Rotating spirals of curvature flows: a center manifold approach

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1. Introduction and main results

Rigidly rotating spiral wave patterns of Archimedean shape are a prominent phenomenon in the spatio-temporal evolution of planar reaction-diffusion systems. A first analysis by Wiener and Rosenblueth [WR46] was motivated by electrical waves in heart tissue. For experimental evidence of Archimedean spirals in Belousov- Zhabotinsky systems and other excitable chemical reaction media see for example [Win72], [Win87], [Win01], [BE93], [UUM93], [RKPD94], [RNOE93]. See also the recent survey [Mik03] and the references there. For related fluid convection experiments see [PB96].

One mathematical approach, initiated already by Wiener, aims at a geometric description of the spatio-temporal dynamics of the sharp wave fronts which seem to emanate from a highly focused "core" region or "tip" and which take the form of an Archimedeam spiral, far away form the tip. This approach, where the wave front is represented simply by a planar curve $z = z(s) \in \mathbb{R}^2$, parameterized over arc length $s \geq 0$, is sometimes called *kinematic theory of spiral waves*, and has been developed on a mostly formal level. See for example [KT92], [Kee92], [MZ91], [Mik03] and the references there. Typically the dynamics of the curve $z(s)$ is then described by a curvature driven flow

$$u = U(\kappa), \tag{1.1}$$

where u indicates the normal velocity and κ denotes the signed curvature of the oriented curve $z(s)$. Here we define the *signed curvature* κ along the negative of the left normal unit vector $\mathbf{n}(s)$ to the unit tangent $z'(s)$. In complex notation for $z(s) \in \mathbb{R}^2 \cong \mathbb{C}$ we therefore have

$$z'' = -\kappa\mathbf{n} = -i\kappa z'. \tag{1.2}$$

We will only consider the case of nonvanishing wave speeds $c := U(0) \neq 0$ of planar wave fronts, in the normal direction $\mathbf{n}(s)$. We will first, and mostly, consider the case

$$c := U(0) > 0. \tag{1.3}$$

Our sign convention for κ in (1.2) is chosen to later accomodate left rotating Archimedean spirals which are winding outwards in clockwise direction, as are observed in the Belousov-Zhabotinsky medium for monotonically decreasing U; see (1.4) below and the proof of Lemma 4.1.

By a *spiral curve* we quite generally mean a non-circular planar curve $z = z(s), s \geq 0$, with everywhere non-zero curvature κ and with uniformly bounded normal velocity u. The special case

$$u = c - D\kappa, \tag{1.4}$$

where the normal velocity is simply an affine linear function of curvature κ, has been proposed for spiral wave descriptions in the Belousov-Zhabotinsky reaction, see for example [Kee89], [MZ91], as well as for surface waves in catalysis, see [IIY98], [Mer92]. The particular case $c = 0$ is known as curve shortening and has also received significant attention; see for example [Ang90], [Ang91], [GH86].

Rather than by its position z, we may also describe any planar curve $z(s)$ of class C^2, alternatively, by specifying its curvature $\kappa = \kappa(s)$ as a function of arc length s. The advantage of such a description lies in the obvious fact that different curves $z(s)$ are represented by the same curvature function $\kappa(s)$ if, and only if, they differ by a rigid planar proper Euclidean motion. Thus $\kappa(s)$ distills the pure "shape" out of $z(s)$. Moreover, a rigidly rotating temporal evolution $z(t, s)$ of wave front curves will be represented by a single t-independent curvature function $\kappa(s)$. Under the curvature flow (1.1), in fact, the resulting nonlinear parabolic partial differential equation for $z(t, s)$ reduces to a (nonlocal) ordinary differential equation, for a rotating wave $\kappa(s)$. Specifically

$$u'(s) + \kappa(s) \int_0^s \kappa u ds = \omega, \tag{1.5}$$

where ω denotes the rotation frequency and $u = U(\kappa)$ may be nonlinear. For the convenience of our reader, we derive (1.5) in section 2 below. See also the companion paper [FGT03] for an alternative, estimate-based treatment of (1.5), as well as the earlier references [Mer92], [IIY98].

Equation (1.5) for rigidly rotating curves $z(s)$, under curvature flow, in fact suggests to consider the normal velocity u, rather than curvature κ, as the dependent variable. We therefore invert the constitutive relation (1.1):

$$\kappa = \Gamma(u) \tag{1.6}$$

with $\Gamma := U^{-1}$. For Γ and its continuous derivative Γ_u, we assume

$$\Gamma \in C^4, \quad \Gamma(c) = 0 \text{ for some } c > 0, \text{ and } \Gamma_u(u) \neq 0 \text{ for all } u. \tag{1.7}$$

Nonmonotone, and hence noninvertible velocity dependence $U(\kappa)$ on curvature has been proposed by [ZMM98], but is excluded in our present analysis. Because spiral curves possess nonzero curvature κ, everywhere, assumption (1.7) restricts our search to global solutions $u = u(s)$ of (1.5) which satisfy

$$u(s) \neq c \text{ for all } s, \quad \sup |u(s)| < \infty, \quad \text{and } u' \not\equiv 0. \tag{1.8}$$

Indeed, $u' \not\equiv 0$ avoids circles $z(s)$ of constant curvature, and the second condition avoids wave fronts of unbounded speeds. In Proposition 3.4 below we will in fact show that the uniform boundedness of $|u|$ is a consequence of the other assumptions, for $u < c$.

In polar coordinates $z = r\exp(i\varphi)$ for curves $z = z(s) \in \mathbb{R}^2 \cong \mathbb{C}$, we can clearly distinguish two different types of spirals, depending on the asymptotic behavior of $(r(s), \varphi(s))$ for arc length $s \to +\infty$. We speak of an *Archimedean spiral*, if

$$\varphi \to -\infty, \quad \frac{dr}{d\varphi} \to a \neq 0, \tag{1.9}$$

possibly after a reflection $z \mapsto \overline{z}$. Note that the limiting value $2\pi a$ can be viewed as the asymptotic radial wave length in the far field $r \to \infty$. In contrast, we speak of a *limit circle spiral*, if

$$\varphi \to -\infty, \quad r \to R > 0, \tag{1.10}$$

possibly after a reflection and a translation $z \mapsto z + z_0$. In particular, as we will see below, any rotating Archimedean spiral necessarily satisfies

$$u(s) \to c \quad \text{and} \quad \kappa(s) \to 0 = \Gamma(c) \tag{1.11}$$

for $s \to \infty$, in view of the constitutive relation (1.6). Similarly, a limit circle spiral must satisfy

$$u(s) \to 0 \quad \text{and} \quad |\kappa(s)| \to \frac{1}{R} = \Gamma(0). \tag{1.12}$$

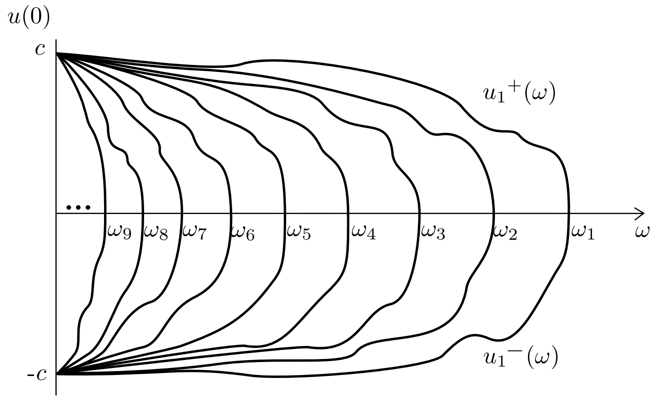

Figure 1.1. *A global bifurcation diagram of rotating Archimedean spirals with angular wave speeds $\omega > 0$ and tip velocity $u(0)$.*

We now formulate our main theorems 1.1 and 1.2, which show that the above two types of spirals are the only ones which can occur as rigidly rotating solutions to the curvature flow (1.1), (1.5) with nonzero angular speed ω.

Theorem 1.1. *Let the constitutive relation $\kappa = \Gamma(u)$ of (1.6) satisfy (1.7), that is, $\Gamma(c) = 0$ for some $c > 0$, and $\Gamma_u(u) \neq 0$ for all u. Consider positive angular rotation speeds ω and monotonically decreasing constitutive laws,*

$$\omega > 0 > \Gamma_u. \tag{1.13}$$

Then there exists a strictly decreasing sequence $\omega_1 > \omega_2 > \cdots \searrow 0$ and associated functions $u_n^\pm(\omega)$, for $0 < \omega \leq \omega_n$, with the following property. A C^3-solution $z = z(s)$, $s \geq 0$, of the curvature flow equation (1.5),(1.6), for which the normal velocity $u = u(s)$ satisfies spiralling assumption (1.8) with $u < c$, exists if, and only if, the tip velocity $u(0)$ and the derivative $u'(0)$ at the tip $s = 0$ satisfy

$$u(0) \in \{u_n^-(\omega), u_n^+(\omega)\}, \quad u'(0) = \omega \tag{1.14}$$

for some positive integer n.

The associated solutions $z = z(s) = r\exp(i\varphi)$ are right winding, left rotating Archimedean spirals, in the sense of (1.9), (1.11). The tip $z(0)$ rotates on a circle of radius $\rho = |u(0)/\omega|$. The asymptotic wave length $2\pi a$ of the Archimedean spiral in the far field $r \to \infty$, in the sense of (1.9), is given by

$$2\pi a = 2\pi c/\omega. \tag{1.15}$$

The functions $\omega \to u_n^\pm(\omega)$ possess the following properties for every $n \in \mathbb{N}$:
(i) the union of the graphs of $u_n^\pm$ forms a C^3-curve, with nonvanishing curvature at $\omega = \omega_n$;
(ii) $u_n^\pm(\omega_n) = 0$, and $u_n^\pm(\omega) \to \pm c$ for $\omega \searrow 0$;
(iii) for all angular rotation speeds $0 < \omega < \omega_n$ we have the strict ordering

$$u_1^+(\omega) > u_2^+(\omega) > ... > u_n^+(\omega) > 0 > u_n^-(\omega) > ... > u_2^-(\omega) > u_1^-(\omega). \tag{1.16}$$

See Figure 1.1 for an illustration of the resulting global bifurcation diagram of rotating Archimedean spirals, and Figure 1.2(b),(c) for a numerical simulation of the Archimedean spirals $z(s) \in \mathbb{R}^2$, themselves. For a more detailed discussion of the interpretation of Figure 1.2 we refer to section 8.

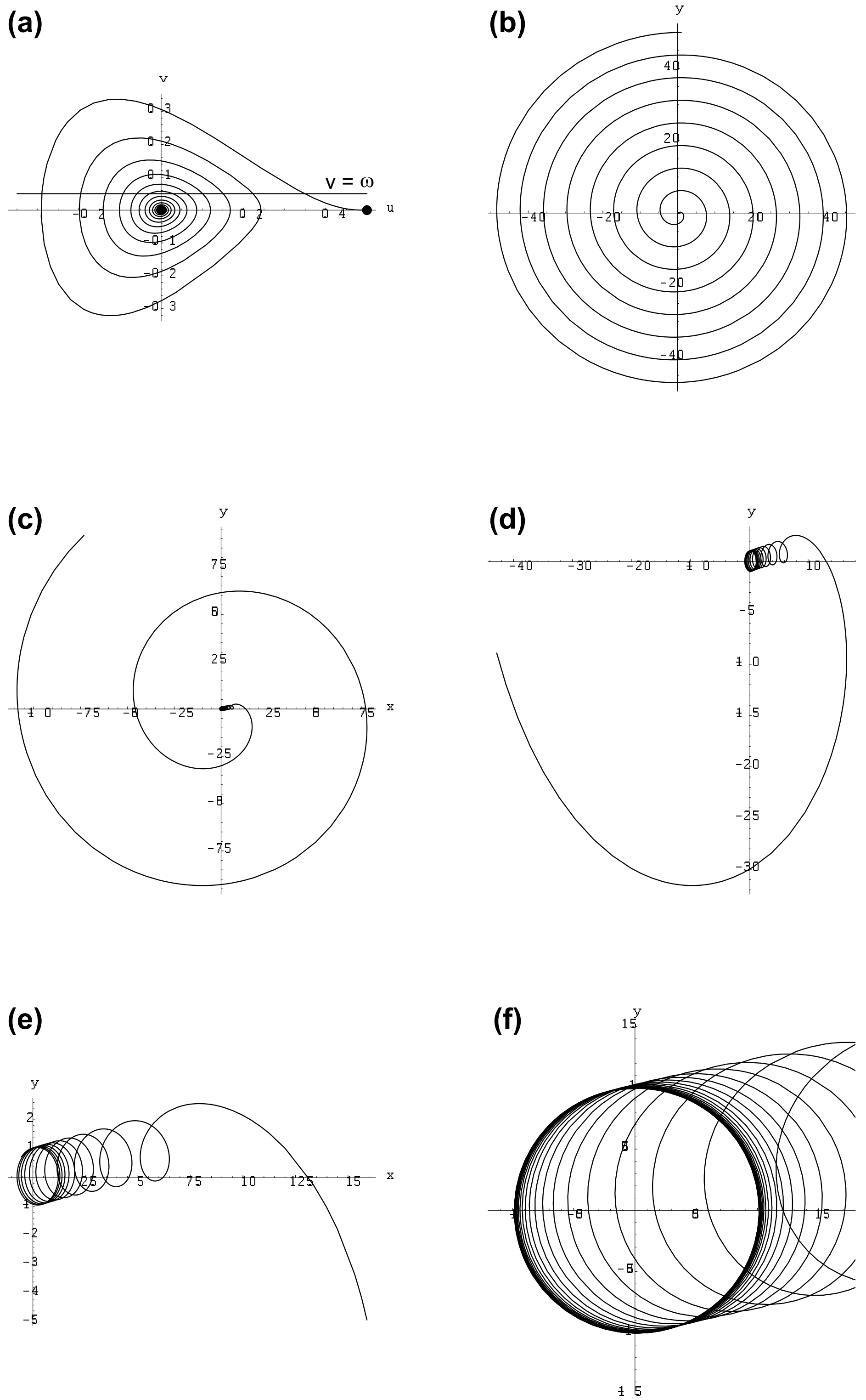

Figure 1.2. *Archimedean* $((b),(c))$ *and circle* $((d)-(f))$ *spirals* $z = z(s) = (x(s), y(s))$ *for* $\Gamma(u) = 1 - 2u,\ \omega = \pm 0.05,$ *by increasing resolution near the tip. For the underlying center manifold in the plane* $(u, v) = (u, u')$ *see (a).*

Theorem 1.2. *Under the assumptions of theorem 1.1, consider negative angular rotation speeds* ω *and monotonically decreasing constitutive laws, such that*

$$\omega < 0, \quad \Gamma_u < 0. \tag{1.17}$$

Then for any tip velocity $u(0) < c$ *there exists a unique rotating solution* $z(s), s \geq 0$*, of the curvature flow equation. The normal velocity is given by the unique bounded solution* $u(s)$ *of (1.5),(1.6) with tip derivative* $u'(0) = \omega$.

The associated solutions $z = z(s) = r\exp(i\varphi)$ *are limit circle spirals, in the sense of (1.10),(1.12). In particular, the tip* $z(0)$ *rotates on a circle of radius* $\rho = |u(0)/\omega|$. *The limit circle, for* $s \to \infty$*, in the sense of (1.10), possesses radius* $R = 1/\Gamma(0)$*, as claimed in (1.12).*

Corollary 1.3.
(i) Assume $\Gamma(c) = 0$ *for some* $c > 0$*, and consider* $u < c$*, but this time for the monotonically increasing case* $\Gamma_u > 0$*. Then Theorems 1.1, 1.2 remain valid, if we replace* $z(s)$ *by its complex conjugate* $\overline{z}(s)$*. The Archimedean spirals, for example, are then left winding and left rotating.*
(ii) If $\Gamma(c) = 0$ *for some* $c < 0$*, and* $u > c$*, then Theorem 1.1 holds for* $\omega < 0 < \Gamma_u$*, whereas Theorem 1.2 applies to* $\omega > 0$*,* $\Gamma_u > 0$*. Sign statements and inequalities involving negative* ω *and* ω_n *then hold for* $-\omega, -\omega_n$ *instead.*
(iii) If $\Gamma(c) = 0$ *for some* $c < 0$*, and* $u > c$*, then Theorem 1.1 also applies to the cases* $\omega < 0$*,* $\Gamma_u < 0$*, while Theorem 1.2 applies to* $\omega > 0 > \Gamma_u$*. In these cases, both the additional remarks of statements (i) and (ii) apply.*

The remaining paper is organized as follows. We begin with the case $\omega > 0 > \Gamma_u$ of Theorem 1.1. In section 2, we derive the integral equation formulation (1.5) for rotating waves under curvature flow, see also [Mer92]. We derive an equivalent second order formulation of (singular) Lienard pendulum equation type. In particular we propose a Lyapunov function $E = E(u, u')$ of energy type, which is singular at $u = c$ and is increasing for $\omega\Gamma\Gamma_u < 0$, but decreasing for $\omega\Gamma\Gamma_u > 0$. In section 3, we use a regularizing time transformation which exhibits spirals, in the sense of our definition (1.8), to lie on the (stable) center manifold of the trivial unstable equilibrium $(u, u') = (c, 0)$, after regularization. The energy function $E(u, u')$, however, remains singular. In particular we show, that solutions outside the center manifold do not qualify as spirals. Moreover we compute the center manifold to leading third order. In section 4, we show that solutions in the left center manifold indeed are Archimedean spirals as claimed in Theorem 1.1. The global bifurcation diagrams as given by the graphs of the functions $u_n^{\pm}(\omega)$ are derived in section 5. Section 6 reviews the results of the previous sections and proves Theorem 1.1. The necessary modifications for the proof of Theorem 1.2 are collected in section 7. In fact we use certain symmetry arguments to also treat the remaining cases mentioned in Corollary 1.3. We conclude, in section 8, by investigating the self-intersection properties of the Archimedean and the limit circle spirals $z(s)$, as established in Theorems 1.1, 1.2, and Corollary 1.3 above.

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2. Equations for rotating wave curves

In this section we briefly derive the integral equation

$$u'(s) + \kappa(s) \int_0^s \kappa(\sigma)u(\sigma)d\sigma = \omega \tag{2.1}$$

for a planar C^2-curve $z = z(s)$ in $\mathbb{R}^2 \cong \mathbb{C}$ which left rotates at constant angular speed $\omega \in \mathbb{R}$ around the origin, under curvature flow; see also [Mer92]. As before, s indicates arclength measured from the tip $s = 0$, u is normal velocity, and κ is signed curvature; see (1.2). For the present paper, we prefer an equivalent singular second order formulation of the Lienard pendulum type, which we derive in (2.13), (2.14), (2.15) below. For an associated singular energy, or Lyapunov function, see Proposition 2.2.

Proposition 2.1. *Let $z = z(s) \in \mathbb{R}^2 \cong \mathbb{C}, s \geq 0$, be a planar C^2-curve rotating at constant angular velocity ω. Then curvature $\kappa(s)$ and normal velocity $u(s)$ satisfy (2.1), as functions of arclength s.*

Proof. We recall that the arclength parametrization satisfies

$$|z'|^2 = z'\overline{z}' \equiv 1, \quad \text{and} \quad z''\overline{z}' + z'\overline{z}'' \equiv 0 \tag{2.2}$$

in complex notation. The scalar product (z_1, z_2) of $z_1, z_2 \in \mathbb{R}^2$ is given by

$$(z_1, z_2) = \mathrm{Re}(z_1\overline{z_2}) = \frac{1}{2}(z_1\overline{z_2} + \overline{z_1}z_2), \tag{2.3}$$

and $\mathbf{n} := iz'$ denotes the left unit normal to the curve $z(s)$.

The normal velocity u of the time dependent curve $Z = e^{i\omega t}z(s)$, which left rotates at constant angular speed $\omega \in \mathbb{R}$, is given by the normal projection

$$u = (Z_t, \mathbf{n}) = (i\omega e^{i\omega t}z, e^{i\omega t}iz') = \omega \mathrm{Re}(z\overline{z}') = \frac{\omega}{2}(z\overline{z}' + \overline{z}z'). \tag{2.4}$$

Using (2.2), the s-derivative u' is given by

$$u' - \omega = \omega \mathrm{Re}(z\overline{z}''). \tag{2.5}$$

At the tip $s = 0$, rigid rotation of $z(s)$ around the origin provides the additional condition

$$0 = (z, \mathbf{n}) = \mathrm{Re}(z \cdot (-i\overline{z}')) = \mathrm{Im}(z\overline{z}'). \tag{2.6}$$

Indeed the curve normal $\mathbf{n}$ must be tangent to the circle $| z(0) |= \rho$ of the tip motion. Hence $z(0)$ and the tangent $z'(0)$ are parallel:

$$|z(0)| = \rho = |\mathrm{Re}(z\overline{z}')|. \tag{2.7}$$

For the normal velocity $u(0)$ at the tip

$$u(0) = \omega \mathrm{Re}(z\overline{z}'), \tag{2.8}$$

two cases arise. If z' points out of the tip circle of radius $\rho = Re(z\overline{z}') \geq 0$, this implies $u(0) = \omega\rho$. If z' points inside the tip circle, however, then $\rho = -Re(z\overline{z}') \geq 0$ and $u(0) = -\omega\rho$ possesses the opposite sign, due to our choice of curve orientation.

We finally recall our definition (1.2) of signed curvature κ. By (2.2) we can therefore write

$$\kappa = iz''/z' = iz''\overline{z}' = -\mathrm{Im}(z''\overline{z}'), \tag{2.9}$$

and κ is of course real.

With these preparations we can now calculate

$$
\begin{aligned}
\int_0^s \kappa u ds &= \int_0^s iz''\overline{z}' \cdot \frac{\omega}{2}(z\overline{z}' + \overline{z}z')ds \\
&= \frac{i\omega}{2}\int_0^s (-\overline{z}''z'z\overline{z}' + z''\overline{z}'\overline{z}z')ds \\
&= \omega \mathrm{Im}\left(\int_0^s \overline{z}''zds\right) \qquad (2.10)\\
&= \omega \mathrm{Im}\left(\left[\overline{z}'z\right]_0^s - \int_0^s \overline{z}'z'ds\right) \\
&= \omega \mathrm{Im}(\overline{z}'(s)z(s)).
\end{aligned}
$$

Besides the defining equations (2.4), (2.9) for u, κ, we have freely used (2.2) and integration by parts here. Not that the imaginary part of $\overline{z}'z' \equiv 1$ vanishes. At the tip $s = 0$, moreover, $\mathrm{Im}(\overline{z}'z) = 0$ by (2.6).

To complete the proof of the integral curvature equation (2.1), we use (2.9), (2.10), (2.2), and recall (2.5) to obtain

$$
\begin{aligned}
\kappa(s)\int_0^s \kappa u ds &= \omega\kappa \mathrm{Im}(\overline{z}'z) \\
&= \frac{\omega}{2i}(iz''\overline{z}')(\overline{z}'z - z'\overline{z}) \\
&= -\frac{\omega}{2}(\overline{z}''z + z''\overline{z}) \qquad (2.11)\\
&= -\omega \mathrm{Re}(z\overline{z}'') \\
&= \omega - u'.
\end{aligned}
$$

This proves (2.1) and the proposition. □

To rewrite (2.1) as a pendulum equation we first note that $u \in C^2$, by (2.1), for C^2-curves z rotating under curvature flow, with $\kappa = \Gamma(u)$ a C^1-function of u. Differentiating (2.1) with respect to s and using (2.1) to replace the integral term, we thus arrive at

$$
u'' + \frac{\kappa'}{\kappa}(\omega - u') + \kappa^2 u = 0. \qquad (2.12)
$$

Substituting $\kappa = \Gamma(u)$, this is equivalent to the singular second order pendulum equation of Lienard type

$$
u'' + \frac{\Gamma_u}{\Gamma}(\omega - u')u' + \Gamma^2 u = 0. \qquad (2.13)
$$

The singularity arises in the plane wave limit $u = c$, of course, where $\kappa = \Gamma(c) = 0$ by assumption (1.7).

As is standard, we may rewrite (2.13) as a singular first order system

$$
\begin{aligned}
u' &= v \qquad (2.14)\\
v' &= -\frac{\Gamma_u}{\Gamma}(\omega - v)v - \Gamma^2 u. \qquad (2.15)
\end{aligned}
$$

For later analysis it will be most convenient to study (2.14), (2.15) after multiplication by the Euler multiplier Γ. If we introduce a new, non-arclength parameter τ such that

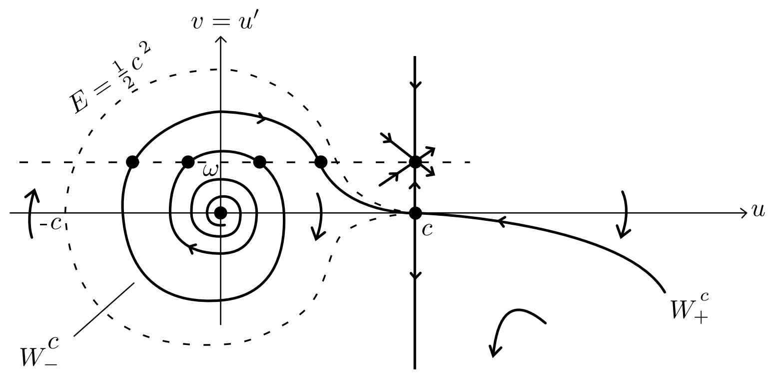

Figure 2.1. *Phase portrait of the curvature flow systems (2.14)-(2.15), (2.16)-(2.17), (3.1)-(3.2), with respect to increasing arclength s.*

$\dot{} = \frac{d}{d\tau} = \Gamma(u(s))\frac{d}{ds} = \Gamma \cdot {}'$, then the singular rotating wave system (2.14), (2.15) takes the regular form

$$\dot{u} = \Gamma v \tag{2.16}$$

$$\dot{v} = -\Gamma_u(\omega - v)v - \Gamma^3 u. \tag{2.17}$$

Specifically our parameter transformation reads

$$s = \int_0^\tau \Gamma(u(\tau))d\tau \tag{2.18}$$

in the new variables $u(\tau)$. Note the sign reversal in τ for negative Γ. Below we use the same letter to denote $u = u(\tau)$ and, barely avoiding confusion,

$$v = v(\tau) = u'(s) = \dot{u}(\tau)/\Gamma(u). \tag{2.19}$$

Whatever parametrization we use, let us consider the following singular energy or Lyapunov function

$$E = E(u, u') = E(u, v) := \frac{1}{2}(v/\Gamma(u))^2 + \frac{1}{2}u^2. \tag{2.20}$$

Proposition 2.2. *Along solutions $(u, u') = (u, v)$ of (2.13)-(2.17) the energy function E of (2.20) satisfies the monotonicity property*

$$E' = -\omega\Gamma_u v^2/\Gamma^3 \tag{2.21}$$

and therefore is a Lyapunov function as long as $\omega\Gamma\Gamma_u \neq 0$. Note how the angular rotation speed ω plays the role of a damping parameter in (2.21). Equivalently to (2.20) we have

$$\dot{E} = -\omega\Gamma_u(v/\Gamma)^2. \tag{2.22}$$

Proof. Differentiating (2.20) with respect to arclength s and using (2.14)-(2.18), the monotonicity property (2.21), (2.22) is immediate. $\square$

3. Center manifold and boundedness

In this section we study the center manifold W^c of the trivial plane wave equilibrium $(u, v) = (c, 0)$ of the regularized rotating wave system

$$\dot{u} = \Gamma v \tag{3.1}$$

$$\dot{v} = -\Gamma_u(\omega - v)v - \Gamma^3 u \tag{3.2}$$

which was derived in (2.16), (2.17). For a phase portrait of (3.1), (3.2), alias (2.16), (2.17) and (2.14), (2.15), see Figure 3.1. See [Van89], [CH82] for a background on center manifolds. In particular we compute the leading third order term of W^c, in Lemma 3.1. In Lemmas 3.2, 3.3, we then describe the central role of the center manifold W^c for our global study of bounded spirals, in the sense of definition (1.8). Throughout we assume that (1.7) holds: $\Gamma \in C^4$, $\Gamma(c) = 0$ for some $c > 0$, and $\Gamma_u \neq 0$. For definiteness we consider the case

$$\Gamma_u < 0 < \omega \tag{3.3}$$

in the present section.

Lemma 3.1. *Under assumptions (1.7), (3.3), the trivial plane wave equilibrium $(u, v) = (c, 0)$ of (3.1), (3.2) possesses a center manifold W^c: $v = v^c(u)$. In the left half-space $u < c$, this center manifold is unique and consists of all initial conditions $(u, v) = (u_0, v_0)$, at $\tau = 0$, for which*

$$\lim_{\tau \to +\infty} (u, v)(\tau) = (c, 0). \tag{3.4}$$

We call this part of the center manifold W^c_-. Locally near $(c, 0)$, the center manifold W^c_{loc} is of differentiability class C^3 and depends C^3 on the parameter $\omega > 0$. The manifold W^c_{loc} is a graph over u and possesses an expansion

$$v^c(u) = \alpha(c - u)^3 + \cdots \tag{3.5}$$

with the nonzero coefficient

$$\alpha = \Gamma_u(c)^2 \cdot c/\omega \tag{3.6}$$

In particular, the flow on W^c is locally given by

$$\dot{u} = \Gamma(u)v^c(u) = -\frac{\Gamma_u(c)^3 \cdot c}{\omega}(c - u)^4 + \cdots \tag{3.7}$$

Globally, we observe that

$$\sup E = \frac{1}{2}c^2 \tag{3.8}$$

on $W^c \setminus \{(c, 0)\}$ holds for the Lyapunov function E of Proposition 2.2.

Proof. The vertical line $u = c$ is invariant under (3.1), (3.2). Linearization at the trivial equilibrium $(u, v) = (c, 0)$ provides the diagonal system

$$\dot{u} = 0 \tag{3.9}$$

$$\dot{v} = -\omega\Gamma_u(c)v \tag{3.10}$$

with eigenvalues 0 and $-\omega\Gamma_u(c) > 0$, by (3.3). The vertical line $u = c$ is the (strong) unstable manifold of $(c, 0)$. By [Van89], for example, system (3.1), (3.2) also possesses a one-dimensional, locally invariant, local center manifold

$$v = v^c(u) = \alpha_m(c-u)^m + \cdots \tag{3.11}$$

of class C^3, for $\Gamma \in C^4$. Here $m \geq 2$, so that W^c is tangent to the horizontal eigenspace of the zero eigenvalue.

To determine α_m, we use (3.11), the $\dot{u}$-equation of (3.1), and invariance of the center manifold to derive

$$\dot{v} = v^c_u(u)\dot{u} = v^c_u \Gamma v^c = m\alpha_m^2 \Gamma_u(c)(c-u)^{2m} + \cdots \tag{3.12}$$

On the other hand, (3.11) and the $\dot{v}$-equation of (3.2) provide

$$\dot{v} = -\Gamma_u v^c(\omega - v^c) - \Gamma^3 u = -\omega\Gamma_u(c)\alpha_m(c-u)^m + \Gamma_u(c)^3 c(c-u)^3 + \cdots \tag{3.13}$$

Comparing leading coefficients in (3.12) and (3.13), we immediately conclude

$$m = 3, \qquad \alpha = \alpha_3 = \Gamma_u(c)^2 \cdot c/\omega \tag{3.14}$$

as was claimed in (3.5), (3.6). Inserting our expression of W^c into the $\dot{u}$-equation of (3.1) we obtain (3.7). In particular sign condition (3.3) implies $\dot{u} > 0$ on the (local) center manifold $v = v^c(u)$, for $u \neq c > 0$. Because the vertical line $u = c$ is flow-invariant, the center manifold W^c never crosses $u = c$, except trivially at $v = 0$. Thus $W^c \setminus \{(c, 0)\}$ decomposes into a stable left branch

$$W^c_- = W^c \cap \{u < c\} \tag{3.15}$$

and an unstable right branch W^c_+, where $u > c$. Note that $W^c_\pm$ are single trajectories, each. Because $\dot{u} > 0$ on W^c_-, locally, solutions on W^c_- satisfy convergence property (3.4) locally, and hence globally.

To prove (3.8), $\sup E = \frac{1}{2}c^2$ on $W^c_\pm$, we note that Proposition 2.2 implies $\dot{E} > 0$ there. Inserting expansion (3.5) locally in (2.20) provides

$$E = \frac{1}{2}c^2 - c(c-u) + \cdots \tag{3.16}$$

It remains to show uniqueness of W^c_-, locally, and the characterization by convergence (3.4). This follow from Shoshitaishvili's theorem; see [Arn83]. Generalizing the Grobman-Hartman theorem, this result asserts that near $(u, v) = (c, 0)$ system (3.1), (3.2) is locally C^0-equivalent to the product flow

$$\dot{u} = \Gamma(u)v^c(u) \tag{3.17}$$

$$\dot{v} = -\omega\Gamma_u(c)v \tag{3.18}$$

of the u-flow on W^c and the linearized hyperbolic remainder v. Because the set of initial conditions with forward convergence property (3.4) is one-dimensional in (3.17), (3.18), coinciding with the part $u \leq c$ of the u-axis, and because that set corresponds to $W^c_- \cup \{(c, 0)\}$, locally, the left part W^c_- of the center manifold is locally – and hence globally – characterized by convergence property (3.4). In particular, W^c_- is unique. This proves Lemma 3.1. ☐

Lemma 3.2. *As in Lemma 3.1, assume*

$$\Gamma_u < 0 < \omega. \tag{3.19}$$

Let $u = u(s) < c, 0 \leq s < \infty$, *be any solution of the rotating curve equation (1.5), alias (2.12)-(2.15). (By sign convention (1.2),* $\Gamma_u < 0$, *and* $\Gamma(c) = 0$ *for some* $c > 0$, *this corresponds to positive* $\kappa = \Gamma(u)$ *and to right winding rotating curves* $z(s)$.*) Then* $u(s)$ *satisfies the boundedness and sign conditions (1.8) if, and only if,* $(u, v) = (u(s), u'(s))$ *lies on the left center manifold* W^c_- *as defined in Lemma 3.1.*

Proof. By Proposition 2.2 and assumption (3.19), the energy $0 \leq E(s) = E(u(s), u'(s)) = \frac{1}{2}(u'/\Gamma(u))^2 + \frac{1}{2}u^2$ is strictly increasing for $u < c$, except at the equilibrium $u = u' = 0$.

Suppose first that $(u, v) \in W^c_-$. Then $u(s) < c$ for all s; see (3.8). Obviously $|u(s)|$ is bounded, because $u(s) \to c$ for $s \to \infty$. Likewise, $u' \not\equiv 0$ because $u \not\equiv c$. This proves (1.8). Now suppose, conversely, that the boundedness and sign conditions (1.8) hold. By $u' \not\equiv 0$, our solution $(u, v) = (u(s), u'(s))$ does not coincide with the equilibrium $u = v = 0$. Consider

$$0 \leq \bar{E} := \limsup E(s) \leq \infty \tag{3.20}$$

We will distinguish the three case

$$\begin{aligned} &\text{(a)} \quad \overline{E} = 0, \\ &\text{(b)} \quad 0 < \bar{E} < \infty, \\ &\text{(c)} \quad \bar{E} = \infty. \end{aligned} \tag{3.21}$$

In case (a), the monotonicity property of E implies $(u, u') \equiv (0, 0)$, which is excluded in (1.8).

In case (b), the finite energy bound $\overline{E} < \infty$ for $E = \frac{1}{2}(v/\Gamma(u))^2 + \frac{1}{2}u^2$ provides bounds on $|u|$, and then on $|v|$. We claim that the Ω-limit set Ω in the regularized system (2.16), (2.17) then satisfies

$$\Omega = \{(c, 0)\}. \tag{3.22}$$

Indeed the construction (2.20) of the energy E and the Lyapunov property (2.22) imply that either $\dot{E} = 0$ on Ω, or else $u \equiv c$ so that E has become "singular". Boundedness $0 \leq E \leq \overline{E} < \infty$ of E implies that $u \equiv c$ implies $v \equiv 0$ in the "singular" case, proving claim (3.22). In the other case, $\dot{E} = 0$ implies $v \equiv 0$ on Ω, by (2.22). Consequently (2.17) implies $0 \equiv \dot{v} = -\Gamma^3 u$. The case $u \equiv 0$ being treated in (a), this leaves $\Gamma(u) \equiv 0$, that is $u \equiv c$ on Ω, again proving claim (3.22).

Therefore (3.22) holds and (u, v) lies on W^c_-, by Lemma 3.1. Inserting expansion (3.5) for W^c_-, we in fact see that $\overline{E} = E(c, 0) = \frac{1}{2}c^2$ in this case.

In case (c) we first claim that v can cross the u-axis at most once, when $E > \frac{1}{2}c^2$ holds for the Lyapunov function $E(u, v) = \frac{1}{2}(v/\Gamma)^2 + \frac{1}{2}u^2$ of Proposition 2.2. Clearly $v = 0$ then implies $|u| > c$. If $u < -c$, then $v = 0$ implies $v' = -\Gamma^2 u > 0$. The case $u > c$ is excluded by assumption. This proves our claim. Therefore v must be eventually positive, or else eventually negative.

If v is eventually positive, then v is bounded, as is u. Indeed $u' = v > 0$ then implies $u \geq -C$. Moreover $u < c$ by assumption, and $-C \leq u < c$ implies

$$v' = -\frac{\Gamma_u}{\Gamma} v(\omega - v) - \Gamma^2 u < 0, \tag{3.23}$$

for large enough $v > \omega$. The Lyapunov property of Proposition 2.2 then implies that $(u(s), v(s))$ converges to the equilibrium (c, ω). Indeed, convergence to $(c, 0)$ is excluded because W^c_- is confined inside the bubble $E \leq \frac{1}{2}c^2$, by (3.8). The convergence to (c, ω)

occurs at an exponential rate, in the τ-variable, by linearization of (2.16), (2.17). Therefore (2.18) implies

$$s \leq \int_0^\infty \Gamma(u(\tau))d\tau < \infty, \tag{3.24}$$

and convergence to $\kappa = \Gamma(c) = 0$ occurs at finite arclength s. This is excluded by assumption (1.8).

We now address the case of eventually negative v. Then $u' = v < 0$ makes u decrease. The bounds $-C \leq u < c$ and $\overline{E} = +\infty$ therefore imply that $|\, v\,|$ must become unbounded, eventually, by getting strongly negative. But then

$$v' \leq -av^2 \tag{3.25}$$

holds for some positive constant a. Hence $-v(s)$ blows up at some finite arclength $s_0 > 0$, proportionally to $(s_0 - s)^{-1}$. Therefore, $|u|$ is unbounded for $s \nearrow s_0$, contradicting our boundedness assumption (1.8). This concludes our analysis of case (c), and proves the Lemma. ☐

As a twin to Lemma 3.2 we now consider the backwards flow $-\infty < s \leq 0$ of the rotating wave equation (1.5). Just as Lemma 3.2 will be associated to the Archimedean spiral solutions of Theorem 1.1, the following lemma will, after a "time" reversal $s \mapsto -s$ lead to the limit circles of Theorem 1.2, via the transformations of section 7.

Lemma 3.3. *Assume $\Gamma_u < 0 < \omega$ as in (3.19) above. Let $u = u(s) < c$, $-\infty < s \leq 0$, be any nonconstant solution of the rotating curve equation (1.5), alias (2.13)-(2.15). Then $u(s)$ satisfies the boundedness and sign conditions (1.8) for $-\infty < s \leq 0$ if, and only if,*

$$\lim_{s \to -\infty} (u(s), u'(s)) = (0, 0). \tag{3.26}$$

Proof. By Proposition 2.2 and assumption (3.19), the energy $0 \leq E(s) = E(u(s), u'(s)) = \frac{1}{2}(u'/\Gamma(u))^2 + \frac{1}{2}u^2$ is strictly decreasing for $u < c$ and for decreasing $s \searrow -\infty$, except at the equilibrium $u = u' = 0$.

Obviously (3.26) implies boundedness (1.8). Suppose conversely that boundedness assumption (1.8) holds. Analogously to the proof of Lemma 3.2, (3.20) and (3.21)(a)-(c), we distinguish the three cases $\underline{E} = 0$, $0 < \underline{E} \leq \frac{1}{2}c^2$, and $\frac{1}{2}c^2 < \underline{E} < \infty$ for $\underline{E} := \underline{\lim} E(s) \geq 0$ and $s \to -\infty$.

In case (a), $\underline{E} = 0$, the conclusion

$$\lim_{s \to -\infty} u(s) = 0 \tag{3.27}$$

is obvious. Since $\Gamma(0) > 0$, by assumption (3.19) and because $c > 0$, we also conclude $u' = v \to 0$ for $s \to -\infty$, from $\underline{E} = 0$. This proves (3.26) in case (a).

We thus have to exclude cases (b) and (c). In case (b), $0 < \underline{E} \leq \frac{1}{2}\, c^2$, the trajectory $(u(s), v(s))$ enters the bounded energy bubble $\{0 \leq E \leq \frac{1}{2}c^2\}$ of the center manifold W_c^-, in backwards "time" s. Since the trivial equilibrium $(u, v) = (c, 0)$ is repelling inside this bubble, in backwards "time" direction, the u-component remains bounded away from c, uniformly:

$$u(s) \leq c - \delta \tag{3.28}$$

for some $\delta > 0$ and all $s \leq 0$. In particular the energy E remains nonsingular. Hence $E' \equiv 0$ on the α-limit set of the backwards bounded trajectory $(u, v)(s)$. Therefore the $\alpha-$limit set coincides with the unique equilibrium $u = v = 0$ inside the energy bubble $\{E \leq \frac{1}{2}c^2\}$. In particular, $\underline{\text{E}} = 0$, contradicting case (b).

In case (c), $\underline{\text{E}} > \frac{1}{2}c^2$, backwards convergence of even a subsequence $(u(s_n), v(s_n))$ to the equilibrium $(c, 0)$ is excluded by the assumption $u(s) < c$. Indeed the (unique) forward unstable manifold of $(c, 0)$ under the regularized flow (2.16), (2.17) is confined to the invariant vertical line $u = c$. By forward stability along the center manifold W^c_-, the unstable manifold coincides with the unstable set, from the left half plane $u < c$. See Figure 3.1. Similarly, the equilibrium $(u, v) = (c, \omega)$ is stable from the left and, therefore, cannot be in the α-limit set of our trajectory.

By the above arguments, bounded parts of the closed α-limit set must be bounded away from $u = c$; see (3.28). Because $E \equiv \underline{\text{E}}$ is therefore nonsingular on bounded parts of the α-limit set, such parts would have to consist of equilibria which, however, have all been excluded above. Therefore, the α-limit set is empty. With $|u|$ being bounded, this implies

$$\lim_{s \to -\infty} |v(s)| = \infty. \tag{3.29}$$

Sign inspection of $v' = -\Gamma_u v(\omega - v)/\Gamma - \Gamma^2 u$ with $\Gamma_u < 0 < \Gamma$ shows that this can only happen, in backwards "time", if $v \to +\infty$. With $|\Gamma_u/\Gamma|$ bounded below, a simple Riccati comparison argument as in Lemma 3.2 shows that this contradicts the global existence assumption in (1.8). This excludes case (c) and proves the Lemma. □

We conclude this section with a comment concerning the boundedness part

$$\sup |u(s)| < \infty \tag{3.30}$$

of our assumption (1.8). The following proposition shows, that the boundedness property (3.30) in fact holds automatically under the remaining assumptions of (1.8), if we consider solutions which are defined globally for $s \in S$, where either $S = [0, \infty)$ or $S = (-\infty, 0]$.

Proposition 3.4. *Assume $\omega > 0 > \Gamma_u$ as in (3.19) above. Let $u = u(s) < c$, $s \in S$, be any solution of the rotating curve equation (1.5), alias (2.13)-(2.15). Then*

$$\sup_{s \in S} |u(s)| < \infty, \tag{3.31}$$

both in case $S = [0, \infty)$ and in case $S = (-\infty, 0]$, provided that

$$\gamma(u) := \sup_{\tilde{u} \leq u} \left(\sqrt{\Gamma_u(\tilde{u})/\tilde{u}} \cdot \Gamma(\tilde{u})^{-3/2} \right) \tag{3.32}$$

is integrable over $u \in (-\infty, -1]$, with finite integral.

Proof. We only consider the case $S = [0, \infty)$, omitting analogous arguments for $S = (-\infty, 0]$.

In the proof of Lemma 3.2, cases (a),(b) of (3.21), we have already observed that a finite energy bound $\overline{E}$ on $E = \frac{1}{2}(v/\Gamma(u))^2 + \frac{1}{2}u^2$ implies a bound on $|u|$. Therefore suppose $\overline{E} = \infty$, as in case (3.21) (c). We have then also noticed that (u, v) remains bounded if v ever becomes positive.

It only remains to consider the case $v < 0$ for all $s \geq 0$. We can therefore assume indirectly

$$u(s) \searrow -\infty, \quad \text{for } s \to \infty. \tag{3.33}$$

Without loss of generality, then, suppose $u \leq -1$ for all $s \geq 0$. By construction of our assumption (3.32), we observe that $\gamma(u) > 0$ is increasing with u, possesses the limit $\gamma(-\infty) = 0$, and hence $\gamma(u(s))$ is decreasing to zero for $s \to \infty$.

We first consider the measurable set

$$S_- := \{s \geq 0;\ v(\omega - v) + \gamma(u)^{-2} < 0\}. \tag{3.34}$$

We claim that S_- possesses finite Lebesgue measure. Indeed $v < 0$ implies

$$-u' = -v > -\frac{\omega}{2} + (\frac{\omega^2}{4} + \gamma(u)^{-2})^{1/2} > 0 \tag{3.35}$$

on S_-. For sufficiently negative u, where the limit $\gamma(-\infty) = 0$ implies $\gamma(u)^{-2} \to +\infty$, we can therefore use (3.35) and estimate the Lebesgue measure of S_- to be finite:

$$|S_-| = \int_{S_-} ds \leq -\int_{-1}^{-\infty} \frac{du}{-v} \leq \int_{-\infty}^{-1} 2(\gamma(u)^{-2})^{-1/2} du \leq 2\int_{-\infty}^{-1} \gamma(u) du < \infty. \tag{3.36}$$

In particular, the complementary set

$$S_+ := [0, \infty) \setminus S_- = \{s \geq 0;\ v(\omega - v) + \gamma(u)^{-2} \geq 0\} \tag{3.37}$$

is nonempty. We claim next that S_+ is positively invariant, i.e., $S_+ = [s_0, \infty)$ for some $s_0 \geq 0$, and $v' \geq 0$ on S_+. Indeed

$$v(\omega - v) \geq -\gamma(u)^{-2} = \sup_{\tilde{u} \leq u}(-\tilde{u}\Gamma(\tilde{u})^3/\Gamma_u(\tilde{u})) \geq -u\Gamma(u)^3/\Gamma_u(u) \tag{3.38}$$

holds for all $s \in S_+$. Hence $\Gamma_u < 0 < \Gamma$ implies the second part of our claim:

$$v' = -\frac{\Gamma_u}{\Gamma} v(\omega - v) - u\Gamma^2 \geq 0. \tag{3.39}$$

To prove the first part, positive invariance of S_+, suppose we are ever in danger of leaving S_+ at $s = s_1$. Then

$$v(\omega - v) + \gamma(u)^{-2} = 0 \tag{3.40}$$

there. Differentiation with respect to s implies

$$\frac{d}{ds}(v(\omega - v) + \gamma(u)^{-2}) = (\omega - 2v)v' - \frac{2}{\gamma^3}\frac{d}{ds}\gamma(u(s)) \geq 0. \tag{3.41}$$

Here we have used (3.39), the monotone decay of $\gamma > 0$ and $v < 0 < \omega$. Therefore $S_+ = [s_0, \infty)$ is nonempty, positively invariant, and $v' \geq 0$ on S_+.

In conclusion, $v \nearrow \bar{v} \leq 0$ is bounded. This contradicts the underlying differential equation (2.15), by which $v < 0 < \omega$ and $u < 0$ imply

$$v' = -\frac{\Gamma_u}{\Gamma} v(\omega - v) - \Gamma^2 u \geq \Gamma^2(\gamma^2 v(\omega - v) + 1) \cdot (-u). \tag{3.42}$$

Since $\Gamma(u) > 0$ is bounded below for $u < 0$, and since $\gamma \to 0$, $u \to -\infty$ for $s \to \infty$, we see from (3.42) that $v' \to +\infty$, contradicting boundedness of v. This proves, indirectly, that v cannot remain negative for all $s \geq 0$. The proof of the proposition is therefore complete. □

4. Archimedean spirals and the left center manifold

In this section we study global solutions $u(s) < c, s \geq 0$, of the rotating curve equation (1.5) in the case

$$c > 0, \quad \omega > 0 > \Gamma_u \tag{4.1}$$

of Theorem 1.1. We show that the associated rotating curves $z = z(s)$ are Archimedean spirals in the sense of (1.9).

Lemma 4.1. *Let $u(s) < c, s \geq 0$, be a nonconstant bounded solution of (1.5), alias (2.13)-(2.15). Then $(u, u')(s)$ lies in the left center manifold W^c_- introduced in Lemma 3.1. The associated planar curve $z = z(s)$ with arc length parameter s and curvature $\kappa(s) = \Gamma^{-1}(u(s))$ is a right-winding Archimedean spiral in the sense of (1.9), (1.11). More precisely, the curve $z(s) = r(s)\exp(i\varphi(s))$ satisfies*

$$\begin{aligned} \lim u &= c \\ \lim \varphi &= -\infty \\ \lim \frac{dr}{d\varphi} &= -c/\omega \end{aligned} \tag{4.2}$$

for $s \to +\infty$, in polar coordinates (r, φ).

Proof. The solution (u, u') lies in W^c_-, by Lemmas 3.1, 3.2. In particular $\lim u(s) = c$, for $s \to +\infty$. To prove the Archimedean property (4.2), we differentiate the polar coordinates $z = r\exp(i\varphi)$:

$$z' = (r'/r + i\varphi')z. \tag{4.3}$$

By (2.4), the normal velocity u is given as

$$u = \omega Re(z\bar{z}') = \omega r r' \tag{4.4}$$

because r, φ in (4.3) are both real. Therefore $\lim u = c$ in W^c_- implies

$$\frac{1}{2}(r^2)' = rr' \to c/\omega > 0 \tag{4.5}$$

for $s \to \infty$. In particular $r \to \infty$.

Because $z = z(s)$ is parametrized by arclength s, we also have

$$1 = |z'|^2 = (r')^2 + (r\varphi')^2 \tag{4.6}$$

Dividing by $(rr')^2$, for large s, and letting $r, s \to \infty$, we obtain

$$\lim \frac{d\varphi}{dr} = \pm\frac{\omega}{c} \tag{4.7}$$

from (4.5), (4.6). Up to a sign, this proves claim (4.2) of the lemma.

To determine the sign in (4.7), we observe that outward oriented right winding spirals are characterized by slope $dr/d\varphi < 0$ and, according to our sign convention (1.2), by signed curvature $\kappa > 0$. In our case, $u \nearrow c$ approaches c from below on the left local center manifold W^c_-. Therefore $\Gamma_u < 0 = \Gamma(c)$ indeed implies $\kappa = \Gamma(u) > 0$, and hence $dr/d\varphi < 0$, for large positive s. (Alternatively, we could have computed the signs of κ and of $dr/d\varphi$ directly from our expansion (3.5), (3.6) of the center manifold.) This proves the lemma. □

5. The global bifurcation diagram for rotating Archimedean spirals

We continue under the assumptions of section 4. In particular, Lemma 4.1 exhibits the Archimedean character of all nonconstant solutions $u(s) < c, s \geq 0$, of the curve equation (1.5), alias (2.13)-(2.15). In addition, such solutions must necessarily lie in the left center manifold W_-^c of $(c, 0)$. By invariance of W_-^c, this is equivalent to

$$(u_0, v_0) \in W_-^c \tag{5.1}$$

for the tip velocity $u_0 = u(0)$ and its derivative $v_0 = v(0) = u'(0)$ with respect to arc length s. The rotation frequency ω of that rotating curve solution is then simply given by

$$v_0 = \omega \tag{5.2}$$

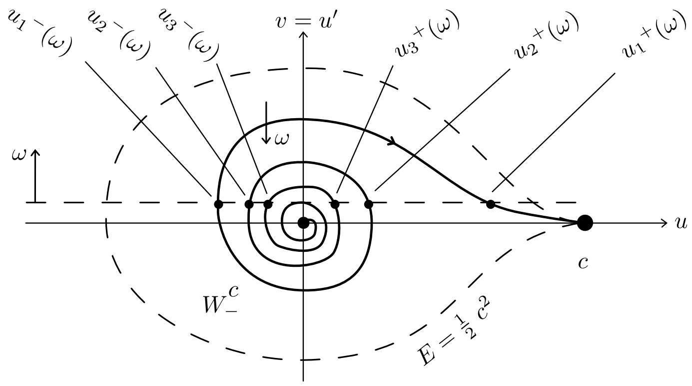

Figure 5.1. *Contracting, spiralling center manifold W_-^c, and upwards moving intersection line $v = \omega$, for increasing rotation frequencies $\omega > 0$.*

as follows trivially from (1.5) at $s = 0$; see also (2.5). Since ω enters the rotating curve system (2.14), (2.15), alias (2.16), (2.17), as a parameter, the center manifold W_-^c in fact depends on ω. In the present section we study the highly nonlinear system (5.1), (5.2) and, more specifically, the global bifurcation diagram associated to the parameter $\omega > 0$. See Figure 1.1.

The geometric intuition behind our result is simple enough; see Figure 5.1. On the one hand, an increasing rotation frequency ω strengthens the friction term $\dot{E} = -\omega\Gamma_u(v/\Gamma)^2$ of the Lyapunov function E; see (2.22). In backwards "time" s, or τ, this contracts the center manifold W_-^c, as it spirals into $u = v = 0$ for $s \to -\infty$, making it decay to zero faster and faster. The boundary condition $v_0 = \omega$ of (5.2), on the other hand, pushes the horizontal intersection line $v = \omega$ of Figure 5.1 upwards. As a result, we see the pairs $u_n^{\pm}(\omega)$ of intersection points move upwards and coalesce, at $\omega = \omega_n$ and with collision values $u_n^{\pm}(\omega_n) = 0$.

Evident as this may sound, we take a more cautious approach for the sake of complete mathematical rigor. We first parametrize the ω-dependent left center manifold W_-^c in the form $u = u(s, \omega),\ v = v(s, \omega)$, with the normalization

$$u(0, \omega) = 0 \quad \text{and} \quad u(s, \omega) > 0 \quad \text{for } s > 0. \tag{5.3}$$

Note that u, v depend C^3-differentiably on (s, ω). In general, the tip $(u, v) = (u_0, v_0)$ does not correspond to $s = 0$ in this normalization.

In Lemmas 5.1, 5.2 below, we prove that zero is a regular value of the function

$$V(s,\omega) := v(s,\omega) - \omega \tag{5.4}$$

This allows us to solve the boundary condition (5.2), alias

$$V(s,\omega) = 0, \tag{5.5}$$

for $s = s_n^\pm(\omega)$, in Lemma 5.3. The two branches $s_n^\pm(\omega)$ coalesce at their common value

$$s_n^\pm(\omega) = s_n, \text{ at } \omega = \omega_n. \tag{5.6}$$

Here $s_n \le 0$ and $\omega_n > 0$ will enumerate all solutions of the system

$$u(s_n,\omega_n) = 0, \quad v(s_n,\omega_n) = \omega_n. \tag{5.7}$$

In Lemma 5.4, finally, we will define

$$u_n^\pm(\omega) := u(s_n^\pm(\omega),\omega) \tag{5.8}$$

for $0 < \omega < \omega_n$, to see that these intersection points of the contracting, spiralling center manifold W_-^c with the upwards moving line $v = \omega$ indeed satisfy the claims of Theorem 1.1. In Figure 5.1, for example, the values $u_n^\pm(\omega_n) = 0$ indicate that the horizontal line $v = \omega_n$ just touches the center manifold spiral W_-^c from above.

Lemma 5.1. *Let $d = d(s,\omega)$ denote the Jacobi determinant of the center manifold solution $(u(s,\omega), v(s,\omega))$ and let $' = \partial_s$ denote the partial derivative with respect to arc length s. Then*

$$d' = (-\Gamma_u/\Gamma)((\omega - 2v)d + v^2), \quad \text{and} \tag{5.9}$$

$$d < 0 \text{ for all } s \in \mathbb{R},\ \omega > 0. \tag{5.10}$$

Proof. Once (5.9) is proved, $d(s,\omega)$ cannot change sign from positive to negative as s increases. It is then sufficient to prove (5.10) for large $s \to +\infty$, where the local expansion (3.5), (3.6) is valid for the center manifold W_-^c. Inserting this expansion into the definition of d yields the expansion

$$d = v^c \partial_\omega v^c = -\frac{\Gamma_u(c)^4 c^2}{\omega^3}(c-u)^6 + \cdots < 0 \tag{5.11}$$

for $s \to +\infty,\ u \to c$.

If remains to prove (5.9). Abbreviating the rotating curve system (2.14), (2.15) by $\mathbf{u}' = f(\mathbf{u},\omega)$ with $\mathbf{u} = (u,v)$, we see that

$$\begin{aligned} d &= \mathbf{u}' \wedge \mathbf{u}_\omega \\ (\mathbf{u}')' &= f_\mathbf{u}\mathbf{u}' \\ (\mathbf{u}_\omega)' &= f_\mathbf{u}\mathbf{u}_\omega + f_\omega \end{aligned} \tag{5.12}$$

A straightforward calculation then shows

$$\begin{aligned} d' &= \mathbf{u}'' \wedge \mathbf{u}_\omega + \mathbf{u}' \wedge \mathbf{u}_\omega' \\ &= ((f_\mathbf{u}\mathbf{u}') \wedge \mathbf{u}_\omega + \mathbf{u}' \wedge (f_\mathbf{u}\mathbf{u}_\omega)) + \mathbf{u}' \wedge f_\omega \\ &= tr(f_\mathbf{u}) \cdot (\mathbf{u}' \wedge \mathbf{u}_\omega) + \mathbf{u}' \wedge f_\omega \\ &= tr(f_\mathbf{u}) \cdot d + \mathbf{u}' \wedge f_\omega \end{aligned} \tag{5.13}$$

Inserting the appropriate partial derivatives of f then proves (5.9) and the lemma. ☐

Lemma 5.2. *Consider $V(s,\omega) = v(s,\omega) - \omega$, as defined in (5.4) for $s \in \mathbb{R},\ \omega > 0$. Then zero is a regular value of V. More specifically, the following two cases arise when $v(s,\omega) = \omega$.*

(i) If $u(s,\omega) \neq 0$, then $V_s = v' \neq 0$.

(ii) If $u(s,\omega) = 0$, then

$$V_s = v' = 0,\ v_\omega < 0, \quad \text{and} \quad v'' = -\Gamma^2(0)\omega < 0. \tag{5.14}$$

Proof. From (2.15) we conclude

$$v' = -\Gamma^2 u, \tag{5.15}$$

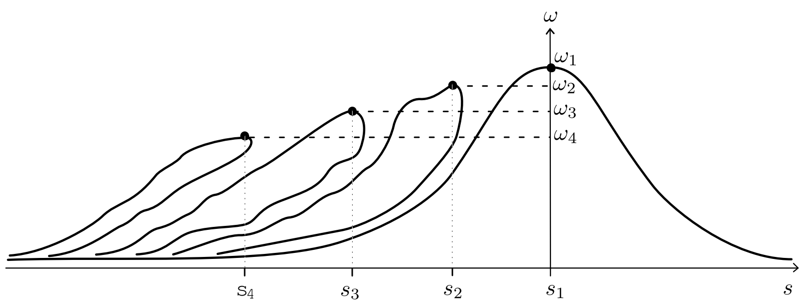

Figure 5.2. *The solution set of $v(s,\omega) = \omega$, according to Lemma 5.3.*

whenever $v = \omega$, alias $V = 0$. Since $\Gamma > 0$ for $u < c$, this proves claim (i). To prove claim (ii) it remains to show $v_\omega < 0$ for $u = 0, v = \omega$. But in this case Lemma 5.1 implies

$$0 > d = \det \begin{pmatrix} u' & u_\omega \\ v' & v_\omega \end{pmatrix} = \det \begin{pmatrix} v & u_\omega \\ 0 & v_\omega \end{pmatrix} = v v_\omega = \omega v_\omega. \tag{5.16}$$

The calculation of v'' is straightforward. This proves the lemma. ☐

From Lemma 5.2 and the implicit function theorem it is clear that the solution set of $V(s,\omega) = 0$ – the primary object of study in our present section – consists of at most countably many disjoint, embedded, planar curves. The following lemma describes these curves as pairs of graphs of functions $s = s_n^\pm(\omega),\ 0 < \omega < \omega_n$, joined at the same limits $s = s_n$ for $\omega \nearrow \omega_n$.

Lemma 5.3. *The solution set of $V(s,\omega) = 0$, for $s \in \mathbb{R},\ \omega > 0$, consists of countably many disjoint connected components. Each component is an embedded planar curve and is characterized uniquely by its maximal value ω_n of $\omega > 0$. These values are given by the solutions of the system*

$$u(s_n, \omega_n) = 0, \quad v(s_n, \omega_n) = \omega_n, \tag{5.17}$$

ordered such that

$$\omega_1 > \omega_2 > \cdots \searrow 0,\ s_n \to -\infty. \tag{5.18}$$

The remaining two pieces of each connected component of solutions $V(s,\omega) = 0$ are pairs of graphs of functions

$$s = s_n^\pm(\omega),\ 0 < \omega < \omega_n. \tag{5.19}$$

These pairs share the same limit

$$s_n = \lim_{\omega \nearrow \omega_n} s_n^{\pm}(\omega) \tag{5.20}$$

and are ordered such that

$$s_1^+ > s_1^- > s_2^+ > s_2^- > s_3^+ > s_3^- > \cdots \tag{5.21}$$

for any $\omega > 0$, as long as $s_n^{\pm}(\omega)$ are defined. See Figure 5.2.

Proof. Because zero is a regular value of V, by Lemma 5.2, the solution set of $V(s, \omega) = 0$ decomposes into (at most) countably many curves. We claim that ω must attain its maximum ω_n on each such curve. Indeed $v \to 0$ for $s \to \pm\infty$ and any $\omega > 0$. Backward convergence of W^c to $(0, 0)$, for $s \to -\infty$, in fact follows analogously to the proof of Lemma 3.3. Moreover, energy increase implies

$$E = \frac{1}{2}(v/\Gamma)^2 + \frac{1}{2}u^2 \leq \frac{1}{2}c^2 \tag{5.22}$$

on W_-^c; see Proposition 2.2 and the analysis of (3.21) (a)-(c) in the proof of Lemma 3.2. This implies a corresponding upper bound for v. These two observations allow us to conclude that $\omega = v > 0$ does attain its maximal value, say ω_n at $s = s_n$, on each such curve. Differentiating implicitly with respect to s then implies

$$0 = V_s + V_\omega \omega'(s_n) = V_s = v' = -\Gamma^2 u \tag{5.23}$$

at $(s, \omega) = (s_n, \omega_n)$, and hence $u = 0$. This proves (5.17).

Conversely, let $(s_n, \omega_n) \in \mathbb{R} \times (0, \infty)$ be any solution of system (5.17). Let $(s(\theta), \omega(\theta))$ be a parameterization of the solution curve $V(s, \omega) = 0$ through (s_n, ω_n) corresponding to $\theta = 0$. Then

$$\dot{\omega}(0) = 0, \quad \ddot{\omega} < 0, \tag{5.24}$$

where the dot $\dot{}$ indicates a derivative with respect to θ. Indeed (5.17) implies $V_s = v' = -\Gamma^2 u = 0$, at $\theta = 0$, and therefore $\dot{\omega}(0) = 0$ by implicit differentiation of $V \equiv 0$ as in (5.23). Differentiating $V \equiv 0$ twice, inserting $v' = V_s = 0$, $u = 0$, $v = \omega$, $\dot{\omega} = 0$ and, without loss of generality, $\dot{s} = 1$, we obtain

$$0 = V_{ss}\dot{s}^2 + V_\omega \ddot{\omega} = v'' - (1 - v_\omega)\ddot{\omega} = -\Gamma^2 \omega - (1 - v_\omega)\ddot{\omega}. \tag{5.25}$$

Because $v_\omega < 0$ at $u = 0$, by (8.15), this implies $\ddot{\omega} < 0$ and proves claim (5.24).

By (5.24), none of the solution curves $V(s, \omega) \equiv 0$ can possess a local minimum ω_n of ω. Moreover the local maximum ω_n of ω at $\theta = 0$ is unique, on each curve, and coincides with its global maximum. Because $\dot{\omega}(\theta) \neq 0$ for $\theta \neq 0$, we may globally and differentiably parameterize each curve over ω,

$$s = s_n^{\pm}(\omega), \tag{5.26}$$

for positive and negative θ, respectively. Since ω does not possess a local minimum and because $\omega = v \to 0$ for $s \to \pm\infty$ on any solution curve, the parameterizations (5.26) are well-defined for all $0 < \omega < \omega_n$. Without loss of generality we may also assume

$$s_n^-(\omega) < s_n^+(\omega) \tag{5.27}$$

for all ω. This proves part of the $s_n^{\pm}$-ordering (5.21). Also $s_n = \lim_{\omega \nearrow \omega_n} s_n^{\pm}(\omega)$ follows by construction, proving (5.20). The full ordering of the transverse intersections times $s = s_n^{\pm}$

of the solution $(u(s,\omega), v(s,\omega)) \in W^c_-$ with the horizontal line $v = \omega$, for any fixed ω, then reflects their ordering along the center manifold W^c_-, as it spirals backwards into $u = v = 0$.

It only remains to prove that system (5.17) indeed possesses countably many solutions (s_n, ω_n), ordered as in (5.18). With the above local analysis, the solutions (s_n, ω_n) of (5.17) are clearly isolated in $\mathbb{R} \times (0, \infty)$; hence their number is at most countable. We claim their number is infinite. Indeed, the left center manifold W^c_- consists of countably many pieces within the half plane $v > 0$. Each such piece gives rise to one solution continuum $(s_n(\theta), \omega_n(\theta))$ of $V \equiv 0$, and to one ω-maximum at (s_n, ω_n). The maximum values ω_n are disjoint, for different continua. Indeed $\omega_n = \omega_m$ implies $s_n = s_m$ because W^c_- is not a periodic solution. Therefore ω_n and ω_m belong to the same (s,ω)-continuum and hence $n = m$. We may therefore order the sequence ω_n such that

$$\omega_1 > \omega_2 > \cdots . \tag{5.28}$$

The limit $\omega_n \searrow 0$ follows because the left center manifold W^c_- can intersect the horizontal line $v = \omega$ at most finitely often, for any fixed $\omega > 0$. The limit $s_n \to -\infty$ follows because the trajectory W^c_- needs time $s_n \to -\infty$ for its n-th crossing of the positive v-axis, in backwards "time" s. This proves the lemma. □

Based on Lemma 5.3, we can now complete our analysis in the case $\omega > 0 > \Gamma_u$ of Theorem 1.1. Based on the parametrization $(u, v) = (u(s, w), v(s, w))$ of the left center manifold W^c_- and on the functions $s = s^\pm_n(\omega)$ of Lemma 5.3, see (5.3) and (5.19), we define

$$u^\pm_n(\omega) := \begin{cases} u(s^\pm_n(\omega), \omega), & \text{for} \quad 0 < \omega < \omega_n \\ 0, & \text{for} \quad \omega = \omega_n \end{cases} \tag{5.29}$$

By construction, the pairs

$$(u_0, v_0) = (u^\pm_n(\omega), \omega), \tag{5.30}$$

enumerate all tip points $(u_0, v_0) \in W^c_-$ for which $v_0 = \omega$, and thus solve the original problem (5.1), (5.2) of the present section. Here $\omega > 0$ is fixed and the pertinent indices n satisfy $\omega_n > \omega$, so that (5.30) is indeed well-defined.

Lemma 5.4. *The C^3-functions $u^\pm_n(\omega)$ defined in (5.29) above satisfy claims (i)–(iii) of Theorem 1.1.*

Proof. To prove differentiability claim (i), we only have to consider the point $\omega = \omega_n$ where the curves $u^\pm_n$ meet:

$$u^-_n(\omega_n) = 0 = u^+_n(\omega_n). \tag{5.31}$$

Because zero is a regular value of $V = v(s,\omega) - \omega$, by Lemma 5.2, and because $\omega = \omega_n$ is a local maximum of the n-th solution curve of $V \equiv 0$, by Lemma 5.3, we can C^3-parametrize the solutions of $V \equiv 0$ by s,

$$\omega = \Omega_n(s), \tag{5.32}$$

locally near $\Omega_n(s_n) = \omega_n$. In this local parametrization,

$$u(s, \Omega_n(s)) = \begin{cases} u^-_n(\Omega_n(s)), & \text{for} \quad s \le s_n \\ u^+_n(\Omega_n(s)), & \text{for} \quad s \ge s_n \end{cases} \tag{5.33}$$

$$\omega = \Omega_n(s) \tag{5.34}$$

is a C^3-parametrization of the joined curves $u_n^\pm$. Differentiating $V \equiv 0$ twice with respect to s, and using $\Omega_n'(s_n) = 0$ at the local maximum $\omega_n = \Omega_n(s_n)$, we obtain

$$\Omega'' = v''/(1 - v_\omega) < 0 \tag{5.35}$$

at $s = s_n$; see (5.25) and Lemma 5.2, (5.14). Differentiating $s \mapsto u(s, \Omega_n(s))$ once with respect to s, at $s = s_n$, we observe

$$\frac{d}{ds} u(s, \Omega_n(s)) = u' + u_\omega \Omega_n' = u' = v = \Omega_n > 0. \tag{5.36}$$

Therefore we may as well use a local parametrization $\omega = \widetilde{\Omega}_n(u)$ of the curve (5.33), (5.34) over u. Note that the curvature at the local maximum $\omega_n = \widetilde{\Omega}_n(0)$ does not vanish, also in this new parametrization, by (5.35), (5.36). This proves claim (i).

To prove claim (iii), we recall the ordering (5.21) of the intersection times $s_n^\pm(\omega)$ of the backwards center manifold trajectory $(u(s,\omega), v(s,\omega)) \in W_-^c$ with the horizontal line $v = \omega > 0$. Since $v' = -\Gamma^2 u$ on this line, we see that

$$v(s, \omega) > \omega, \quad \text{for} \ \ s_n^-(\omega) < s < s_n^+(\omega), \tag{5.37}$$

describes the arcs of W_-^c above $v = \omega$. In particular this implies $u_n^+(\omega) > 0 > u_n^-(\omega)$, for $0 < \omega < \omega_n$, by the sign of v' at these intersection points. The ordering $u_1^+(\omega) > u_2^+(\omega) > \cdots$ follows from the time ordering (5.21) of the intersection times $s_n^\pm(\omega)$ and the strict nesting of the arcs (5.37) of W_-^c above the line $v = \omega$. This proves claim (iii).

To prove claim (ii), we first observe that $u_n^\pm(\omega_n) = 0$ holds by definition (5.29). To study the limit of $u_n^\pm(\omega)$ for $\omega \searrow 0$ we note that $\omega = 0$ is the integrable limit of conserved energy $E = \frac{1}{2}(v/\Gamma)^2 + \frac{1}{2}u^2$; see Proposition 2.2. Any fixed finite number of arcs (5.37) of W_-^c above the line $v = \omega$ converges to the level line

$$E(u, v) = E(c, 0) = \frac{1}{2}c^2 \tag{5.38}$$

in this limit, see also case (b) of the proof Lemma 3.2. Indeed (2.21) of Proposition 2.2 implies

$$E' = -\omega(\Gamma_u/\Gamma)(2E - u^2) \tag{5.39}$$

along any trajectory. Applied to any level set $E(u, v) = E_0 < \frac{1}{2}c^2$, a periodic orbit for $\omega = 0$, we see from (5.39) how the energy decay along the corresponding orbit converges to zero for $\omega \searrow 0$, uniformly during any finite number n of revolutions. Because trajectories do not intersect, this proves convergence of the arcs of W_-^c above the line $v = \omega$ to the level set $E = \frac{1}{2}c^2$. A forteriori, each intersection point $(u_n^\pm(\omega), \omega) \in W_-^c$ converges to the intersection of this level set with the u-axis $v = 0$, for $\omega \searrow 0$. These latter intersections are trivially given by the u-values $u = \pm c$. This proves claim (ii) and the lemma. ☐

6. Proof of Theorem 1.1

In this section we recollect the ingredients to the proof of Theorem 1.1, as provided in sections 2-5 above.

We seek to determine all solutions $z = z(s)$ of the curvature flow equation (1.1), (1.5), (1.6) which rotate at constant angular velocity ω by moving with normal velocity u at signed curvature $\kappa = \Gamma(u)$. We consider the case $\omega > 0 > \Gamma_u$. We assume $u < c$, where $c > 0$ is the unique zero of Γ. Moreover $u(s)$ is nonconstant and bounded for $0 \leq s < \infty$ by assumption (1.8).

In section 2 we have observed that $u = u(s)$ then is a global, forward bounded solution of the singular second order system (2.13), alias (2.14), (2.15), of Lienard pendulum type. This system can also be rescaled to the form

$$\dot{u} = \Gamma_v \tag{6.1}$$
$$\dot{v} = -\Gamma_u(\omega - v)v - \Gamma^3 u \tag{6.2}$$

by a transformation $s \mapsto \tau$ from arc length s to a new "time" τ; see (2.16)-(2.18). In Proposition 2.2 we have stated an energy increasing property of system (6.1), (6.2) for the singular energy $E = \frac{1}{2}(v/\Gamma)^2 + \frac{1}{2}u^2$.

In section 3 we have studied the center manifold W^c of the degenerate equilibrium $(u, v) = (c, 0)$ of (6.1), (6.2); see Lemma 3.1. Lemma 3.2 then proves that $u(s) < c$ satisfies the boundedness condition (1.8) if, and only if, $(u, v) = (u, u')$ lies on the unique left part W^c_- of the center manifold W^c. Because dim $W^c_- = 1$, this is a single trajectory of (6.1), (6.2).

Lemma 4.1 of section 4 shows that the trajectory of W^c_- gives rise to a right-winding Archimedean spiral $z(s)$, asymptotically for $\tau, s \to +\infty$. This fact does not depend on the choice of the tip point $(u_0, v_0) \in W^c_-$, which corresponds to $s = 0$. Because bounded solutions $u(s) < c$, $s \geq 0$, must lie on W^c_-, by section 3, this proves our claims on the Archimedean property of rotating solutions to curvature flow, as made in Theorem 1.1.

Section 5 observes that necessarily $u'(0) = v_0 = \omega$ for any rotating solution of our curvature flow equation; see (5.1), (5.2) and also (2.5). This proves the second part of claim (1.14) of Theorem 1.1. In Lemmas 5.3 and 5.4 we then study the solution set $V \equiv 0$, which correspond precisely to the intersections of the trajectory W^c_- with the horizontal line $v = \omega$ in the (u, v)-plane. Alternatively, by (5.1) and (5.2), the u-values of these intersection points correspond 1-1 to the tip velocities $u(0)$ of the solutions $z(s)$ to curvature flow which rotate at constant angular velocity ω. The complete characterization of the bifurcation diagram of these u-values for varying positive parameter ω, in Lemma 5.4, therefore completes the proof of Theorem 1.1. ☐

Remark 6.1. We repeat and emphasize that the boundedness assumption (1.8) in Theorem 1.1 can be weakened, for example according to Proposition 3.4. There it was proved that solutions $u(s) < c$, $s \geq 0$, are uniformly bounded, a priori, provided that $\Gamma(u)$ satisfies a growth restriction expressed by integrability of the function γ defined in (3.32). The integrability condition holds, for example, for any at least algebraic growth $\Gamma(u) \sim |u|^p$, $p > 0$, for $u \to -\infty$. The integrability condition fails to hold, for example, for only logarithmic growth $\Gamma(u) \sim (log|u|)^p$ with $p \leq \frac{1}{2}$.

7. Proofs of Theorem 1.2 and Corollary ??

Theorem 1.2 addresses the same nonlinearity $\Gamma_u < 0$ with a positive zero $u = c$ of $\Gamma(u)$ but, in contrast to Theorem 1.1, with negative angular velocity, $\omega < 0$. As the resulting limit circle spirals $u(s)$ differ drastically from the Archimedean spirals of Theorem 1.1, we cannot expect to prove Theorem 1.2 by a simple reflection $z \mapsto \overline{z}$ in the complex z-plane. We prove Theorem 1.2 by a transformation $\omega \mapsto \widehat{\omega} := -\omega$, and a "time" reversal $s \mapsto \widehat{s} := -s$, in the second order Lienard equation (2.13). This will allow us to freely use the results of sections 2-5, as recalled in our proof of Theorem 1.1 in section 6. Similar transformations will then prove the remaining cases, as summarized in Corollary 1.3.

In the case $\omega < 0$, $\Gamma_u < 0$, $c > 0$ of Theorem 1.2 we apply the "time" reversal transformation

$$\begin{aligned} \omega &\mapsto \widehat{\omega} = -\omega \\ s &\mapsto \widehat{s} = -s \\ u &\mapsto \widehat{u} = u \\ v &\mapsto \widehat{v} = -v \end{aligned} \tag{7.1}$$

to the Lienard second order system (2.14), (2.15). Denoting by $'$ the derivative with respect to $\widehat{s}$ again and omitting "$\wedge$" for typographic convenience, we obtain the transformed system

$$u' = v \tag{7.2}$$

$$v' = -\frac{\Gamma_u}{\Gamma}(\omega - v)v - \Gamma^2 u \tag{7.3}$$

which is identical to the original form of (2.14), (2.15). Moreover ω, in fact $\widehat{\omega}$, has become positive once again. The only remaining difference to our previous analysis in sections 2-6 is that we now have to consider bounded solutions $u(s) < c$ of (7.2), (7.3) in backwards "time" $-\infty < s \leq 0$, rather than forward time. This is due to the "time" reversal (7.1). Moreover

$$v(0) = \omega \tag{7.4}$$

must hold at the tip $s = 0$. Indeed this follows from the integral equation (1.5) for rotating solutions of curvature flow by inserting $s = 0$, just as in our derivation of (5.2).

Lemma 7.1. *Under the sign conditions $\omega < 0$, $\Gamma_u < 0$, $c > 0$ of Theorem 1.2, consider any solution $(u, v) = (u(s), v(s))$ of the singular Lienard system (7.2), (7.3) with initial condition (u_0, v_0) at $s = 0$ satisfying*

$$u_0 < c, \quad v_0 = \omega. \tag{7.5}$$

Then the solution (u, v) exists globally in backwards "time" $-\infty < s \leq 0$ and spirals into zero at an exponential rate.

Proof. By Proposition 2.2, the energy $E = \frac{1}{2}(v/\Gamma)^2 + \frac{1}{2}u^2$ decays in backwards "time" s. In particular, E is bounded, and so are (u, v). Lemma 3.3 then proves convergence of (u, v) to $(0, 0)$ for $s \to -\infty$ and, in particular, global backwards existence – not only in the regularized "time" variable τ but also in the arc length variable s itself. Exponential convergence follows by linearization at the (forward) unstable focus $u = v = 0$. This proves the lemma. ☐

We now return to forward convergent solutions $u(s) \to 0$ for $s \to +\infty$, in the original arc length variable s, reverting transformation (7.1). The following lemma provides the analogue of Lemma 4.1 for limit cycle spirals $z = z(s)$.

Lemma 7.2. *Let $u(s)$ converge to zero exponentially for $s \to +\infty$. Then the associated planar curve $z = z(s)$ with arc length parameter s and curvature $\kappa(s) = \Gamma(u(s))$ is a right-winding limit circle spiral in the sense of (1.10), (1.12). More precisely, the curve $z(s) = r(s)exp(i\varphi(s))$ satisfies*

$$\lim \varphi' = -1/R, \quad \lim r = R := 1/\Gamma(0), \quad \lim \kappa = 1/R, \tag{7.6}$$

for $s \to +\infty$, in polar coordinates (r, φ).

Proof. Trivially $u \to 0$ implies $\kappa = \Gamma(u) \to \Gamma(0) =: 1/R > 0$. ¿From the proof of Lemma 4.1 we recall the general facts

$$\omega(\frac{1}{2}r^2)' = u, \tag{7.7}$$

$$(r')^2 + (r\varphi')^2 = |z'|^2 = 1; \tag{7.8}$$

see (4.4), (4.6). Exponential convergence $u \to 0$ and $\kappa \to 1/R$ therefore implies exponential convergence $r' \to 0$, by (7.7), and $r \to R = 1/\Gamma(0)$. Therefore (7.8) implies exponential convergence $\varphi' \to \pm 1/R$. A look at signed curvature, according to $z'' = -i\kappa z'$ in (1.2), then shows $\varphi \to -1/R < 0$ and $\varphi \to -\infty$. This proves (7.6) and the lemma. □

Lemmas 7.1 and 7.2 together prove Theorem 1.2 except for the claim $\rho = |u(0)/\omega|$ on the tip radius $\rho = |z(0)|$. This follows trivially because the normal velocity $u(0)$ at the tip is tangential to the tip circle $|z| = \rho$, traversed at angular velocity ω; see (2.6), (2.7). This proves Theorem 1.2. □

To prove Corollary 1.3 we first address the case $\Gamma_u > 0$ and $\Gamma(c) = 0$ for some $c > 0$. Replacing $\Gamma < 0$ by

$$\widehat{\Gamma}(u) := -\Gamma(u) > 0 \tag{7.9}$$

neither changes integral equation (2.1), where $\kappa = \Gamma(u)$ enters quadratically, nor the singular Lienard system (2.14), (2.15). Therefore our analysis of solutions $(u(s), v(s))$ of these equations, which led to the proof of Theorems 1.1, 1.2, remains valid verbatim. (Cautious readers will carefully apply a regularizing time transformation $\dot{} = -\Gamma \cdot'$ here, with $-\Gamma > 0$, to preserve the "time" direction in the regularized system (2.16), (2.17) and its subsequent center manifold analysis.) The corresponding rotating solution $z(s)$ of the curvature flow equation, however, now satisfies

$$z'' = -i\Gamma z' = +i\widehat{\Gamma} z', \tag{7.10}$$

instead of the original form (1.2). Passing to complex conjugates

$$\overline{z}'' = -i\widehat{\Gamma}\overline{z}' \tag{7.11}$$

then re-establishes the original form (1.2), for $\overline{z}$ instead of z, and then proves Corollary 1.3(i).

To prove Corollary 1.3(ii) where $\Gamma(c) = 0$ for some $c < 0$, and $u > c$, we replace u by $-u$ according to the following transformation. Let

$$\widehat{u} = -u, \quad \widehat{v} = -v, \quad \widehat{\omega} = -\omega, \quad \widehat{\Gamma}(\widehat{u}) = \Gamma(u) = \Gamma(-\widehat{u}). \tag{7.12}$$

Then $(\widehat{u}, \widehat{v})$ satisfy the Lienard system (2.14), (2.15) with $\widehat{\omega} = -\omega$ and $\widehat{\Gamma}(\widehat{u}) = \Gamma(-\widehat{u})$ replacing ω and Γ. Since $\widehat{\omega}$, $\widehat{\Gamma}$ satisfy the assumptions of Theorems 1.1 and 1.2, in the respective cases, and since

$$z'' = -i\kappa z' = -i\Gamma(u)z' = -i\widehat{\Gamma}(\widehat{u})z' \tag{7.13}$$

the previous analysis applies verbatim, up to the obvious reversals of inequalities caused by $\widehat{u} = -u$ and $\widehat{\omega} = -\omega$.

To prove Corollary 1.3(iii), we only have to combine the above transformations (7.9) and (7.12). This proves Corollary 1.3. □

The totality of the four cases, three in the corollary and one in the theorem, corresponds quite simply to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the commutative transformations

(7.9) and (7.12). Theorems 1.1 and 1.2 are related, less trivially, by a "time" reversal $s \mapsto -s$ as in (7.1), for (2.14), (2.15), and by its dynamic consequences for the associated curves $z(s)$.

8. Self-intersections

So far, we have proved Theorems 1.1 and 1.2 on rotating spiral curves $z = z(s) \in \mathbb{C}$, parametrized over arc length s in the complex plane and left rotating at angular velocity ω. We now address the issue of self-intersections of the resulting spirals $z(s)$. This is a particularly relevant issue with respect to our original motivation by singular front positions $z(s)$ in singular limits $\varepsilon \searrow 0$ of excitable reaction diffusion systems; see section 1 and the references there. If we interpret fronts $z(s)$ as level curves of solutions to such systems, for small $\varepsilon > 0$, we may in fact expect any such intersection to disconnect immediately, at least for generic initial data. See [FM00] for a detailed justification of this claim. In particular, any self-intersecting rotating wave curve, if at all existent, should become unstable with respect to perturbations of initial data, if at all existent, for small $\varepsilon > 0$.

We continue to consider the case $\Gamma(c) = 0$ for some $c > 0$. In Proposition 8.1 below we show that all limit circle spirals of Theorem 1.2, $\omega < 0$, $\Gamma_u < 0$, are unfortunately self-intersecting. Proposition 8.2 then deals with the branches $u(0) \in \{u_n^{\pm}(\omega)\}$, $u'(0) = \omega$ of Archimedean spirals $z(s)$, which are asserted to exist for $\omega > 0 > \Gamma_u$ in Theorem 1.1. We show that these spirals do not intersect themselves, for the primary branch $n = 1$. For small $\omega > 0$ and high branch numbers n, in contrast, we do encounter self-intersections again.

Proposition 8.1. *The limit circle spirals $z = z(s)$ of Theorem 1.2 and Corollary 1.3 possess infinitely many self-intersections for $s \to +\infty$.*

Proof. We recall from (1.2) and (2.14), (2.15) that

$$z'' = -i\kappa z'; \tag{8.1}$$

$$u' = v, \tag{8.2}$$

$$v' = -\frac{\Gamma_u}{\Gamma}(\omega - v)v - \Gamma^2 u, \tag{8.3}$$

where $\kappa = \Gamma = \Gamma(u)$ denotes curvature. Limit circle spirals, according to the proof of Theorem 1.2 and Corollary 1.3 in section 7, correspond to solutions $(u(s), v(s))$ converging to zero exponentially for $s \to \pm\infty$. The direction of $s \to -\infty$ should be taken, for example, when we now choose to apply transformation (7.1) and work with $\omega > 0$ in the proof of Proposition 8.1 for Theorem 1.2.

By C^1-linearization of (8.2), (8.3) at zero, for example according to Belitskii [Bel73], we can expand

$$u(s) = \mathrm{Re}(\widetilde{a}e^{\mu s}) + o(e^{\mu s}) \tag{8.4}$$

for $s \to -\infty$. Here μ denotes the complex eigenvalue of the linearizatoin of system (8.2), (8.3) at zero and $\widetilde{a}$ denotes some nonzero complex constant. Note $\mathrm{Re}\mu > 0$ and $\mathrm{Im}\mu \neq 0$. Inserting expansion (8.4) into $\kappa = \Gamma(u)$, we get a similar expansion

$$\kappa = \kappa_0 + \mathrm{Re}(ae^{\mu s}) + \cdots, \tag{8.5}$$

omitting terms of order $o(e^{\mu s})$ from now on. Note that $\kappa_0 = \Gamma(0) = 1/R$ is the curvature of the limit circle.

Next we define

$$\exp(i\psi) := z' \in S^1 \tag{8.6}$$

and rewrite (8.1) as

$$\psi' = -\kappa. \tag{8.7}$$

Inserting expansion (8.5) and integrating (8.7) we obtain

$$\psi(s) = -\kappa_0 s + \psi_0 - \mathrm{Re}(\frac{a}{\mu}e^{\mu s}) + \cdots \tag{8.8}$$

for a suitable constant ψ_0. Choosing $s = 0$ appropriately, we may assume $a/\mu = 1$ in (8.8). Integrating (8.6) via expansion (8.8), and also expanding the exponential $\exp(i\psi)$ with respect to the (backwards) exponentially decaying term $Re(\exp(\mu s))$ we calculate

$$z(s) = z_0 + e^{i\psi_0}\left(\frac{1}{i\kappa_0}(1 - e^{-i\kappa_0 s}) - i\int_0^s e^{-i\kappa_0\sigma}\mathrm{Re}(e^{\mu\sigma})d\sigma + \cdots\right). \tag{8.9}$$

To simplify (8.9) we observe that equation (8.1) is invariant with respect to, both, translations and complex multiplications of the curve $z(s)$ by constants. To study self-intersections of $z(s)$ we may therefore omit additive and multiplicative constants in (8.9). The additive constant is in fact zero, necessarily, because the resulting limit circle, for $s \to -\infty$, must rotate around its center; see Lemma 7.2. This reduction allows us to study the curve

$$z(s) = e^{-i\kappa_0 s} + \kappa_0\int_s^\infty e^{-i\kappa_0\sigma}\mathrm{Re}(e^{\mu s})d\sigma + \cdots \tag{8.10}$$

instead. Elementary integration in (8.10) shows

$$z(s) = e^{-i\kappa_0 s}\left(1 - \frac{1}{2}(\frac{1}{\mu - i\kappa_0}e^{\mu s} + \frac{1}{\bar{\mu} - i\kappa_0}e^{\bar{\mu}s}) + \cdots\right), \tag{8.11}$$

and therefore the asymptotically oscillatory behavior

$$1 - |z|^2 = 2\mathrm{Re}\Big(\frac{\mu}{\mu^2 + \kappa_0^2}e^{\mu s} + \cdots\Big) \to 0 \tag{8.12}$$

for $s \to -\infty$.

To complete the proof we only claim that expansion (8.12) entails infinitely many self-intersections of the curve $z(s)$. Clearly the right hand side possesses infinitely many simple zeros, for $s \to -\infty$, due to the nonvanishing imaginary part of μ. Suppose now, indirectly, the curve $z(s) = r\exp(i\varphi)$ was not self-intersecting for large enough negative s. Then $r = |z| \to 1$, in our scaling (8.12), and $\lim \varphi' = -1$ by (7.6). We now argue as in the proof of the Poincaré-Bendixson theorem. The non-selfintersecting Jordan curve $z(s)$ must approach its limit circle monotonically : numbering by $s_1 > s_2 > \cdots \to -\infty$ the passage of $r(s)$ at any particular fixed angle φ_0, the sequence $r_n = r(s_n) \to 1$ must be eventually monotone. Note that the direction of this monotonicity does not depend on the choice of φ_0. This observation clearly contradicts the infinitely many sign changes of (8.12), for $s \to -\infty$. Therefore the proposition is proved. □

Proposition 8.2. *Consider the Archimedean spirals with tip velocities* $u(0) \in \{u_n^-(\omega), u_n^+(\omega)\}$, $u'(0) = \omega$, *for* $0 < \omega \le \omega_n$, *as derived in Theorem 1.1 and Corollary 1.3. For* $n = 1$ *these Archimedean spirals are not self-intersecting. For sufficiently large* n *and sufficiently small* ω, *however, these Archimedean spirals do possess self-intersections.*

Proof. We only address Theorem 1.1, where $\omega > 0 > \Gamma_u$ and $\Gamma(c) = 0$ for some $c > 0$, with analogous arguments for Corollary 1.3. We first address the case $n = 1$ of Theorem 1.1. By Lemma 5.3 and in particular by ordering (5.21), the "times" $s_1^{\pm}(\omega)$ are the last positive intersections "times" of the center manifold solution $(u(s), v(s)) \in W_-^c$ of system (8.2), (8.3) with the axis $v = \omega$. Our phase plane analysis of W_-^c in section 3 then shows that

$$u' = v > 0 \tag{8.13}$$

for $s_1^-(\omega) \leq s < +\infty$. Therefore the curvature $\kappa(s) = \Gamma(u(s))$ of the asymptotically Archimedean spiral $z(s)$ is strictly decreasing along solutions, in case $n = 1$.

The monotonicity of $\kappa(s)$ excludes self-intersections. Indeed any self-intersection would produce a closed loop, with possibly infinite curvature at the intersection. Curvature along

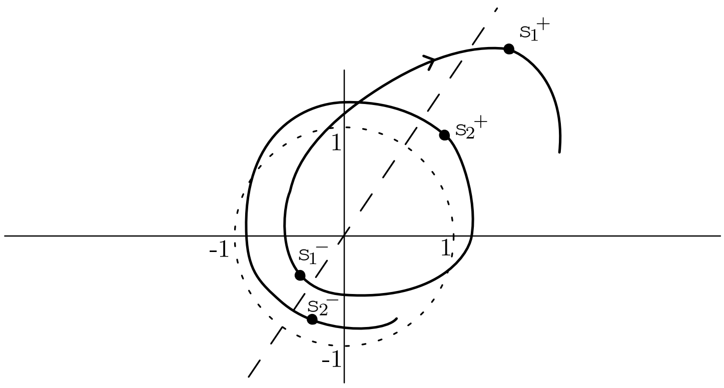

Figure 8.1. *Self-intersections of Archimedean spirals near the core region.*

a differentiable closed loop, however, possesses at least two local maxima and two local minima; see for example [Ang99]. Even with one maximum being infinite, the remaining curvature $\kappa(s)$ cannot be strictly monotone. This contradiction excludes self-intersections, in case $n = 1$.

We now choose $\omega > 0$ sufficiently small and prove self-intersections of $z(s)$ corresponding to tip conditions $u(0) \in \{u_n^{\pm}(\omega)\}$, $u'(0) = \omega$ with large enough n. In fact we choose n so large that $(u(s), v(s))$ remains near the unstable focus equilibrium $u = v = 0$ for at least two cycles. The cycle time is controlled by the uniformly nonzero imaginary part

$$\nu = \sqrt{\Gamma^2 - \left(\frac{\Gamma_u}{2\Gamma}\omega\right)^2} = \Gamma - \frac{1}{8}\frac{\Gamma_u^2}{\Gamma^3}\omega^2 + \cdots \tag{8.14}$$

of the eigenvalue $\mu = \eta + i\nu$ of the linearization of (8.2), (8.3) at $u = v = 0$. Here the expansion refers to small $\omega > 0$ and Γ, Γ_u are evaluated at $u = 0$. Similarly, the unstable read part $\eta > 0$ of μ is given by

$$\eta = \left(-\frac{\Gamma_u}{2\Gamma}\right)\omega. \tag{8.15}$$

By linearizing the flow (8.2), (8.3) over a uniformly finite time horizon of two cycle times, approximately $4\pi/\nu \approx 4\pi/\Gamma(0)$, we now repeat the derivation of the oscillatory behavior (8.12) verbatim. We recall that in our derivation (8.4)-(8.11) of (8.12) we have rescaled a/μ to 1, by shifting s in $\exp(\mu s)$ to sufficiently negative values. With both $|a|$, alias $|u(0)|$, and $v(0) = \omega$ uniformly small we may therefore assume (8.12) to remain valid with remainder terms of order $o(|u(0)| + \omega) = o(|\exp(\mu s)|)$ and with large negative s. Validity does not extend to $s \to -\infty$, however, because Belitskii linearization cannot be invoked to hold uniformly in the integrable limit $\omega \searrow 0$. Validity over two cycle times, however, follows by simple linearization of the finite time nonlinear flow itself.

Why are two cycles of $(u, v)(s)$ sufficient to prove self-intersection of $z(s)$? The reason is a one-to-one resonance between the cycling frequency ν of $(u, v)(s)$ and the approximate cycling frequency $\kappa \approx \Gamma(0)$ of $z'' = -i\kappa z'$ for $\kappa = \Gamma(u)$ and u near zero. See the expansion (8.14) for ν. In passing we note that this resonance is also responsible for the drifting motion of the approach to a limit circle in Figure 1.2(f). Choose

$$s_1^+ > s_1^- > s_2^+ > s_2^- \tag{8.16}$$

to denote alternately strict local maxima (+) and minima (−) of (8.12), all within two cycles of (u, v). For small $\omega > 0$, the spacing of these times is approximately one half period, $\pi/\Gamma(0)$, apart. In polar coordinates $z(s_j^\pm) = r_j^\pm \exp(i\varphi_j^\pm)$, the resonance $\varphi'(s) \approx -\Gamma(0)$ implies that the angles $\varphi_j^\pm$ are approximately π apart, so that φ_j^+ and φ_j^- point in two antipodal directions. Moreover the right winding curve satisfies

$$r_1^+ > r_2^+ > 1 > r_2^- > r_1^-, \tag{8.17}$$

by (8.12). See Figure 8.1. Such a curve $z(s)$ must intersect itself. This proves Proposition 8.2. □

References

[Ang90] S. Angenent. Parabolic equations for curves on surfaces. I: curves with p-integrable curvature. *Ann. Math.*, **132**:451–483, (1990).

[Ang91] S. Angenent. Parabolic equations for curves on surfaces. II: Intersections, blow-up and generalized solutions. *Ann. Math.*, **133**:171–251, (1991).

[Ang99] S. Angenent. Inflection points, extatic points and curve shortening. Simó, Carles (ed.), Hamiltonian systems with three more degrees of freedom. Dordrecht: Kluwer Academic Publishers. NATO ASI Ser. C, Math. Phys. Sci., **533**:3–10, 1999.

[Arn83] V.I. Arnol'd. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer-Verlag, New York, 1983.

[BE93] M. Braune and H. Engel. Compound rotation of spiral waves in a light-sensitive Belousov-Zhabotinsky medium. *Chem. Phys. Lett.*, **204**:257–264, (1993).

[Bel73] G.R. Belitskii. Functional equations and conjugacy of local diffeomorphisms of a finite smoothness class. *Funct. Analysis Appl.*, 7:268–277, (1973).

[CH82] S.-N. Chow and J. K. Hale. *Methods of Bifurcation Theory.* Springer-Verlag, New York, 1982.

[FGT03] B. Fiedler, J.-S. Guo, and J.-C. Tsai. Existence of steadily rotating spiral waves. Preprint, 2003.

[FM00] B. Fiedler and R.-M. Mantel. Crossover collision of core filaments in three-dimensional scroll wave patterns. *Doc. Math.*, **5**:695–731, (2000).

[GH86] M. Gage and R.S. Hamilton. The heat equation shrinking convex plane curves. *J. Diff. Geom.*, **23**:69–96, (1986).

[IIY98] R. Ikota, N. Ishimura, and T. Yamaguchi. On the structures of steady solutions for the kinematic model of spiral waves in excitable media. *Japan J. Indust. Appl. Math.*, **15**:317–330, (1998).

[Kee89] J.J. Tyson, J.P. Keener. Singular perturbation theory of spiral waves in excitable media. *Physica D*, **32**:327–361, (1989).

[Kee92] J.P. Keener. The core of the spiral. *SIAM J. Appl. Math.*, **52**:1370–1390, (1992).

[KT92] J.P. Keener and J.J. Tyson. The dynamics of scroll waves in excitable media. *SIAM Rev.*, **34**:1–39, (1992).

[Mer92] E. Meron. Pattern formation in excitable media. *Physics Reports*, **218**:1–66, (1992).

[Mik03] A.S. Mikhailov. Modeling pattern formation in excitable media: The Legacy of Norbert Wiener. In: Epilepsy as a Dynamic Disease. *J. Milton & P. Jung (eds.), Springer-Verlag*, 2003.

[MZ91] A.S. Mikhailov and V.S. Zykov. Kinematic theory of spiral waves in excitable media: comparison with numerical simulations. *Physica D*, **52**:379–397, (1991).

[PB96] B.B. Plapp and E. Bodenschatz. Core dynamics of multiarmed spirals in Rayleigh-Bénard convection. *Physica Scripta*, **67**:111–117, (1996).

[RKPD94] B. Rudovics, P. de Kepper, J.-J. Perraud, and E. Dulos. Experimental study of stationary Turing patterns and their interaction with traveling waves in a chemical system. *Int. J. Bifurcation Chaos*, **4**:1215–1231, (1994).

[RNOE93] H.H. Rotermund, S. Nettesheim, A. von Oertzen, and G. Ertl. Reaction diffusion patterns in the catalytic CO-oxidation on Pt(110) – front propagation and spiral waves. *J. Chem. Phys.*, **98**:9977–9985, 1993.

[UUM93] Z. Ungvarai-Nagy, J. Ungvarai, and S.C. Mueller. Complexity in spiral wave dynamics. *Chaos*, **3**:15–19, (1993).

[Van89] A. Vanderbauwhede. Center manifolds, normal forms and elementary bifurcations. *Dynamics Reported*, **2**:89–169, (1989).

[Win72] A.T. Winfree. Spiral waves of chemical activity. *Science*, **175**:634–636, (1972).

[Win87] A.T. Winfree. *When Time Breaks down*. Princeton University Press, Princeton, NJ, 1987.

[Win01] A.T. Winfree. *The geometry of biological time*. Springer-Verlag, Berlin-New York, 2001.

[WR46] N. Wiener and A. Rosenblueth. The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle. *Arch. Inst. Cardiol. Mexico* **16**:205–265, (1946).

[ZMM98] V.S. Zykov, A.S. Mikhailov, and S.C. Mueller. Wave instabilities in excitable media with fast inhibitor diffusion. *Phys. Rev. Lett.*, **81**:2811–2814, (1998).