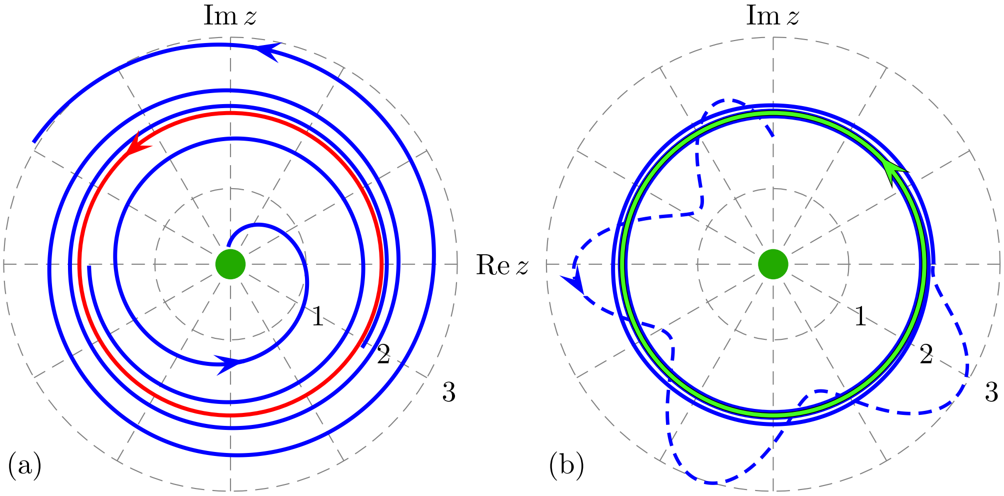


Figure 5.1: Phase portraits for the subcritical Hopf normal-form, with (a) *and without* (b) *a stabilizing control applied. Note the bi-stability of the equilibrium and the periodic orbit in* (b). *The parameters* $\lambda = -4$, $\gamma = 1{+}10\mathrm{i}$, $\theta = \frac{3}{4}$, $k = -1.5{+}1.66\mathrm{i}$ *and* $\ell = -0.57{+}0.057\mathrm{i}$ *were chosen, leading to an unstable Floquet multiplier* $\tilde{a} \approx 0.106$ *(*$a \approx 0.92$*). This choice of* k *and* ℓ *leads approximately to the most-stabilizing* $(b_0^\star, b_1^\star, b_2^\star)$*. The stable objects are colored green, unstable ones are indicated in red. Intermediate orbits are blue, with the prehistory for* $t \in [-\theta T, 0]$ *indicated by a dashed line.*

control-parameters let us see which parameters $(k, l) \in \mathbb{C}^2$ corresponds to a choice of $(b_0, b_1, b_2) \in \mathbb{R}^3$:

$$\operatorname{Re} k \;:=\; -(2\theta T)^{-1} b_1 \tag{5.4}$$

$$\operatorname{Im} k \;:=\; \big(-\operatorname{Re}\gamma \operatorname{Re} k - r_*^2 \operatorname{Re}\left(\gamma\bar{\ell}\right) + \tfrac{1}{2}(r_*\theta T)^{-2} b_0\big) / \operatorname{Im}\gamma \tag{5.5}$$

$$|\ell| \;:=\; r_*^{-2}\sqrt{\big||k|^2 - (\theta T)^{-2} b_2\big|} \tag{5.6}$$

Note that, by choosing appropriate $\arg(\ell)$, the scalar product

$$\operatorname{Re}\left(\gamma\bar{\ell}\right) = |\ell|\,\big(\cos\big(\arg(\ell)\big)\operatorname{Re}\gamma + \sin\big(\arg(\ell)\big)\operatorname{Im}\gamma\big) \tag{5.7}$$

can be chosen freely between $-|\gamma||\ell|$ and $+|\gamma||\ell|$. If we choose $\operatorname{Re}(\gamma\bar{\ell}) = 0$, then k is defined by b_0 and b_1, while b_2 defines $|\ell|$. Of course, we can also choose any other $\operatorname{Re}(\gamma\bar{\ell})$ and use the resulting control-parameters as defined in (5.3).

The Main Theorem now tells us that stabilization is possible for θ chosen small enough. Additionally Theorem 2 gives us a suggestion of what control parameters to choose. We can achieve any b_0, b_1 and b_2 by choosing appropriate k and ℓ, so especially the most-stabilizing control parameters can be chosen. In figure 5.1 one can see a numerical simulation of a successful stabilization with a most-stabilizing control-scheme of form (CH).

Remark. If we restrict ourselves to controls with $\ell = 0$, we find that $b_2 = |k\theta T|^2 > 0$ is prescribed. We are then not able to choose b_2 freely anymore, which hinders our ability to control the system. Numerical observations suggest that, for $\tilde{\eta} > 6/\theta$, there does not exist any stabilization region with $b_2 \geq 0$. This is in accordance with [9].

6 Application to a coupled system

As an example for the scope of our theorems, we will apply the Main Theorem to a system of coupled oscillators in Hopf normal-form. In [23], Isabelle Schneider and I have investigated in detail a ring of n oscillators with bi-directional nearest neighbor coupling, which is a specific type of network. Now, we consider arbitrary networks and extend the results to periodic solutions which are further away from the bifurcation point. The form of the periodic orbits – rotating wave solutions – and thus the form of the control carries over from the example with only minor modifications. The resulting periodic orbits will (after a coordinate transformation) fall under the scope of the Main Theorem and thus can be considered regardless of their distance from bifurcation point.

6.0.2 Model system

Consider the system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}, \lambda) + A\mathbf{z}, \tag{OS}$$

with $\mathbf{z} \in \mathbb{C}^n$ representing the single oscillator. Here $\mathbf{F}$ is a vector-version of the Hopf normal-form, with each component satisfying

$$F_j(\mathbf{z}, \lambda) = F(z_j, \lambda) = (\lambda + \mathrm{i} + \gamma |z_j|^2) z_j. \tag{6.1}$$

The coupling matrix $A \in \mathbb{C}^{n \times n}$ represents the links between the oscillators.

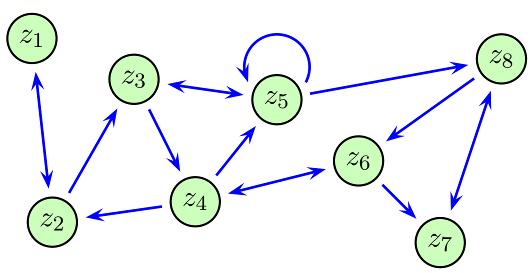
Remark. In order to represent oscillators with different bifurcation point and frequency, i.e.

$$F_j(\mathbf{z}, \lambda) = (\lambda - \Delta\lambda_j + \mathrm{i}\omega_j + \gamma |z_j|^2) z_j \tag{6.2}$$

one can add self-coupling terms $a_{jj} = -\Delta\lambda_j + \mathrm{i}(\omega_j - 1)$ to the matrix. Thus our system can describe a large group of oscillator networks.

We also require our coupling matrix to be diagonizable, i.e. we need to have a linear transformation $S \in \mathbb{C}^{n \times n}$ such that $SAS^{-1} = \mathrm{diag}(-\lambda_1, -\lambda_2, \ldots, -\lambda_n)$.

The transformation matrix S plays an important role in finding the periodic solutions and designing the control. Its columns are the eigenvectors $\mathbf{e}_j$ to the eigenvalues λ_j of A. To simplify the notation we will rescale the vectors, such that the shape of periodic solutions is simplified.



$$A = \begin{pmatrix} \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 5+\mathrm{i} & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & 6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 7 & \cdot & \cdot & 3 & \cdot & \cdot \\ \cdot & \cdot & 8 & -4 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2\mathrm{i} & \cdot & \cdot & \cdot & 5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 4 & \cdot & 9\mathrm{i} & \cdot \end{pmatrix}$$

Figure 6.1: Example for a network of 8 coupled oscillators. On the right a corresponding coupling matrix A is depicted. The dots denote zero entries. The eigenvalues can be approximated as $\lambda_1, \ldots, \lambda_n \approx -6.123 + 0.294\mathrm{i}, -4.077 + 2.477\mathrm{i}, -3.358 - 0.962\mathrm{i}, -0.670 - 4.137\mathrm{i}, 6.995 + 0.108\mathrm{i}, 4.669 + 3.099\mathrm{i}, 1.567 + 0.488\mathrm{i}, 2.996 - 1.367\mathrm{i}$. *In particular the matrix is diagonizable.*

Definition. We say the transformation matrix S is **normalized** if for each of its column-vectors $\mathbf{e}_j$, the j-th component is either 1 or 0. If the entry is zero we call the column j of S **degenerate**.

6.0.3 Periodic orbits

Let us have a look at periodic orbits in this system. The following proposition states which special form they have.

Proposition 1. *There exist up to n periodic orbits of* (OS) *given by*

$$\mathbf{z}^{(s)}(t) := S\mathbf{x}^{(s)}(t) \quad \textit{for } s = 1, \ldots, n. \tag{6.3}$$

The orbit with index s exists if the column s of S is not degenerate and

$$\lambda \lessgtr \operatorname{Re}\lambda_j \quad \textit{for} \quad \operatorname{Re}\gamma \lessgtr 0 \quad \textit{respectively.} \tag{6.4}$$

The components of $\mathbf{x}^{(s)}$ are defined by

$$x_j^{(s)}(t) := \begin{cases} r_j \exp\left(\Omega_j t\right) & \textit{for } j = s \\ 0 & \textit{for } j \neq s \end{cases} \tag{6.5}$$

with the radius and the angular velocity given by

$$\begin{aligned} r_j &:= \sqrt{\frac{\operatorname{Re}\lambda_j - \lambda}{\operatorname{Re}\gamma}} \quad \textit{and} \\ \Omega_j &:= 1 - \operatorname{Im}\lambda_j + r_j^2 \operatorname{Im}\gamma. \end{aligned} \tag{6.6}$$

To prove this, we will look at the system in transformed coordinates

$$\mathbf{x} := S^{-1}\mathbf{z} \tag{6.7}$$

such that the coupling becomes diagonal:

$$\dot{\mathbf{x}} = S^{-1}\dot{\mathbf{z}} = S^{-1}\mathbf{f}(S\mathbf{x},\lambda) + \operatorname{diag}(-\lambda_1, -\lambda_2, \dots, -\lambda_n)\,\mathbf{x} \tag{6.8}$$

with all $\lambda_j \in \mathbb{C}$ and $\operatorname{Re}\lambda_1 \leq \operatorname{Re}\lambda_2 \leq \ldots \leq \operatorname{Re}\lambda_n$. Thus the equation for the j-th component becomes

$$\dot{x}_j = \left(\lambda + \mathrm{i} + \gamma|(S\mathbf{x})_j|^2 - \lambda_j\right) x_j \tag{6.9}$$

as the linear terms in the vectorized Hopf normal form commute with S.

We see that the term $S^{-1}\mathbf{f}(S\mathbf{x},\lambda)$ is no longer diagonal, because the nonlinearity $\gamma|z_j|^2 z_j$ does not commute with S and S^{-1}. To obtain our periodic solutions, we thus look for solutions in the subspaces

$$X_s = \{\mathbf{x}(t) \mid x_j(t) \equiv 0 \text{ for } j \neq s\}, \quad s = 1, \dots, n \tag{6.10}$$

in which the periodic orbits lie. As only the component x_s will be non-zero, we can replace $(S\mathbf{x})_j$ by $(S_{js}x_s)$ in (6.9). As long as the column s of S is not degenerated we will have a 1 on the diagonal leading to

$$\dot{x}_j = \begin{cases} \left(\lambda - \operatorname{Re}\lambda_j + (1 - \operatorname{Im}\lambda_s)\,\mathrm{i} + \gamma|x_s|^2\right) x_s & \text{for } j = s \\ 0 & \text{for } j \neq s \end{cases}. \tag{6.11}$$

This means the subspaces X_j with $j = 0, \dots, n$ are dynamically invariant, i.e. taking initial values in one subspace results in solutions staying in it for all times.

The bifurcation points of the Hopf normal-form are shifted to $\operatorname{Re}\lambda_j$ and the frequency is not scaled to unity if $\operatorname{Im}\lambda_j \neq 0$. Apart from that the system stays in Hopf normal form in each component. Therefore, we know that periodic solutions in X_s have the form $x_s^*(t) = r_s \exp(\Omega_s t)$, with r_s and Ω_s defined as in (6.6). □

6.0.4 Controlling the system

We now apply equivariant Pyragas control to this system. We add a time-delayed feedback term such that the system (OS) becomes

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z},\lambda) + A\mathbf{z} + K\left[\mathbf{z} - h\mathbf{z}(t - \theta T)\right] + L\mathbf{z}^2 \cdot \left[\bar{\mathbf{z}} - h\bar{\mathbf{z}}(t - \theta T)\right], \tag{COS}$$

where $\mathbf{z}^2$, the multiplication, and the complex conjugation are to be taken componentwise:

$$\left(\mathbf{z}^2 \cdot \left[\bar{\mathbf{z}} - h\bar{\mathbf{z}}(t - \theta p)\right]\right)_j = z_j^2\left[\bar{z}_j - (h\bar{\mathbf{z}}(t - \theta p))_j\right] \tag{6.12}$$

The *control-matrices* $K, L \in \mathbb{C}^{n\times n}$ are still left to choose. For this method of control the following theorem holds:

Theorem 3. *Select an orbit* $\mathbf{x}^{(s)}$ *as introduced in Proposition 1 that is unstable. Let* $T = 2\pi/\Omega_s$ *be its minimal period. Choose* θ *and* $h = h(\theta)$ *such that*

$$\theta < 9/(\tilde{a}T) \tag{6.13}$$

where $\tilde{a}$ *is the unstable-Floquet multiplier of* x_s^**. Further, let* θ *satisfy*

$$\theta < \theta_{\max}(A, \gamma, \lambda) \tag{6.14}$$

Then there exist control-matrices K *and* L *such that the periodic orbit is stabilized non-invasively by equivariant Pyragas control of the form* (COS)*.*

To prove this we propose – in a similar fashion as in [23] – control-matrices that diagonalize in our transformed $\mathbf{x}$-coordinates:

$$K = S^{-1} \operatorname{diag}(k_1, k_2, \ldots, k_n)\, S \tag{6.15}$$

$$L = S^{-1} \operatorname{diag}(\ell_1, \ell_2, \ldots, \ell_n)\, S \tag{6.16}$$

This type of structure for a matrix K was originally proposed in [25]. The second control-matrix L uses the same structure and mimics the behavior of the ℓ-term of the method (CH) used to stabilize the Hopf normal form in chapter 5. Our approach is indeed an extension of (CH) in for a vector-valued system, as our aim is to apply the Main Theorem. The equation for the x_s-component is a rotationally symmetric complex system and thus falls under the scope of the Main Theorem, giving us the existence of b_s and c_s suitable for stabilization.

It remains to show that the other components of $\mathbf{x}$ can also be stabilized. For those we have to prove stabilization at $x_j = 0$ only.

The Theorem on "‘the upper threshold on the remaining eigenvalues"’ in [25] provides a solution for this. Let us check the assumptions.

From (6.9) and (6.11) we can see how the linearization near a rotating wave solution in the other components x_j for $j \neq s$ looks like without control:

$$\dot{\delta x_j} = \left(\lambda - \lambda_j + \mathrm{i} + \gamma r_s^2 - \lambda_j\right) \delta x_j \tag{6.17}$$

We have the S^1-symmetry as the group, our F is componentwise rotationally symmetric, i.e. equivariant with respect to this group. If we restrict ourselves to controls with $\ell_j = 0$ for $j \neq s$, the theorem applies. An additional corollary tells us that, if we choose our θ below a certain bound there exist k_j providing stabilization.

Since we imposed the restriction (6.14) in our theorem, there exists a stabilizing-scheme. □

6.0.5 Example of a ring-structure

The ring of oscillators with nearest-neighbor coupling used as a model system in [23] leads to a symmetric (tridiagonal) matrix

$$A_{\mathrm{Ring}} = a_{\mathrm{cpl}} \begin{pmatrix} -2 & 1 & 0 & \ddots & 0 & 0 & 1 \\ 1 & -2 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \ddots & 1 & -2 & 1 \\ 1 & 0 & 0 & \ddots & 0 & 1 & -2 \end{pmatrix}. \tag{6.18}$$

Thus it is diagonizable and we can apply Theorem 3. In the paper the transformation matrix was calculated and the proposed stabilization matrix is of the form (6.15). All considered control-terms were linear, leading to $L = 0$.

7 Discussion

In this thesis we proved that for a control-method of the form

$$\dot{x}(t) = f\big(x(t),\, g[x(t), hx(t-\theta T)]\big) \tag{3.4}$$

in a rotationally symmetric two-dimensional system we can achieve stabilization. An upper bound on the delay-time θT that can be used was obtained. Also we introduced the concept of most-stabilizing control-schemes and proved an implicit formula for them.

We considered a rotationally symmetric two-dimensional system and made use of the rotational symmetry to turn the periodic orbit in the original systems into a ring of equilibria in the controlled system. For other symmetries which allow a similar transformation this might also be possible. For systems without such a symmetry we would not have this option.

For our approach the characteristic equation needed to have ‘enough‘ control-parameters – for a two-dimensional system b_0, b_1 and b_2. With those parameters we were able to characterize eigenvalue-solutions with fixed real part. In many publications (e.g. [18]) a single parameter is considered. This of course simplifies the parameter space, but in this thesis the three parameters were necessary to unfold the Takens-Bogdanov bifurcation. A similar approach for other problems with known characteristic equations might also be successful.

One direction for further research might be the consideration of multiple time-delays (see e.g. [23]). With multiple delays we still have similar characteristic equations. As mentioned in [9], small time delays – which equivariant schemes might require to work – are often not obtainable for control. In this case considering control-methods of the form

$$\dot{\mathbf{x}} = \mathbf{f}\big(x, g[x(t-\tau), hx(t-\tau-\theta T)]\big) \tag{7.1}$$

with a non-invasive control-scheme g and additional time-delay $\tau > 0$ could be considered. The control still vanishes on periodic orbits with the chosen spatio-temporal symmetry

$$\mathbf{x}_*(t) = h\,\mathbf{x}_*(t-\theta T) \tag{1.2}$$

but τ can be chosen arbitrarily. This might account for (unknown) time-intervals between measurements and feedback. Whether this control-method can lead to better stability properties than those with a single delay, still needs to be explored.

In the coupled-oscillator-system of chapter 6 we had to pose certain restriction on our control to cite the existing theorem in [25]. To get results for all control-schemes one could try to investigate the full characteristic equation

$$0 = \eta - a + b_0 u + b_1 u\eta + b_2 u^2 \eta + c/\eta \tag{2.18}$$

obtained in Lemma 1. Note that for periodic orbits we only needed the case $c = 0$.

We used the rotational symmetry for an equivariant control-method in this thesis. We could additionally explore other symmetries (e.g. the index-shift from [23]), especially in systems of higher dimensionality like the coupled oscillator system. As we only used the S^1-symmetry, the control approach presented in chapter 6 is not necessarily *selective* in its stabilization. It is non-invasive on all rotationally symmetric orbits with equal period. So if two or more periodic solutions with the same period exist, we might stabilize multiple orbits at the same time. This might be undesirable if we need the system to converge to a specific one.

Our perturbations of the system only stay small near the periodic orbit we chose to stabilize. As suggested in [21] one might only 'turn on' the control when near to the periodic orbit. Otherwise no control is applied. This method aims at making the chosen orbit an attractor of the whole system. This will only succeed if even far away orbits get near enough to the periodic orbit after some time. This is the behavior if the unstable periodic orbit lies in a chaotic attractor [22]. None of the systems we considered are chaotic though – solutions tend to equilibria, periodic orbits or infinity. Applying our an extended control to a chaotic system might thus be a worthwhile task.

In conclusion, this thesis shows a new constructive method of stabilizing a periodic orbit via time-delayed feedback control using equivariance.

Glossary of Notation

To those who might be lost in the notation, this list of symbols provides a short explanation and a reference to the page where the symbol is introduced.

symbol	**description**	**p.**
A	Complex coupling-matrix for the n oscillators	37
$a, \tilde{a}$	Rescaled or unscaled unstable eigenvalue of the uncontrolled system, respectively	11
$\mathcal{B}(a, \rho)$	ρ-Region of control-parameters (b_0, b_1, b_2)	23
b_0, b_1, b_2	Control-parameters in the characteristic equation	11
$b_0^\star, b_1^\star, b_2^\star$	Most-stabilizing control-parameters	11
$\mathbf{b}_{\mathrm{H}}(\omega, \rho)$	Hopf-curve of complex eigenvalue solutions	25
$\mathbf{b}_{\mathrm{S}}(b_1, \rho)$	Saddle-curve of real eigenvalue solutions	24
c	Coefficient in the characteristic equation, which is zero for periodic orbits	11
D	Differential operator for the first derivative	/
$\delta \ldots$	Small change in a variable	21
E	Abbreviation for $(1 - \exp(-\rho))^{-1}$	29
$\mathbf{e}_j$	Eigenvector of A to the eigenvalue λ_j	38
$\eta, \tilde{\eta}$	Rescaled or unscaled Floquet exponent, respectively	11
$\mathbf{F}$	n-dimensional version of the Hopf normal-form	37
F	Hopf normal-form	13
$\mathbf{f}$	General right hand side of a delay differential equation	7
f	Rotationally symmetric right hand side (in the complex)	15
g	Control-scheme keeping the rotational symmetry	15
γ	Parameter of the cubic term of the Hopf normal-form	13
h	Group element from $SO(2, \mathbb{R})$, chosen corresponding with θ	15
Id	Identity-matrix	/
i	Imaginary unit	/
j	Index (of the oscillators in the coupled system), from 1 to n	37
K, L	Complex control-matrices for the n oscillators	39
k, ℓ	Complex parameters for the control in the Hopf normal form case	33
λ	Bifurcation parameter in the Hopf normal-form	13

symbol	description	p.
$\lambda_1, \ldots, \lambda_n$	Eigenvalues of the coupling matrix A whose real parts are also bifurcation points for λ	37
M_0, M_τ	Matrices in the general, linearized system with one delay	10
μ	Floquet muliplier (which satisfies $\mu = \exp(\tilde{\eta})$)	9
n	Complex dimension of the considered system	37
$\mathbf{P}$	Poincaré map	8
π	Infinite sum of $4\,\mathrm{i}^{2j}/(2j+1)$ for $j = 1, \ldots, \infty$	/
R	Abbreviation for $\exp(-\rho)$	29
r_*	(Constant) radius of the periodic orbit	13
ρ	Real part of $\eta = \rho + \mathrm{i}\omega$	23
$\rho^\star$	Real eigenvalue produced by most-stabilizing control	27
$\mathcal{S}$	Poincaré section	8
S	(Normalized) transformation matrix, diagonalizing A	37
s	Index of the *selected* rotating wave solution	38
T	Minimal period of the periodic orbit	3
t	Time	3
θ	Normalized time-delay	4
u, $\tilde{u}$	Abbreviation for $(1 - \exp(-\eta))/\eta$, resp. $(1 - \exp(-\tilde{\eta}\theta T))$	11
$\mathbf{x}$	n-dimensional (complex- or real-valued) state variable	8
x	Complex one-dimensional variable representing two dimensions	15
x_*	Periodic orbit in the complex	15
$\mathbf{x}_*$	Periodic orbit in $\mathbb{R}^n$	4
$\mathbf{z}$	Vector in $\mathbb{C}^n$, with each component describing one oscillator in Hopf normal-form	37
$\mathbf{z}^{(s)}$	Periodic orbit, which is a rotating wave solution, in $\mathbb{C}^n$	38
z	The variable x, transformed to co-rotating coordinates	20
ω	Imaginary part of $\eta = \rho + \mathrm{i}\omega$	24
Ω	Angular velocity of a periodic solution	20

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