Bifurcations without parameters:
some ODE and PDE examples

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Abstract

Bifurcation theory studies dynamical systems depending on one or several real parameters λ. Frequently, trivial equilibria or fixed points $x = 0$ are assumed to exist. The constant parameter λ provides a trivial foliation, transverse to these equilibria.

In contrast, we consider vector fields with lines and planes of trivial equilibria in absence of any parameters. We address the failure of normal hyperbolicity, due to transverse zero or purely imaginary eigenvalues. In particular we indicate the resulting dissipative Hopf and Takens-Bogdanov bifurcations, along with some reversible and integrable variants, in absence of parameters.

We illustrate our results by dynamic uncouplings of coupled oscillators, oscillatory viscous shock profiles of stiff nonlinear hyperbolic balance laws, and spatially non-periodic stationary waves in the Kolmogorov problem of fluid flows in planar channels.

1 Applied motivation

In this article we sketch and illustrate some elements of the nonlinear dynamics near equilibrium manifolds. Denoting the equilibrium manifold by $x = 0$, in local coordinates $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$, we consider systems

$$
\begin{aligned}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{aligned}
\tag{1.1}
$$

and assume

$$
f(0, y) = g(0, y) = 0,
\tag{1.2}
$$

for all y. For simplicity we will only address the cases $k = 1$ of lines of equilibria, and $k = 2$ of equilibrium planes. Sufficient smoothness of f, g is assumed. The occurrence of equilibrium manifolds is infinitely degenerate, of course, in the space of all vector fields (f, g) – quite like many mathematical structures are: equivariance under symmetry groups, conservation laws, integrability, symplectic structures, and many others. The special case

$$
g \equiv 0
\tag{1.3}
$$

in fact amounts to standard bifurcation theory, in the presence of a trivial solution $x = 0$; see for example [5]. Note that condition (1.3), which turns the k-dimensional variable y into a preserved constant parameter, is infinitely degenerate even in our present setting (1.2) of equilibrium manifolds. Due to the analogies of our results and methods with bifurcation theory, we call our emerging theory *bifurcation without parameters*. This terminology emphasizes the intricate dynamics which arises when normal hyperbolicity of the equilibrium manifold fails; see the sections below.

To motivate assumption (1.2), we present several examples. First, consider an "octahedral" graph Γ of $2(m + 1)$ vertices $\{\pm 1, \ldots, \pm(m + 1)\}$. The graph Γ results from the complete graph by eliminating the "diagonal" edges, which join the antipodal vertices $\pm j$, for $j = 1, \ldots, m + 1$. For $m = 1$ we obtain the square, for $m = 2$ the octahedron, and so on. Consider the system

$$
\dot{u}_j = f_j \Big(u_j, \sum_{k \neq \pm j} u_k \Big)
\tag{1.4}
$$

of oscillators $u_j \in \mathbb{R}^{n'}$ on Γ, additively coupled along the edges by f_j. We assume an antipodal oddness symmetry of the individual oscillator dynamics

$$
f_{-j}(-u_j, 0) = -f_j(u_j, 0).
\tag{1.5}
$$

As a consequence, the antipode space

$$
\Sigma := \{ u = (u_j)_{j \in \Gamma}; \ u_{-j} = -u_j \}
\tag{1.6}
$$

is invariant under the flow (1.4). Moreover, the flow on Σ completely decouples into a direct product flow of the $m + 1$ diagonally antipodal, decoupled pairs

$$\dot{u}_{\pm j} \;=\; f_{\pm j}(u_{\pm j}, 0). \tag{1.7}$$

For the square case $m = 1$, this decoupling phenomenon was first observed in [2]. For more examples see also [3].

An m-plane of equilibria arises from periodic solutions of the decoupled system (1.7). Assume (1.7) possesses time periodic orbits $u_j(t + \varphi_j)$ of equal period $T_j = 2\pi$, for $j = 1, \ldots, m + 1$. Choose arbitrary phases $\varphi_j \in S^1$ and let $u_{-j}(t) := -u_j(t)$, $\varphi_{-j} = \varphi_j$. Then

$$u^\varphi(t) \;:=\; (u_j(t + \varphi_j))_{j \in \Gamma} \;\in\; \Sigma \tag{1.8}$$

is a 2π-periodic solution of (1.4), (1.7), for arbitrary phases $\varphi \in T^{m+1}$. Eliminating one phase angle φ_{m+1} by passing to an associated Poincaré map, an m-dimensional manifold of fixed points arises, parametrized by the remaining m phase angles. Assuming, in addition to the diagonal oddness symmetry (1.5), equivariance of (1.4) with respect to an S^1-action, the Poincaré map can in fact be obtained as a time-2π map of an autonomous flow within the Poincaré section. In suitable notation, $y = \{\varphi_1, \ldots, \varphi_m\}$, the fixed point manifold then becomes an m-dimensional manifold of equilibria, as presented in (1.1), (1.2) above. For more detailed discussions of this example in the context of bifurcations without parameters see [15, 10, 9].

As a second example of equilibrium manifolds we consider viscous profiles $u = u((\xi - st)/\varepsilon)$ of systems of nonlinear hyperbolic conservation laws and stiff balance laws

$$\partial_t u + \partial_\xi F(u) + \varepsilon^{-1} G(u) \;=\; \varepsilon \partial_\xi^2 u. \tag{1.9}$$

Viscous profiles then have to satisfy an ε-independent ODE system

$$\ddot{u} \;=\; (F'(u) - s \cdot \mathrm{id})\dot{u} + G(u). \tag{1.10}$$

Standard conservation laws, for example, require $G \equiv 0$. The presence of m conservation laws corresponds to nonlinearities G with range in a manifold of codimension m in u-space. Typically, then, $G(u) = 0$ describes an equilibrium manifold of dimension m of pairs $(u, \dot{u}) = (u, 0)$, in the phase space of (1.10). For an analysis of this example in the context of bifurcations without parameters see [8, 16]. For another example, which relates binary oscillations in central difference discretizations of hyperbolic balance laws with diagonal uncoupling of coupled oscillators, see [11].

We conclude our introductory excursion with a brief summary of some further examples. In [7], lines of equilibria have been observed for the dynamics of models of competing populations. This included a first partial analysis of failure of normal hyperbolicity.

A topologically very interesting example in compact three-dimensional manifolds involves contact structures $\eta(\xi)$ (i.e., nonintegrable plane fields and gradient vector fields $\dot{\xi} = -\nabla V(\xi) \in \eta(\xi)$. See [6] for an in-depth analysis. Examples include mechanical systems with nonholonomic constraints. Notably, level surfaces of regular values of the potential V consist of tori. Under a nondegeneracy assumption, equilibria form embedded circles, that is, possibly linked and nontrivial knots.

For a detailed study of plane Kolmogorov fluid flows in the presence of a line of equilibria with a degeneracy of Takens-Bogdanov type and an additional reversibility symmetry, see [1].

As a caveat we repeat that lines of equilibria, which are transverse to level surfaces of preserved quantities λ do *not* provide bifurcations, without parameters. Rather, $\dot{y} = 0$ for $y := \lambda$ exhibits this problem as belonging to standard bifurcation theory; see (1.3).

2   Sample vector fields and resulting flows

In this section we collect relevant example vector fields (1.1), (1.2) with lines and planes of equilibria $x = 0$; see [10, 9, 1] for further details. We illustrate and comment the resulting flows.

Normally hyperbolic equilibrium manifolds admit a transverse C^0-foliation with hyperbolic linear flows in the leaves. See for example [19], [20] and the ample discussion in [4]. As a first example, we therefore consider

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= x. \end{aligned} \tag{2.1}$$

Note the loss of normal hyperbolicity at $x = y = 0$, due to a nontrivial transverse eigenvalue zero of the linearization. Clearly $dx/dy = y$, and the resulting flow lines are parabolas; see Figure 2.1. For comparison with standard bifurcation theory, where $y = \lambda$, we draw the y-axis of equilibria horizontally.

As a second example, consider

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= \pm x^2. \end{aligned} \tag{2.2}$$

Again, a transverse zero eigenvalue occurs – this time with an additional reflection symmetry $y \mapsto -y$. Dividing by the Euler multiplier x, the reflection becomes a time reversibility. See the left parts of Figure 2.2 for the resulting flows. Note the resulting integrable, harmonic oscillator case which originates from the elliptic sign $\dot{y} = -x^2$.
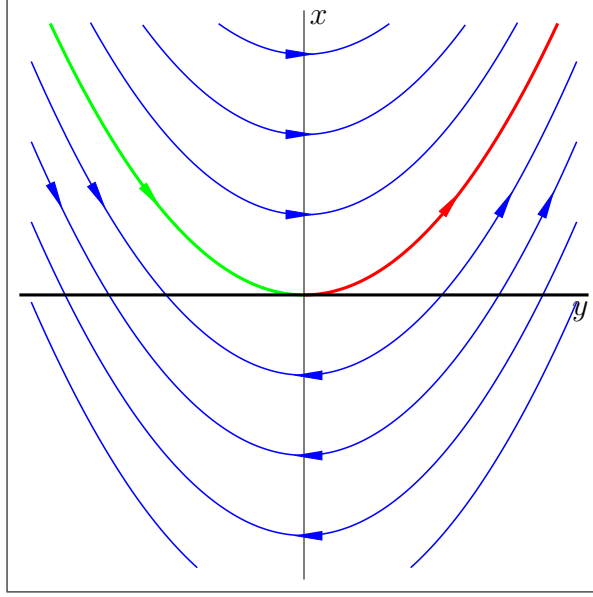
Figure 2.1: A line of equilibria (y-axis) with a nontrivial transverse eigenvalue zero.

As a third example, we consider $x = (x_1, x_2) \in \mathbb{R}^2$, $y \in \mathbb{R}$ with a line $x = 0$ of equilibria and a purely imaginary nonzero eigenvalue $i\omega$ at $x = 0$. Normal-form theory, for example as in [21], then generates an additional S^1-symmetry by the action of $\exp(i\omega t)$ in the x-eigenspace. This equivariance can be achieved, successively, up to Taylor expansions of any finite order. In polar coordinates (r, φ) for x, an example of leading order terms is given by

$$\begin{aligned} \dot{r} &= ry \\ \dot{y} &= \pm r^2 \\ \dot{\varphi} &= \omega. \end{aligned} \tag{2.3}$$

Since the first two equations in (2.3) coincide with (2.2), the dynamics is then obtained by simply rotating the left parts of Figure 2.2 around the y-axis at speed w. The right parts of Figure 2.2 provide three-dimensional views of the effects of higher-order terms which do not respect the S^1-symmetry of the normal forms. In the elliptic case (b), all nonstationary orbits are heteroclinic from the unstable foci, at $y > 0$, to stable foci at $y < 0$. The two-dimensional respective strong stable and unstable manifolds will split, generically, to permit transverse intersections.
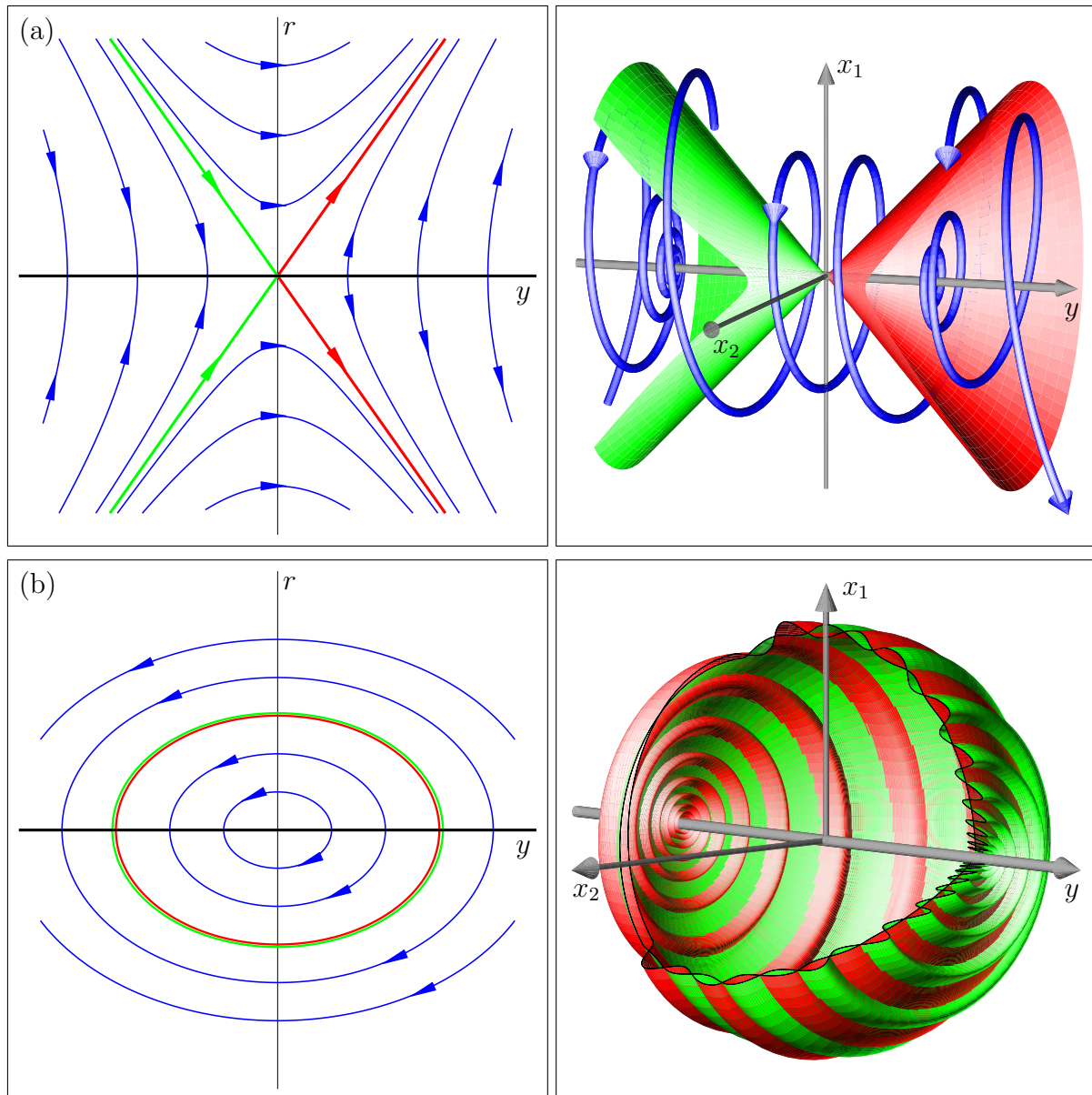
Figure 2.2: Lines of equilibria (y-axis) with imaginary eigenvalues: Hopf bifurcation without parameters. Case (a) hyperbolic; case (b) elliptic. Red: strong unstable manifolds; green: stable manifolds.

Our fourth example addresses Takens-Bogdanov bifurcations without parameters. In suitable rescaled form it reads

$$\ddot{y} + y\dot{y} \;\; = \;\; \varepsilon((\lambda - y)\ddot{y} + b\dot{y}^2) \tag{2.4}$$

with fixed parameters b, λ and ε. The y-axis as equilibrium line is complemented by the two transverse directions $x = (\ddot{y}, \dot{y})$. Note the algebraically triple zero eigenvalue, double in the transverse x-directions, for $\lambda = y = 0$. Two examples of the resulting dynamics for small positive ε are summarized in Figure 2.3.

The coordinates τ and $\tilde{H}$ in Figure 2.3 are adapted to the completely integrable case $\varepsilon = 0$. Indeed, obvious first integrals are then given by $\Theta = \ddot{y} + \frac{1}{2}y^2$ and $H = \frac{1}{2}\dot{y}^2 - y\ddot{y} - \frac{1}{3}y^3$. Coordinates are $\tau = \log \Theta$ and $\tilde{H} = \Theta^{-3/2}H$, not drawn to scale. Parameters are $\varepsilon, \lambda > 0$ and, for the hyperbolic case, $-17/12 < b < -1$. For the elliptic case we consider $b > -1$. The equilibrium y-axis, a cusp in (Θ, H) coordinates, transforms to the top (saddles) and bottom (foci) horizontal boundaries $\tilde{H} = \pm\frac{2}{3}\sqrt{2}$, with $y = 0$ relegated to $\tau = -\infty$. Since τ and $\tilde{H}$ are constants of the flow, for $\varepsilon = 0$, they represent slow drifts on the unperturbed periodic motion, for small $\varepsilon > 0$ and $\mid \tilde{H} \mid < \frac{2}{3}\sqrt{2}$. The top value $\tilde{H} = +\frac{2}{3}\sqrt{2}$ also represents homoclinics to the saddles, for $\varepsilon = 0$.

Along the focus line $\tilde{H} = -\frac{2}{3}\sqrt{2}$ we observe Hopf bifurcations without parameters, corresponding to $y = \lambda > 0$. The value of b distinguishes elliptic and hyperbolic cases. In addition, lines of saddle-focus heteroclinic orbits and isolated saddle-saddle heteroclinics are generated, for $\varepsilon > 0$, by breaking the homoclinic sheets of the integrable case. Note in particular the infinite swarm of saddle-saddle heteroclinics, in the hyperbolic case.

As a final, fifth example we consider a reversible Takens-Bogdanov bifurcation without parameters:

$$\ddot{y} + (1 - 3y^2)\dot{y} \;\; = \;\; ay\ddot{y} + b\dot{y}^2. \tag{2.5}$$

Here we fix a, b to be small. Again $x = (\ddot{y}, \dot{y})$ denotes the directions transverse to the equilibrium y-axis. Time reversibility generates solutions $-y(-t)$ from solutions $y(t)$. For two examples of the resulting dynamics see Figure 2.4. Coordinates are the obvious first integrals, for $\varepsilon = 0$, given by $\Theta = \ddot{y} + y - y^3$ and $H = -\ddot{y}y + \frac{1}{2}\dot{y}^2 + \frac{3}{4}y^4 - \frac{1}{2}y^2$. Note the two Takens-Bogdanov cusps, separated by a Hopf point along the lower arc of the equilibrium "triangle". Compare Figures 2.2, 2.3. The elliptic Hopf point (b) arises for $a \cdot (a - b) > 0$, whereas $a \cdot (a - b) < 0$ in the hyperbolic case (a). Also note the associated finite and infinite swarms of saddle-saddle heteroclinics, respectively.
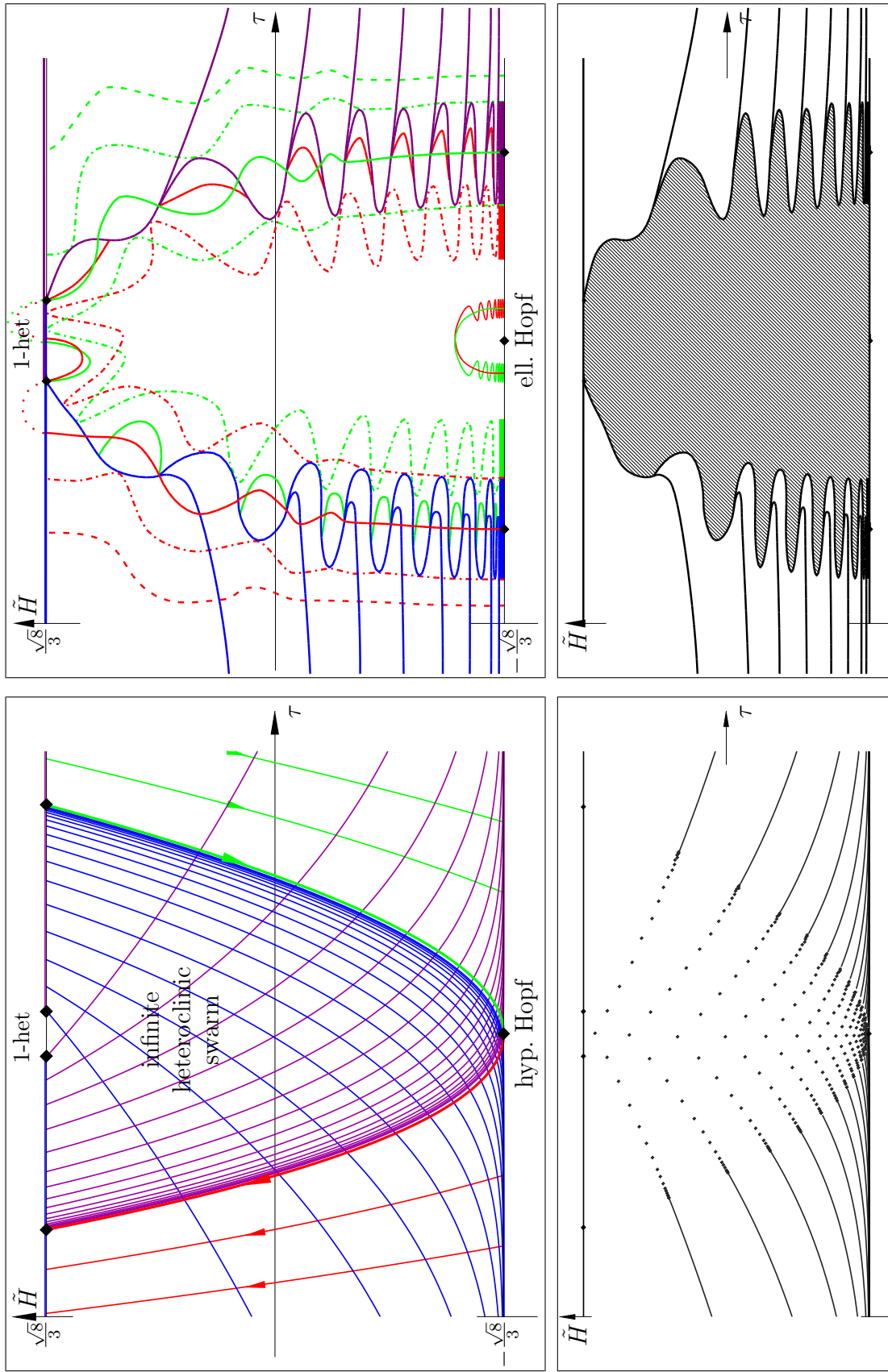
Figure 2.3: Takens-Bogdanov bifurcations without parameters. Case (a) hyperbolic; case (b) elliptic. Top: stable and unstable manifolds; bottom: invariant sets. For coordinates and fixed parameters see text.
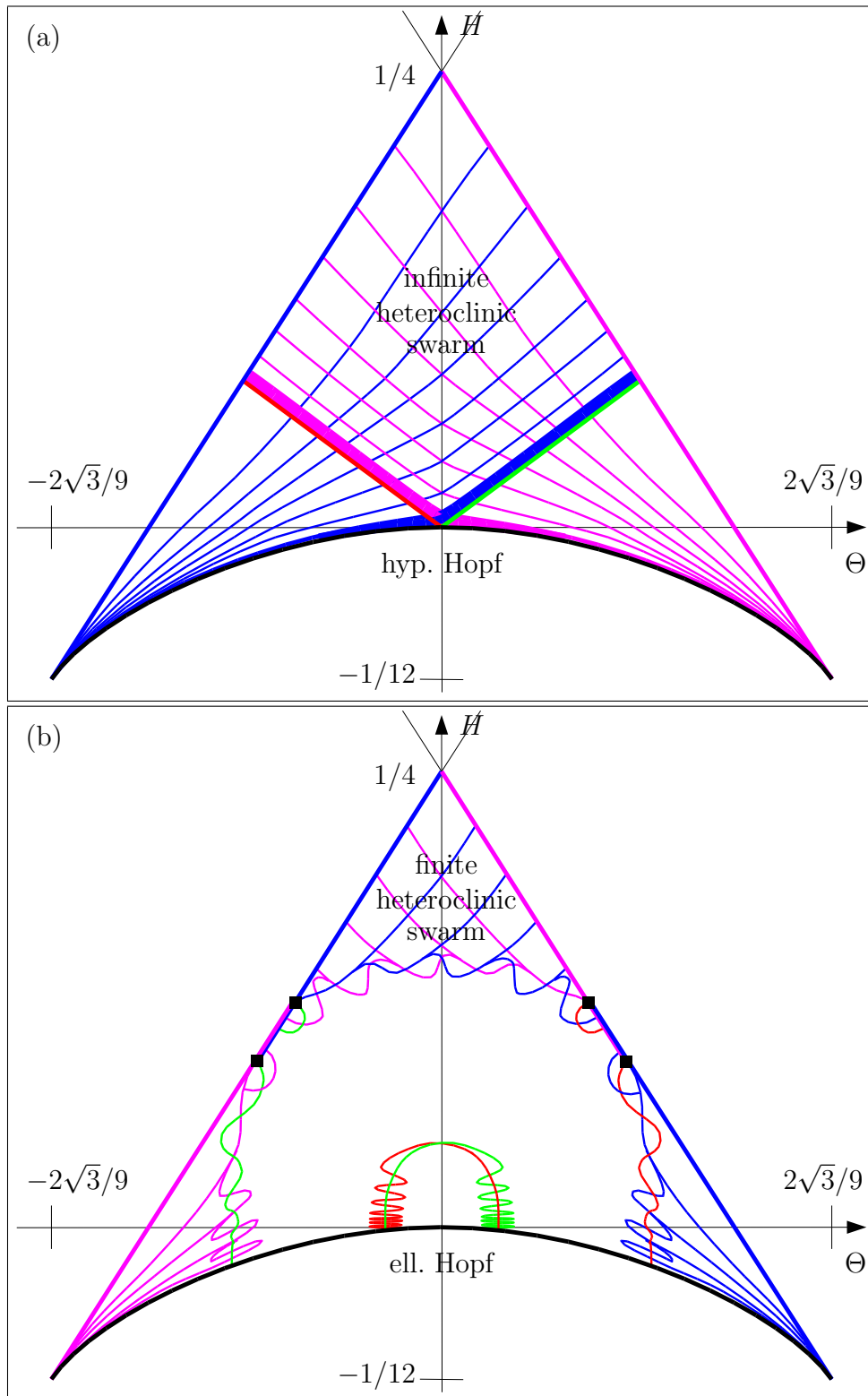
Figure 2.4: Reversible Takens-Bogdanov bifurcations without parameters. Case (a) hyperbolic; case (b) elliptic. For coordinates and fixed parameters see text.

3 Methods

Pictures are not proofs. What has been proved, then, and how? We use ingredients involving algebra, analysis, and numerical analysis, as we outline in this section. For further details see [16, 10, 9, 1].

In a first algebraic step, we derive normal forms for vector fields with lines or planes of equilibria, assuming spectral degeneracies of the linearization A in transverse directions. The spectral assumptions on A in fact coincide with those established for parameter-dependent matrix families in standard bifurcation theory. This is reflected in the naming of the five examples of section 2.

There are more or less standard procedures to derive normal forms of vector fields. By suitable polynomial diffeomorphisms, certain Taylor coefficients of the vector field are eliminated, successively, for higher and higher order. See for example [21] for a systematic choice of normal forms, particularly apt for introducing equivariance of the nonlinear normal-form terms under the action of $\exp(A^T t)$. Normal forms are, however, nonunique in general and other choices are possible.

In the present cases, we adapt the normal-form procedure to preserve the locally flattened equilibrium manifolds. Although the approach in [21] can be modified to accommodate that requirement, it did not provide vector fields convenient for subsequent analysis of the flow. A systematic approach to this combined problem is not known, at present. All examples (2.1)–(2.5) represent truncated normal forms. For the derivation of specific normal forms, for example of (2.4), to any order, and of example (2.5), to third order, see [9], [1], respectively.

Subsequent analysis of the normal-form vector fields is based on scalings, alias blow-up constructions. This is the origin, for example, of the small scaling parameter ε in the Takens-Bogdanov example (2.4). In passing, we note a curious coincidence of two view points for (2.4), concerning the roles of the equilibrium coordinate $y \in \mathbb{R}$ and the "fixed" real parameter $\lambda \in \mathbb{R}$. First, we may consider λ as a parameter, with a line of equilibria y associated to each fixed λ. Then (2.4) describes the collision of a transverse zero eigenvalue at $y = 0$, as in (2.1), with imaginary Hopf eigenvalues at $y = \lambda > 0$, as in (2.3), as λ decreases through zero. Alternatively, we may consider normal forms for a plane $\mathbf{y} = (y_1, y_2)$ of equilibria with a transverse double zero eigenvalue, at $\mathbf{y} = 0$. It turns out that the two cases coincide, after a scaling blow-up, up to second order in ε, via the correspondence $y = y_1, \lambda = y_2$.

The core of any successful flow analysis in bifurcation theory is an integrable vector field; see again section 2. The issues of nonintegrable perturbations, by small $\varepsilon > 0$, and

of omitted higher order terms, not in normal form, both ensue. Since the underlying integrable dynamics is periodic or homoclinic, in examples (2.3)–(2.5), averaging procedures apply. Indeed, $\varepsilon > 0$ then introduces a periodically forced, slow flow on first integrals, like (Θ, H), characteristic of $\varepsilon = 0$. We therefore derive an appropriate, but autonomous Poincaré flows, on (Θ, H), such that the associated true Poincaré map can be viewed as a time discretization of first order and step size ε. In the unperturbed periodic region, this amounts to averaging, while the Poincaré flow indicates Melnikov functions at homoclinic or heteroclinic boundaries. The exponential averaging results by Neishtadt [18], for example, then imply that the separatrix splittings in the elliptic Hopf case (b), indicated in Figure 2.2, are exponentially small in the radius r of the split sphere, for analytic vector fields. See also [12].

Lower bounds of separatrix splittings have not been established, in our settings. This problem is related to the very demanding Lazutkin program of asymptotic expansions for exponentially small separatrix splittings. For recent progress, including the case of Takens-Bogdanov bifurcations for analytic maps, see [13] and the references there. In absence of rigorous lower bounds, our figures indicate only simplest possible splitting scenarios.

While the splitting near elliptic Hopf points are exponentially small, the discretization of the Poincaré flow also exhibits splittings of the unperturbed saddle homoclinic families, which are of first order in the perturbation parameter ε, in example (2.4), or in the small parameters a, b, in example (2.5). Explicit expressions have been derived for the Melnikov functions associated to these homoclinic splittings, in terms of elliptic function in case (2.4), and even of elementary functions in case (2.5). Simplicity and uniqueness of zeros of the Melnikov functions, however, has only been confirmed numerically. While this does not, strictly speaking, match an analytic proof, it still at least supports the validity of the scenarios summarized in Figures 2.3 and 2.4.

4   Interpretation and perspective

We indicate some consequences of the above results for the examples of coupled oscillators, viscous shock profiles, and Kolmogorov flows indicated in section 1. We conclude with a few remarks on the future perspective of bifurcations without parameters.

We first return to the example (1.4)–(1.8) of a coupled oscillator square, $m = 1$. Equilibria y of the Poincaré flow then indicate decoupled antipodal periodic pairs, say with phase difference $y + c$. The case of a transverse zero eigenvalue, (2.1) and Figure 2.1, then indicates a 50% chance of recovery of decoupling with a stable phase difference $y < 0$, locally, even when the stability threshold $y = 0$ has been exceeded. The hyper-

bolic Hopf case (2.2), Figure 2.2 (a), illustrates immediate oscillatory loss of decoupling stability by transverse imaginary eigenvalues. A 100% recovery of decoupling stability, in contrast, occurs at elliptic Hopf points; see (2.2), Figure 2.2 (b). The exponentially small Neishtadt splitting of separatrices indicates a very delicate variability in the asymptotic phase relations of this recovery, for $t \to \pm\infty$. See [10]. The Takens-Bogdanov cases (2.4), Figures 2.3 (a), (b) can then be viewed as consequences of a mutual interaction, of a transverse zero eigenvalue with either Hopf case, for recovery of stable decoupling.

In the example (1.9), (1.10) of stiff balance laws, elliptic Hopf bifurcation without parameters as in (2.4), Figure 2.3 (b), indicates *oscillatory* shock profiles $u(\tau), \tau = (\xi - st)/\varepsilon$. Such profiles in fact contradict the Lax condition, being over-compressive, and violate standard monotonicity criteria. For small viscosities $\varepsilon > 0$, weak viscous shocks in fact turn out unstable, in any exponentially weighted norm, unless they travel at speeds s exceeding all characteristic speeds. The oscillatory profiles can be generated, in fact, by the interaction of inherently non-oscillatory gradient flux functions $F(u)$ with inherently non-oscillatory gradient-like kinetics $G(u)$ in systems of dim ≥ 3. See [16].

The problem of plane stationary Kolmogorov flows asks for stationary solutions of the incompressible Navier-Stokes equations in a strip domain $(\zeta, \eta) \in \mathbb{R} \times [0, 2\pi]$, under periodic boundary conditions in η; see [17]. An η-periodic external force $(F(\eta), 0)$, is imposed, acting in the unbounded ζ-direction. Kolmogorov chose $F(\eta) = \sin\eta$. The Kirchgässner reduction [14] captures all bounded solutions which are nearly homogeneous in ζ, in a center manifold spirit which lets us interpret ζ as "time". The resulting ordinary differential equations in $\mathbb{R}^6$ reduce to $\mathbb{R}^3$, by fixing the values of three first integrals. A line of ζ-homogeneous equilibria appears, in fact, and Kolmogorov's choice corresponds to example (2.5) with $a = b = 0$, to leading orders. In particular note the double reversibility, then, under $y(t) \mapsto \pm y(-t)$ which is generated by

$$
\begin{aligned}
F(\eta) &= -F(\eta + \pi), \qquad \text{and} \\
F(\eta) &= -F(-\eta).
\end{aligned}
\tag{4.1}
$$

As observed by Kolmogorov, an abundance of spatially periodic profiles results. The sample choice $F(\eta) = \sin\eta + c\sin 2\eta$, in contrast, which breaks the first of the symmetries in (4.1), leads to (2.5) with $b = 0 < a$, alias an elliptic reversible Takens-Bogdanov point without parameters; see Figure 2.4 (b). In particular, the set of near-homogeneous bounded velocity profiles of the incompressible, stationary Navier-Stokes system is then characterized by an abundance of oscillatory heteroclinic wave fronts, which decay to different asymptotically homogeneous ζ-profiles, for $\zeta \to \pm\infty$. The PDE stability of these heteroclinic profiles is of course wide open.

As for perspectives of our approach, we believe to have examples at hand, from suf-

ficiently diverse origin, to justify further development of a theory of bifurcations without parameters. In fact, transverse spectra $\{0, \pm i\omega\}$ and $\{\pm i\omega_1, \pm i\omega_2\}$ still await investigation before we can claim any insight into nonhyperbolicity of even the simple case of an equilibrium plane. This assumes the absence of further structural ingredients like symplecticity, contact structures, symmetries, and the like. Certainly our example collecting activities are far from complete, at this stage.

In addition, we have not addressed the issue of perturbations, so far, which could destroy the equilibrium manifolds by small drift terms. Examples arise, for example, when slightly detuning the basic frequencies $2\pi/T_j$ of our uncoupled oscillators or, much more generally, in the context of multiple scale singular perturbation problems. Feedback and input from our readers will certainly be most appreciated!

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