Generic Hopf bifurcation from lines of equilibria without parameters: III. Binary oscillations *

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Abstract

We consider discretized systems of hyperbolic balance laws. Decoupling of the flow, associated to a central difference scheme, can lead to binary oscillations — even and odd numbered grid points, separately, provide time-evolutions of two distinct, different, separate profiles.

Investigating the stability of this decoupling phenomenon, we encounter Hopf-like bifurcations in the absence of parameters. With some computer algebra assistance, we describe the qualitative behavior near these bifurcation points. In particular we observe distinct even/odd profiles which oscillate periodically in time and, for arbitrarily fine discretization, exhibit preferred, nonzero phase relationships between adjacent discretization points.

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1 Introduction

Binary oscillations have been observed, both numerically and analytically, in certain discretizations of systems of nonlinear hyperbolic conservation laws; see [LL96]. More specifically, we consider systems of hyperbolic balance laws of the form

$$u_t + f(u)_x = g(u). \tag{1.1}$$

Semidiscretization of $x \in \mathbb{R}$ with step size $\epsilon/2 > 0$ can, for example, be performed by a central difference scheme

$$\dot{u}_k + \epsilon^{-1}(f(u_{k+1}) - f(u_{k-1})) = g(u_k) \tag{1.2}$$

As a cautioning remark we hasten to add that we do *not* recommend this particular discretization for numerical purposes. Rather, it is our goal to investigate peculiar short range oscillation phenomena of system (1.2). Rescaling time, we obtain the equivalent system

$$\dot{u}_k = \epsilon g(u_k) - f(u_{k+1}) + f(u_{k-1}) \tag{1.3}$$

Note how this system decouples into a direct product flow, if $u_{k+2} = u_k$, for all $k \in \mathbb{Z}$. Indeed, any solution $u_0(t)$ of $\dot{u}_0 = \epsilon g(u_0)$ gives rise to a solution of (1.3) satisfying

$$\begin{aligned} u_1(t) &= u_0(t+\chi) \\ u_{k+2}(t) &= u_k(t), \qquad \text{for all } k \end{aligned} \tag{1.4}$$

for any fixed choice of $\chi \in \mathbb{R}$.

It is the goal of our present paper to investigate loss of stability of this decoupling phenomenon. In general, u_{2k} and u_{2k-1} can define consistently smooth, but different profiles $x \mapsto u(t,x)$. We consider only the simplest case

$$k \ (\mathrm{mod} 4) \tag{1.5}$$

of u_k defining the corners of a square. In this case, decoupling phenomena as above have been discovered by [AA86] in the slightly different context of periodic orbits of linearly coupled oscillators. For more intricate, nonplanar graphs of coupled oscillators supporting such decoupling effects see [AF89].

For simplicity, we consider systems of two balance laws, $u_i \in \mathbb{R}^2$. To facilitate our calculations further, we impose an S^1 equivariance condition

$$\begin{aligned} f(R_\varphi u) &= R_\varphi f(u), \\ g(R_\varphi u) &= R_\varphi g(u) \end{aligned} \tag{1.6}$$

under all rotation matrices

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{1.7}$$

Denoting the Euclidean norm by $|u|$, we can therefore write

$$\begin{array}{rcl} f(u) & = & a(|u|^2)u \\ g(u) & = & b(|u|^2)u. \end{array} \tag{1.8}$$

where the values of a, b are scalar multiples of rotation matrices.

We assume that the decoupled vector field $\dot{u} = g(u)$ of the reaction term alone possesses an exponentially stable periodic orbit. Assuming

$$b(|u|^2) = \begin{pmatrix} 1-|u|^2 & -\Omega \\ \Omega & 1-|u|^2 \end{pmatrix}, \tag{1.9}$$

we normalize the periodic orbit to $|u| = 1$, its frequency to Ω, and its exponential rate of attraction to -2. On the slow time scale $\dot{u} = \epsilon g(u)$ these latter values become $\epsilon\Omega$ and -2ϵ, of course.

On the square (1.3), (1.5) decoupling produces an invariant 2-torus foliated by these periodic orbits; see (1.4). Normalizing the time shift χ, these solutions are given explicitly by

$$T^2: \qquad \begin{array}{rcl} u_0(t) & := & R_{\epsilon\Omega t}e_1 \\ u_1(t) & := & u_0(t+(\epsilon\Omega)^{-1}\chi) \;=\; R_\chi u_0(t) \\ u_2(t) & := & u_0(t) \\ u_3(t) & := & u_1(t) \;=\; R_\chi u_2(t) \end{array} \tag{1.10}$$

Here $\chi \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, due to S^1-equivariance, and $e_1 \in \mathbb{R}^2$ denotes the first unit vector. Our investigation of binary oscillations will focus on the detailed dynamics near the decoupled 2-torus (1.10).

Our assumption (1.6) on S^1-equivariance allows us to eliminate one variable from our eight-dimensional vector field (1.3), (1.5). Indeed, the flow of (1.3), (1.5) maps S^1-orbits onto S^1-orbits, by equivariance under the S^1-action

$$(R_\varphi \mathbf{u})_i := R_\varphi u_i \tag{1.11}$$

on $\mathbf{u} = (u_0, \ldots, u_3) \in \mathbb{R}^8$. The induced flow on the space of group orbits can be computed in explicit coordinates, representing a cross section to the group orbits; see (1.12) below.

For specific calculations, we will use polar coordinates (r_k, φ_k) for $u_k \in \mathbb{R}^2$; see sections 3 and 4.

Relating back to dynamics, consider the Poincaré return map to any Poincaré cross section X through any of the periodic orbits on our 2-torus T^2. In particular, the section X is also transverse to the S^1-action (1.11) which is free near the 2-torus. We can therefore rewrite the Poincaré map as the time $t = 2\pi(\epsilon\Omega)^{-1}$ map of a suitable associated flow

$$\dot{x} = F(x) \tag{1.12}$$

representing the induced flow on the seven-dimensional Poincaré cross section X. The fixed Poincaré return time can in fact be achieved by incorporating a scalar Euler multiplier into the induced flow on X. We will make an explicit choice for X later, based on polar coordinates.

In the coordinates $x \in X$, the periodic 2-torus T^2 from (1.10) becomes a one-dimensional curve of equilibria. Indeed, time action and S^1-action coincide on T^2. Therefore the fixed points of the Poincaré return map given by $T^2 \cap X$ coincide with equilibria of the induced flow $\dot{x} = F(x)$ on X. In other words, the relative equilibria on T^2, relative to the S^1-action, become equilibria on the local space X of S^1-orbits.

As long as the curve of equilibria remains normally hyperbolic, the local dynamics has been clarified by [Sho75], [Fen77], [HPS77], and others; see also [Shu87], [Wig94]. Locally, the dynamics is fibered into invariant leaves over each equilibrium, with dynamics in each leaf governed by hyperbolic linearization.

Bifurcations from lines of equilibria in absence of parameter have been investigated in [FLA98] from a theoretical view point. We briefly recall the pertinent result, for convenience. As in (1.12), now consider general C^5 vector fields $\dot{x} = F(x)$ with $x \in X = \mathbb{R}^n$. We assume a one parameter curve of equilibria

$$0 = F(x(\chi)) \tag{1.13}$$

tangent to $x'(\chi_0) \neq 0$ at $\chi = \chi_0,\ x(\chi_0) = x^0$. At $\chi = \chi_0$, we assume the Jacobi matrix $F'(x^0)$ to be hyperbolic, except for a trivial kernel vector along the direction of $x'(\chi_0)$ and a complex conjugate pair of simple purely imaginary eigenvalues $\mu(\chi), \overline{\mu(\chi)}$ crossing the imaginary axis transversely as χ increases through $\chi = \chi_0$:

$$\begin{aligned} \mu(\chi_0) &= i\omega_0, \qquad \omega_0 > 0 \\ \operatorname{Re}\mu'(\chi_0) &\neq 0 \end{aligned} \tag{1.14}$$

Let E be the two-dimensional real eigenspace of $F'(x^0)$ associated to $\pm i\omega_0$.

Coordinates in E are chosen as coefficients of the real and imaginary parts of the complex eigenvector associated to $i\omega_0$. Note that the linearization acts as a rotation with respect to these not necessarily orthogonal coordinates. Let P_0 be the one-dimensional eigenprojection onto the trivial kernel along the direction $x'(\chi_0)$. Our final nondegeneracy assumption then reads

$$\Delta_E P_0 F(x^0) \neq 0. \tag{1.15}$$

Here the Laplacian Δ_E is evaluated with respect to the above eigenvector coordinates in the eigenspace E of $\pm i\omega_0$. Fixing positive χ-orientation, we can consider $\Delta_E P_0 F(x_0)$ as a real number. Depending on the sign

$$\eta \;:=\; \mathrm{sign}(\mathrm{Re}\,\mu'(\chi_0)) \cdot \mathrm{sign}(\Delta_E P_0 F(x^0)) \tag{1.16}$$

we call the "bifurcation" point $x = x^0$ *elliptic* if $\eta = -1$, and *hyperbolic* for $\eta = +1$.

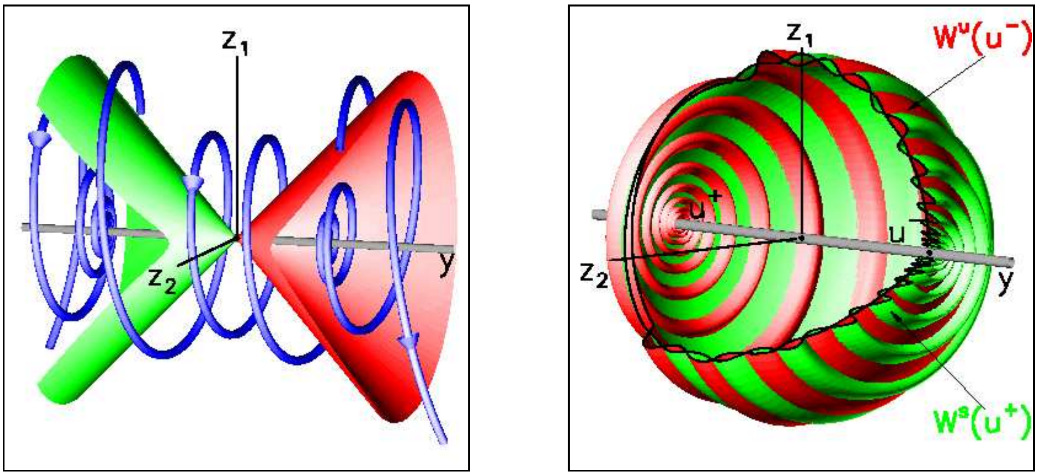
The following result from [FLA98] investigates the qualitative behavior of solutions in a normally hyperbolic three-dimensional center manifold to $x = x^0$.

Theorem 1.1
Let assumptions (1.13)-(1.15) hold for the C^5 vector field $\dot{x} = F(x)$ along the curve $x(\chi)$ of equilibria. Then the following holds true in a neighborhood U of $x = x^0$ within a three-dimensional center manifold to $x = x^0$.

In the hyperbolic case, $\eta = +1$, all nonequilibrium trajectories leave the neighborhood U in positive or negative time direction (possibly both). The stable and unstable sets of $x = x^0$, respectively, form cones around the positive/negative direction of $x'(\chi_0)$, with asymptotically elliptic cross section near their tips at $x = x_0$. These cones separate regions with different convergence behavior. See fig. 1.1a).

In the elliptic case, $\eta = -1$, all nonequiblibrium trajectories starting in U are heteroclinic between equilibria $x^{\pm} = x(\chi_{\pm})$ on opposite sides of $\chi = \chi_0$. If $F(x)$ is real analytic near $x = x^0$, then the two-dimensional strong stable and strong unstable manifolds of $x^{\pm}$ within the center manifold intersect at an angle which possesses an exponentially small upper bound in terms of $|x^{\pm} - x^0|$. See fig. 1.1b).

Our main result is theorem 2.1 in section 2. It provides specific examples of flux functions f and reaction terms g which realize the elliptic variant $\eta = -1$ of theorem 1.1



Case a) hyperbolic, $\eta = +1$. Case b) elliptic, $\eta = -1$.

Figure 1.1: Dynamics near Hopf bifurcation from lines of equilibria.

in binary oscillation systems (1.3), (1.5), only. The result holds in the limit $\epsilon \searrow 0$ of small discretization steps. Both the hyperbolic and the elliptic variants occur in viscous profiles of systems of hyperbolic balance laws, see [FL98]. For applications to binary oscillations in Ginzburg-Landau and nonlinear Schrödinger equations as well as a more global study of decoupling in squares of additively coupled oscillators, see [AF98].

The proof of theorem 2.1 is spread over the remaining two sections. In section 3 we present a detailed analysis of the linearization near the curve $x(\chi)$ of equilibria on X. In particular, we verify assumptions (1.14) on transverse crossing of simple, purely imaginary eigenvalues. The nondegeneracy condition (1.15) is verified in section 4, completing the proof of theorem 2.1.

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2 Result

Setting up for our main result, theorem 2.1, we further specify our square binary oscillation system

$$\dot{u}_k = \epsilon g(u_k) + f(u_{k-1}) - f(u_{k+1}), \quad k(\mathrm{mod} 4). \tag{2.1}$$

In the S^1-equivariant formulation (1.8), we have already completely specified

$$g(u) \;=\; b(|u|^2)u \;=\; \begin{pmatrix} 1-|u|^2 & \Omega \\ \Omega & 1-|u|^2 \end{pmatrix} u; \tag{2.2}$$

see (1.9). To avoid formula overkill by chain and product rules, we specify the derivative

$$A(u) = f'(u) = (a(|u|^2)u)' \tag{2.3}$$

at $u = e_1 = (1, 0)$ to be given by the symmetric indefinite matrix

$$A(e_1) = \begin{pmatrix} c & -1 \\ -1 & 0 \end{pmatrix} \tag{2.4}$$

In terms of $a(|u|^2)$; this choice corresponds to

$$a(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad a'(1) = \begin{pmatrix} \frac{1}{2}c & 1 \\ -1 & \frac{1}{2}c \end{pmatrix} \tag{2.5}$$

To ensure transverse crossing (1.14) of purely imaginary eigenvalues in the limit $\epsilon \searrow 0$, we assume $|c| < 1$ is nonzero:

$$|c| < 1, \quad c \neq 0 \tag{2.6}$$

The nondegeneracy condition (1.15) on $\Delta_E P_0 P(0)$ will hold due to the same assumption (2.6).

Theorem 2.1
Consider the square binary oscillation system (2.1) with specific nonlinearities (2.2)-(2.6). Then, for $0 < \epsilon < \epsilon_0$ small enough, Hopf points of elliptic type occur at the periodic solution through $u = u(\chi)$ given by $u_0 = u_2 = e_1$, $u_1 = u_3 = R_{\chi(\epsilon)} e_1$.

Here $\chi = \chi(\epsilon) \in (0, \pi/4)$ satisfies

$$\cos(2\chi(\epsilon)) = \frac{2-\epsilon^2}{2+c^2} \tag{2.7}$$

More precisely, the induced flow $\dot{x} = F(x)$ on the space of S^1-orbits, represented by a Poincaré cross section X to the above periodic orbit, satisfies assumptions and conclusions of theorem 1.1 for the hyperbolic / elliptic types $\eta = \pm 1$ in a neighborhood $U = U_\epsilon$ of $u = u(\chi(\epsilon))$. The type determining sign η is given by

$$\eta = -1, \tag{2.8}$$

independently of the choice of c in (2.6). In particular the stability region of decoupling into separate, phase-related periodic solutions on odd-labeled / even-labeled discretization points includes a full neighborhood U_ϵ of $u(\chi(\epsilon))$ with a preferred sign $\chi - \chi(\epsilon)$ *for the phase shift χ of decoupling periodic solutions $u(\chi)$.*

The proof of this theorem consists of checking the transverse crossing assumption (1.14) and the nondegeneracy condition (1.15) of theorem 1.1. In the limit $\epsilon \searrow 0$, these two conditions are checked in sections 3 and 4, respectively.

3 Eigenvalue crossing

In this section we provide the linear analysis for Hopf points of purely imaginary eigenvalues along the 2-torus T^2 of decoupled periodic orbits $u(t) = (u_0(t), ..., u_3(t))$ given by

$$\begin{array}{rcccl} u_2(t) & = & u_0(t) & = & R_{\epsilon\Omega t} e_1 \\ u_3(t) & = & u_1(t) & = & R_\chi u_0(t) \end{array} \tag{3.1}$$

see (1.10). Passing to polar coordinates

$$u_k = r_k R_{\varphi_k} e_1 \tag{3.2}$$

we explicitly factor out the S^1-action

$$(R_\varphi \mathbf{u})_k = r_k R_{\varphi_k + \varphi} e_1 \tag{3.3}$$

This explicitly converts the 2-torus T^2 into a line of equilibria $x(\chi)$ of $\dot{x} = F(x)$ in a Poincaré cross section X; see (1.12), (3.9). We compute the linearization along the equilibria $x(\chi)$ and determine the location of Hopf points at $\chi = \chi(\epsilon)$, in the limit $\epsilon \searrow 0$. We then determine an explicit expression for the crossing direction

$$\mathrm{Re}\ \mu'(\chi(\epsilon)) \tag{3.4}$$

of the Hopf eigenvalues. For later use, we also determine $\epsilon-$expansions for the Hopf eigenvalues $\mu = \mu(\chi(\epsilon)) = i\omega(\epsilon)$ and the Hopf eigenvectors $v = v(\epsilon)$.

Calculations in this and the following section were performed with Mathematica and Maple; any other symbolic calculation package should also do.

We begin with a transformation to polar coordinates (3.2). In variables (r_k, φ_k), $k(\mathrm{mod} 4)$, equations (2.1) for binary oscillations mod 4 read

$$\begin{array}{rcll} \dot r_k &=& \epsilon r_k(1-r_k^2) & +\rho(r_{k-1})\cos(\varphi_{k-1}-\varphi_k+\psi(r_{k-1})) \\ &&& -\rho(r_{k+1})\cos(\varphi_{k+1}-\varphi_k+\psi(r_{k+1})), \\ \dot\varphi_k &=& \epsilon\Omega & +r_k^{-1}\rho(r_{k-1})\sin(\varphi_{k-1}-\varphi_k+\psi(r_{k-1})) \\ &&& -r_k^{-1}\rho(r_{k+1})\sin(\varphi_{k+1}-\varphi_k+\psi(r_{k+1})). \end{array} \tag{3.5}$$

Here the new nonlinearities $\rho(r), \psi(r)$ are related to the flux function $f(u) = a(|u|^2)u$ by $a(r^2) = r^{-1}\rho(r)R_{\psi(r)}$. In fact assumptions (2.5) on $a(1), a'(1)$ translate as

$$\begin{array}{rclcrcl} \rho(1) &=& 1; && \rho'(1) &=& -1; \\ \psi(1) &=& \pi/2; && \psi'(1) &=& -c. \end{array} \tag{3.6}$$

The 2-torus (3.1) of decoupled binary oscillations becomes

$$\begin{array}{rcl} r_k &\equiv& 1, \\ \varphi_1 &\equiv& \varphi_0 + \chi, \\ \varphi_{k+2} &\equiv& \varphi_k \end{array} \tag{3.7}$$

in polar coordinates, with flow

$$\dot\varphi_k = \epsilon\Omega. \tag{3.8}$$

Note that the right hand side of (3.8) is proportional to the infinitesimal generator of the S^1–action (3.3). We choose an orthogonal section

$$X \;=\; \{(\mathbf{r}, \boldsymbol{\varphi}) \,|\, \varphi_0 + ... + \varphi_3 = 0\} \;= <\mathbf{e}>^\perp \tag{3.9}$$

in coordinates $\mathbf{r} = (r_0, ..., r_3)$, $\boldsymbol{\varphi} = (\varphi_0, ..., \varphi_3)$, $\mathbf{e} = (0, ..., 0, 1, ..., 1)$. The vector field

$$\dot x = F(x) \tag{3.10}$$

of the induced flow on S^1–orbits, represented by $x \in X$, is then given by orthogonal projection of (3.5) onto the section X. In particular, the term $\epsilon\Omega$ disappears in this projection and the line of equilibria is given by

$$x(\chi): \qquad r_k = 1, \quad \varphi_k = (-1)^{k+1}\chi/2. \tag{3.11}$$

To simplify our calculations, we restrict the time shift χ to the interval

$$\chi \in [0, \tfrac{1}{2}\pi]. \tag{3.12}$$

This can be done without loss of generality due to the D_2–symmetry of the square ring coupling of our system (1.3),(1.5). Indeed, the D_2–symmetry is generated by the rotation α and reflection β

$$\begin{array}{rcccc} \alpha & : & (0,1,2,3) & \mapsto & (1,2,3,0) \\ \beta & : & (0,1,2,3) & \mapsto & (2,1,0,3) \end{array} \tag{3.13}$$

of the indices. This leads to an equivariance of the system (1.3),(1.5) under

$$\begin{array}{rcl} \alpha & : & u_k \mapsto u_{k+1}, \quad k \ (\mathrm{mod} 4) \\ \beta & : & u_0 \mapsto -u_2,\ u_2 \mapsto -u_0,\ u_1 \mapsto u_1,\ u_3 \mapsto u_3, \end{array} \tag{3.14}$$

when we recall that f and g are odd by (1.6). In terms of the time shift χ these transformations are

$$\begin{array}{rcccl} \alpha & : & \chi & \mapsto & 2\pi - \chi \qquad \text{and} \\ \beta & : & \chi & \mapsto & \chi + \pi. \end{array} \tag{3.15}$$

This immediately gives the fundamental domain (3.12).

The linearization of F at $x(\chi)$ is given by restriction and projection of the 8×8 linearization $L(\chi)$ of the original polar coordinate vector field (3.5) at the relative equilibrium $x(\chi)$. Rather than writing $L(\chi)$ out explicity, we recall equivariance of (3.5) and invariance of $x(\chi)$ under the action $k \mapsto k+2$ of shifting indices k by 2. The four-dimensional representation subspaces $V^\pm$ under this action are given by

$$V^\pm := \{r_{k+2} = \pm r_k,\ \varphi_{k+2} = \pm\varphi_k\}. \tag{3.16}$$

By equivariance, these are invariant subspaces of the linearization $L(\chi)$. Due to decoupling, V^+ is in fact also invariant under the nonlinear flow.

Let $L^{\pm}(\chi)$ denote the respective restrictions of $L(\chi)$; explicitly

$$\begin{aligned} L^{+}(\chi) &= 2\begin{pmatrix} -\epsilon & 0 & & \\ 0 & 0 & & \\ & & -\epsilon & 0 \\ & & 0 & 0 \end{pmatrix}, \\ L^{-}(\chi) &= 2\begin{pmatrix} -\epsilon & 0 & -c\cos\chi-\sin\chi & \cos\chi \\ 0 & 0 & -c\sin\chi+\cos\chi & \sin\chi \\ c\cos\chi-\sin\chi & -\cos\chi & -\epsilon & 0 \\ -c\sin\chi-\cos\chi & \sin\chi & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.17}$$

with respect to coordinates $(r_0, \varphi_0, r_1, \varphi_1)$ in $V^{\pm}$.

Obviously only $L^{-}(\chi)$ can carry purely imaginary eigenvalues. In the following, we therefore restrict our attention to V^{-}. The characteristic polynomial p of $L^{-}(\chi)$ on V^{-} is given by

$$\begin{aligned} 0 &= p(\mu, \epsilon, \gamma) := \\ &= \mu^4 + 4\epsilon\mu^3 + 2((c^2+4)\gamma + c^2 + 2\epsilon^2)\mu^2 + \\ &\quad +16\epsilon\gamma\mu + 8(2 + \epsilon^2(\gamma - 1)) \end{aligned} \tag{3.18}$$

with the abbreviation

$$\gamma = \cos(2\chi). \tag{3.19}$$

Decomposing into real and imaginary parts, we immediately see that imaginary eigenvalues $\mu = i\omega$ satisfy

$$\omega^2 = 4\gamma > 0 \tag{3.20}$$

and only occur for ϵ, γ satisfying

$$\gamma = \gamma(\epsilon) = \frac{2-\epsilon^2}{2+c^2}. \tag{3.21}$$

This proves (2.7) of theorem 2.1. Before computing the associated eigenspace, we address the transverse crossing condition (3.4) for the eigenvalues $\mu = \mu(\epsilon, \gamma)$ near Hopf points. Note that

$$\mathrm{Re}\ \mu'(\chi) = -2\sqrt{1-\gamma^2}\,\mathrm{Re}\ \partial_\gamma \mu(\epsilon, \gamma) \tag{3.22}$$

at $\gamma = \gamma(\epsilon)$, by (3.19) and the chain rule. At $\epsilon = 0, \gamma = \gamma(0), \mu = i\omega(0)$ we have

$$\partial_\mu p = 8\sqrt{2} i c^2 (4 + c^2)(2 + c^2)^{-3/2} \neq 0. \tag{3.23}$$

Hence the implicit function theorem applies:

$$\partial_\gamma \mu(\epsilon, \gamma) = -\partial_\gamma p / \partial_\mu p \tag{3.24}$$

At $\epsilon = 0$ we obtain

$$\partial_\gamma \mu = -\sqrt{2} i c^{-2} \sqrt{2+c^2} \tag{3.25}$$

with vanishing real part. Totally differentiating (3.24) with respect to ϵ along the path $\epsilon \geq 0, \gamma = \gamma(\epsilon), \mu = i\omega(\epsilon)$, we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \partial_\gamma \mu \;=\; -\partial_\epsilon(\partial_\gamma p / \partial_\mu p) \;=\; -4\frac{(c^2+2)^2}{c^4(c^2+4)}. \tag{3.26}$$

This yields the expansion

$$\mathrm{Re}\ \mu'(\chi) \;=\; 8\frac{c^2+2}{|c|^3(c^2+4)^{1/2}}\epsilon + O(\epsilon^2). \tag{3.27}$$

In particular

$$\mathrm{sign\ Re}\ \mu'(\chi) = +1 \tag{3.28}$$

for small $\epsilon > 0$.

For later use, we also provide an expansion

$$v(\epsilon) = v^0 + \epsilon v^1 + O(\epsilon^2) \tag{3.29}$$

for the complex eigenvector $v(\epsilon)$ associated to the imaginary eigenvalue

$$\mu = i\omega(\epsilon) = i\omega_0 + O(\epsilon^2). \tag{3.30}$$

Normalization of $v(\epsilon)$ will not be necessary. We decompose

$$L_\epsilon^-(\chi(\epsilon)) = L_0 + \epsilon L_1 + ... \tag{3.31}$$

where in fact $L_0 = L_0^-(\chi(0)),\ \epsilon L_1 = L^+$. Comparing coefficients of ϵ^0, ϵ^1 in

$$(L_0 + \epsilon L_1 + ...)(v^0 + \epsilon v^1 + ...) = (i\omega_0 + ...)(v^0 + \epsilon v^1 + ...) \tag{3.32}$$

we immediately see

$$\begin{aligned} (L_0 - i\omega_0)v^0 &= 0, \\ (L_0 - i\omega_0)v^1 &= -L_1 v^0. \end{aligned} \tag{3.33}$$

With the abbreviation $\tilde{c} := \sqrt{c^2+4}$ and some substitutions $c^2 = \tilde{c}^2 - 4$, explicit solutions are given by

$$\begin{aligned} v^0 &= \begin{pmatrix} -i(c^2\tilde{c} + c|c|) \\ -i(c^3\tilde{c} + c\tilde{c} - 2|c|) \\ c^3 - |c|\tilde{c} + 2c \\ c^2(c^2+3) \end{pmatrix} \\ v^1 &= \frac{\sqrt{2}}{4c\sqrt{c^2+2}(c^2+4)} \begin{pmatrix} -c^6|c| + 3c^5\tilde{c} - 6c^4|c| + 6c^3\tilde{c} - 20c^2|c| + 8c\tilde{c} - 16|c| \\ -c^7|c| - 3c^5|c| - 2c^4\tilde{c} + 6c^3|c| + 24c|c| \\ i(c^5|c|\tilde{c} + 3c^6 + 4c^3|c|\tilde{c} + 10c^4 + 8c|c|\tilde{c} + 8c^2) \\ i(c^6|c|\tilde{c} + c^4|c|\tilde{c} - 4c^5 - 6c^2|c|\tilde{c} - 4c^3 - 8|c|\tilde{c} + 16c) \end{pmatrix} \end{aligned} \tag{3.34}$$

Note that v^0, v^1 are complex orthogonal.

4 Nondegeneracy

In this section we check the nondegeneracy condition

$$\Delta_E P_0 R(x) \neq 0 \tag{4.1}$$

in the limit $\epsilon \searrow 0$; see (1.15). Here the Hopf point $x = x^\epsilon$, given in polar coordinates $(r_k^\epsilon, \varphi_k^\epsilon)$, lies in the section $X = \langle \underline{\mathbf{e}} \rangle^\perp = \{\varphi_0 + \ldots + \varphi_3 = 0\}$ and satisfies

$$\begin{aligned} r_k^\epsilon &= 1 \\ \varphi_k^\epsilon &= \tfrac{1}{2}(-1)^{k+1}\chi^\epsilon \\ \gamma^\epsilon &= \cos 2\chi^\epsilon \;=\; \tfrac{2-\epsilon^2}{2+c^2} \end{aligned} \tag{4.2}$$

see (3.9), (3.11), (3.21). The projection P_0 is the eigenprojection onto the one-dimensional kernel of the linearization in X. By our $V^\pm-$ decomposition (3.16), (3.17), the full linearization $L(\chi^\epsilon)$ possesses kernel only in V^+. Indeed, the characteristic polynomial $p = p(\mu, \epsilon, \gamma^\epsilon)$ on V^- does not possess zero eigenvalues; see (3.18). The two-dimensional kernel of $L^+(\chi^\epsilon)$ in V^+ is given by

$$r_k = 0, \qquad \varphi_{k+2} = \varphi_k \tag{4.3}$$

for small $\epsilon > 0$; see (3.17) again. By restriction and orthogonal projection to X, we see that $P_0 F(x)$ is given by

$$P_0 F(x) = \frac{1}{2}(-F_0^\varphi + F_1^\varphi - F_2^\varphi + F_3^\varphi) \cdot \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \tag{4.4}$$

Here we have written the polar coordinate components of the original vector field (3.5) in the form

$$\dot{\varphi}_k = F_k^\varphi \tag{4.5}$$

Note that the unit vector in (4.4) points along the φ-components of the line $x(\chi)$ of equilibria in positive χ-direction. Moreover P_0 does not depend on ϵ. We can therefore consider $\Delta_E P_0 F(x)$ to be given by the real number

$$\Delta^\epsilon := \frac{1}{2}\Delta_{E^\epsilon}(-F_0^\varphi + F_1^\varphi - F_2^\varphi + F_3^\varphi)(x^\epsilon), \tag{4.6}$$

with only the Hopf point x^ϵ and the Hopf eigenspace E^ϵ depending on ϵ. Equivariance with respect to index change $k \mapsto k+2$ further simplifies expression (4.6) for Δ^ϵ. Indeed $E^\epsilon \subset V^-$, because the Hopf eigenspace E^ϵ results from $L^-(\chi^\epsilon)$; see (3.17). Restricted to V^-, the quadratic Hessian forms of F_k^φ and F_{k+2}^φ at x^ϵ coincide. Therefore (4.6) simplifies to

$$\Delta^\epsilon = \Delta_{E^\epsilon}(-F_0^\varphi + F_1^\varphi)(x^\epsilon) \tag{4.7}$$

Expanding Δ^ϵ to including first order terms in ϵ, we immediately notice that

$$\Delta^\epsilon = \Delta_{E^\epsilon}(-F_0^\varphi + F_1^\varphi)(x^0) + \mathcal{O}(\epsilon^2) \tag{4.8}$$

Indeed $x^\epsilon = x(\chi^\epsilon)$ depends only to second order on ϵ, as does χ^ϵ itself; see (3.19), (3.21). Therefore we only have to consider dependence of the Hopf eigenspace

$$E^\epsilon = \text{span}\,\{\text{Re}\ v(\epsilon), \text{Im}\ v(\epsilon)\} \tag{4.9}$$

on ϵ, to first order. We recall that a first order expansion

$$v(\epsilon) = v^0 + \epsilon v^1 + \dots \tag{4.10}$$

of complex eigenvectors $v(\epsilon)$ to the simple eigenvalue $\mu(\epsilon) = i\omega(\epsilon)$ near $i\omega(0)$ was derived in section 3; see (3.34). Also recall that $v(\epsilon)$ are not normalized.

Denoting second derivatives by D^2, we abbreviate the Hessian by

$$H_0 = -D^2 F_0^\varphi(x^0) + D^2 F_1^\varphi(x^0) \tag{4.11}$$

and expand

$$\begin{aligned} \Delta^\epsilon &= H_0[v(\epsilon), \bar{v}(\epsilon)] + \mathcal{O}(\epsilon^2) \\ &= H_0[v_0, \bar{v}_0] + 2\,\text{Re}\,(H_0[v_1, \bar{v}_0])\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.12}$$

It is worth noting here that indeed Δ_{E^ϵ} has to be evaluated with respect to the eigenbasis Re $v(\epsilon)$, Im $v(\epsilon)$ of E^ϵ and not with respect to an orthonormal basis. This follows from the proof of theorem 1.1 in [FLA98]. Indeed, the term Δ^ϵ arises in the normal form process after a linear transformation of the linearization to pure rotation in the Hopf eigenspace. The length of $v(\epsilon)$ is irrelevant in that analysis: only the sign of Δ^ϵ enters into the final result.

After these preparations we find

$$H_0[v_0, \bar{v}_0] \equiv 0. \tag{4.13}$$

The term of order ϵ can be considerably simplified to

$$2\,\text{Re}\,(H_0[v_1, \bar{v}_0]) \;=\; -8c^2((c^2+1)\sqrt{c^2+4}|c| - 2c) \;<\; 0. \tag{4.14}$$

From (4.12) - (4.14) we finally obtain

$$\operatorname{sign} \Delta^{\epsilon} = -1, \tag{4.15}$$

for small $\epsilon > 0$.

Proof of theorem 2.1:
Theorem 2.1 follows from theorem 1.1, proved in [FLA98]. The location (2.7) of Hopf points was derived in (3.21). Transverse crossing (1.16) of purely imaginary eigenvalues has been established in (3.27), (3.28) with

$$\operatorname{sign} \operatorname{Re} \mu'(\chi) = +1, \tag{4.16}$$

for small $\epsilon > 0$. Nondegeneracy condition (1.15) has been verified in (4.15) with

$$\operatorname{sign} \Delta_{E^{\epsilon}} P^0 F(x^{\epsilon}) = \operatorname{sign} \Delta^{\epsilon} = -1, \tag{4.17}$$

again for small $\epsilon > 0$. We therefore have shown that the assumptions of theorem 1.1 and the conclusions of theorem 2.1 hold with elliptic type η determined by

$$\eta = -1. \tag{4.18}$$

This completes the proof of theorem 2.1. $\bowtie$

As a concluding remark, we note that the existence of hyperbolic Hopf points in central difference schemes (1.2) has not been established yet. Further investigations of more general nonlinearities are necessary to clarify the possibility of such bifurcation points.

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