Generic Hopf bifurcation from lines of equilibria without parameters: II. Systems of viscous hyperbolic balance laws *

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Abstract

We investigate viscous shock profiles of the Riemann problem for systems of hyperbolic balance laws. Even strictly hyperbolic flux terms together with a non-oscillating kinetic part can lead to oscillating viscous shock profiles. They appear near a Hopf-like bifurcation point of the traveling wave equation.

Keywords: viscous profiles, oscillating viscous shocks, Riemann problem, hyperbolic balance laws

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1 Introduction

Searching for viscous shock profiles of the Riemann problem, we consider systems of hyperbolic balance laws of the form

$$u_t + f(u)_x = \epsilon^{-1} g(u) + \epsilon\delta u_{xx}, \tag{1.1}$$

with $u = (u_0, u_1, \ldots, u_N) \in \mathrm{IR}^{N+1}, f \in C^3, g \in C^2, \delta > 0$, and with real time t and space x. We assume strict hyperbolicity, that is, the Jacobian $A(u) = f'(u)$ possesses simple real distinct eigenvalues

$$\sigma_0, \sigma_1, \ldots, \sigma_N \in \text{spec } A(u) \tag{1.2}$$

The case of conservation laws, $g \equiv 0$, has been studied extensively. See, for example, [Smo83] for a background. Viscous profiles are traveling wave solutions of the form

$$u = u\left(\frac{x - st}{\epsilon}\right) \tag{1.3}$$

with wave speed s. Here (1.3) provides a solution of (1.1) if

$$-s\dot{u} + A(u)\dot{u} = g(u) + \delta\ddot{u}. \tag{1.4}$$

Here $A(u) = f'(u)$ denotes the Jacobian and $\dot{} = \frac{d}{d\tau}$ with $\tau = (x - st)/\epsilon$. Note that (1.4) is independent of $\epsilon > 0$. Any solution of system (1.4) for which

$$\lim_{\tau \to \pm\infty} u(\tau) = u^{\pm} \tag{1.5}$$

exists gives rise, for $\epsilon \searrow 0$, to a solution of the Riemann problem of (1.1) with values $u = u^{\pm}$ connected by a shock traveling with shock speed s. We call solutions $u(\cdot)$ of (1.4), (1.5) *viscous profiles.*

We rewrite the viscous profile equation (1.4) as a second order system

$$\begin{array}{rcl} \dot{u} &=& v, \\ \delta\dot{v} &=& -g(u) + (A(u) - s)v. \end{array} \tag{1.6}$$

Note that any viscous profile must satisfy

$$g(u^{\pm}) \;=\; 0. \tag{1.7}$$

In other words, the asymptotic states $u^{\pm}$ must be equilibria of the reaction term $g(u)$. In the conservation law case, $g \equiv 0$, this condition does not impose any constraint on the Riemann values $u^{\pm}$. In the other extreme of a reaction term g with unique equilibrium, we obtain $u^+ = u^-$ and traveling shock profiles of Riemann type do not exist. In the case of one conservation law mixed with N balance laws, one expects curves of equilibria $g(u^\pm) = 0$. For example, reaction terms $g(u)$ typically depend on concentrations or temperature, but not on velocity.

Addressing a simple case, for demonstration purposes, we therefore assume that the u_0-component does not contribute to the reaction terms and still all reaction components vanish, say at $u = 0$. Specifically, we assume throughout this paper that

$$g = g(u_1, \ldots, u_N) = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{pmatrix} \tag{1.8}$$

is independent of u_0 and satisfies

$$g(0) = 0. \tag{1.9}$$

This gives rise to a line of equilibria

$$u_0 \in \mathrm{IR}, \quad u_1 = \cdots = u_N = 0, \quad v = 0, \tag{1.10}$$

of our viscous profile system (1.6).

The asymptotic behavior of viscous profiles $u(\tau)$ for $\tau \to \pm\infty$ depends on the linearization L of (1.6) at $u = u_\pm$, $v = 0$. In block matrix notation corresponding to coordinates (u, v) we have

$$L = \begin{pmatrix} 0 & \mathrm{id} \\ -\delta^{-1} g' & \delta^{-1}(A - s) \end{pmatrix}. \tag{1.11}$$

Here $A = A(u)$, and $g' = g'(u)$ describes the Jacobi matrix of the reaction term g at $u = u^\pm$. In (1.11) we write s rather than $s \cdot \mathrm{id}$ for brevity. In the case $g \equiv 0$ of pure conservation laws, the linearization L possesses an $(N+1)$-dimensional kernel corresponding to the then arbitrary choice of the equilibrium $u \in \mathrm{IR}^{N+1}$, $v = 0$. Normal hyperbolicity of this family of equilibria, in the sense of dynamical systems [HPS77], [Fen77], [Wig94], is ensured for wave speeds s not in the spectrum of the strictly hyperbolic Jacobian $A(u)$:

$$s \notin \mathrm{spec}\ A(u) = \{\sigma_0, \ldots, \sigma_N\}. \tag{1.12}$$

Indeed, (1.12) ensures that additional zeros do note arise in the real spectrum

$$\mathrm{spec}\ L = \{0\} \cup \delta^{-1}\,\mathrm{spec}\,(A(u) - s). \tag{1.13}$$

In the present paper we investigate the failure of normal hyperbolicity of L along the line of equilibria $u = (u_0, 0, \ldots, 0), v = 0$ given by (1.10). Although our method applies in complete generality, we present just a simple specific example for which purely imaginary eigenvalues of L arise when $\delta > 0$ is fixed small enough. Specifically, we consider three-dimensional systems, $N = 2$, satisfying

$$\begin{aligned} A(u_0,0,0) &= A_0 + u_0 \cdot A_1, \\ A_0 &= \begin{pmatrix} \alpha & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \alpha \neq 0, \quad A_1 \text{ symmetric}, \\ g'(0) &= \begin{pmatrix} 0 & & \\ & \gamma & 1 \\ & 1 & \gamma \end{pmatrix}, \quad |\gamma| < 1, \end{aligned} \tag{1.14}$$

with omitted entries being zero. Note that these data can arise from flux functions f which are gradient vector fields, still giving rise to purely imaginary eigenvalues. At the end of this paper we present a specific example where the reaction terms $\dot{u} = g(u)$ *alone*, likewise, do not support even transient oscillatory behavior; see (3.12). The interaction of flux and reaction, in contrast, is able to produce purely imaginary eigenvalues of the linearization L, as follows.

Proposition 1.1 *Consider the linearization $L = L(u_0)$ along the line $u_0 \in \mathrm{IR}, u_1 = u_2 = 0$ of equilibria of the viscous profile system (1.6) in IR^3; see (1.11). Assume (1.14) holds.*

For $\delta \searrow 0$ and small $|s|, |u_0|$, the spectrum of L then decouples into two parts:

(i) an unboundedly growing part $\mathrm{spec}_\infty(L) = \delta^{-1}\,\mathrm{spec}\,(A - s) + O(1)$

(ii) a bounded part $\mathrm{spec}_{bd}(L) = \mathrm{spec}\,((A-s)^{-1}g') + O(\delta)$

Here A, g are evaluated at $u = (u_0, 0, 0)$.

For $\delta = 0, s = 0, |\gamma| < 1$, the bounded part $\mathrm{spec}_{bd}(L)$ at $u_0 = 0$ limits onto simple eigenvalues $\mu_0 \in \{0, \pm i\omega_0\}$, $\omega_0 = \sqrt{1-\gamma^2}$ with eigenvectors $\binom{\tilde{u}}{\tilde{v}}$ given by $\tilde{v} = \mu_0 \tilde{u}$ and

$$\begin{aligned} \tilde{u} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \textit{for } \mu_0 = 0, \\ \tilde{u} &= \begin{pmatrix} 0 \\ -\gamma - i\omega_0 \\ 1 \end{pmatrix} \qquad \textit{for } \mu_0 = +i\omega_0. \end{aligned} \tag{1.15}$$

Proof :
Regular perturbation theory applies to the scaled block matrix

$$\delta L = \begin{pmatrix} 0 & 0 \\ -g' & A-s \end{pmatrix} + \delta \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}, \tag{1.16}$$

which becomes lower triangular for $\delta = 0$. This provides us with the unbounded part $\operatorname{spec}_\infty(L)$ of the spectrum, generated in v-space alone with $u = 0$.

Moreover, δL possesses three-dimensional kernel, at $\delta = 0$, given by

$$g'u = (A-s)v. \tag{1.17}$$

On this kernel, the eigenvalue problem for L reduces to

$$\mu_0 u \;=\; v \;=\; (A-s)^{-1} g' u. \tag{1.18}$$

The characteristic polynomial of (1.18) at $u_0 = 0$ is given by

$$p_0(\mu) = \left(\mu^2 - \frac{2\gamma s}{1-s^2}\mu + \frac{1-\gamma^2}{1-s^2} \right) \mu. \tag{1.19}$$

Direct calculation completes the proof. $\bowtie$

Note that our choices of A_0 and $g'(0)$ are normalized, such that the bifurcation occurs at a shock speed $s = 0$. For more general systems the Hopf point may occur at nonzero values of the shock speed parameter.

For explicit calculations here and below, we have used and recommend assistance by symbolic packages like Mathematica, Maple, etc.

Bifurcations from lines of equilibria in absence of parameters have been investigated in [Lie97], [FLA98a] from a theoretical view point. We briefly recall that result for convenience. Consider C^5 vector fields

$$\dot{\mathbf{u}} = F(\mathbf{u}) \tag{1.20}$$

with $\mathbf{u} = (u_0, u_1, \ldots, u_n) \in \mathbb{R}^{n+1}$. We assume a line of equilibria

$$0 = F(u_0, 0, \ldots, 0) \tag{1.21}$$

along the u_0-axis. At $u_0 = 0$, we assume the Jacobi matrix $F'(u_0, 0, \ldots, 0)$ to be hyperbolic, except for a trivial kernel vector along the u_0-axis and a complex conjugate pair of simple, purely imaginary, nonzero eigenvalues $\mu(u_0), \overline{\mu(u_0)}$ crossing the imaginary axis transversely as u_0 increases through $u_0 = 0$:

$$\begin{aligned} \mu(0) &= i\omega(0), \quad \omega(0) > 0, \\ \text{Re } \mu'(0) &\neq 0. \end{aligned} \tag{1.22}$$

Let Z be the two-dimensional real eigenspace of $F'(0)$ associated to $\pm i\omega(0)$. By Δ_Z we denote the Laplacian with respect to variations of u in the eigenspace Z. Coordinates in Z are chosen as coefficients of the real and imaginary parts of the complex eigenvector associated to $i\omega(0)$. Note that the linearization acts as a rotation with respect to these not necessarily orthogonal coordinates. Let P_0 be the one-dimensional eigenprojection onto the trivial kernel along the u_0-axis. Our final nondegeneracy assumption then reads

$$\Delta_Z P_0 F(0) \neq 0. \tag{1.23}$$

Fixing orientation along the positive u_0-axis, we can consider $\Delta_Z P_0 F(0)$ as a real number. Depending on the sign

$$\eta := \text{sign } (\text{Re } \mu'(0)) \ \cdot \text{sign } (\Delta_Z P_0 F(0)), \tag{1.24}$$

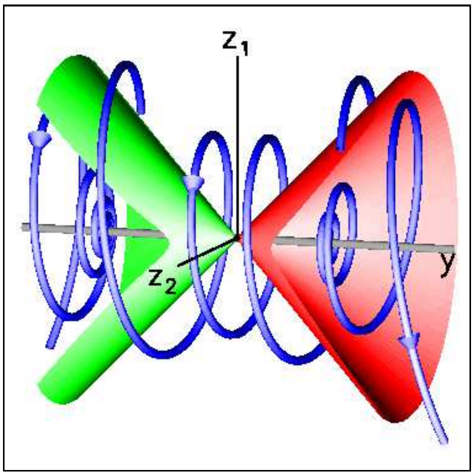
we call the "bifurcation" point $u_0 = 0$ *elliptic* if $\eta = -1$ and *hyperbolic* for $\eta = +1$.

The following result from [FLA98a] investigates the qualitative behavior of solutions in a normally hyperbolic three-dimensional center manifold to $\mathbf{u} = 0$.

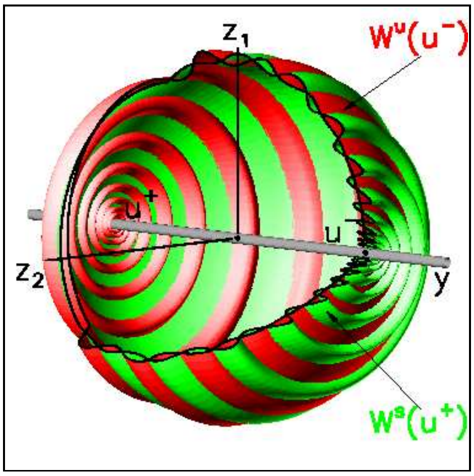
The results for the hyperbolic case $\eta = +1$ are based on normal form theory and a spherical blow-up construction inside the center manifold. The elliptic case $\eta = -1$ is based on Neishtadt's theorem on exponential elimination of rapidly rotating phases [Nei84]. For a related application to binary oscillators in discretized systems of hyperbolic balance laws see [FLA98b]. For an application to square rings of additively coupled oscillators see [AF98].

Theorem 1.1 *Let assumptions (1.21) – (1.23) hold for the C^5 vector field $\dot{\mathbf{u}} = F(\mathbf{u})$. Then the following holds true in a neighborhood U of $\mathbf{u} = 0$ within a three-dimensional center manifold to $\mathbf{u} = 0$.*

In the hyperbolic case, $\eta = +1$, all nonequilibrium trajectories leave the neighborhood U in positive or negative time direction (possibly both). The stable and unstable sets of $\mathbf{u} = 0$, respectively, form cones around the positive/negative u_0-axis, with asymptotically



Case (a) hyperbolic, $\eta = +1$.



Case (b) elliptic, $\eta = -1$.

Figure 1.1: Dynamics near Hopf bifurcation from lines of equilibria.

elliptic cross section near their tips at $\mathbf{u} = 0$. *These cones separate regions with different convergence behavior. See Fig. 1.1 (a).*

In the elliptic case all nonequilibrium trajectories starting in U *are heteroclinic between equilibria* $\mathbf{u}^{\pm} = (u_0^{\pm}, 0, \ldots, 0)$ *on opposite sides of* $u_0 = 0$. *If* $F(\mathbf{u})$ *is real analytic near* $\mathbf{u} = 0$, *then the two-dimensional strong stable and strong unstable manifolds of* $\mathbf{u}^{\pm}$ *within the center manifold intersect at an angle which possesses an exponentially small upper bound in terms of* $|\mathbf{u}^{\pm}|$. *See Fig. 1.1 (b).*

In the present paper, we apply theorem 1.1 to the problem of zero speed viscous profiles of systems of hyperbolic balance laws near Hopf points as in proposition 1.1. Nonzero shock speeds can be treated completely analogously, absorbing them into the flux term.

Theorem 1.2 *Consider the problem (1.5)–(1.7) of finding viscous profiles with shock speed* $s = 0$ *to hyperbolic balance laws (1.1). Let assumptions (1.8)–(1.10), (1.14) hold, so that a pair of purely imaginary simple eigenvalues occurs for the linearization* L, *in the limit* $\delta \to 0$.

Then there exist nonlinearities $A(u) = f'(u)$ *and* $g(u)$, *compatible with the above assumptions, such that the assumptions and conclusions of theorem 1.1 are valid for the viscous profile system (1.6). Both the elliptic and the hyperbolic case occur; see Fig. 1.1.*

Since both conditions are open, the results persist, in particular, for small nonzero *shock speeds s, even when f, g remain fixed.*

Specific choices of flux $f(u)$ and reaction terms $g(u)$ are presented in corollary 3.3; see (3.12). In the elliptic case $\eta = -1$, we nevertheless observe (at least) pairs of weak shocks with oscillatory tails, connecting u_- and u_+. In the hyperbolic case $\eta = +1$, viscous profiles leave the neighborhood U and thus represent large shocks. At $u_0 = 0$, their profiles change discontinuously and the role of the u_0-axis switches from providing the left to providing the right asymptotic state with oscillatory tail.

This paper is organized as follows. In section 2 we check transversality condition (1.22) for the purely imaginary eigenvalues. We also compute an expansion in terms of δ for the eigenprojection P_0 onto the trivial kernel along the u_0-axis. In section 3, we check nondegeneracy condition (1.23) for $\Delta_Z P_0 F(0)$, in the limit $\delta \searrow 0$, completing the proof of theorem 1.2 by reduction to theorem 1.1.

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2 Linearization and transverse eigenvalue crossing

In this section we continue our linear analysis of the linearization

$$L^\delta(u_0) = \begin{pmatrix} 0 & \mathrm{id} \\ -\delta^{-1}g' & \delta^{-1}A \end{pmatrix}, \tag{2.1}$$

with $A = A(u)$, $g' = g'(u)$ evaluated along the line of equilibria $u = (u_0, 0, 0)$. See (1.11) with $s = 0$ and proposition 1.1. In the limit $\delta \searrow 0$, we address the issue of transverse crossing of purely imaginary eigenvalues in lemma 2.1. In lemma 2.2, we explicitly compute the one-dimensional eigenprojection P_0^δ onto the trivial kernel of $L^\delta(0)$.

Throughout this section we fix the notation

$$A(u_0, 0, 0) = A_0 + u_0 A_1 = \begin{pmatrix} \alpha & & \\ & 1 & \\ & & -1 \end{pmatrix} + u_0 \cdot (a^1_{jk})_{0 \le j,k \le 2}, \tag{2.2}$$

with $a^1_{jk} = a^1_{kj}$ symmetric; see (1.14). We also assume that the linearized reaction term

$$g'(u_0, 0, 0) = g'(0) = \begin{pmatrix} 0 & & \\ & \gamma & 1 \\ & 1 & \gamma \end{pmatrix}, \qquad |\gamma| < 1, \tag{2.3}$$

which is independent of u_0 by assumption (1.8), possesses a vanishing g_0-component.

By proposition 1.1, purely imaginary eigenvalues of $L^\delta(u_0)$ arise from an $O(\delta)$ perturbation of the matrix

$$A^{-1}g' = (A_0 + u_0 A_1)^{-1} g'(0) \tag{2.4}$$

with spectrum $\mathrm{spec}_{bd}\,(L)$. Let

$$\mu(u_0), \quad \bar{\mu}(u_0) \tag{2.5}$$

denote the continuation of the simple, purely imaginary eigenvalues

$$\mu(0) = i\omega_0, \quad \bar{\mu}(0) = -i\omega_0$$

with u_0-derivatives $\mu'(u_0), \bar{\mu}'(u_0)$.

Lemma 2.1 *In the above setting and notation,*

$$\mathrm{Re}\; \mu'(0) = -\frac{\gamma}{2}(a^1_{11} + a^1_{22}) + a^1_{12}. \tag{2.6}$$

Proof :
Since the unit vector e_0 in u_0-direction is a trivial kernel vector of $g'(0)$ and since the remaining eigenvalues of $A^{-1}g'$ remain conjugate complex for small $|u_0|$, we have

$$\mathrm{Re}\; \mu(u_0) = \frac{1}{2}\; \mathrm{trace}\; (A^{-1}g'). \tag{2.7}$$

In particular, trace $A_0^{-1}g' = 0$. With the u_0-expansion

$$A^{-1} = (A_0 + u_0 A_1)^{-1} = A_0^{-1} - u_0 A_0^{-1} A_1 A_0^{-1} + \cdots \tag{2.8}$$

we immediately obtain

$$\mathrm{Re}\; \mu'(u_0) = -\frac{1}{2}\; \mathrm{trace}\,(A_0^{-1} A_1 A_0^{-1} g'). \tag{2.9}$$

Inserting A_0, A_1, g' proves the lemma. $\bowtie$

By regular perturbation of $\mathrm{spec}_{bd}(L)$, the result Re $\mu'(0) \neq 0$ of lemma 2.1 extends to small positive δ.

We now turn to an expansion for the eigenprojection P_0^δ onto the one-dimensional kernel of the 6×6-matrix $L^\delta(u_0)$ at $u_0 = 0$; see (2.1), (2.2). Aligning the notations of proposition 1.1 and of theorem 1.2, we decompose

$$\mathbf{u} = (u, v) \in \mathrm{I\!R}^6 = \mathrm{I\!R}^3 \times \mathrm{I\!R}^3.$$

Again, $e_0^T = (1, 0, 0)$ denotes the first unit vector in $\mathrm{I\!R}^3$ and $\mathbf{e}_0^T = (e_0^T, 0)$ the first unit vector in $\mathrm{I\!R}^6$.

Lemma 2.2 *In the above setting and notation*

$$\begin{aligned} P_0^\delta &= \mathbf{e}_0 \cdot \mathbf{e}_\delta^T, \quad \text{with} \\ \mathbf{e}_\delta^T &= (1 + (\tfrac{\delta}{\alpha})^2)^{-1/2}(e_0^T, -\tfrac{\delta}{\alpha} e_0^T). \end{aligned} \tag{2.10}$$

Proof :
Kernel and co-kernel of $L^\delta(u_0)$ are one-dimensional, corresponding to the simple zero eigenvalue of $L^\delta(u_0)$. Obviously

$$\ker L^\delta(u_0) = \mathbf{e}_0, \tag{2.11}$$

because $g'(u_0, 0, 0)e_0 = 0$. At $u_0 = 0$, the co-kernel of $L^\delta(u_0)$ is given by

$$0 = \mathbf{e}_\delta^T \cdot \begin{pmatrix} 0 & \mathrm{id} \\ -\delta^{-1} g' & \delta^{-1} A_0 \end{pmatrix}. \tag{2.12}$$

Inserting A_0 from (2.2) and g' from (2.3) proves the lemma. $\bowtie$

3 Higher order nondegeneracy

In this section we complete the proof of theorem 1.3. In view of theorem 1.2, we have already checked transverse crossing of purely imaginary eigenvalues, assumption (1.22), in lemma 2.1. Letting

$$F(\mathbf{u}) = F(u, v) = \begin{pmatrix} v \\ -\delta^{-1} g(u) + \delta^{-1} A(u) v \end{pmatrix} \tag{3.1}$$

it remains to check the nondegeneracy assumption $\Delta_Z P_0 F \neq 0$; see (1.23). In lemma 3.1, we check this assumption in the limit $\delta \searrow 0$. In corollary 3.2, we provide explicit expressions for the type determining sign $\eta = \pm 1$ defined in (1.24). In particular, we show in corollary 3.3 that both the hyperbolic case $\eta = +1$ and the elliptic case $\eta = -1$ can be realized by our nonlinear hyperbolic balance laws, even with gradient flux terms. This then completes the proof of theorem 1.2.

To check nondegeneracy condition (1.23) on $\Delta_Z P_0 F$ in the limit $\delta \searrow 0$, we use the following notation. By transverse eigenvalue crossing at $\delta = 0$, lemma 2.1, we also obtain purely imaginary eigenvalues $\pm i\omega^\delta$ at equilibria $\mathbf{u}^\delta = (u_0^\delta, 0, \ldots, 0) = (u^\delta, 0)$ on the u_0-axis, for small $\delta > 0$. Let Z^δ denote the corresponding eigenspace. We recall our expression for the eigenprojection P_0^δ onto the trivial kernel,

$$P_0^\delta = (1 + (\tfrac{\delta}{\alpha})^2)^{-1/2}\ \mathbf{e}_0 \cdot (e_0^T, -\tfrac{\delta}{\alpha} e_0^T) \tag{3.2}$$

with $\mathbf{e}_0 = (e_0^T, 0)^T$, see lemma 2.2. Note that $\mathbf{u}^\delta, \omega^\delta, P_0^\delta$, and Z^δ vary differentiably with δ.

Lemma 3.1 *In the above setting and notation we have*

$$\Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) = (1 + (\tfrac{\delta}{\alpha})^2)^{-1/2}\ \frac{1}{\alpha} g_0''(u^\delta)[\tilde{u}^\delta, \bar{\tilde{u}}^\delta]\ \mathbf{e}_0 \tag{3.3}$$

at the Hopf point $\mathbf{u}^\delta = (u^\delta, 0)$ *with complex eigenvector* $(\tilde{u}^\delta, \tilde{v}^\delta)$ *of* $i\omega^\delta$.

Consider in particular quadratic forms $g_0''(0)$*, which are strictly positive/negative definite on* (u_1, u_2)*-space, with* $\Gamma = \pm 1$ *indicating the sign of definiteness. Then*

$$\text{sign}\, \Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) = \Gamma \cdot \text{sign}\ \alpha, \tag{3.4}$$

for all small $\delta > 0$.

Proof :
By lemma 2.2 we have

$$(1 + (\tfrac{\delta}{\alpha})^2)^{1/2} P_0^\delta = \mathbf{e}_0 \cdot \mathbf{e}_0^T - \tfrac{\delta}{\alpha} \mathbf{e}_0 \cdot (0, e_0^T). \tag{3.5}$$

The explicit form (3.1) of the nonlinearity F implies

$$\Delta_{Z^\delta} P_0^0 F(\mathbf{u}^\delta) = \mathbf{e}_0 \Delta_{Z^\delta} v_0 = 0 \tag{3.6}$$

on any subspace Z^δ and for any $\mathbf{u}^\delta$, simply because the u-component of F is linear. With P_0^δ instead of P_0^0 we obtain more generally

$$\begin{aligned} (1 + (\tfrac{\delta}{\alpha})^2)^{1/2} \mathbf{e}_0^T \Delta_{Z^\delta} P_0^\delta F(\mathbf{u}) &= -\Delta_{Z^\delta} \left(0\,,\ -\tfrac{\delta}{\alpha} e_0^T\, \delta^{-1}(-g(u) + A(u)v)\right) \\ &= -\tfrac{1}{\alpha} \Delta_{Z^\delta} \left(-g_0(u) + (A(u)v)_0\right). \end{aligned} \tag{3.7}$$

Here $(A(u)v)_0$ denotes the zero-component of $A(u)v$. We treat this term first, using the notation

$$\tilde{\mathbf{u}}^\delta = \begin{pmatrix} \tilde{u}^\delta \\ \tilde{v}^\delta \end{pmatrix} \tag{3.8}$$

for the complex eigenvector of the purely imaginary Hopf eigenvalue $\mu^\delta = i\omega^\delta$ at $u = u^\delta, v = 0$. Then $Z^\delta = \text{span}\{\text{Re}\,\tilde{\mathbf{u}}^\delta, \text{Im}\,\tilde{\mathbf{u}}^\delta\}$. Denoting by $\Delta_\beta = \partial^2_{\beta_1} + \partial^2_{\beta_2}$ the standard Laplacian, evaluated at $\beta = 0$, and inserting $\tilde{v}^\delta = \mu^\delta \tilde{u}^\delta$ yields

$$\begin{aligned} \Delta_{Z^\delta}(A(u)v)_0 = {} & \Delta_\beta \Big(A(u^\delta + \beta_1 \text{Re}\,\tilde{u}^\delta + \beta_2 \text{Im}\,\tilde{u}^\delta)\,(\beta_1\,\text{Re}\,\tilde{v}^\delta + \beta_2 \text{Im}\,\tilde{v}^\delta)\Big)_0 \\ = {} & 2\Big((A'(u^\delta)\text{Re}\,\tilde{u}^\delta)\text{Re}\,\tilde{v}^\delta + (A'(u^\delta)\text{Im}\,\tilde{u}^\delta)\text{Im}\,\tilde{v}^\delta\Big)_0 \\ = {} & 2\,\text{Re}\,((A'(u^\delta)\tilde{u}^\delta)\bar{\tilde{v}}^\delta)_0 \\ = {} & 2\,\text{Re}\,(\bar{\mu}^\delta)\,(f''(u^\delta)[\tilde{u}^\delta, \bar{\tilde{u}}^\delta])_0 \\ = {} & 2\,\text{Re}\,(\mu^\delta)\,f_0''(u^\delta)[\tilde{u}^\delta, \bar{\tilde{u}}^\delta] \;=\; 0 \end{aligned} \tag{3.9}$$

all along the Hopf curve $u = u^\delta, v = 0$. Here we have used $A(u) = f'(u)$ for the flux function and the fact that the Hessian matrix $f_0''(0)$ is symmetric.

Therefore, we can conclude from (3.7), (3.9) that

$$(1 + (\tfrac{\delta}{\alpha})^2)^{1/2}\mathbf{e}_0^T \Delta_{Z^\delta} P_0^\delta F(\mathbf{u}^\delta) = \frac{1}{\alpha}\Delta_{Z^\delta} g_0(u) = \frac{1}{\alpha} g_0''(u^\delta)[\tilde{u}^\delta, \bar{\tilde{u}}^\delta]. \tag{3.10}$$

This proves (3.3) and the lemma. $\bowtie$

Corollary 3.2 *Combining lemmata 2.1 and 3.1, the* sign $\eta = \pm 1$ *distinguishing elliptic from hyperbolic Hopf bifurcation along our line of equilibria is given explicitly by*

$$\begin{aligned} \eta = {} & \text{sign Re}\;\mu'(0) \cdot \text{sign}\,\Delta_Z P_0 F(0) \\ = {} & \text{sign}(a_{12} - \tfrac{\gamma}{2}(a_{11} + a_{22})) \cdot \text{sign}\,\alpha \cdot \Gamma, \end{aligned} \tag{3.11}$$

for $\delta > 0$ small enough. Here derivatives are evaluated at $u = 0$ and are assumed to be chosen such that $\eta \neq 0$. The sign $\Gamma = \pm 1$ *indicates positive/negative definiteness of $g_0''(0)$ on (u_1, u_2)-space. Obviously, both signs of η can be realized.*

Corollary 3.3 *Theorems 1.1, 1.2 hold true for $\eta = \pm 1$ with the following specific choices of a gradient flux term $f(u) = \nabla\Phi(u)$ and a reaction term $g(u)$:*

$$\begin{aligned} g(u) &= \begin{pmatrix} u_1^2 & + & u_2^2 \\ -\tfrac{1}{2}u_1 & + & u_2 \\ u_1 & - & \tfrac{1}{2}u_2 \end{pmatrix}, \\ \Phi(u) &= u_0^2 + \tfrac{1}{2}(u_1^2 - u_2^2) + \eta u_0 u_1^2. \end{aligned} \tag{3.12}$$

These choices correspond to $\alpha = 2, \gamma = -\frac{1}{2}, \Gamma = +1$.

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