EXISTENCE OF STEADILY ROTATING SPIRAL WAVES

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Abstract. In this paper, we study the existence of steadily rotating spiral waves in the kinematic theory of excitable media. The spiral wave is assumed to be a family of curves. It rotates along a circle with a constant speed such that the tip of the curve neither grows nor retracts in the tangential direction. Using an analytical approach, we are able to derive the existence and non-existence of spiral waves with different speeds. Moreover, there is a monotone decreasing positive sequence tending to zero such that the number of spiral waves for each given speed can be computed exactly and the number can be any positive integers.

1. Introduction

In this paper, we are interested in the following equation:

$$\kappa_t + v_{yy} + (\kappa \int_0^y \kappa v d\xi)_y + G\kappa_y = 0, \tag{1.1}$$

where y denotes the arc length, $\kappa := \kappa(y,t)$ is the curvature of a family of curves in $\mathbf{R}^2$, $v := v(y,t)$ is the normal velocity, and $G := G(t)$ is the tangential velocity at the tip. The equation (1.1) can be dervied from the definitions of the normal and tangent vectors, normal and tangential velocities, and the Frenet-Serret Theorem in the plane.

In the study of steadily rotating spiral wave in the kinematic theory of excitable media (cf. [1, 4, 5]), we assume that the family of curves (keeping the same shape for all t) rotates along a circle with a constant speed such that the tip of the curve neither grows nor retracts in the tangential direction (i.e., $G \equiv 0$) and the normal velocity v satisfies the relation $v = c - D\kappa$. Then (1.1) reduces to

$$-D\kappa''(y) + \left(\kappa(y) \int_0^y \kappa(\xi)[c - D\kappa(\xi)]d\xi\right)' = 0,\ y > 0, \tag{1.2}$$

By integrating (1.2) once, we obtain that κ satisfies the equation

$$-D\kappa'(y) + \kappa(y) \int_0^y \kappa(\xi)[c - D\kappa(\xi)]d\xi = \omega, \tag{1.3}$$

where ω is the constant angular frequency of the wave. We also assume that κ satisfies the following condition

$$\kappa(0) = \kappa_0, \quad \kappa(\infty) = 0, \tag{1.4}$$

where κ_0 is the curvature at the tip.

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Following [3], by differentiating (1.3) with respect to y once, we obtain that κ satisfies the equation

$$-D\kappa'' + \kappa^2(c - D\kappa) + \frac{\kappa'}{\kappa}(\omega + D\kappa') = 0. \tag{1.5}$$

We are interested in finding a spiral wave with *positive curvature*. Setting $w := \ln(\kappa)$ and noting that

$$-D\kappa'(0) = \omega, \tag{1.6}$$

we end up with the following initial value problem (P_η):

$$w'' + g(w) = \eta e^{-w} w',\ y > 0, \tag{1.7}$$

$$w(0) = \ln(b),\ w'(0) = -\eta/b, \tag{1.8}$$

where $g(w) = e^{2w} - ae^w$, a, b are positive constants, and η is a real constant. Indeed

$$a := \frac{c}{D},\ b := \kappa_0,\ \eta := \frac{\omega}{D}.$$

The local existence and uniqueness of solutions of (P_η) is trivial.

We are interested in the existence of global solution w of (P_η) satisfying the property

$$\lim_{y\to\infty} w(y) = -\infty. \tag{1.9}$$

In [2], we studied a simplified equation of (1.3), namely,

$$-D\kappa'(y) + \kappa(y) \int_0^y c\kappa(\xi)d\xi = \omega$$

and obtained a family of steadily rotating spiral waves. In [3], they studied the equation (1.7) with the initial condition

$$w(0) = w_0, \quad w'(0) = 0.$$

They obtained many interesting results. In particular, for $\eta = 0$ the solution is periodic if $b \in (0, 2a) \setminus \{a\}$; a constant if $b = a$; and is monotone decreasing with $w(y) \to -\infty$ as $y \to \infty$ if $b \geq 2a$. There are many interesting questions left for the equation (1.7), especially for any arbitrary $\eta > 0$. We shall study the case when $\eta \neq 0$ in this paper. Using a different method from [3], we are able to classify the structure of solutions of (P_η).

This paper is organized as follows. In §2, we first give some preliminary results. In §3, we prove that there is no global solution of (P_η) satisfying (1.9) when $\eta < 0$. See also Theorem 1 in [3]. The case $\eta > 0$ is treated in §4. We prove that there is a critical value $\tilde{\eta} > 0$ such that a spiral wave exists if and only if $\eta \in (0, \tilde{\eta}]$. Moreover, we are able to count the exact number of spiral waves for any given $\eta \in (0, \tilde{\eta}]$.

2. Preliminary

For a local solution w of (1.7), we define the energy function by

$$E(y) := \frac{1}{2}[w'(y)]^2 + G(w(y)), \quad G(w) := \frac{1}{2}e^{2w} - ae^w. \tag{2.1}$$

It follows from

$$E'(y) = \eta e^{-w(y)}[w'(y)]^2 \tag{2.2}$$

that E is monotone increasing (decreasing) if $\eta > 0$ ($\eta < 0$, respectively).

Recall the following result from [3].

Lemma 2.1. *Let $\eta \in \mathbb{R}$. If $E(y)$ is bounded above, then w exists globally for all $y > 0$.*

Note that from (1.7) it is easy to see that a critical point y of w is a maximum point (minimum point) if $w(y) > \ln(a)$ ($w(y) < \ln(a)$, respectively).

Lemma 2.2. *Let $\eta \in \mathbb{R}$. Suppose that w is a global solution of (1.7) such that $l := \lim_{y\to\infty} w(y)$ exists and $l > -\infty$ (l may be $+\infty$). Then $l = \ln(a)$.*

Proof. Suppose that $l = +\infty$. Then there is a $y_0 > 0$ such that $w > 2\ln(a)$ and $w' > 0$ in $[y_0, \infty)$. By integrating (1.7) from y_0 to $y > y_0$, we obtain that

$$w'(y) + \eta e^{-w(y)} - w'(y_0) - \eta e^{-w(y_0)} = -\int_{y_0}^{y} g(w(s))ds. \tag{2.3}$$

By letting $y \to \infty$ in (2.3), we reach a contradiction.

Suppose that $l \in (-\infty, \ln(a)) \cup (\ln(a), \infty)$. Then there is a $y_0 > 0$ such that $(-\ln(a) + 3l)/2 < w < (\ln(a) + l)/2$ in $[y_0, \infty)$. Also, we can find a sequence $\{y_n\}$ in $[y_0, \infty)$ such that $y_n \to \infty$ and $w'(y_n) \to 0$ as $n \to \infty$. By integrating (1.7) from y_0 to y_n, we obtain that

$$w'(y_n) + \eta e^{-w(y_n)} - w'(y_0) - \eta e^{-w(y_0)} = -\int_{y_0}^{y_n} g(w(s))ds. \tag{2.4}$$

This is impossible if we let $n \to \infty$ in (2.4). This proves the lemma. □

3. The case $\eta < 0$

In this section, we always assume that η is a fixed negative constant. It follows from Lemma 2.1 and (2.2) that any solution of (P_η) must be globally defined. Notice that $w'(0) > 0$.

The following lemma follow from Lemma 2.2 directly.

Lemma 3.1. *Suppose that $w' > 0$ in $[0, \infty)$. Then $w(y) \to \ln(a)$ as $y \to \infty$.*

Lemma 3.2. *Suppose that there is a $y_0 > 0$ such that $w'(y_0) = 0$ and $w' < 0$ in (y_0, ∞). Then $w(y) \to \ln(a)$ as $y \to \infty$.*

Proof. It follows from Lemma 2.2 that either $w(y) \to -\infty$ or $w(y) \to \ln(a)$ as $y \to \infty$.

Suppose that $w(y) \to -\infty$ as $y \to \infty$. Note that $w(y_0) > \ln(a)$. Let $y_1 > y_0$ be the point with $w(y_1) = \ln(a)$. From (2.3) it follows that

$$\eta e^{-w(y)} = \eta e^{-w(y_0)} - \int_{y_0}^{y} g(w(s))ds - w'(y) \tag{3.1}$$

$$\geq \eta e^{-w(y_0)} - \int_{y_0}^{y_1} g(w(s))ds. \tag{3.2}$$

This is impossible, since $\eta e^{-w(y)} \to -\infty$ as $y \to \infty$. The lemma is proved. □

For the case when w has at least 2 critical points, we have

Lemma 3.3. *Suppose that w has at least 2 critical points. Then w is bounded.*

Proof. By assumption, w has at least one minimum point, say, $y_0 > 0$. Then $E(y_0) = G(w(y_0)) < 0$, since $w(y_0) < \ln(a)$. Also, by the definition of G there are constants m, M such that

$$G(m) = G(M) = E(y_0), \quad -\infty < m \leq \ln(a) \leq M < \ln(2a). \tag{3.3}$$

By the decreasing property of E, we have $E(y) \leq E(y_0)$ for all $y \geq y_0$. Since $G(w(y)) \leq E(y)$, it follows that $m \leq w(y) \leq M$ in $[y_0, \infty)$. Therefore, w is bounded. □

From Lemmas 3.1-3.3, we conclude that there is no global solution of (P_η) with the property (1.9).

4. The case $\eta > 0$

4.1. **Some properties.** In this section, we always assume that η is a positive constant. Let w be a solution of (1.7) and let $[0, R)$, $R \leq \infty$, be the maximum existence interval of w. Note that $E(y) = G(w(y)) < 0$ for any minimum point y of w, since $w(y) < \ln(a)$. On the other hand, $E(y) \geq 0$ for any maximum point y of w such that $w(y) \geq \ln(2a)$.

Recall that $E(y)$ is monotone increasing in y. The first part of the following lemma is similar to Lemma 2 in [3].

Lemma 4.1. *Suppose that $E(y_0) \geq 0$ for some maximum point $y_0 \geq 0$ or some point $y_0 \geq 0$ with $w'(y_0) < 0$. Then $w' < 0$ in (y_0, R) and $w(y) \to -\infty$ as $y \to R^-$.*

Proof. For contradiction, let $y_1 > y_0$ be the first critical point. Then y_1 must be a minimum point. This is impossible, since $E(y_1) > E(y_0) \geq 0$. Therefore, $w' < 0$ for $y > y_0$ as long as w exists.

Let $l := \lim_{y \to R^-} w(y)$. Suppose that $l > -\infty$. If $R = \infty$, then $l = \ln(a)$ by Lemma 2.2. Also, there is a sequence $\{y_n\}$ such that $y_n \to \infty$ and $w'(y_n) \to 0$ as $n \to \infty$. We obtain that $E(y_n) \to G(\ln(a)) < 0$ as $n \to \infty$, a contradiction. On the other hand, if $R < \infty$ then from

$$w'(y) + \eta e^{-w(y)} - w'(y_0) - \eta e^{-w(y_0)} = -\int_{y_0}^{y} g(w(s))ds \tag{4.1}$$

it follows that w' is bounded. This implies that w can be continued beyond R, a contradiction. We conclude that $l = -\infty$. This completes the proof. □

Lemma 4.2. *Suppose that $E(y_0) \geq 0$ for some $y_0 \geq 0$. Then $w' < 0$ in (y_1, R) for some $y_1 \geq y_0$ and $w(y) \to -\infty$ as $y \to R^-$.*

Proof. By Lemma 4.1, it suffices to consider the case when $w'(y_0) \geq 0$ and $w' > 0$ in a right neighborhood of y_0. If $w'(y_1) = 0$ for some $y_1 \in (y_0, R)$, then $w' < 0$ in (y_1, R) and $w(y) \to -\infty$ as $y \to R^-$ by Lemma 4.1.

If $w' > 0$ in (y_0, R), then $R < \infty$. Otherwise, $w(y) \to \ln(a)$ as $y \to \infty$ by Lemma 2.2. This implies that $E(y) \to G(\ln(a)) < 0$ as $y \to \infty$, a contradiction. Hence $R < \infty$. Let $l := \lim_{y\to R^-} w(y)$. Then $l > -\infty$. If $l < \infty$, then by (4.1) w' is bounded and so w can be continued beyond R. This contradicts with the definition of R. If $l = \infty$, then by (4.1) again w' is bounded. This implies that w is bounded, a contradiction. Therefore, we have $w'(y) = 0$ for some $y \in (y_0, R)$. Hence the proof is completed. □

Let $L := \lim_{y\to R^-} E(y)$ be the energy limit of w. The limit exists since E is monotone increasing. If $L < \infty$, then $R = \infty$ by Lemma 2.1. It is clear that $E(y_0) \geq 0$ for some $y_0 \geq 0$, if $L = \infty$. Conversely, we have

Lemma 4.3. *Suppose that $E(y_0) \geq 0$ for some $y_0 \geq 0$. Then $L = \infty$ and $w(y), w'(y) \to -\infty$ as $y \to R^-$.*

Proof. Clearly, $L > 0$. From Lemma 4.2 it follows that $w(y) \to -\infty$ as $y \to R^-$. Using the relation

$$\frac{1}{2}[w'(y)]^2 = E(y) - G(w(y))$$

we deduce that $w'(y) \to -\sqrt{2L}$ as $y \to R^-$. Moreover, it follows from (2.2) that $L = \infty$. This completes the proof. □

Lemma 4.4. *There is no solution of (1.7) with $L < 0$ except the trivial solution $w \equiv \ln(a)$.*

Proof. Suppose that there is a solution w_0 of (1.7) with the energy limit $L_0 < 0$ and $w_0 \not\equiv \ln(a)$. Set $c = L_0$. Let $(w_1, 0)$, $(w_2, 0)$ be the (two) points on $\Gamma_c := \{(w, v) \mid v^2/2 + G(w) = c\}$ with $w_1 < w_2$. By the theory of continuous dependence on initial data and parameter, there is a positive constant δ such that the trajectory of any solution w of (1.7) reaching the line $S := \{(w, 0) \mid w_2 - \delta < w < w_2 + \delta\}$ will leave Γ_c so that the energy limit L satisfying $L > c$.

Note that w_0 is bounded, since $G(w_0(y)) \leq E(y) < c$ for all y. Since $c > G(\ln(a))$, it follows from Lemma 2.2 that w cannot be monotone ultimately. Then there is a sequence $\{y_n\}$ of critical points of w_0 such that $y_n \to \infty$ and $w_0(y_n) \to w_2$ as $n \to \infty$. Therefore, the trajectory of w_0 reaches S at y_n for all n sufficiently large. This implies that $L_0 > c$, a contradiction. The lemma follows. □

It follows from Lemmas 4.3 and 4.4 that either $L = \infty$ or $L = 0$ for the energy limit L of any non-constant solution of (1.7). Also, from the increasing property of E it follows that $E(y) < 0$ for all $y \geq 0$ and $E(y) \to 0$ as $y \to \infty$, if $L = 0$. For convenience, we call a solution with $L = 0$ a *Type I solution*; and a solution with $L = \infty$ a *Type II solution*. We remark that w is a Type II solution if and only if $E(y_0) \geq 0$ for some $y_0 \geq 0$.

The following lemma shows that any Type II solution is non-global.

Lemma 4.5. *Suppose that $L = \infty$. Then $R < \infty$.*

Proof. For contradiction, we assume that $R = \infty$. Recall that $w(y), w'(y) \to -\infty$ as $y \to \infty$. It follows from (1.7) that $w''(y) \to -\infty$ as $y \to \infty$. Then by applying l'Hôpital's rule we compute that

$$\begin{aligned} \lim_{y\to\infty} \{y^2 e^{w(y)}\} &= \lim_{y\to\infty} \frac{y^2}{e^{-w(y)}} \\ &= \lim_{y\to\infty} \frac{2y}{-e^{-w(y)}w'(y)} \\ &= \lim_{y\to\infty} \frac{2}{-e^{-w(y)}w''(y) + e^{-w(y)}[w'(y)]^2} \\ &= 0. \end{aligned}$$

Hence there is a constant $y_0 \geq 1$ such that

$$e^{w(y)} \leq \frac{1}{y^2} \tag{4.2}$$

for all $y \geq y_0$.

On the other hand, from (1.7) it follows that

$$w'(y) + \eta e^{-w(y)} + \int_{y_0}^{y} g(w(s))ds = A := w'(y_0) + \eta e^{-w(y_0)}$$

and so

$$e^{w(y)}w'(y) + \eta + e^{w(y)} \int_{y_0}^{y} g(w(s))ds = Ae^{w(y)}.$$

By an integration again we end up with

$$e^{w(y)} - e^{w(y_0)} + \eta(y - y_0) + \int_{y_0}^{y} \left\{ e^{w(\xi)} \int_{y_0}^{\xi} g(w(s))ds \right\} d\xi = A \int_{y_0}^{y} e^{w(s)}ds. \tag{4.3}$$

Taking y_0 sufficiently large so that $|g(w(s))| \leq ae^{w(s)}$ for all $s \geq y_0$. Then it is easy to show that the integrals in (4.3) are uniformly bounded for all $y \geq y_0$. This contradicts the assumption $R = \infty$. The lemma is proved. □

If $L = 0$ and w has infinitely many critical points, then by the increasing property of E the sequence of maximum points (minimum points) is increasing and tends to $\ln(2a)$ (is decreasing and tends to $-\infty$, respectively).

We say that w is *monotone ultimately* if w is monotone for all y sufficiently large.

Lemma 4.6. *If $L = 0$, then $w'(y) < 0$ for all y sufficiently large. Moreover, $w(y) \to -\infty$ and $w'(y) \to 0$ as $y \to \infty$.*

Proof. First, we claim that w is monotone ultimately. By the theory of continuous dependence on initial data and parameters, there is a positive constant δ such that any solution w of (1.7) with $w(0) \in (\ln(2a) - \delta, \ln(2a))$ and $w'(0) = 0$ has the energy limit $L = \infty$. Let $c := G(\ln(2a) - \delta)$. Note that $c < 0$. Consider the closed curve $\Gamma_c := \{(w, v) \mid v^2/2 + G(w) = c\}$. Suppose that w has infinitely many critical points. Then there is a sequence of maximum points $\{y_n\}$ such that $w(y_n) \to \ln(2a)$ as $n \to \infty$, since $L = 0$. Then we have $w(y_N) \in (\ln(2a) - \delta, \ln(2a))$ for some $N \geq 1$. This implies that $L = \infty$, a contradiction. Therefore, w is monotone ultimately.

Next, we claim that $w'(y) < 0$ for all y sufficiently large. For contradiction, we suppose that $w'(y) > 0$ for all y sufficiently large. Then $w(y) \to \ln(a)$ as $y \to \infty$. Take a sequence $\{y_n\}$ such that $y_n \to \infty$ and $w'(y_n) \to 0$ as $n \to \infty$. Then $E(y_n) \to G(\ln(a)) < 0$ as $n \to \infty$, a contradiction. Therefore, we must have $w' < 0$ for all y sufficiently large.

If $l := \lim_{y\to\infty} w(y) > -\infty$, then $l = \ln(a)$, by Lemma 2.2, and so there is a sequence $\{y_n\}$ such that $y_n \to \infty$ and $w'(y_n) \to 0$ as $n \to \infty$. Then $E(y_n) \to G(\ln(a)) < 0$ as $n \to \infty$. This contradicts the assumption $L = 0$. Hence $l = -\infty$.

Finally, from the relation

$$\frac{1}{2}[w'(y)]^2 = E(y) - G(w(y))$$

it follows that $w'(y) \to 0$ as $y \to \infty$. Hence the lemma is proved. □

Remark 4.1. *Suppose that there is a solution w with $L = 0$. Then $w(y) \to -\infty$ and $w'(y) \to 0$ as $y \to \infty$. Since $ae^w \geq -g(w)$, it follows from (1.7) that*

$$a \int_0^y e^{w(y)} dy \geq -\int_0^y g(w(y)) dy = w'(y) + \eta e^{-w(y)} \to \infty$$

as $y \to \infty$. Recall that $\kappa(y) = e^{w(y)}$. Hence the rotation number is $+\infty$ and the corresponding curve is a spiral wave.

4.2. **Existence.** In the sequel, we denote $w(y; \eta, b)$ the solution of (P_η) to specify the dependence of w on the parameters η and/or b.

We consider the problem (Q_η):

$$w'' + g(w) = \eta e^{-w} w' \tag{4.4}$$

$$w(0) = \ln(a),\ w'(0) = -\eta/a. \tag{4.5}$$

Let $w(y; \eta) = w(y; \eta, a)$ be the solution of (Q_η). Set

$$A_1 := \{\eta > 0 \mid w'(y; \eta) = 0 \text{ for some } y > 0\}.$$

Note that A_1 is an open set. We shall claim that $(0, \eta_0) \subset A_1$ for some positive constant η_0.

For this, we set

$$u := w - \ln(a), \quad s = ay.$$

Then w satisfies (4.4)-(4.5) if and only if u satisfies the problem:

$$\ddot{u} - ke^{-u}\dot{u} + (e^{2u} - e^u) = 0 \tag{4.6}$$

$$u(0) = 0,\ \dot{u}(0) = -k, \tag{4.7}$$

where $k := \eta/a^2$ and the dot denotes the differentiation with respect to s.

Now, we consider the following equivalent system:

$$\dot{u} = z$$
$$\dot{z} = ke^{-u}z + (e^u - e^{2u})$$

with the initial condition $(u(0), z(0)) = (0, -k)$. Set $u = r\cos\theta$ and $z = r\sin\theta$. Then we compute that

$$\dot{r} = kr\sin^2\theta + f_1(r,\theta), \tag{4.8}$$
$$\dot{\theta} = -1 + k\sin 2\theta/2 + f_2(r,\theta), \tag{4.9}$$

where

$$f_1(r,\theta) := k\sin^2\theta(e^{-u} - 1)r + \sin\theta(e^u - e^{2u}) + r\sin\theta\cos\theta,$$
$$f_2(r,\theta) := k\sin 2\theta(e^{-u} - 1)/2 + \cos\theta(e^u - e^{2u})/r + \cos^2\theta.$$

Fix $\alpha \in (0,2)$ such that $4\pi\alpha/(1-\alpha/2) < \ln 2$ and let $k \in (0,\alpha)$. Note that $f_2(r,\theta) = O(r)$ as $r \to 0^+$ uniformly in k. Hence we can choose $r_0 \in (0,1)$ (independent of k) such that

$$|f_2(r,\theta)| < \frac{1}{2}(1 - \frac{\alpha}{2}) \text{ for all } r \in (0, r_0] \text{ and } \theta \in \mathbb{R}. \tag{4.10}$$

Therefore, we obtain that

$$\dot{\theta} \leq -(1 - \alpha/2)/2 \tag{4.11}$$

as long as $r \in (0, r_0]$. Moreover, we may take $r_0 < 1/(40\pi)$ (independent of k) small enough such that

$$|f_1(r,\theta)| \leq 5r^2 \text{ for all } r \in (0, r_0] \text{ and } \theta \in \mathbb{R}. \tag{4.12}$$

Set $s_0 := 4\pi/(1 - \alpha/2)$. Then we have

$$1/2 - e^{ks}/4 > 0 \text{ for all } s \in (0, s_0] \text{ and for all } k \in (0,\alpha). \tag{4.13}$$

Since

$$\lim_{\alpha\to 0^+} \frac{1/2 - e^{ks_0}/4}{5s_0e^{ks_0}/4} = \frac{1}{20\pi},$$

we may further assume that α is small enough such that

$$\frac{1/2 - e^{ks_0}/4}{5s_0e^{ks_0}/4} > \frac{1}{40\pi}. \tag{4.14}$$

From now on, we fix the constants r_0 and α so that all the above conditions hold.

Lemma 4.7. *The solution (r,θ) of (4.8) and (4.9) with $r(0) \leq r_0/4$ and $\theta(0) \in \mathbb{R}$ satisfies $r(s) \in (0, r_0/2)$ for all $s \in [0, s_0]$.*

Proof. Integrating the equation (4.8) from 0 to s and using (4.12), we obtain that

$$r(s) \leq e^{ks}r(0) + se^{ks}5r_0^2/4 \tag{4.15}$$

as long as $r \le r_0/2$. For contradiction, we assume that $r(s) = r_0/2$ for some $s \le s_0$. Then it follows from (4.15) that

$$r_0 \ge \frac{1/2 - e^{ks}/4}{5se^{ks}/4} \ge \frac{1/2 - e^{ks_0}/4}{5s_0e^{ks_0}/4},$$

by using (4.13) and the decreasing property of the function

$$K(s) := \frac{1/2 - e^{ks}/4}{5se^{ks}/4}, \quad s \in (0, s_0].$$

Hence we reach a contradiction and the lemma is proved. □

Lemma 4.8. *The set* A_1 *contains* $(0, \eta_0)$ *for some small positive constant* η_0*.*

Proof. By Lemma 4.7, we can integrate (4.11) from 0 to s_0 to obtain that

$$\theta(s_0) - \theta(0) \le -2\pi. \tag{4.16}$$

Then the lemma follows by setting $\eta_0 := \min\{r_0/4, \alpha\}a^2$. □

Remark 4.2. *It follows from (4.16) that the trajectory* (u, z) *in the phase plane goes around* $(0, 0)$ *clockwise once back to the line* $\{u = 0,\ z < 0\}$ *if* $\eta \in (0, \eta_0)$*. Indeed, we can obtain that the trajectory goes around* $(0, 0)$ *as many times as we want by assuming the* η *small enough.*

Note that the equation (4.4) is equivalent to the system (S_η):

$$\begin{aligned} w' &= v, \\ v' &= \eta e^{-w} v - g(w). \end{aligned}$$

By a simple phase plane analysis, it is easy to see that any trajectory (w, v) of (S_η), with $(w(0), v(0)) = (\ln(a), v_0)$ for some $v_0 < 0$ and $(w(y_0), v(y_0)) = (w_0, 0)$ for some $y_0 > 0$ and $w(y_0) < \ln(a)$, goes around $(\ln(a), 0)$ clockwise back to the line $\{w = \ln(a),\ w' < 0\}$. In particular, it is interesting to see the property of the vector field on the initial curve $\Gamma := \{(\ln(b), -\eta/b) \mid b > 0\}$ for a given fixed $\eta > 0$. Let D_1 (D_2, respectively) denote the region lying above (below, respectively) Γ. Set $v = w'$. Then $(v, 1)$ is a normal vector of the initial curve Γ at the point (w, v). Since the inner product of (w', v') and $(v, 1)$ is given by $-\eta(av + \eta)/v^2$, the vector field on Γ is pointed inward to D_2 if $w > \ln(a)$; to D_1 if $w < \ln(a)$.

Let $\Gamma_0^- := \{(w, v) \mid v < 0, w \le \ln(a), v^2/2 + G(w) = 0\}$. Recall that the energy E is strictly increasing along any trajectory. Fix $\eta = a^2$. Then $(\ln(a), -\eta/a) \in \Gamma_0^-$ and the region lying below Γ_0^- and above $\Gamma \cap \{w \le \ln(a)\}$ is a positively invariant region. Thus the solution w of (Q_η) is monotone decreasing to $-\infty$ and so A_1 is bounded above by a^2. Then $\hat{\eta} := \sup A_1$ is well-defined and $\hat{\eta} < a^2$.

As before, we set $L(\eta) := \lim_{y \to R^-} E(y; \eta)$, where $E(y; \eta) = [w'(y; \eta)]^2/2 + G(w(y; \eta))$. Note that $L(\eta) = \infty$ if $E(y; \eta) \ge 0$ for some $y \ge 0$. Introduce

$$A_2 := \{\eta > 0 \mid w'(y; \eta) < 0 \text{ in } [0, R) \text{ and } L(\eta) = \infty\}.$$

Similarly, we can see that for any given $\eta > a^2$ the region lying below Γ_0^-, above $\Gamma \cap \{w \le \ln(a)\}$, and to the left of $\{w = \ln(a), -\eta/a \le v \le -a\}$ is a positively invariant region. Thus

$[a^2,\infty) \subset A_2$. Hence $\tilde\eta := \inf A_2$ is well-defined and $a^2 > \tilde\eta \ge \hat\eta$. Note that A_2 is an open set. Moreover, for any $\eta \in [\hat\eta,\tilde\eta]$ the solution $w(y;\eta)$ of (Q_η) is of type I such that $w' < 0$ for all $y \ge 0$.

We shall claim that $\tilde\eta = \hat\eta$ and that there is a spiral wave solution if and only if $\eta \in (0,\tilde\eta]$. To do this, we need the following comparison lemma.

Lemma 4.9. *Suppose that v_i is the solution of the following initial value problem*

$$\frac{dv}{dw} = \eta_i e^{-w} + \frac{ae^w - e^{2w}}{v}, \tag{4.17}$$

$$v(\ln(b)) = -c_i, \tag{4.18}$$

for $i = 1,2$, where $0 < b \le a$, $c_2 > c_1 > 0$, and $\eta_2 > \eta_1 > 0$. Suppose that $v_1v_2 \neq 0$ on $(R,\ln(b)]$ for some $R < \ln(b)$. Then $v_1 > v_2$ and $v_1' < v_2'$ on $(R,\ln(b)]$.

Proof. Note that $v_1(\ln(b)) > v_2(\ln(b))$. For contradiction, we assume that there is $w \in (R,\ln(b))$ such that $v_1 > v_2$ on $(w,\ln(b)]$ and $v_1(w) = v_2(w)$. Then we have $v_1'(w) \ge v_2'(w)$. However, from (4.17), for $i = 1,2$, it follows that

$$v_1'(w) = \eta_1 e^{-w} + \frac{ae^w - e^{2w}}{v_1(w)} < \eta_2 e^{-w} + \frac{ae^w - e^{2w}}{v_2(w)} = v_2'(w), \tag{4.19}$$

a contradiction. Therefore, we obtain that $v_1 > v_2$ on $(R,\ln(b)]$.

Note that $v_2 < v_1 < 0$ on $(R,\ln(b)]$ by assumption. Also, $ae^w - e^{2w} \ge 0$ for $w \in (R,\ln(b)]$, since $b \le a$. Hence $v_1' < v_2'$ on $(R,\ln(b)]$ by (4.19). The proof is completed. □

Now, we claim that $\tilde\eta = \hat\eta$. Suppose that $\hat\eta < \tilde\eta$. Set $w_1(y) := w(y;\hat\eta)$ and $w_2(y) := w(y;\tilde\eta)$. Then w_i is of type I such that $v_i(y) := w_i'(y) < 0$ on $[0,\infty)$ for $i = 1,2$. Hence we can view v_i as function of w_i, $i = 1,2$. Therefore, v_i $(i = 1,2)$ is the solution of (4.17)-(4.18) with $b = a$, $\eta_1 := \hat\eta$, and $\eta_2 := \tilde\eta$. It follows from Lemma 4.9 that $v_1 > v_2$ on $(-\infty,\ln(a)]$. Also, from (4.17) it follows that

$$\frac{d(v_1 - v_2)}{dw} = (\eta_1 - \eta_2)e^{-w} + (ae^w - e^{2w})\left(\frac{1}{v_1} - \frac{1}{v_2}\right) < 0 \quad \text{on } (-\infty,\ln(a)). \tag{4.20}$$

By integrating (4.20) from $-\infty$ to $w \le \ln(a)$, we obtain that

$$(v_1 - v_2)(w) - (v_1 - v_2)(-\infty) < 0$$

for all $w \le \ln(a)$, a contradiction. Hence we conclude that $\tilde\eta = \hat\eta$.

Lemma 4.10. *For a fixed $\eta > 0$, the problem (P_η) has at most one type I solution (up to translations).*

Proof. For contradiction, we assume that w_1 and w_2 are two type I solutions of (P_η). Since, by Lemma 4.6, $v_i(y) = w_i'(y) < 0$ for all sufficiently large y, we can view v_i as a function of w for $i = 1,2$. Moreover, since Eq. (1.7) is autonomous, we may assume that $v_1 > v_2$ for all $w < w_0$ for some $w_0 < \ln(a)$. Then, as in (4.20) with $\eta_1 = \eta_2 = \eta$, we obtain that $(v_1 - v_2)'(w) < 0$ for all $w < w_0$. This leads to a contradiction and the proof is completed. □

We denote the solution $w(y;\eta,b)$ of (P_η) by $w_b(y;\eta)$ or simply $w_b(y)$. Also, set $v_b(y) := w_b'(y)$. Note that (w_b,v_b) satisfies the system (S_η).

Theorem 1. *There is a type I solution of (P_η) if and only if $\eta \in (0, \tilde{\eta}]$.*

Proof. First, we assume that $\eta > \tilde{\eta}$. Then we have $w_a'(y) < 0$ for all $y \in [0, R_a)$ and $w_a(y), w_a'(y) \to -\infty$ as $y \to R_a^-$ for some $R_a < \infty$. Let γ be the trajectory of (w_a, v_a) in the phase plane. Note that γ lies above Γ. Then any trajectory (w_b, v_b) starting at $(\ln(b), -\eta/b)$, with $b < a$, remains in the region below γ and above Γ. Hence $w_b'(y) < 0$ for all $y \in [0, R_b)$ and $w_b(y), w_b'(y) \to -\infty$ as $y \to R_b^+$ for some $R_b < \infty$, i.e., w_b is of type II. For $b > a$, the corresponding trajectory either remains below Γ or reaches Γ at some $y > 0$ with $w_b(y) < \ln(a)$ by the phase plane analysis. Therefore, it is also of type II.

Now, given a fixed $\eta < \tilde{\eta}$. Then either $L(\eta) = 0$ or $L(\eta) = \infty$.

Suppose that $L(\eta) = \infty$. Let $y_0 = 0$ and y_i be the y-value of the ith intersection point of the trajectory (w_a, v_a) with the line $\{w = \ln(a),\ v < 0\}$, $i = 1, \ldots, N$, such that $-\eta/a = w_a'(y_0) > w_a'(y_1) > \cdots > w_a'(y_N)$. Note that $1 \le N < \infty$. Consider the set $\{\alpha \in (w_a'(y_N), w_a'(y_{N-1}))\}$ such that the trajectory starting from the point $(\ln(a), \alpha)$ has the property that $L = \infty$. Then it is easy to see that the supremum α_0 of this set has the property that the trajectory (w_0, v_0) starting from the point $(\ln(a), \alpha_0)$ is of type I with $v_0(y) < 0$ for all $y \ge 0$. Therefore, we have at least $2N$ distinct solutions of (P_η) such that they are all of type I.

For the case $L(\eta) = 0$, it follows from Lemma 4.6 that there is a finite integer $N \ge 1$ such that the problem (P_η) has at least $2N+1$ distinct type I solutions. The theorem follows. □

In the following, we shall study the exact number of type I solutions for each $\eta \in (0, \tilde{\eta}]$.

Lemma 4.11. *Suppose that (w_i, v_i) is the solution of (S_{η_i}) with $(w_i(y_i), v_i(y_i)) = (d_i, 0)$ and $(w_i, v_i) \in (-\infty, \ln(a)) \times (0, +\infty)$ for $y \in (y_i, z_i)$, where $\eta_2 > \eta_1 > 0$, $\ln(a) > d_1 > d_2$, and z_i satisfies that $w(z_i) = \ln(a)$ for $i = 1, 2$. Then there does not exist $t_i \in [y_i, z_i]$, $i = 1, 2$, such that $(w_1(t_1), v_1(t_1)) = (w_2(t_2), v_2(t_2))$. Moreover, if we view the w_i as a function of v for $i = 1, 2$, then we have $w_1 > w_2$ on $[0, V]$, where $V := v_1(z_1)$.*

Proof. Since $v_i > 0$ for $y \in (y_i, z_i]$, we can view w_i as a function of v, and so w_i satisfies the following initial value problem

$$\frac{dw}{dv} = \frac{v}{ae^w - e^{2w} + \eta_i e^{-w} v}, \tag{4.21}$$

$$w(0) = d_i \tag{4.22}$$

for $i = 1, 2$. Note that $d_1 > d_2$. For contradiction, we assume that there exists $v_0 \in (0, V]$ such that $w_1 > w_2$ on $[0, v_0)$ and $w_1(v_0) = w_2(v_0)$. Therefore, $w_1'(v_0) \le w_2'(v_0)$. Note that $ae^w - e^{2w} + \eta_i e^{-w} v > 0$ for $(w, v) \in (-\infty, \ln(a)] \times (0, +\infty)$ and $i = 1, 2$. Then we have

$$\begin{aligned} w_1'(v_0) &= \frac{v_0}{ae^{w_1(v_0)} - e^{2w_1(v_0)} + \eta_1 e^{-w_1(v_0)} v_0} \\ &> \frac{v_0}{ae^{w_2(v_0)} - e^{2w_2(v_0)} + \eta_2 e^{-w_2(v_0)} v_0} \\ &= w_2'(v_0), \end{aligned}$$

a contradiction. This completes the proof. □

Lemma 4.12. *Given $\eta_0 > 0$, there exists $\delta > 0$ and $W_1 < \ln(a)$ such that for any $\eta \in (\eta_0 - \delta, \eta_0 + \delta)$, the solution $(w(y), v(y))$ of the system (S_η) with the initial condition $w(0) = w_1 < W_1$ and $v(0) = 0$, intersects the line $\{w = \ln(a),\ v > 0\}$ at $(\ln(a), v(z))$ for some $z > 0$ such that $v > 0$ on $(0, z]$ and $E(z) = [v(z)]^2/2 + G(\ln(a)) > 0$.*

Proof. Recall that for any fixed $\eta > 0$, the solution $(w(y), v(y))$ of the system (S_η) with the initial condition $(w(0), v(0)) = (r, 0)$ for some $r < \ln(a)$ intersects the line $\{w = \ln(a),\ v > 0\}$ at $(\ln(a), v(z)) = (\ln(a), s)$ for some $s > 0$ and $z > 0$ such that $v > 0$ on $(0, z]$. It is clear that s is a decreasing function of r and $s = s(r)$ maps $(-\infty, \ln(a))$ onto $(0, +\infty)$. Therefore, given $\eta_0 > 0$ and $s_0 > 0$ with $s_0^2/2 + G(\ln(a)) > 1/2$, there exists $r_0 < \ln(a)$ such that the solution $(w_0(y), v_0(y))$ of the system (S_η) with the initial condition $(w_0(0), v_0(0)) = (r_0, 0)$ intersects the line $\{w = \ln(a),\ v > 0\}$ at $(\ln(a), v_0(z_0)) = (\ln(a), s_0)$ for some $z_0 > 0$ and $v > 0$ on $(0, z_0]$. Using the standard theory of continuous dependence on initial condition and parameter and noting that the decreasing property of the function s, we can find a $\delta > 0$ and $W_1 < \ln(a)$ such that the conclusion of the lemma holds. The proof is completed. □

Lemma 4.13. *Suppose that (w_i, v_i) is a solution of (S_{η_i}) with $(w_i(y_i), v_i(y_i)) = (\ln(a), d_i)$ and $(w_i, v_i) \in (\ln(a), +\infty) \times (0, +\infty)$ for $y \in (y_i, z_i)$, where $\eta_2 > \eta_1 > 0$, $d_2 > d_1 > 0$, and $v_i(z_i) = 0$ for $i = 1, 2$. Then there does not exist $t_i \in [y_i, z_i]$ for $i = 1, 2$ such that $(w_1(t_1), v_1(t_1)) = (w_2(t_2), v_2(t_2))$. Moreover, if we view the v_i as a function of w for $i = 1, 2$, then we have $v_2 > v_1$ on $[\ln(a), W]$, where $W := w_1(z_1)$.*

Proof. Since $v_i > 0$ for $y \in [y_i, z_i)$, we can view v_i as a function of w, and so v_i satisfies the following initial value problem

$$\frac{dv}{dw} = \eta_i e^{-w} + \frac{ae^w - e^{2w}}{v}, \tag{4.23}$$

$$v(\ln(a)) = d_i, \tag{4.24}$$

for $i = 1, 2$. Note that $d_2 > d_1$. For contradiction, we assume that there exists $w_0 \in (\ln(a), W]$ such that $v_2 > v_1$ on $[\ln(a), w_0)$ and $v_1(w_0) = v_2(w_0)$. Therefore, $v_1'(w_0) \geq v_2'(w_0)$. On the other hand, from (4.23) it follows that

$$\begin{aligned}
& (v_2 - v_1)(w_0^-) - (v_2 - v_1)(\ln(a)) \\
= & \int_{\ln(a)}^{w_0^-} \{(\eta_2 - \eta_1)e^{-w} + (ae^w - e^{2w})(1/v_2 - 1/v_1)\} dw \\
> & \int_{\ln(a)}^{w_0^-} (\eta_2 - \eta_1)e^{-w} dw \\
> & \ 0,
\end{aligned}$$

where we have used the facts that $v_2 > v_1 > 0$ on $[\ln(a), w_0)$ and $ae^w - e^{2w} < 0$ for $w \in (\ln(a), w_0]$. This implies that $v_1(\ln(a)) > v_2(\ln(a))$, a contradiction. The lemma follows. □

Lemma 4.14. *Suppose that (w_i, v_i) is the solution of (S_{η_i}) with $(w_i(y_i), v_i(y_i)) = (d_i, 0)$ and $(w_i, v_i) \in (\ln(a), +\infty) \times (-\infty, 0)$ for $y \in (y_i, z_i)$, where $\eta_2 > \eta_1 > 0$, $d_2 > d_1 > \ln(a)$, and z_i satisfies that $w(z_i) = \ln(a)$ for $i = 1, 2$. Then there does not exist $t_i \in [y_i, z_i]$ for $i = 1, 2$*

such that $(w_1(t_1), v_1(t_1)) = (w_2(t_2), v_2(t_2))$*. Moreover, if we view the* w_i *as a function of* v *for* $i = 1, 2$*, then we have* $w_2 > w_1$ *on* $[V, 0]$*, where* $V := v_1(z_1)$*.*

Proof. Proceed as in Lemma 4.11 □

Combining Lemma 4.9 and Lemmas 4.11-4.14, the following lemma follows.

Lemma 4.15. *Suppose that* (w_i, v_i) *is the solution of* (S_{η_i}) *with* $(w_i(y_i), v_i(y_i)) = (0, d_i)$ *for* $i = 1, 2$*, where* $\eta_2 > \eta_1 > 0$*,* $0 > d_1 > d_2$*, and* y_1, y_2 *are real numbers. If* (w_2, v_2) *goes around the point* $(\ln(a), 0)$ *clockwise and has at least* m *intersection points with the line* $\{w = \ln(a),\ v < 0\}$ *for* $y \in [y_2, R_2)$*, then so does for* (w_1, v_1) *in* $[y_1, R_1)$*.*

We are ready to prove one of the main theorem of this paper as follows.

Theorem 2. *There exists a sequence of positive numbers*

$$\tilde{\eta} = \eta_0 > \eta_1 > \eta_2 > \cdots > \eta_m > \cdots > 0$$

such that (P_η) *has exactly* $2m$ *type I solutions, if* $\eta \in (\eta_m, \eta_{m-1})$*; and* (P_η) *has exactly* $2m+1$ *type I solutions, if* $\eta = \eta_m$*.*

Proof. We shall denote the solution of (Q_η) by $w_\eta(y)$. Also, set $v_\eta(y) := w'_\eta(y)$. First, we note that $w_{\tilde{\eta}}$ is the only type I solution of $(P_{\tilde{\eta}})$.

Next, given a fixed $\eta < \tilde{\eta}$. Let $L(\eta) := \lim_{y \to R^-} E(y; \eta)$, where $E(y; \eta) = [w'_\eta(y)]^2/2 + G(w_\eta(y))$. Then either $L(\eta) = 0$ or $L(\eta) = \infty$. Consider the solution $(w_{\tilde{\eta}}, v_{\tilde{\eta}})$ of $(S_{\tilde{\eta}})$. Using Lemma 4.9 and the uniqueness of type I solution of (Q_η) among all positive numbers η, the solution (w_η, v_η) of (S_η) never intersect $(w_{\tilde{\eta}}, v_{\tilde{\eta}})$ and (w_η, v_η) hits the w-axis at $(w_0, 0)$ for some $w_0 < \ln(a)$. By theory of continuous dependence and Lemma 4.12, there exists $\delta > 0$ and $W_1 < \ln(a)$ such that for all $\eta \in (\tilde{\eta} - \delta, \tilde{\eta})$, the solution (w_η, v_η) of (Q_η) will hit the w-axis at $(w_0, 0)$ for some $w_0 < W_1$, then go into the region $(-\infty, \ln(a)) \times (0, +\infty)$, and intersect the line $\{w = \ln(a),\ v > 0\}$ at the point $(0, v_0)$ for some $v_0 > 0$ such that

$$E(y_0; \eta) = v_0^2/2 + G(\ln(a)) > 0,$$

where y_0 satisfies that $v(y_0) = v_0$. Therefore, (w_η, v_η) will hit the line $\{w = \ln(a),\ v < 0\}$ exactly two times and there exists $y_1 > y_0$ such that $(w_\eta, v_\eta) \in (-\infty, \ln(a)) \times (-\infty, 0)$ for all $y \in (y_1, R_\eta)$ and $(w_\eta, v_\eta) \to (-\infty, -\infty)$ as $y \to R_\eta^-$, and so $L(\eta) = +\infty$. Define B_1 to be the set of all positive numbers $\eta \in (0, \tilde{\eta})$ such that (w_η, v_η) hits the line $\{w = \ln(a),\ v < 0\}$ exactly two times and $L(\eta) = +\infty$. Note that from Lemma 4.9 and Lemmas 4.11-4.14 it follows that if $r \in B_1$, then $\eta \in B_1$ for all $\eta \in [r, \tilde{\eta})$. By Remark 4.2 and the above discussion, the set B_1 is nonempty and bounded below. Therefore, $\eta_1 := \inf B_1$ exists. Furthermore, (w_{η_1}, v_{η_1}) hits the line $\{w = \ln(a),\ v < 0\}$ exactly two times and $(w_{\eta_1}, v_{\eta_1}) \to (-\infty, 0)$ as $y \to +\infty$. By a similar argument, we can find a sequence of positive numbers

$$\tilde{\eta} > \eta_1 > \eta_2 > \cdots > \eta_m > \cdots > 0$$

such that

(1) (w_η, v_η) hits the line $\{w = \ln(a),\ v < 0\}$ exactly $m + 1$ times and $(w_\eta, v_\eta) \to (-\infty, -\infty)$ as $y \to R_\eta^-$ for all $\eta \in (\eta_m, \eta_{m-1})$.

(2) (w_{η_m}, v_{η_m}) hits the line $\{w = \ln(a),\ v < 0\}$ exactly $m+1$ times and $(w_{\eta_m}, v_{\eta_m}) \to (-\infty, 0)$ as $y \to +\infty$.

Now, we turn to the problem (P_η) for a fixed $\eta \in [\eta_m, \eta_{m-1})$. Firstly, we suppose that $\eta \in (\eta_m, \eta_{m-1})$. Let $y_0 = 0$ and y_i be the y-value of the ith intersection point of the trajectory (w_η, v_η) with the line $\{w = \ln(a),\ v < 0\}$, $i = 1, \ldots, m$, such that $-\eta/a = w'_\eta(y_0) > w'_\eta(y_1) > \cdots > w'_\eta(y_m)$, where we have used the fact that the energy is increasing. Note that $1 \le m < \infty$. Consider the set $\{\alpha \in (w'_\eta(y_m), w'_\eta(y_{m-1}))\}$ such that the trajectory starting from the point $(\ln(a), \alpha)$ has the property that $L = \infty$. Then it is easy to see that the supremum α_0 of this set has the property that the trajectory (w_0, v_0) starting from the point $(\ln(a), \alpha_0)$ is of type I with $v_0(y) < 0$ for all $y \ge 0$. It is easy to see that the set $\{(w_0(y), v_0(y)) | y \in R\}$ intersect the initial curve Γ at exactly $2m$ points. Therefore, from Lemma 4.10 it follows that we have exactly $2m$ distinct solutions of (P_η) such that they are all of type I.

For the case $\eta = \eta_m$, it follows that the set $\{(w_\eta(y), v_\eta(y)) | y \in R\}$ intersect the initial curve Γ at exactly $2m+1$ points. Therefore, from Lemma 4.10 it follows that we have exactly $2m+1$ distinct solutions of (P_η) such that they are all of type I. The theorem follows. □

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