Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory *

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Abstract

Motivated by decoupling effects in coupled oscillators, by viscous shock profiles in systems of nonlinear hyperbolic balance laws, and by binary oscillation effects in discretizations of systems of hyperbolic balance laws, we consider vector fields with a one-dimensional line of equilibria, even in the absence of any parameters. Besides a trivial eigenvalue zero we assume that the linearization at these equilibria possesses a simple pair of nonzero eigenvalues which cross the imaginary axis transversely as we move along the equilibrium line.

In normal form and under a suitable nondegeneracy condition, we distinguish two cases of this Hopf-type loss of stability: *hyperbolic* and *elliptic*. Going beyond normal forms we present a rigorous analysis of both cases. In particular α-/ω-limit sets of nearby trajectories consist entirely of equilibria on the line.

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1 Introduction

A peculiar infinite degeneracy has been observed, more than a decade ago, in a square ring of four additively coupled oscillators

$$\dot{u}_k = F(u_k, u_{k-1} + u_{k+1}), \quad k(\mathrm{mod}4); \tag{1.1}$$

see [AA86] and [AF89]. A more specific example arises in one-dimensional complex Ginzburg-Landau equations, alias nonlinear Schrödinger equations, discretized by symmetric finite difference. See for example [Hak87] and [Kur84]. The mod 4 spatial period corresponds to (artificial) discretization effects.

Suppose the nonlinearity F is odd. Then the linear space

$$u_2 = -u_0, \quad u_3 = -u_1 \tag{1.2}$$

of anti-phase motions is a flow-invariant subspace of (1.1). Moreover, the dynamics on this subspace is governed by the totally decoupled system

$$\dot{u}_k = F(u_k, 0), \quad k = 0, 1. \tag{1.3}$$

Suppose, for example, that (1.3) with $k = 0$ possesses an exponentially attracting time periodic solution $u_0(t)$, say with period 2π. Then we obtain an invariant 2-torus of (1.1), foliated by the 2π-periodic solutions

$$u_1(t) = u_0(t + \chi) \tag{1.4}$$

with arbitrarily fixed phase angle $\chi \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In a Poincaré cross section, we obtain a line of fixed points of the Poincaré return map. Understanding the possible transitions from stability to instability along the decoupled dynamics on the 2-torus is one of the motivating examples driving the results of the present paper.

To simplify our analysis let us assume $u_k \in \mathbb{R}^2 \cong \mathbb{C}$ is real two-dimensional and F commutes with complex rotations $e^{i\varphi}$, that is

$$F(e^{i\varphi}u, e^{i\varphi}v) = e^{i\varphi}F(u, v). \tag{1.5}$$

We can then assume that the periodic solution

$$u_0(t) = e^{i\omega t}u_0(0) \tag{1.6}$$

is a relative equilibrium to the group action of $e^{i\varphi}$, provided that

$$F(u_0(0), 0) = i\omega \tag{1.7}$$

in complex notation. In example (1.4) above, we have normalized $\omega \neq 0$ to $\omega = 1$.

In the S^1-equivariant case it turns out that the Poincaré return map near the invariant, decoupled 2-torus can be written as the time $= 2\pi/\omega$ map of an autonomous flow

$$\dot{x} = f(x) \tag{1.8}$$

on the Poincaré cross section. See [AF98], [FL98], [FLA98] for complete details. In particular, the line of fixed points of the Poincaré return map becomes a line of equilibria of the vector field (1.8). Note how this line of equilibria is induced by the additive nearest-neighbor coupling in (1.1), together with S^1-equivariance condition (1.5). Similar decoupling phenomena in more complicated graphs of coupled oscillators have been observed in [AF89]. For an in-depth analysis of decoupling in the square ring see [AF98].

For now, we consider general vector fields (1.8) with a line of equilibria. With this degeneracy at hand, we investigate loss of stability along the line of equilibria under additional *non*degeneracy conditions. We first consider the real case, where stability is lost by a simple, nontrivial real eigenvalue crossing zero, along the equilibrium line. See theorems 1.1, 1.2 below. In theorems 1.4 and 1.5, we then address the more complicated complex case where the loss of stability is caused by a pair of simple, nonzero, purely imaginary eigenvalues.

As a warm-up, we first consider the case of real loss of stability. Restricting to a real two-dimensional center manifold, we can assume

$$x = (y, z) \in \mathbb{R}^2. \tag{1.9}$$

We choose coordinates y, z, without loss of generality, such that

$$\begin{aligned} \dot{y} &= f^y(y, z) \\ \dot{z} &= f^z(y, z) \end{aligned} \tag{1.10}$$

with $f = (f^y, f^z) \in C^2$ satisfying the three conditions

$$\begin{aligned} 0 &= f(y, 0) \\ 0 &= \partial_z f^z(0, 0) \\ 0 &\neq \partial^2_{yz} f^z(0, 0) \end{aligned} \tag{1.11}$$

Note how the first condition straightens out the line of equilibria to coincide with the y-axis. The second condition indicates that the nontrivial second eigenvalue of the linearization vanishes at $y = 0$. Indeed $\partial_z f^z(y, 0)$ is the second eigenvalue. The third

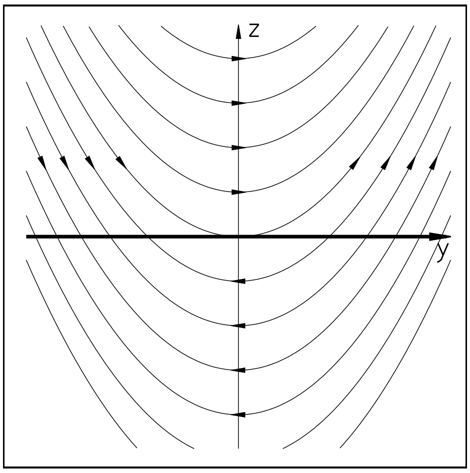


Figure 1.1: Real loss of stability of a line of equilibria; see (1.13).

condition ensures that this second eigenvalue crosses through zero at nonvanishing speed, as y increases through $y = 0$. We impose a final nondegeneracy condition

$$0 \neq \partial_z f^y(0,0). \tag{1.12}$$

This condition provides minimal degeneracy of the double zero eigenvalue at $x = 0$: a 2×2 Jordan block occurs.

Theorem 1.1 *Consider a line of equilibria in* $\mathbb{R}^2$ *with real loss of stability according to conditions (1.11), (1.12) above.*

Then there exists a C^1*-diffeomorphism which maps orbits of the flow (1.8) to orbits of the normal form*

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= zy \end{aligned} \tag{1.13}$$

locally near $x = (y, z) = 0$*; see fig. 1.1. The time orientation of orbits is preserved.*

Proof :
Because $f(y, 0) = 0$, by assumption (1.11), we can write our vector field in the form

$$\dot{x} = z\tilde{f}(x), \tag{1.14}$$

with $\tilde{f} \in C^1$, locally near $z = 0$. Note that (1.14) differs from the rescaled vector field

$$x' = \tilde{f}(x) \tag{1.15}$$

by a scalar Euler multiplier z which vanishes along the y-axis and provides the line of equilibria. Conditions (1.11), (1.12) now read

$$\begin{aligned} y' &= \tilde{f}^y \neq 0 \\ z' &= \tilde{f}^z = 0 \\ z'' &= \partial_y \tilde{f}^z \cdot \tilde{f}^y \neq 0 \end{aligned} \tag{1.16}$$

at $x = 0$. Because $\tilde{f}(0)$ is nonzero, we can transform the flow (1.15) to become

$$\begin{aligned} \tilde{y}' &= 1 \\ \tilde{z}' &= 0 \end{aligned} \tag{1.17}$$

By (1.16), the y-axis transforms to a curve

$$\tilde{z} = p(\tilde{y}) = a\tilde{y}^2 + \ldots \tag{1.18}$$

with $a \neq 0$, in these coordinates.

To prove our theorem, it is now sufficient to perform a fiber preserving C^1-diffeomorphism

$$\begin{aligned} \tilde{\tilde{y}} &= q(\tilde{y}) \\ \tilde{\tilde{z}} &= \tilde{z} \end{aligned} \tag{1.19}$$

such that the transformed y-axis becomes a parabola $\tilde{\tilde{z}} = \pm\tilde{\tilde{y}}^2$. This is achieved by putting

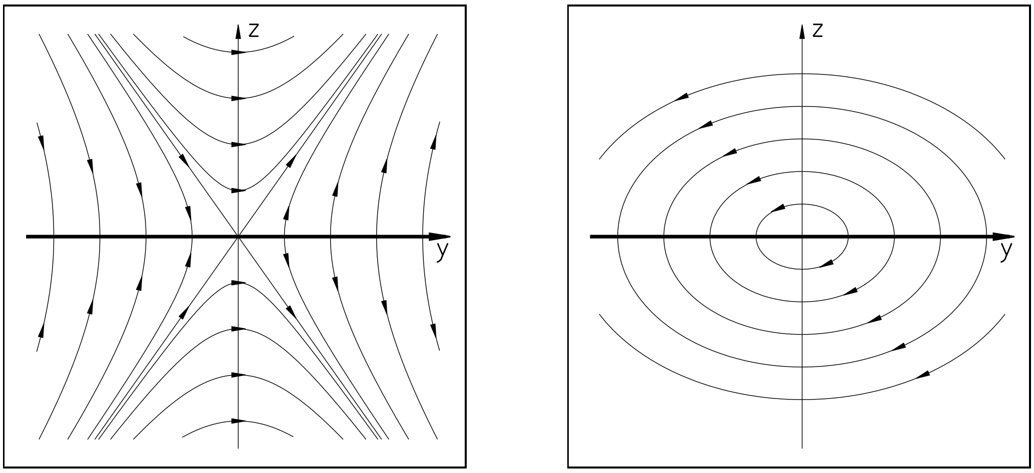
$$q(\tilde{y}) := \tilde{y}\sqrt{|p(\tilde{y})|\tilde{y}^{-2}} = |a|^{1/2}\tilde{y} + \ldots \tag{1.20}$$

The resulting vector field for $(\tilde{\tilde{y}}, \tilde{\tilde{z}})$ is then clearly orbit equivalent to (1.13), and the theorem is proved. $\bowtie$

As a preparation for the case of complex eigenvalues, we now consider a $\mathbb{Z}_2$-symmetric variant of the previous theorem. The role of z, here, will later be played by the radius variable of polar coordinates within the eigenspace to the purely imaginary eigenvalue – in normal form. Eliminating the effects of higher order terms, not in normal form, will be the main technical problem to be overcome in the present paper.

To be specific we again consider planar C^2-vector fields (1.8) such that $f = (f^y, f^z)$ satisfies the $\mathbb{Z}_2$-symmetry condition

$$\begin{aligned} f^y(y, -z) &= f^y(y, z) \\ f^z(y, -z) &= -f^z(y, z) \end{aligned} \tag{1.21}$$



Case a) hyperbolic, $\delta = +1$. Case b) elliptic, $\delta = -1$.

Figure 1.2: Real loss of stability with $\mathbb{Z}_2$-symmetry; see (1.23).

In other words, $x = (y(t), z(t))$ is a solution if and only if $(y(t), -z(t))$ is. Note that $\partial_z f^y = 0$, at $x = 0$, because f^y is even in z: nondegeneracy condition (1.12) fails. Instead, we assume

$$\delta := -\operatorname{sign}(\det \partial_x \partial_z f) \neq 0 \tag{1.22}$$

at $x = 0$.

Theorem 1.2 *Consider a line of equilibria in $\mathbb{R}^2$ with $\mathbb{Z}_2$-symmetric real loss of stability according to conditions (1.11), (1.21), (1.22) above.*

Then there exists a C^1-diffeomorphism which maps orbits of the flow (1.8) to orbits of the normal form

$$\begin{aligned} \dot{y} &= \tfrac{1}{2}\delta z^2 \\ \dot{z} &= zy \end{aligned} \tag{1.23}$$

locally near $x = (y, z) = 0$; see figure 1.2. The time orientation of orbits is preserved. We call $\delta = +1$ the hyperbolic *and $\delta = -1$ the* elliptic *case.*

Proof :
As in the proof of theorem 1.1, the x-orbits of $\dot{x} = f(x) = z\tilde{f}(x)$ are related to orbits of $x' = \tilde{f}(x)$ by an Euler multiplier z, see (1.14), (1.15). The symmetry conditions (1.21) imply time reversibility

$$\tilde{f}(\mathbf{R}x) = -\mathbf{R}\tilde{f}(x) \tag{1.24}$$

of the vector field $\tilde{f} \in C^1$ with respect to the involution $\mathbf{R}(y, z) = (y, -z)$. Conditions (1.11), (1.21) imply

$$\begin{aligned} \tilde{f}(0) &= 0 \\ \delta &= -\operatorname{sign}\det \partial_x \tilde{f}(0) \neq 0. \end{aligned} \tag{1.25}$$

Rescaling y, z, if necessary, we can assume

$$A := \partial_x \tilde{f}(0) = \begin{pmatrix} 0 & \frac{\delta}{2} \\ 1 & 0 \end{pmatrix} \tag{1.26}$$

without loss of generality.

We consider the hyperbolic case ($\delta = +1$) first. In this case we can linearize $\tilde{f}$ by a local C^1-diffeomorphism $\Phi(x) = x + \ldots$, due to Belitskii's theorem [Bel73]. Note that $\mathbf{R}$-equivariance of Φ,

$$\Phi(\mathbf{R}x) = \mathbf{R}\Phi(x), \tag{1.27}$$

can be assumed. Indeed, reversibility of A implies that the $\mathbf{R}$-averaged diffeomorphism

$$\tilde{\Phi}(x) := (\Phi(x) + \mathbf{R}^{-1}\Phi(\mathbf{R}x))/2 \tag{1.28}$$

also linearizes the flow. By $\mathbf{R}$-equivariance (1.27), the diffeomorphism Φ maps the y-axis, alias the fixed points of $\mathbf{R}$, into the y-axis. So, Φ automatically preserves the equilibrium line. This proves that Φ is an orbit equivalence between $\dot{x} = f(x)$ and the normal form (1.23), in the hyperbolic case.

In the elliptic case, we can invoke Hopf bifurcation for the reversible planar system $x' = \tilde{f}(x)$; see [Van89]. This provides us with a local family $x(y;t)$ of periodic orbits surrounding the origin. We normalize the family such that $x(y;0) = (y, 0)$, for $y > 0$. The minimal period of $x(y;\cdot)$ is given by $2\pi/\omega(y)$ with a differentiable function $\omega(y) > 0$. The transformation

$$(r, \varphi) \mapsto x(r, \omega(r)\varphi) \tag{1.29}$$

then conjugates the harmonic oscillator, written in polar coordinates (r, φ), diffeomorphically to the flow of $x' = \tilde{f}(x)$.

By reversibility, $x(r, \omega(r)\pi)$ also lies on the y-axis: half a period of $x' = \tilde{f}(x)$ is spent above and half a period below the reversibility axis $z = 0$, respectively. Therefore transformation (1.29) maps the y-axis, the equilibrium line of $\dot{x} = z\tilde{f}(x)$, into itself, providing an orbit equivalence between $\dot{x} = f(x)$ and the normal form (1.23) in the elliptic case as well. This proves the theorem. $\bowtie$

We now turn to the complex case, where loss of stability along our line of equilibria occurs by a pair of complex eigenvalues crossing the imaginary axis. Reducing to a three-dimensional center manifold, we keep the notation (1.8)-(1.10) for the vector field $\dot{x} = \tilde{f}(x),\ x = (y, z),\ y \in \mathbb{R}$, this time with real two-dimensional $z \in \mathbb{R}^2 = \mathbb{C}$. We again assume the equilibrium line to coincide with the y-axis, and (y, z) to be an eigenspace decomposition. In other words, assumption (1.11) will be replaced by

$$\begin{aligned} 0 &= f(y,0) \\ 0 &= \operatorname{Re}\lambda(0) \\ 0 &\neq \partial_y \operatorname{Re}\lambda(0) \end{aligned} \tag{1.30}$$

where we have written the linearization at $(y, 0)$ as

$$\partial_{(y,z)} \begin{pmatrix} f^y \\ f^z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda(y) \end{pmatrix} \tag{1.31}$$

in complex notation, with eigenvalues $\lambda(y) \in \mathbb{C} \setminus \mathbb{R}$. Denoting the usual real Laplacian with respect to the coordinate $z \in \mathbb{C} = \mathbb{R}^2$ by Δ_z, we finally require the nondegeneracy condition

$$\Delta_z f^y(0,0) \neq 0. \tag{1.32}$$

The following definition is similar, in spirit, to the $\mathbb{Z}_2$-symmetric case treated in theorem 1.2 above.

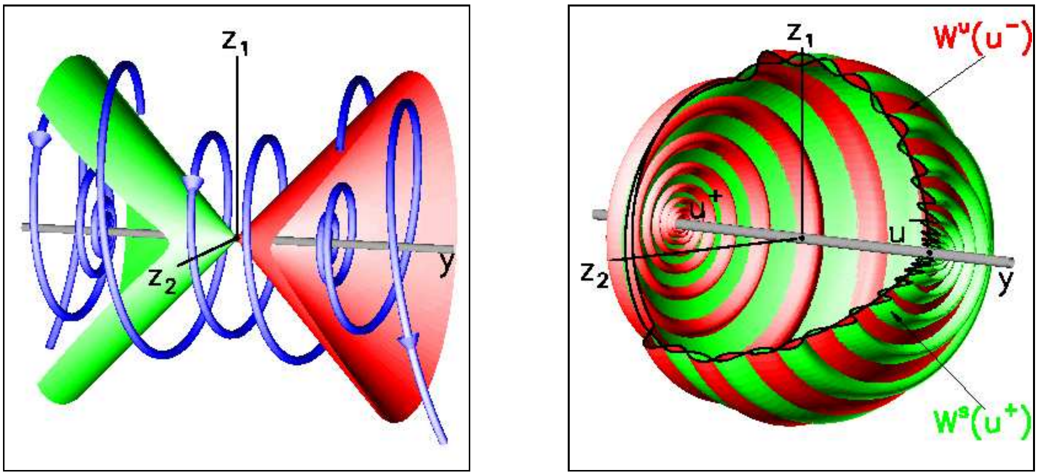
Definition 1.3 *For a complex loss of stability along a line of equilibria satisfying (1.30)-(1.32) above, let*

$$\delta := \operatorname{sign}\left((\partial_y \operatorname{Re}\lambda) \cdot (\Delta_z f^y)\right) = \pm 1 \qquad \text{at} \ \ (y,z) = (0,0). \tag{1.33}$$

We call the loss of stability hyperbolic, *if $\delta = +1$, and* elliptic, *if $\delta = -1$.*

Theorem 1.4 *Consider a line of equilibria in $\mathbb{R}^3$ with complex loss of stability according to conditions (1.30)-(1.32) above.*

Then the normal form, truncated at finite order and expressed in polar coordinates $z = re^{i\varphi}$, becomes equivariant with respect to rotations in φ. In particular, the resulting truncated differential equations in $(y, r) \in \mathbb{R}^2$ become independent of φ and satisfy assumptions and conclusions of theorem 1.2. Hyperbolic and elliptic cases correspond, respectively. The angle variable φ superimposes a rotation $\dot{\varphi} \approx \operatorname{Im}\lambda(0)$, locally.



Case a) hyperbolic, $\delta = +1$. Case b) elliptic, $\delta = -1$.

Figure 1.3: Dynamics near Hopf bifurcation from lines of equilibria.

Theorem 1.5 *Let the assumptions of theorem 1.4 hold, but now consider the original vector field $\dot{x} = f(x)$ near $x = 0$, of differentiability class at least C^5 and with higher order terms not necessarily in normal form.*

Then there exists $\epsilon > 0$ such that any solution $x(t)$ which stays in an ϵ-neighborhood of $x = 0$ for all positive or negative times (possibly both) converges to a single equilibrium on the y-axis.

In the hyperbolic case, all nonequilibrium trajectories leave the neighborhood U in positive or negative time directions (possibly both). The asymptotically stable and unstable sets of $x = 0$, respectively, form cones with tip regions tangent to the rotated images of the corresponding normal form lines of figure 1.2a); see figure 1.3a). These cones separate regions with different behavior of convergence.

In the elliptic case, all nonequilibrium trajectories starting sufficiently close to $x = 0$ are heteroclinic between equilibria $(y_\pm, 0)$ on opposite sides of $y = 0$. The two-dimensional strong stable and strong unstable manifolds of such equilibria $(y_\pm, 0)$ intersect at an angle with exponentially small upper bound in terms of $|y_\pm|$, provided f is real analytic; see figure 1.3b).

As a disclaimer we add two cautioning remarks on situations where our results do not apply: geometric singular perturbation theory, and reversibility in odd dimensions.

Geometric singular perturbation theory is an important and powerful method, where lines of equilibria appear. In the formal limit $\epsilon \to 0$ of

$$\begin{aligned} \dot{y} &= \epsilon f^y(y,z) \\ \dot{z} &= f^z(y,z) \end{aligned} \tag{1.34}$$

called the fast time system, an equilibrium curve appears, say $f^z(y,0) = 0$. For $\epsilon = 0$, the coordinate y indeed becomes a parameter and usual Hopf bifurcation applies. Note how both our elliptic and our hyperbolic case differ from the scenario of Hopf bifurcation. In fact, we do not assume invariant foliation of $\mathbb{R}^3$ given by the planes $y = \text{const.}$, for $\epsilon = 0$. For $\epsilon > 0$, on the other hand, the usual assumption $f^y = O(1)$ takes effect. It induces a slow drift along the invariant y-axis which leads to the phenomenon of *delayed bifurcation*; see [Arn94], ch. I.4.4, and the references there. Note how normal hyperbolicity breaks down at bifurcation, and the line of equilibria disappears into a slow drift. If the $\epsilon = 0$ equilibrium curve is tilted with respect to the (y,z)-decomposition, then even the invariant manifold breaks up for $\epsilon > 0$, with interesting dynamic consequences. See [Arn94] for recent progress.

Time reversibility in odd dimensions is another example, where lines of equilibria do appear canonically but our results do not apply. To be specific, consider the involution $\mathbf{R}$ in $\mathbb{R}^{2N+1}$ given by

$$\mathbf{R}(y,z) := (y,-z) \tag{1.35}$$

with $y \in \mathbb{R}^{N+1}$, $z \in \mathbb{R}^N$. Let $\dot{x} = f(x)$ with $x = (y,z)$ be time reversible: $f(\mathbf{R}x) = -\mathbf{R}f(x)$. Then $f^y(y,0) = 0$, by reversibility, and $(y,0)$ is an equilibrium if and only if

$$f^z(y,0) = 0 \in \mathbb{R}^N \tag{1.36}$$

holds for some $y \in \mathbb{R}^{N+1}$. Generically (1.36) nicely produces equilibrium curves in y-space. The linearization, however, satisfies

$$\partial_x f = -\mathbf{R}^{-1}\partial_x f \mathbf{R} \tag{1.37}$$

at points $x = (y,0)$ which are fixed under $\mathbf{R}$. In particular, the spectrum of the linearization $\partial_x f$ is point symmetric to the origin in $\mathbb{C}$ with a trivial eigenvalue located at zero, of course. A complex loss of stability as studied in the present paper, caused by a pair of simple complex eigenvalues crossing the imaginary axis, is therefore excluded.

Our paper is organized as follows. Section 2 proves theorem 1.4, by normal form reduction to theorem 1.2. Preparing for the proof of theorem 1.5 we also perform a spherical blow-up of the coordinates $x = (y,z) \in \mathbb{R}^3$ at $x = 0$. In other words, we introduce spherical polar coordinates. In section 3 we prove theorem 1.5 by explicit rescaling arguments. For applications of these results to coupled oscillators, alias complex Ginzburg-Landau equations, to viscous profiles of systems of hyperbolic balance laws, and to binary oscillations in discretizations of hyperbolic balance laws we refer to [AF98], [FL98], [FLA98].

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2 Normal form and spherical blow-up

In this section we begin our analysis of complex loss of stability, aiming at theorems 1.4 and 1.5. Throughout this section we therefore fix assumptions (1.30) - (1.32). Smoothness of f and of all transformations will be assumed as required by the desired order of normal forms. We derive the normal form for semisimple spectrum $\{0, \pm i\}$ of the linearization of

$$\dot{x} = f(x) \tag{2.1}$$

at $x = (y, z) = 0$, see proposition 2.1. Our normal form preserves the $y-$axis as a line of equilibria. We then prove theorem 1.4, establishing the relation of the formally truncated normal form with $\mathbb{Z}_2-$symmetric real loss of stability as established in theorem 1.2. Finally, in proposition 2.3 we recast our normal form into spherical polar coordinates, including higher order terms not in normal form. This prepares the proof of theorem 1.5 in the following section.

Proposition 2.1 *Under the assumptions of theorem 1.4 there exists a local diffeomorphism transforming $\dot{x} = f(x)$ to normal form, near $x = 0$ and up to any finite order K. In coordinates $x = (y, z) \in \mathbb{R} \times \mathbb{C}$ and polar coordinates $z = re^{i\varphi}$, the normal form of order K has the form*

$$\begin{array}{rcrcr} \dot{y} &=& r^2\, g^y(y, r^2) &+& \eta^y(y, re^{i\varphi}) \\ \dot{r} &=& ry\, g^r(y, r^2) &+& \eta^r(y, re^{i\varphi}) \\ \dot{\varphi} &=& g^\varphi(y, r^2) &+& r^{-1}\eta^\varphi(y, re^{i\varphi}) \end{array} \tag{2.2}$$

The normal form functions g^y, g^r, g^φ are polynomials in y and r^2, satisfying

$$\begin{array}{rcl} g^y(0,0) &=& \frac{1}{4}\Delta_z f^y(0,0) \\ g^r(0,0) &=& \partial_y \operatorname{Re}\lambda(0) \\ g^\varphi(0,0) &=& \operatorname{Im}\lambda(0) \end{array} \tag{2.3}$$

The error terms $\eta^y, \eta^r, \eta^\varphi$ are of order

$$O((|y|+|r|)^{K+1}). \tag{2.4}$$

Along points $(y,0)$ on the $y-$axis they satisfy

$$\eta^y = 0, \quad \eta^r = 0, \quad r^{-1}\eta^\varphi = 0 \tag{2.5}$$

Proof :
We use normal form theory as presented, for example in [Van89]. For semisimple spectrum, the remaining part is given simply by the average

$$\tilde{h}(x) := \frac{1}{2\pi}\int_0^{2\pi} \exp(-A\varphi)h(\exp(A\varphi)x)d\varphi \tag{2.6}$$

in each normal form step. Here h represents n-th order terms before normal form transformation, which become $\tilde{h}$, in normal form. The necessarily associated transformation in x, however, modifies higher order terms in each step. Therefore h does not coincide with f, in general, except for terms of second order. The linearization $A = \partial_x f(0)$ is diagonal in (y,z)-coordinates and takes the form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \tag{2.7}$$

if we normalize the eigenvalue $\lambda(0) = i$. Since the S^1-action of $\exp(A\varphi)$ leaves the $y-$axis fixed, the averaging (2.6) preserves equilibria on the $y-$axis. Moreover the vector field g is equivariant with respect to this S^1-action:

$$g(\exp(A\varphi)x) = \exp(A\varphi)g(x). \tag{2.8}$$

We now write out the original vector field $\dot{x} = f(x)$ in polar coordinates $x = (y,z),\ z = re^{i\varphi}$. An elementary calculation yields

$$\begin{array}{rcl} \dot{y} &=& f^y(y, re^{i\varphi}) \\ \dot{r} &=& \operatorname{Re}(e^{-i\varphi} f^z(y, re^{i\varphi})) \\ \dot{\varphi} &=& r^{-1}\operatorname{Im}(e^{-i\varphi} f^z(y, re^{i\varphi})) \end{array} \tag{2.9}$$

where $\dot{z} = f^z$ is written in complex notation. In complex coordinates $g = (\tilde{g}^y, \tilde{g}^z)$ the equivariance property (2.8) of the truncated normal form becomes

$$\begin{aligned} \tilde{g}^y(y, re^{i\varphi}) &= r^2 g^y(y, r^2) \\ re^{-i\varphi}\tilde{g}^z(y, re^{i\varphi}) &= r^2 g^z(y, r^2) \end{aligned} \tag{2.10}$$

Indeed the left hand sides, polynomial in $y, z, \bar{z}$ after truncation, are invariant under the S^1-action, and the right hand sides, polynomial in y, r^2, provide the general polynomial invariants for this action with vanishing constant terms. The absence of constant terms is caused by the line $g(y, 0) = 0$ of equilibria, preserved by normal form averaging (2.6).

We now decompose $g^z = yg^r + ig^\varphi$ into real and imaginary parts. Note that $\lambda(y) = g^z(y, 0)$ and $\partial_y \operatorname{Re}\lambda(y) = \partial_y \operatorname{Re} g^z(y, 0) = g^r$, at $y = 0$. Inserting g into (2.9) now proves the normal form (2.2).

To prove (2.3), we first observe that normal form averaging (2.6) does not change the linear part. This proves $yg^r + ig^\varphi = \lambda(y)$, at $r = 0$. It remains to compute g^y at $x = 0$. By normal form averaging (2.6) we have in real notation

$$\begin{aligned} g^y &= \tfrac{1}{2}\partial_r^2\left(\tfrac{1}{2\pi}\int_0^{2\pi} f^y(y, r\cos\varphi, r\sin\varphi)d\varphi\right) \\ g^y(0,0) &= \tfrac{1}{4}\Delta_z f^y(0,0) \end{aligned} \tag{2.11}$$

as defined in (1.32). This proves (2.3).

The error estimates (2.4) are immediate from (2.9) after putting terms up to order K into normal form. Similarly, (2.5) follows because the normal form transformation fixes the $y-$axis pointwise and the linearization is accounted for by (2.3). This proves the proposition. $\bowtie$

We are now ready to prove normal form theorem 1.4.

Proof (Theorem 1.4):

We have to study the truncated normal form given by (2.2) with identically vanishing error terms $\eta^y, \eta^r, \eta^\varphi$, that is

$$\begin{aligned} \dot{y} &= r^2\, g^y(y, r^2) \\ \dot{r} &= ry\, g^r(y, r^2) \\ \dot{\varphi} &= g^\varphi(y, r^2) \end{aligned} \tag{2.12}$$

Equivariance with respect to the S^1-action $\varphi \mapsto \varphi + \varphi_0$ is obvious, as is $\mathbb{Z}_2-$symmetry with respect to $r \mapsto -r$. To apply theorem 1.2 (with z there replaced by r), we have to check the sign condition (1.22) in the notation of the first two equations of (2.12). At $y = r = 0$ we compute

$$\begin{aligned} \det \partial_{(y,r)} \partial_r \begin{pmatrix} r^2 g^y \\ ry\, g^r \end{pmatrix} &= \det \begin{pmatrix} 0 & 2g^y \\ g^r & 2y\partial_r g^r \end{pmatrix} \\ &= -2g^y(0)g^r(0) \\ &= -\tfrac{1}{2}\Delta_z f^y(0,0) \cdot \partial_y \operatorname{Re} \lambda(0) \neq 0 \end{aligned} \tag{2.13}$$

Here we have used (2.3) and assumptions (1.30)–(1.32). Comparing the two definitions of the sign δ, namely (1.33) for the complex case and (1.22) for the $\mathbb{Z}_2-$symmetric real case, we note equality by (2.13). This completes the proof of theorem 1.4. ⋈

Corollary 2.2 *Under the assumptions of theorem 1.4, the normal form (2.2) of proposition 2.1 simplifies to*

$$\begin{aligned} \dot{y} &= \tfrac{1}{2}\delta r^2 + \eta^y \\ \dot{r} &= ry + \eta^r \\ \dot{\varphi} &= 1 \end{aligned} \tag{2.14}$$

with suitably rescaled time and with error terms η^y, η^r satisfying (2.4), (2.5) as before.

Proof :

Apply the transformations of the proof of theorem 1.2 to the normal form of proposition 2.1. Then rescale time, dividing the right hand side by an Euler multiplier $g^\varphi + r^{-1}\eta^\varphi$. This proves the corollary. ⋈

For spherical blow-up near $x = (y, re^{i\varphi}) = 0$, we now introduce spherical polar coordinates (R, ϑ, φ), $0 \leq \vartheta \leq \pi$, by

$$\begin{aligned} y &= R\cos\vartheta \\ r &= R\sin\vartheta \end{aligned} \tag{2.15}$$

Proposition 2.3 *In spherical polar coordinates (2.15), the normal form (2.14) of corollary 2.2 reads*

$$\begin{aligned} \dot{R} &= (R\sin\vartheta)^2\cos\vartheta \cdot (1 + \tfrac{\delta}{2}) + R^2\sin\vartheta \cdot \eta^R \\ \dot{\vartheta} &= (R\sin\vartheta)(\cos^2\vartheta - \tfrac{\delta}{2}\sin^2\vartheta) + R\sin\vartheta \cdot \eta^\vartheta \\ \dot{\varphi} &= 1 \end{aligned} \tag{2.16}$$

with error terms η^R, η^ϑ of order $O(R^{K-1})$ for $R \searrow 0$.

Proof :
Immediate consequence of corollary 2.2. The factor $\sin\vartheta$ in front of η^R, η^ϑ accounts for the vanishing of these error terms along the $y-$axis $r = 0$. $\bowtie$

3 Proof of theorem 1.5

Throughout this section, let assumptions (1.30)–(1.32) of theorem 1.5 hold. Our proof is based on the form (2.16) of our normal form in spherical polar coordinates (R, ϑ, φ), including error terms; see proposition 2.3. For $\delta = \pm 1$ as defined in (1.33), we distinguish the hyperbolic case $\delta = +1$ and the elliptic case $\delta = -1$. We first address the hyperbolic case, introducing a new angle coordinate $\psi = R^2\varphi$ and a rescaled time $d\tau = Rdt$. We then turn to the elliptic case, invoking Neishtadt's theorem [Nei84] to prove exponentially small splittings of strong stable/unstable manifolds.

We introduce rescaled variables

$$\begin{array}{rcl} \psi &=& R^2\varphi \\ d\tau &=& Rdt \end{array} \tag{3.1}$$

Here the angle φ is considered in the universal cover $\mathbb{R}$ rather than $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and R is taken to be positive. Denoting $' = d/d\tau$ and abbreviating $c := \cos\vartheta$, $s := \sin\vartheta$, the normal form equations (2.16) now read

$$\begin{array}{rcl} R' &=& Rs(cs(1+\frac{\delta}{2}) + \eta^R) \\ \vartheta' &=& s(c^2 - \frac{\delta}{2}s^2 + \eta^\vartheta) \\ \psi' &=& R + 2\psi s(cs(1+\frac{\delta}{2}) + \eta^R) \end{array} \tag{3.2}$$

The error terms are now rapidly oscillating in ψ of "period" $2\pi R^2$,

$$(\eta^R, \eta^\vartheta) = (\eta^R, \eta^\vartheta)(R, \vartheta, R^{-2}\psi), \tag{3.3}$$

but still vanish of order $O(R^{K-1})$ for $R \searrow 0$.

In these variables, the $\psi-$axis $s = 0, R = 0$ still consists of equilibria. In the elliptic case $\delta = -1$ further equilibria of (3.2) do not exist for small R. In the hyperbolic case $\delta = +1$, we find additional equilibria only at $(R, \vartheta, \psi) = (0, \vartheta^*_\pm, 0)$ with $\vartheta^*_\pm \in (0, \pi)$ given by one of the two solutions of

$$\cos^2\vartheta^*_\pm = \frac{1}{2}\sin^2\vartheta^*_\pm \tag{3.4}$$

alias $\sin\vartheta^*_\pm = \sqrt{2/3}$. Note that $\vartheta^*_+ \in (0, \frac{\pi}{2})$ relates to the asymptotic opening angle of the conical stable/unstable set of $x = 0$, in original variables. For normal form order $K \geq 4$ the linearization at these additional equilibria $\vartheta^*_\pm$ is given by

$$\begin{pmatrix} \pm\frac{1}{3}\sqrt{3} & 0 & 0 \\ * & \mp\frac{2}{3}\sqrt{3} & 0 \\ 1 & 0 & \pm\frac{2}{3}\sqrt{3} \end{pmatrix}. \tag{3.5}$$

In particular, these equilibria are strictly hyperbolic with associated stable and unstable manifolds. The saddle point property also holds: solutions sufficiently close to these equilibria converge in forward or backward time, or else get ejected along the unstable/stable manifold.

Convergence to equilibrium is analyzed next.

Proposition 3.1 *In both the hyperbolic and the elliptic case, there exists $\epsilon > 0$ such that any solution $(R(t), \vartheta(t), \varphi(t))$ of (2.16) with $0 < \vartheta(0) < \pi$, $R(0) > 0$, and*

$$R(t) < \epsilon \quad \textit{for all } t \geq 0 \tag{3.6}$$

satisfies

$$\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} R(t)\sin\vartheta(t) = 0 \tag{3.7}$$

Proof :
We work with coordinates (R, ϑ, ψ) and with τ instead of t; see (3.1), (3.2). Note that $t \to +\infty$ implies $\tau \to +\infty$. Indeed $\dot\tau(t) = R(\tau(t)) > 0$, and by (3.2) $R(\tau)$ cannot converge down to zero in finite time $\tau \geq 0$. It is therefore sufficient to prove (3.7) for τ instead of t. We also abbreviate $\sin\vartheta(\tau)$ by $s(\tau)$. Note that $s(\tau)$ is positive, for all $\tau \geq 0$.

Suppose $s(\tau_k) \to 0$ for a subsequence $\tau_k \to \infty$. Then $s(\tau)$ converges to zero monotonically, by (3.2), for ϵ chosen small enough, and the proposition is proved.

We can therefore assume next that $s(\tau) \geq \epsilon' > 0$, uniformly for all $\tau \geq 0$. Fix $\epsilon'' > 0$ arbitrarily small and consider the case

$$c^2 - \frac{\delta}{2}s^2 \geq \epsilon'' \tag{3.8}$$

for all $\tau \geq 0$. For $\delta = -1$ elliptic, this condition holds automatically. For $\delta = +1$ hyperbolic, it requires our solution to stay away from the equilibrium zones $R = 0, \vartheta = \vartheta^*_\pm$.

Since $|\vartheta'|$ is uniformly bounded below in both cases, by $0 < R < \epsilon, \quad s \geq \epsilon'$ and (3.8), we reach a contradiction to $\vartheta \in [0, \pi]$ being bounded.

We can therefore assume that $\delta = +1$ is hyperbolic and that

$$\lim \vartheta(\tau_k) = \vartheta^*_\pm \tag{3.9}$$

for some sequence $\tau_k \to +\infty$.

The S^1-symmetry $\varphi \mapsto \varphi + 2\pi$ induces a nonlinear symmetry of the transformed equations under $\psi \mapsto \psi + 2\pi R^2$. Because $0 < R < \epsilon$ with ϵ chosen arbitrarily small, we can therefore also assume $\psi(\tau_k)$ to be close to $\psi = 0$ of order $2\pi\epsilon^2$. In other words, $(R, \vartheta, \psi)(\tau)$ is as close to one of the hyperbolic equilibria $R = 0, \vartheta = \vartheta^*_\pm, \psi = 0$ as we please, for $\tau = \tau_k$.

Consider the 2-dimensionally unstable equilibrium at $\vartheta = \vartheta^*_+$ first; see (3.5). Since $R(\tau_k) > 0$ provides a nonvanishing component in the slow unstable eigendirection with eigenvalue $+\frac{1}{3}\sqrt{3}$, the trajectory has to leave the region $0 < R < \epsilon$ in finite time after $\tau = \tau_k$. This contradicts $0 < R(\tau) \leq \epsilon$, for all $\tau > 0$.

To complete the proof of our proposition, it only remains to analyze the passage near the one-dimensionally unstable equilibrium $\vartheta = \vartheta^*_-$. If $(R, \vartheta, \psi)(\tau_k)$ happens to lie in the two-dimensional local stable manifold of ϑ^*_-, we have

$$\lim_{\tau \to +\infty} R(\tau) = 0 \tag{3.10}$$

and the proposition is proved. If on the other hand $(R, \vartheta, \psi)(\tau_k)$ misses the local stable manifold, then our trajectory gets ejected along the unstable manifold which coincides with the $\vartheta-$axis $R = \psi = 0$. Such a trajectory either approaches $\vartheta = \pi$, in contradiction to $s(\tau) \geq \epsilon'$, or else approaches a sufficiently small neighborhood of $R = 0, \vartheta = \vartheta^*_+, \psi = 0$ with subsequent ejection to $R \geq \epsilon$ as discussed above. These final contradictions complete the proof of the proposition. $\bowtie$

Proposition 3.1 proves the first claim of theorem 1.5, for positive times t, because $|x(t)| < \epsilon$ is equivalent to $R(t) < \epsilon$. Convergence for negative times follows, because the assumptions of theorem 1.5 are invariant under time reversal $t \mapsto -t$.

In the hyperbolic case, we next consider trajectories $x(t)$ which converge to an equilibrium x^* on the $y-$axis for $t \to -\infty$. If $x^* \neq 0$, then x^* is normally hyperbolic and $x(t)$ lies in the unstable manifold. Therefore x^* must be an equilibrium with two-dimensional strong unstable manifold. Note the x^* corresponds to a trajectory $\vartheta = 0, R \equiv R^*, \psi' = R^*$ of (3.2), with strong unstable manifold extending to $\vartheta > 0$. In forward time, our trajectory must therefore leave the region $|x| = R < \epsilon$, forced by the two-dimensional unstable manifold of $R = 0, \vartheta = \vartheta^*_+, \psi = 0$. Similarly, non-equilibrium trajectories converging to $x^* \neq 0$ in forward time must leave $|x| < \epsilon$ in backwards time. The same conclusion holds at $x^* = 0$, by the analysis of the unstable/stable sets of $x^* = 0$ in proposition 2.1: these sets correspond to the unstable/stable manifolds of $R = 0, \vartheta = \vartheta^*_\pm, \psi = 0$. The values of $\vartheta^*_\pm$ indicate the asymptotic opening angles of these sets.

To study the convergence behavior inside the left cone, alias to the right of the stable manifold of $R = 0, \vartheta = \vartheta^*_-, \psi = 0$, we observe that eventually ϑ increases monotonically in this region and converges to $\vartheta = \pi$; see (3.2). The radius R, on the other hand, decreases monotonically to $R^* > 0$. The remaining regions of ϑ can be analyzed similarly, exhibiting backwards convergence to equilibrium inside the ϑ^*_+-cone, and forward as well as backwards escape from $\epsilon-$neighborhoods outside the closures of both cones. This completes the proof of theorem 1.5 in the hyperbolic case.

In the elliptic case, all trajectories with $0 < R(0) < \epsilon$ converge:

$$\lim_{|t|\to+\infty} R(t) = R^*_\pm, \qquad \lim_{|t|\to+\infty} s(t) = 0 \tag{3.11}$$

Indeed, (3.8) is automatically satisfied for $\delta = -1$ and (3.9) cannot occur. Solutions with $R(0), \psi(0)$ small therefore follow the heteroclinic solution of

$$\vartheta' = s(c^2 + \frac{1}{2}s^2) \tag{3.12}$$

on the $\vartheta-$axis from $\vartheta = 0$ to $\vartheta = \pi$ for a long time. With ϑ close to $0, \pi$ for large $|t|$, we observe monotonic convergence of R in (3.2). Of course, the limits $R^*_\pm$ depend on the initial conditions.

In polar coordinates (2.2), (2.14) the two dimensional strong stable and unstable manifolds W^s_+ and W^u_- associated to the limits $y = \mp R^*_\pm$, $r = 0$, $\varphi \in S^1$, intersect along the orbit $(y(t), r(t), \varphi(t))$ associated to the solution $(R(t), \vartheta(t), \varphi(t))$ of (2.16). In truncated normal form, where the remainder terms η^y, η^r of order $O(R^{k+1})$ vanish identically, these manifolds in fact coincide and are given by a pair of heteroclinic orbits from $y = R^*_-$, $r = 0$ to $y = -R^*_+$, $r = 0$ in the $(y, r)-$plane. The angle variable $\varphi \in S^1$ provides the remaining dimension.

Including remainder terms η^y, η^r and passing to the time $t = 2\pi$ Poincaré map associated to $\dot\varphi = 1$, we obtain a diffeomorphism in the $(y, r)-$plane. This can cause a splitting of $W_{\pm}^{s/u} \cap \{\varphi = 0\}$, which we now estimate to be exponentially small in ϵ, for analytic vector fields $\dot{x} = f(x)$.

We use Neishtadt's fundamental result [Nei84]. Consider a vector field

$$\dot{\xi} = \epsilon f(\xi, t, \epsilon), \tag{3.13}$$

$2\pi-$periodic in t, continuous, and in ξ real analytic with uniform domain of convergence for $\epsilon < \epsilon_0, |\xi| \leq c_0$. Then after $O(1/\epsilon)$ averaging steps, pushing explicit $t-$dependence to higher and higher orders of ϵ, we arrive at an autonomous vector field

$$\dot{\zeta} = \epsilon g(\zeta, \epsilon) \tag{3.14}$$

such that the time $t = 2\pi$ maps of (3.13) and (3.14) differ by at most

$$c_1 \exp(-c_2/\epsilon). \tag{3.15}$$

The constants $c_1, c_2 > 0$ can be chosen uniformly in the domain under consideration.

We now apply Neishtadt's exponential averaging theorem to our problem of separatrix splitting in (y, r, φ) coordinates; see (2.2), (2.14). As in corollary 2.2, we rescale time such that $\dot{\varphi} = 1$. We also rescale the ball $R = \epsilon$ to size $R = 1$ by

$$\epsilon\zeta := (y, r). \tag{3.16}$$

Then the normal form (2.2) for $\zeta = (\zeta_1, \zeta_2)$ reads

$$\begin{aligned} \dot{\zeta}_1 &= \epsilon(\zeta_2^2 g^y(\epsilon\zeta_1, \epsilon^2\zeta_2^2) + \epsilon^K \tilde{\eta}^y(\epsilon\zeta_1, \epsilon\zeta_2 e^{it})) \\ \dot{\zeta}_2 &= \epsilon(\zeta_1\zeta_2 g^r(\epsilon\zeta_1, \epsilon^2\zeta_2^2) + \epsilon^K \tilde{\eta}^r(\epsilon\zeta_1, \epsilon\zeta_2 e^{it})) \end{aligned} \tag{3.17}$$

The right hand side satisfies the assumptions of Neightadt's theorem, provided $\epsilon > 0$ is chosen small enough. Choosing $K = O(1/\epsilon)$, we also see that the first K averaging steps amount to void identity transformations, because the lower order terms are already independent of t. More precisely, each step of Neishtadt's averaging procedure is equivalent to a step of the normal form procedure of proposition 2.1; see [Nei84], [Van89]. We can therefore conclude that the time $t = 2\pi$ map of our normal form (2.2), alias (2.14), truncated at order $K = O(1/\epsilon)$, differs from the full time $t = 2\pi$ map by an exponentially small term $c_1 \exp(-c_2/\epsilon)$. The same statement holds true for the variational equation. Moreover, the $y-$axis of fixed points is preserved by the normal form transformations. Their (local) strong stable and unstable manifolds are therefore moved by only exponentially small terms and their splitting angles are likewise exponentially small. This completes the proof of theorem 1.5, and the paper. $\bowtie$

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