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QUANTITATIVE HOMOGENIZATION OF ANALYTIC SEMIGROUPS AND REACTION–DIFFUSION EQUATIONS WITH DIOPHANTINE SPATIAL FREQUENCIES

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Abstract. Based on an analytic semigroup setting, we first consider semilinear reaction–diffusion equations with spatially quasiperiodic coefficients in the nonlinearity, rapidly varying on spatial scale ε. Under periodic boundary conditions, we derive quantitative homogenization estimates of order ε^γ on strong Sobolev spaces H^σ in the triangle

$$0 < \gamma < \min(\sigma - n/2, 2 - \sigma).$$

Here n denotes spatial dimension. The estimates measure the distance to a solution of the homogenized equation with the same initial condition, on bounded time intervals. The same estimates hold for C^1 convergence of local stable and unstable manifolds of hyperbolic equilibria. As a second example, we apply our abstract semigroup result to homogenization of the Navier–Stokes equations with spatially rapidly varying quasiperiodic forces in space dimensions 2 and 3. In both examples, a Diophantine condition on the spatial frequencies is crucial to our homogenization results. Our Diophantine condition is satisfied for sets of frequency vectors of full Lebesgue measure. In the companion paper [7], based on L^2 methods, these results are extended to quantitative homogenization of global attractors in near-gradient reaction–diffusion systems.

1. Introduction

This paper investigates the behavior of solutions u^ε and of local invariant manifolds, for example for scalar reaction diffusion equations

$$u^\varepsilon_t = \Delta u^\varepsilon + f^\varepsilon(x, \omega x/\varepsilon, u^\varepsilon), \qquad u^\varepsilon(t = 0) = u_0 \tag{1.1}$$

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in the limit $\varepsilon \searrow 0$. The Navier–Stokes system in spatial dimensions $n = 2, 3$, as treated in Section 5, will provide a more advanced example. For simplicity, we consider (1.1) under periodic boundary conditions

$$x \in T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n \tag{1.2}$$

in dimensions $n = 1, 2, 3$. Our main interest is the rapid spatial oscillations in the variable $y = \omega x/\varepsilon$. We assume these rapid oscillations to be quasiperiodic; more specifically

$$f : [0, \varepsilon_0) \times T^n \times T^N \times \mathbb{R} \to \mathbb{R}, \quad (\varepsilon, x, y, u) \mapsto f^\varepsilon(x, y, u) \tag{1.3}$$

for any fixed $\varepsilon \geq 0$, and ω is an $N \times n$ frequency matrix with rationally independent entries. Note however that we represent $x \in T^n$ by components in $[0, 2\pi)$, when evaluating y. As a side effect, this produces spatial discontinuities in (1.1).

An obvious candidate for a homogenized version of equation (1.1) is the formally spatially averaged equation

$$u_t^0 = \Delta u^0 + f^0(x, u^0), \text{ where} \tag{1.4}$$

$$f^0(x, u) := (2\pi)^{-N} \int_{T^N} f^0(x, y, u)\, dy. \tag{1.5}$$

Naively we might expect the local solutions of (1.1) to converge in a suitable weak, or even strong sense:

$$u^\varepsilon(t) \to u^0(t), \qquad 0 \leq t \leq T(u_0) \tag{1.6}$$

with u^0 solving the homogenized equation (1.4), (1.5) under the same initial values and boundary conditions. Not surprisingly this expectation proves to be correct. See [3], [24] for a background on homogenization results, in particular for the periodic case $\omega = id$. *Quantitative homogenization*, however, aims at determining a specific rate of convergence. Theorem 1.1 below asserts that

$$\|u^\varepsilon(t) - u^0(t)\|_{H^\sigma} \leq C\varepsilon^\gamma, \tag{1.7}$$

where $n/2 < \sigma < 2$ specifies spatial regularity in terms of the fractional Sobolev spaces H^σ, and γ is suitably chosen below.

Being interested in quantitative strong convergence not only of individual trajectories but also of global attractors, in the dissipative case, we are also providing *fractional order homogenization* estimates of type (1.7) for (local) stable and unstable manifolds. In fact the global attractor of the infinite-dimensional gradient dynamical system (1.1), consists entirely of equilibria, which we may assume to be hyperbolic, and of intersections of their stable and unstable manifolds. See [4], [8], [12], [18], and [21] for a general background on global attractors for dissipative systems.

We now proceed to prepare and precisely state our results on fractional homogenization for the scalar reaction diffusion case; see Theorems 1.1 and 1.2 below. We assume f to be polynomial in u with for simplicity of presentation smooth coefficients

$$\begin{aligned} f^\varepsilon(x,y,u) &= \sum_{m=0}^{d} a_m^\varepsilon(x,y)u^m \\ a_m : [0,\varepsilon_0) \times T^n \times T^N &\longrightarrow \mathbb{R} \quad \text{smoothly}, \\ (\varepsilon, x, y) &\longmapsto a_m^\varepsilon(x,y). \end{aligned} \tag{1.8}$$

For precise smoothness requirements on the coefficients a_m see Section 4, in particular (4.10), (4.12)–(4.14). Note that f^ε and (1.1), (1.4) are dissipative, if d is odd and $a_d^\varepsilon < 0$; in particular, regular solutions then exist for all positive times.

As the homogenized nonlinearity $f^0(x,u)$ at $\varepsilon = 0$ we define

$$f^0(x,u) := \sum_{m=0}^{d} a_m^0(x)u^m, \quad a_m^0(x) := (2\pi)^{-N} \int_{T^N} a_m^0(x,y)\,dy \tag{1.9}$$

as introduced in (1.5). For our results on local stable and unstable manifolds we also assume that (1.1) possesses a trivial equilibrium $u \equiv 0$, for all ε, and the linearization A of the homogenized equation (1.4) at $u = 0$ is hyperbolic:

$$a_0^\varepsilon(x,y) \equiv 0, \quad \text{and} \quad 0 \notin (\operatorname{spec} A), \quad \text{where} \quad Au = \Delta u + a_1^0(x)u. \tag{1.10}$$

In addition, we require a *Diophantine condition* for the columns ω_ϱ, $\varrho = 1, \ldots, n$, of the frequency matrix $\omega = (\omega_1, \ldots, \omega_n)$ of the rapid spatial oscillations:

$$\min_{\varrho=1,\ldots,n} |\, k^T \omega_\varrho \,| \quad \geq \quad c|k|^{-(N-1)-\vartheta} \tag{1.11}$$

for some $c, \vartheta > 0$ and all $k \in \mathbb{Z}^N \setminus \{0\}$. Such Diophantine conditions are ubiquitous in modern dynamical systems, and in particular in the small denominator problems of KAM-theory in celestial mechanics and Hamiltonian systems; see for example [14] and [1] and the references there. We recall that Diophantine conditions (1.11) hold for a set of frequencies ω_ϱ of full Lebesgue measure in $\mathbb{R}^{Nn}$; see for example [5].

As a final ingredient, we recall how fractional Sobolev spaces $H^\sigma(T^n) \subseteq L^2(T^n)$ measure spatial regularity. Let $\sigma \geq 0$ and let

$$u(x) = \sum_{j \in \mathbb{Z}^n} u_j e^{ij^T x} \tag{1.12}$$

denote the Fourier series of $u \in L^2(T^n)$. Then $H^\sigma(T^n)$ consists of those $u \in L^2$ for which the H^σ norm

$$\| u \|_{H^\sigma} := \sum_{j \in \mathbb{Z}^n} (1 + j^2)^{\sigma/2} \mid u_j \mid^2 \tag{1.13}$$

is finite. Here we abbreviate $j^2 = j^T j$. Note that H^σ coincides with the classical Sobolev spaces, for $\sigma \in \mathbb{N}$. Also note that $H^\sigma \hookrightarrow C^0$ for $\sigma > n/2$, by Sobolev embedding.

Theorem 1.1. *Let assumptions* (1.8) *on the nonlinearity f hold and fix a frequency matrix ω satisfying the Diophantine condition* (1.11). *Choose an initial condition $u_0 \in H^\sigma(T^n)$, where $\frac{n}{2} < \sigma < 2$ and in particular the dimension n is restricted to values $n = 1, 2, 3$. Choose σ, γ in the triangle*

$$0 < \gamma < \min(\sigma - \frac{n}{2}, 2 - \sigma). \tag{1.14}$$

Denote by $u^\varepsilon(t)$ and $u^0(t)$ the solutions of (1.1) *and of the homogenized equation* (1.4), *respectively, with the same initial condition $u_0(x)$ at time $t = 0$. Let $u^0(t)$ exist for $0 \leq t \leq T$. Then there exists $\varepsilon_0 > 0$ and a constant C, all depending on the data, on u_0, and on N, c, ϑ, σ, and γ, such that the following fractional estimate holds, uniformly for $0 < \varepsilon < \varepsilon_0$ and $0 \leq t \leq T$:*

$$\| u^\varepsilon(t) - u^0(t) \|_{H^\sigma(T^n)} \leq C\varepsilon^\gamma. \tag{1.15}$$

We remark that by (1.14) the optimal fractional rate γ of convergence towards the homogenized solutions allowed by our theorem is achieved for the choice

$$\sigma = 1 + n/4. \tag{1.16}$$

In fact the limiting fractional rate of convergence $\gamma^* = \gamma^*(n)$ allowed by our result is then given by

$$0 < \gamma < \gamma^*(n) := 1 - \frac{n}{4}. \tag{1.17}$$

We repeat that for any fixed Diophantine constant $\vartheta > 0$ and some $c = c(\omega) > 0$, the Diophantine condition (1.11) is satisfied for almost every choice of columns $\omega_\varrho \in \mathbb{R}^N$ of the frequency matrix ω, in the Lebesgue sense. In other words, each column $\omega_\varrho \in \mathbb{R}^N$ can be chosen from a set of full Lebesgue measure in $\mathbb{R}^N$.

Going beyond mere initial value problems, we consider (local) stable and unstable manifolds next. Summarizing loosely, we strongly recommend [9] and [22] for a technical background. The *unstable manifold* W_ε^u of a hyperbolic equilibrium, say $u \equiv 0$, of (1.1) consists of those $u_0 \in H^\sigma(T^n)$ which possess an associated solution $u^\varepsilon(t)$, defined for all *negative* times, such that

$$\lim_{t \to -\infty} u^\varepsilon(t) = 0. \tag{1.18}$$

These u_0 form W_ε^u, a finite-dimensional C^1 manifold immersed in $H^\sigma(T^n)$. The *local unstable manifold*, an embedded submanifold of a δ-neighborhood of $u \equiv 0$, consists of those u_0 for which the solution $u^\varepsilon(t)$ stays near $u \equiv 0$ for all negative times. By hyperbolicity of $u \equiv 0$ we have in particular $u_0 \in W^u$. Similarly, the *local stable manifold* W_ε^s is given by u_0 in a δ-neighborhood of $u \equiv 0$, for which the forward solution $u^\varepsilon(t), t \geq 0$ stays near $u \equiv 0$. In particular, the forward solution is global and

$$\lim_{t \to +\infty} u^\varepsilon(t) = 0, \tag{1.19}$$

by hyperbolicity of $u \equiv 0$. The dimension of W_ε^s is infinite, with codimension given by $\dim W_\varepsilon^u$.

Section 2 below can also be read as a rather complete technical exposition proving existence and fractional-order convergence of local stable/unstable manifolds for $\varepsilon \searrow 0$.

For gradient systems, like (1.1), the unstable manifold is globally embedded. Consider the dissipative case, where the polynomial nonlinearity f^ε has odd degree d in u and the highest-order coefficients $a_d^\varepsilon(x, y)$ are uniformly negative. Moreover, assume all equilibria to be hyperbolic. Then the *global attractor* $\mathcal{A}_\varepsilon$ of (1.1) consists of the (finite) union of all unstable manifolds. The global attractor is defined here as the maximal compact invariant set or, equivalently, as the smallest set attracting all bounded sets. Upper semicontinuity of the global attractor is known for regular perturbation families $\mathcal{A}_\varepsilon$. Lower semicontinuity holds, provided the stable and unstable manifolds intersect transversely. See [4], [8], and [18]. Fractional estimates for $\mathcal{A}_\varepsilon$, in the context of quantitative homogenization, are our main motivation for investigating the convergence behavior of stable and unstable manifolds in Theorem 1.2 below. For further results in this direction see also the companion paper [7].

Theorem 1.2. *Let f satisfy* (1.8) *and hyperbolicity assumption* (1.10) *at the trivial ε-independent equilibrium $u \equiv 0$. Again fix Diophantine frequencies ω satisfying* (1.11) *and choose σ, γ in the triangle* (1.14).

Then there exists an ε-independent δ neighborhood of $u \equiv 0$ in the Sobolev space $H^\sigma(T^n)$ and local stable and unstable C^1 manifolds $W^s_\varepsilon, W^u_\varepsilon$ of equation (1.1) *in this neighborhood, which for $\varepsilon \searrow 0$ converge with fractional order ε^γ to the corresponding local stable and unstable manifolds W^s_0, W^u_0 of the formally homogenized equation* (1.4)*. Convergence of these manifolds in fact occurs in the topology of C^1 manifolds, that is, in the topology of uniform convergence of both the manifolds and their tangent spaces.*

More precisely, these manifolds are given as graphs of functions $w^s_\varepsilon, w^u_\varepsilon$ over the tangent spaces of W^s_0, W^u_0 at $u \equiv 0$, locally. The differences $w^s_\varepsilon - w^s_0$ and $w^u_\varepsilon - w^u_0$, measured in the topology of $H^\sigma(T^n)$, converge to zero uniformly with fractal order ε^γ, together with their first derivatives. See Theorem 2.1 *and Corollaries* 2.2, 2.3, *and* 2.5 *below.*

We remark that fractional-order convergence of unstable manifolds, as established in Theorem 1.2, was a crucial ingredient to an abstract result on fractional-order convergence of global attractors of gradient systems in pioneering work by Hale and Raugel, [10]. See also the recent survey [18]. Going beyond our fractional homogenization estimate (1.15) of Theorem 1.1, which holds for bounded times only, an estimate

$$\|u^\varepsilon(t) - u^0(t)\|_{H^\sigma(T^n)} \leq C\varepsilon^\gamma e^{\varrho t} \tag{1.20}$$

was required to hold, uniformly for large $t > 0$. In our companion paper [7] we in fact derive such an estimate for near-gradient reaction–diffusion systems and establish fractional-order homogenization results for global attractors in $L^2(\Omega), \dim \Omega = 3$, under Dirichlet boundary conditions.

The remaining sections are organized as follows. In Section 2 we rephrase our motivating example (1.1), (1.4) in the language of analytic semigroups

$$u_t = Au + F^\varepsilon(u) \tag{1.21}$$

with not necessarily self-adjoint infinitesimal generator A on a Banach space X. The crucial convergence assumption is phrased in terms of norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{-\beta}$ in spaces X^α and $X^{-\beta}$ as

$$\|F^\varepsilon(u) - F^0(u)\|_{-\beta} \ \leq \ h(\|u\|_\alpha) \cdot \varepsilon^\gamma; \tag{1.22}$$

see (2.14), (2.42) and (2.49). The fractional-power spaces X^α are domains of definition of fractional powers $(-A_0)^\alpha$, where $A_0 = A - \lambda_0 \cdot id$ is a shifted infinitesimal generator. Passing to negative exponents $-\beta$ provides a regularization of the rapid spatial oscillation, which lends itself to quantitative homogenization. Under the abstract assumption (1.22), we prove the stable and unstable manifold theorems which our proof of Theorem 1.2 is based on;

see Theorem 2.1 and Corollaries 2.2, 2.3, and 2.5. In Corollary 2.4 we indicate the simplifications and modifications for preparing a proof of Theorem 1.1.

Section 3 is devoted to a careful analysis of the convergence properties for $\varepsilon \searrow 0$ of functions $b^\varepsilon(x) := B(x, \omega x/\varepsilon)$. For suitable $B(x, y)$, periodic in x, y with zero y average, sufficiently smooth in y and of regularity X^α in x, with $X := L^2(T^n)$, we prove an estimate

$$\|b^\varepsilon\|_{-\beta} \leq C\varepsilon^{2\beta} \tag{1.23}$$

in Proposition 3.1. This proposition forms a bridge between the abstract semigroup results of Section 2 and our specific example (1.1), (1.4).

In Section 4, we cross this bridge and return to our original example. Using estimates like (1.23), we prove estimates like (1.22), for $\gamma := 2\beta$. Translating $H^\sigma(T^n) = X^{\sigma/2}$, $\alpha := \sigma/2$, the technical constraints for α and β yield the crucial ε^γ convergence in the $H^\sigma(T^n)$ norm of Theorems 1.1 and 1.2, under the triangle condition

$$0 < \gamma < \min(\sigma - n/2,\ 2 - \sigma) \tag{1.24}$$

as stated in (1.14) above.

In Section 5, finally, we return to the case of the Navier–Stokes system with spatially rapidly oscillating quasiperiodic external forces. In Theorem 5.1 and Corollary 5.3 we present quantitative homogenization results based on the abstract semigroup results of Theorem 2.1 and Corollary 2.4.

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2. Invariant manifolds

In this section we prove an abstract invariant-manifold theorem which is adapted to the specific homogenization problem (1.1), (1.4) above. We work in an analytic semigroup setting, largely following [9]. In particular, we use Perron's method to construct the local stable manifold W_ε^s of a fixed, uniformly hyperbolic equilibrium $u = 0$. The results for local unstable manifolds W_ε^u are completely analogous and will be stated briefly along with the considerably simpler issue of convergence of solutions of initial value problems.

The analytic semigroup setting for our specific homogenization problem (1.1), (1.4) will be

$$u_t = Au + F^\varepsilon(u) \tag{2.1}$$

with assumptions on the nonlinearity $F^\varepsilon(u)$ detailed below. The generator A of the analytic semigroup $\exp(At)$, $t \geq 0$, on a Banach space X is assumed to be sectorial and independent of ε.

Fractional powers of A and their domains of definition will be an essential tool of our quantitative result. See [9], [19], or [17] for a technical background. We briefly digress to remind the reader of the basic definitions. Since A is sectorial, we may fix $\lambda_0 \geq 1 + Re(\mathrm{spec} A)$. Decomposing $A = A_0 + \lambda_0 \cdot id$, we have $Re(\mathrm{spec}(-A_0)) \geq 1$. Integrating along a suitably oriented complex sector Γ of opening angle less than $\pm\pi/2$ centered around the negative real axis, we may thus define the fractional powers

$$(-A_0)^\alpha := \frac{1}{2\pi i} \int_\Gamma (-\lambda)^\alpha (\lambda - A_0)^{-1}\, d\lambda \tag{2.2}$$

for $\alpha < 0$. Convergence holds by the resolvent estimate for A_0. We take principal values for the root $(-\lambda)^\alpha$. In particular,

$$(-A_0)^{-\alpha_1}(-A_0)^{-\alpha_2} = (-A_0)^{-(\alpha_1+\alpha_2)} \tag{2.3}$$

for $\alpha_1, \alpha_2 > 0$. Extensions to positive α can be defined via

$$(-A_0)^\alpha := \left((-A_0)^{-\alpha}\right)^{-1}. \tag{2.4}$$

All these fractional powers $(-A)^\alpha$ are closable with maximal domains of definition

$$X^\alpha := \mathcal{D}\left((-A_0)^{-\alpha}\right), \quad \alpha \in \mathbb{R}. \tag{2.5}$$

With the norm

$$\|u\|_\alpha := \|(-A_0)^\alpha u\|_X \tag{2.6}$$

the completion spaces X^α become Banach spaces, isometrically isomorphic to $X = X^0$ with isomorphism $(-A_0)^\alpha$. Also note the embeddings

$$X^\alpha \subseteq X^{\alpha'} \tag{2.7}$$

for $\alpha \geq \alpha'$. The special case $\alpha = 1$ reproduces $A = A^1$, $X^1 = \mathcal{D}(A)$, and $\alpha = -1$ indeed provides the inverse of A. More generally, (2.3) extends to real α_1, α_2 on $X^{\alpha_1+\alpha_2}$.

The use of Banach spaces $X^{-\beta}$ with norms $\|\cdot\|_{-\beta}$, for suitable $0 < \beta < 1$, will be crucial for our quantitative analysis of homogenization. In specific function spaces like $X = L^2(T^n)$ these spaces can be described in terms of distributions or in terms of weighted l^2-spaces of Fourier coefficients. In our abstract, more general situation, the above abstract definition by completion spaces is just as viable. For example, $(-A_0)^\beta$ extends canonically to a closed, densely defined operator on the completion space $X^{-\beta}$ with domain X dense in $X^{-\beta}$, all with respect to the topology of $\|\cdot\|_{-\beta}$ of course. Similarly the analytic semigroup $\exp(A_0 t)$, generated by A_0 on X, extends to $X^{-\beta}$.

Although we do not require A_0 to be self-adjoint or normal, in our specific example (1.1) we will choose $A_0 = \Delta - id$ to be the shifted Laplacian with periodic boundary conditions on $X = L^2(T^n)$. In terms of Fourier coefficients

$$u(x) = \sum_{j\in\mathbb{Z}^n} u_j \exp(ij^T x) \tag{2.8}$$

the norms $\|\cdot\|_\alpha$ associated to $X = L^2(T^n)$ then take the explicit form

$$\|u\|_\alpha = \Big(\sum_{j\in\mathbb{Z}^n} (1+j^2)^{2\alpha}|u_j|^2\Big)^{1/2} \tag{2.9}$$

where we abbreviate $j^T j = j^2$. For example $X^1 = H^2(T^n),\ X^{1/2} = H^1(T^n)$, and $H^\sigma(T^n) = X^{\sigma/2}$ for $\sigma\in\mathbb{R}$. More generally, for $X = L^p(\Omega)$ and $\alpha > 0$ we note the Sobolev embeddings $X^\alpha \hookrightarrow W^{k,q}(\Omega)$ for $k - n/q < 2\alpha - n/p,\ q \geq p$ and the Hölder embeddings $X^\alpha \hookrightarrow C^{k,\vartheta}(\Omega)$ for $k + \vartheta < 2\alpha - n/p$, when $\Omega \subseteq \mathbb{R}^n$ is a smooth domain.

In the abstract setting (2.1) we require regularity and continuous dependence for the nonlinearities

$$F^\varepsilon : X^\alpha \longrightarrow X^{-\beta} \tag{2.10}$$

and for suitable choices

$$\alpha,\beta \geq 0, \qquad \alpha+\beta < 1. \tag{2.11}$$

In specific examples, it is precisely the weaker norm of $X^{-\beta}$ which will provide us with the Hölder estimates of F^ε with respect to ε, as required below. Indeed the isometric isomorphism $(-A_0)^{-\beta} : X^{-\beta} \to X$ allows us to formulate all assumptions on F in terms of

$$(-A_0)^{-\beta} F^\varepsilon : X^\alpha \longrightarrow X. \tag{2.12}$$

The regularizing properties of $(-A_0)^{-\beta}$ will then homogenize rapid spatial oscillations of nonlinearities of the type

$$\begin{aligned} (F^\varepsilon(u))(x) &:= f(\varepsilon, x, x/\varepsilon, u(x)), \quad &\text{for } \varepsilon > 0\\ (F^0(u))(x) &:= \langle f(0,x,\cdot,u(x))\rangle, \quad &\text{for } \varepsilon = 0 \end{aligned} \tag{2.13}$$

in our specific examples.

Abstractly, our assumptions on the nonlinearities F^ε are the following, uniformly for $0 \le \varepsilon < \varepsilon_0$:

$$\begin{aligned} &F^\varepsilon(0) = 0; \\ &\|F^\varepsilon(u_1) - F^\varepsilon(u_2)\|_{-\beta} \le \eta \cdot \|u_1 - u_2\|_\alpha \ ; \\ &\|F^\varepsilon(u) - F^0(u)\|_{-\beta} \le h(\|u\|_\alpha) \cdot \varepsilon^\gamma \end{aligned} \tag{2.14}$$

uniformly for all $u, u_1, u_2 \in X^\alpha$, $0 \le \varepsilon < \varepsilon_0$, suitable positive constants η, γ, and a nondecreasing function $h > 0$. The uniform global Lipschitz constant η will be required to be small – an assumption to be achieved by suitable local cut-off. This cut-off is the standard approach to proving local invariant manifold theorems; see for example [22]. Although we do not require differentiability of F^0, strictly speaking, the Lipschitz assumption will in practice be achieved by subsuming the linearization of F^0 at $u = 0$ into the fixed generator A. See also Corollary 2.5 below for differentiability issues. The Hölder rate of convergence γ will be related to the regularization β by

$$\gamma = 2\beta < 2(1 - \alpha) \tag{2.15}$$

in the specific quasiperiodic example of Section 1.

Uniform hyperbolicity of the equilibrium $u = 0$ can be expressed in terms of the ε-independent generator A on our Hilbert space X, for small enough Lipschitz constants η of the nonlinearity. Assuming the spectrum of A to have nonzero real part, we have spectral projections $P_\pm$ onto subspaces $X_\pm$ of X associated with the positive/negative part of spec(A). We decompose

$$id_{X^{\alpha'}} = P_+ + P_-, \quad X^{\alpha'} = X_+^{\alpha'} \oplus X_-^{\alpha'}, \quad u = (u_+, u_-) \tag{2.16}$$

accordingly, for all real α'. Indeed all spaces $X^{\alpha'}$ are densely embedded into each other, so that the projections $P_\pm$ are well-defined on each of them. Also note that the norms of these projections are independent of $\alpha' \in \mathbb{R}$, since $P_\pm$ commute with A and all $(-A_0)^{\alpha'}$.

We are now ready to state and prove the invariant manifold result of this section.

Theorem 2.1. *Let assumptions* (2.10), (2.11), (2.14) *and hyperbolicity assumption* (2.16) *hold. For small-enough perturbations* $0 \le \varepsilon < \varepsilon_0$ *and small enough Lipschitz constants* $0 \le \eta < \eta_0$*, the semilinear semigroup* (2.1) *then possesses a unique stable manifold* W^s_ε *of the trivial hyperbolic equilibrium* $u = 0$*.*

In terms of the spectral projections $P_\pm$ of the linearization A, the manifold W^s_ε is given globally as the graph of a function

$$w^s_\varepsilon : X^\alpha_- \longrightarrow X^\alpha_+ = X_+ \tag{2.17}$$

with small Lipschitz constant η' proportional to η. As usual, W^s_ε = graph (w^s_ε) is the set of all initial conditions $u^0 \in X^\alpha$ such that there exists a uniformly bounded global forward solution $u(t) \in X^\alpha,\ t \geq 0$, through $u(0) = u^0$ which satisfies

$$\lim_{t\to+\infty} \|u(t)\|_\alpha = 0. \tag{2.18}$$

The manifolds W^s_ε are uniformly close to W^s_0 of order $0(\varepsilon^\gamma)$: there exist constants $C > 0,\ \eta' > 0$ such that the fractional convergence estimate

$$\|w^s_\varepsilon(u_-) - w^s_0(u_-)\|_\alpha \leq Ch(\eta'\|u_-\|_\alpha)\varepsilon^\gamma \tag{2.19}$$

holds, uniformly for $0 \leq \varepsilon < \varepsilon_0$ and $u_- \in X^\alpha_-$. Here γ is the regularized Hölder exponent with respect to ε, as was specified in assumption (2.14).

Our proof below is only a slight adaptation of standard Perron-type proofs of invariant (or inertial) manifold theorems for analytic semigroups as detailed, for example, in [9], [2], [22] [6]. Typically, such theorems are formulated for the case $\beta = 0$, which does not lend itself to our goal of quantitative spatial homogenization for equations like (1.1), when we choose $X = L^2(\Omega)$. Before giving a detailed proof, for the convenience of the reader we sketch the underlying idea, which is just a simple regularization by the fractional operator $(-A_0)^{-\beta}$.

Indeed, the original semigroup equation (2.1) transforms into

$$\tilde{u}_t = A\tilde{u} + \tilde{F}^\varepsilon(\tilde{u}) \tag{2.20}$$

under the regularization

$$\tilde{u} := (-A_0)^{-\beta}u, \quad \tilde{F}^\varepsilon(\tilde{u}) := (-A_0)^{-\beta}F^\varepsilon((-A_0)^\beta\tilde{u}). \tag{2.21}$$

We consider (2.20) as an equation on the space X. Assumptions (2.14) then translate into

$$\begin{array}{rcl} \tilde{F}^\varepsilon(0) & = & 0 \\ \|\tilde{F}^\varepsilon(\tilde{u}_1) - \tilde{F}^\varepsilon(\tilde{u}_2)\|_X & \leq & \eta \cdot \|\tilde{u}_1 - \tilde{u}_2\|_{X^{\tilde{\alpha}}} \\ \|\tilde{F}^\varepsilon(\tilde{u}) - \tilde{F}^0(\tilde{u})\|_X & \leq & h(\|\tilde{u}\|_{\tilde{\alpha}}) \cdot \varepsilon^\gamma, \end{array} \tag{2.22}$$

where $0 \leq \tilde{\alpha} = \alpha + \beta < 1$, by assumption (2.11) above. Except for the slightly unusual fractional dependence on ε^γ, this is the standard setting for an invariant manifold theorem on $X, X^{\tilde{\alpha}}$. The central importance of this fractional dependence for our quantitative averaging result, however, motivates us to give a detailed proof from scratch, rather than backtracking to $\tilde{u} \in X$.

Proof. We begin with an outline of the proof. Following Perron's approach to invariant manifolds, we solve the following fixed point problem:

$$\begin{aligned} u_+(t) &= \int_t^{+\infty} e^{A(t-\tau)} P_+ F^\varepsilon(u_+(\tau), u_-(\tau))\, d\tau \\ u_-(t) &= e^{At} u_-^0 + \int_0^t e^{A(t-\tau)} P_- F^\varepsilon(u_+(\tau), u_-(\tau))\, d\tau. \end{aligned} \tag{2.23}$$

Abbreviating $u(t) = (u_+(t), u_-(t))$, and writing $\Phi = \Phi(\varepsilon, u_-^0, u(\cdot))$ for the right-hand side of (2.23) as a function of time t, we have to solve

$$u(\cdot) = \Phi(\varepsilon, u_-^0, u(\cdot)) \tag{2.24}$$

for $u(\cdot) = u(\cdot; \varepsilon, u_-^0) \in BC^0([0, \infty), X^\alpha) =: \mathcal{X}^\alpha$, by Banach's fixed-point theorem. By the variation-of-constants formula (2.23), the graph of

$$w_\varepsilon^s(u_-^0) := u_+(0; \varepsilon, u_-^0) \tag{2.25}$$

will be an invariant manifold under the analytic semigroup (2.20). Standard hyperbolicity estimates identify graph(w_ε^s) as the stable manifold W_ε^s of the origin. With the fixed point $u(\cdot) \in \mathcal{X}^\alpha$ of (2.24) depending Lipschitz continuously on u_-^0 and $\mathcal{O}(\varepsilon^\gamma)$ on ε, the same is true of w_ε^s and W_ε^s.

The proof is based on hyperbolicity estimates of the linear analytic semigroup $\exp(At)$. Specifically, for any $\alpha' \geq -\beta'$ there exist positive constants M, ϑ such that

$$\begin{aligned} &\|e^{As} P_+\|_{\alpha', -\beta'} \leq M e^{\vartheta s} && \text{for } s \leq 0 \\ &\|e^{As} P_-\|_{\alpha', -\beta'} \leq M s^{-(\alpha'+\beta')} e^{-\vartheta s} && \text{for } s > 0. \end{aligned} \tag{2.26}$$

Here $\| \cdot \|_{\alpha', -\beta'}$ denotes the operator norm from $X^{\alpha'}$ to $X^{-\beta'}$. We have also used $(-A_0)^{-\beta'} P_\pm = P_\pm (-A_0)^{-\beta'}$, so that the norms of $P_\pm$ on $X^{-\beta'}$ are independent of β'. These estimates follow from standard linear semigroup estimates on the linearly invariant subspaces $X_\pm^{\alpha'}$ and the fact that $\exp(As)$ commutes with all fractional powers $(-A_0)^{\alpha'}$. Note that A is boundedly invertible on the unstable eigenspace X_+; hence all the spaces $X_+^{\alpha'} = X_+$ coincide.

For the fixed point map Φ we now show that

$$\Phi(\varepsilon, u_-^0, \cdot) : \mathcal{X}^\alpha \to \mathcal{X}^\alpha \tag{2.27}$$

is a contraction for all $0 < \varepsilon < \varepsilon_0$, all $u_-^0 \in X_-^\alpha$ and sufficiently small Lipschitz constant η. Since $\exp(A\cdot) u_-^0 \in \mathcal{X}^\alpha$, by exponential decay on X_-^α, and

$$\Phi(\varepsilon, 0, 0) = 0, \tag{2.28}$$

and since, by assumption (2.14), $F^\varepsilon(0) = 0$, the range of the map Φ is then automatically contained in $\mathcal{X}^\alpha$.

We estimate contraction only for the u_- component Φ_- of $\Phi = (\Phi_+, \Phi_-)$, the X_+ estimate being even more innocent. For any $t > 0$, and any pair $u^i(\cdot) = (u^i_+(\cdot), u^i_-(\cdot)) \in \mathcal{X}^\alpha$ we have

$$\begin{aligned}
&\|\Phi_-(\varepsilon, u^0_-, u^2(\cdot))(t) \;-\; \Phi_-(\varepsilon, u^0_-, u^1(\cdot))(t)\|_\alpha \\
&= \| \int_0^t e^{A(t-\tau)} P_- \left(F^\varepsilon(u^2(\tau)) \;-\; F^\varepsilon(u^1(\tau))\right) d\tau \|_\alpha \\
&\le \int_0^t \|e^{A(t-\tau)} P_- \left(F^\varepsilon(u^2(\tau)) - F^\varepsilon(u^1(\tau))\right) \|_\alpha \, d\tau \\
&\le \int_0^t \|e^{A(t-\tau)} P_-\|_{\alpha,-\beta} \|F^\varepsilon(u^2(\tau)) - F^\varepsilon(u^1(\tau))\|_{-\beta} \, d\tau \\
&\le M \int_0^t (t-\tau)^{-(\alpha+\beta)} e^{-\vartheta(t-\tau)} d\tau \cdot \eta \cdot \sup_{\tau \ge 0} \|u^2(\tau) - u^1(\tau)\|_\alpha \\
&\le \kappa \cdot \|u^2 - u^1\|_{\mathcal{X}^\alpha}.
\end{aligned} \tag{2.29}$$

Here we have used the linear semigroup estimate (2.26) for $\alpha' = \alpha,\ \beta' = \beta$. The integrals are finite because $0 \le \alpha + \beta < 1$, by assumption (2.11). Our contraction constant κ is given explicitly by

$$\kappa := M\Gamma(1 - (\alpha + \beta))\vartheta^{\alpha+\beta-1}\eta \tag{2.30}$$

in terms of Euler's gamma function Γ. We repeat that the estimate for the difference of the Φ_+ components proceeds analogously to (2.29) and is omitted. Combining both estimates, we see that for small-enough Lipschitz constants $\eta > 0$ we obviously obtain a contraction rate $\kappa < 1$, and hence a fixed point $u(\cdot) = u(\cdot; \varepsilon, u^0_-)$ of Φ, as has been promised in (2.24) above.

The Lipschitz constant η' of $w^s_\varepsilon(u^0_-) = u(0, \varepsilon, u^0_-)$ can be estimated from the fixed point equation (2.24). With the abbreviation $\tilde{u} = u(\cdot; \varepsilon, \tilde{u}^0_-)$, we obtain

$$\begin{aligned}
&\|\tilde{u} - u\|_{\mathcal{X}^\alpha} = \|\Phi(\varepsilon, \tilde{u}^0_-, \tilde{u}) - \Phi(\varepsilon, u^0_-, u)\|_{\mathcal{X}^\alpha} \\
&\le \sup_{t \ge 0} \| \exp(At) P_-(\tilde{u}^0_- - u^0_-)\|_\alpha \quad + \quad \kappa \|\tilde{u} - u\|_{\mathcal{X}^\alpha}.
\end{aligned} \tag{2.31}$$

Inserting the linear semigroup estimate (2.26) with $\alpha' = -\beta' = \alpha$ then proves the Lipschitz estimate

$$\|w^s_\varepsilon(\tilde{u}^0_-) - w^s_\varepsilon(u^0_-)\|_\alpha \le \|\tilde{u} - u\|_{\mathcal{X}^\alpha} \le \eta' \cdot \|\tilde{u}^0_- - u^0_-\|_\alpha, \tag{2.32}$$

with $\eta' := M/(1-\kappa)$. In other words, the fixed point of Φ depends on Lipschitz parameters as Φ itself does.

Dependence on ε can be estimated analogously. Abbreviating the fixed point $u(\cdot;\varepsilon,u_-^0) \in \mathcal{X}^\alpha$ by $\mathbf{u}^\varepsilon$, we immediately see

$$\begin{aligned}\|w_\varepsilon^s(u_-^0) - w_0^s(u_-^0)\|_\alpha &\leq \|\mathbf{u}^\varepsilon - \mathbf{u}^0\|_{\mathcal{X}^\alpha} \\ &\leq \frac{1}{1-\kappa}\|\Phi(\varepsilon,u_-^0,\mathbf{u}^0) - \Phi(0,u_-^0,\mathbf{u}^0)\|_{\mathcal{X}^\alpha}.\end{aligned} \tag{2.33}$$

Indeed, the term $(1-\kappa)^{-1}$ arises from the contraction estimate, if we use the fixed-point properties of $\mathbf{u}^\varepsilon, \mathbf{u}^0$ and insert the terms $\pm\Phi(\varepsilon,u_-^0,\mathbf{u}^0)$ in the resulting difference.

As in (2.29), we estimate the difference of the X_-^α components Φ_- to be

$$\begin{aligned}&\frac{1}{1-\kappa}\|\Phi(\varepsilon,u_-^0,\mathbf{u}^0)(t) - \Phi(0,u_-^0,\mathbf{u}^0)(t)\|_\alpha \\ &\leq \frac{M}{1-\kappa}\int_0^t (t-\tau)^{-(\alpha+\beta)}e^{-\vartheta(t-\tau)}d\tau \sup_{\tau\geq 0}\|F^\varepsilon(\mathbf{u}^0(\tau)) - F^0(\mathbf{u}^0(\tau))\|_{-\beta} \\ &\leq Ch(\|\mathbf{u}^0\|_{\mathcal{X}^\alpha})\varepsilon^\gamma \leq Ch(\eta'\cdot\|u_-^0\|_\alpha)\cdot\varepsilon^\gamma.\end{aligned} \tag{2.34}$$

Here we have used assumption (2.14) to estimate the ε-dependence of F^ε, and the Lipschitz estimate (2.32) with $\tilde{u} \equiv 0$ at $\tilde{u}_-^0 = 0$, to estimate $\|\mathbf{u}^0\|_{\mathcal{X}^\alpha}$. Our constant C is given explicitly as

$$C = \frac{M}{(1-\kappa)}\Gamma(1-(\alpha+\beta))\vartheta^{\alpha+\beta-1} \tag{2.35}$$

with contraction $\kappa < 1$ as in (2.30). This proves the Hölder estimate (2.19).

It remains only to prove that the manifold $W_\varepsilon^s = \operatorname{graph}(w_\varepsilon^s)$ is invariant under the semiflow (2.1) and is characterized by the convergence property (2.18), $\|u(t)\|_\alpha \to 0$ for $t \to +\infty$. Convergence of the integrals (2.23) and the variation-of-constants formula for the Lipschitz function F^ε imply that the fixed point $t \mapsto \mathbf{u}^\varepsilon(t)$ is a strong solution of the differential equation (2.1) in $BC^1([0,\infty),X^\alpha)$ satisfying $\mathbf{u}_-^\varepsilon(0) = u_-^0$. Conversely, any uniformly bounded strong solution $\mathbf{u}(t)$ of (2.1) in $BC^1([0,\infty),X^\alpha)$ satisfies the integral equation (2.23) and therefore coincides with a fixed point $\mathbf{u}^\varepsilon(\cdot)$ of (2.23) with

$$u_-^0 := P_-\mathbf{u}(0). \tag{2.36}$$

In other words, W_ε^s consists of all initial conditions $u(0) = (u_+^0, u_-^0) \in X^\alpha$ such that the solution $u(t) \in X^\alpha$ of (2.8) remains bounded for all $t \geq 0$. This characterization of W_ε^s also proves forward time invariance. Moreover, W_ε^s trivially contains all solutions such that $\lim \|u(t)\|_\alpha = 0$ for $t \to +\infty$.

It remains to show, conversely, that

$$\lim \|\mathbf{u}^\varepsilon(t)\|_\alpha = 0 \tag{2.37}$$

for $t \to +\infty$ and any fixed point $\mathbf{u}^\varepsilon$ of (2.23). By forward invariance of $w^s_\varepsilon = \text{graph}(w^s_\varepsilon)$ and Lipschitz continuity (2.32) of w^s_ε we have

$$\|\mathbf{u}^\varepsilon_+(t)\|_\alpha = \|w^s_\varepsilon(\mathbf{u}^\varepsilon_-(t)) - w^s_\varepsilon(0)\| \le \eta' \|\mathbf{u}^\varepsilon_-(t)\|_\alpha, \tag{2.38}$$

and therefore

$$\|\mathbf{u}^\varepsilon(t)\|_\alpha \le (1+\eta')\|\mathbf{u}^\varepsilon_-(t)\|_\alpha. \tag{2.39}$$

For any $0 < \delta < \vartheta$, we can therefore estimate

$$\begin{aligned} &\|\mathbf{u}^\varepsilon_-(t)e^{\delta t}\|_\alpha \le M e^{-(\vartheta-\delta)t}\|u_0^-\|_\alpha \\ &+ M\int_0^t (t-\tau)^{-(\alpha+\beta)} e^{-(\vartheta-\delta)(t-\tau)} d\tau\, \eta(1+\eta') \sup_{0\le\tau\le t} \|\mathbf{u}^\varepsilon_-(\tau)e^{\delta\tau}\|_\alpha \\ &\le M\|u_0^-\|_\alpha + \kappa' \sup_{0\le\tau\le t} \|\mathbf{u}^\varepsilon_-(\tau)e^{\delta\tau}\|_\alpha. \end{aligned} \tag{2.40}$$

For Lipschitz constants $\eta_0 > 0$ and decay rates $\delta > 0$ small enough such that κ' can be chosen to lie in $(\kappa, 1)$, we immediately obtain the uniform bound

$$\sup_{0\le\tau<\infty} \|\mathbf{u}^\varepsilon_-(\tau)e^{\delta\tau}\|_\alpha \le \frac{M\|u_0^-\|_\alpha}{1-\kappa'}. \tag{2.41}$$

In view of (2.38), this shows exponential decay of $\|\mathbf{u}^\varepsilon(t)\|_\alpha$ and completes the proof of Theorem 2.1. □

Corollary 2.2. *Let assumptions* (2.10), (2.11), (2.14), *and hyperbolicity assumption* (2.16) *of Theorem* 2.1 *hold, but strengthen* (2.14) *by the additional differentiability requirements* $F^\varepsilon \in C^1(X^\alpha, X^{-\beta})$ *with*

$$\begin{aligned} &\|D_u F^\varepsilon(u)\|_{\alpha,-\beta} \le \eta \\ &\|D_u F^\varepsilon(u) - D_u F^0(u)\|_{\alpha,-\beta} \le h(\|u\|_\alpha)\varepsilon^\gamma \end{aligned} \tag{2.42}$$

for all $u \in X^\alpha,\ 0 \le \varepsilon < \varepsilon_0$, *suitable positive constants* η, γ, *and a nondecreasing function* $h > 0$. *For small enough Lipschitz constants* $0 \le \eta < \eta_0$ *and small enough perturbations* $0 \le \varepsilon < \varepsilon_0$, *the stable manifold* $W^s_\varepsilon = graph(w^s_\varepsilon)$ *constructed in Theorem* 2.1 *is then continuously differentiable with uniform bounds*

$$\begin{aligned} &\|D_{u_-} w^s_\varepsilon(u_-)\|_{\alpha,\alpha} \le \eta' \\ &\|D_{u_-} w^s_\varepsilon(u_-) - D_{u_-} w^s_0(u_-)\|_{\alpha,\alpha} \le Ch(\eta'\|u_-\|_\alpha)\varepsilon^\gamma. \end{aligned} \tag{2.43}$$

As before, η' *is proportional to* η.

Proof. Differentiating the fixed point form (2.23), (2.24) with respect to u_-^0, the estimates of the proof of Theorem 2.1 apply. □

Corollary 2.3. *Theorem* 2.1 *and Corollary* 2.2 *hold, likewise, for the unstable manifold* $W_\varepsilon^u = \mathrm{graph}(w_\varepsilon^u)$, *which is characterized to consist of all solutions* $u(t)$ *of* (2.1) *which are defined for all* $t \leq 0$ *and satisfy* $\lim \|u(t)\|_\alpha = 0$ *for* $t \to -\infty$.

Proof. The Perron fixed-point formulation for the unstable manifold

$$w_\varepsilon^u(u_+^0) := u(0; \varepsilon, u_+^0) \tag{2.44}$$

analogous to (2.23)–(2.25) is given by

$$\begin{aligned} u_+(t) &= e^{At} u_+^0 + \int_0^t e^{A(t-\tau)} P_+ F^\varepsilon(u_+(\tau), u_-(\tau))\, d\tau \\ u_-(t) &= \int_{-\infty}^t e^{A(t-\tau)} P_- F^\varepsilon(u_+(\tau), u_-(\tau))\, d\tau \end{aligned} \tag{2.45}$$

on the space $u = (u_+, u_-) \in BC^0((-\infty, 0], X^\alpha)$. The proof then proceeds analogously to the proofs of Theorem 2.1 and Corollary 2.2 above. □

Corollary 2.4. *Let assumptions* (2.10), (2.11), *and* (2.14) *hold with the following modifications: we do not require* $F^\varepsilon(0) = 0$ *and we drop the assumption that the local Lipschitz constant* η *is small.*

Then for any $u_0 \in X^\alpha$ *and for any time* $T = T(u_0) > 0$ *not exceeding the maximal time of existence for* $\varepsilon = 0$, *there exists* $\varepsilon_0 > 0$, *and a constant* $C > 0$ *such that the solutions* $u^\varepsilon(t), u^0(t)$ *of* (2.1) *satisfy the fractional estimate*

$$\|u^\varepsilon(t) - u^0(t)\|_\alpha \leq C\varepsilon^\gamma \tag{2.46}$$

uniformly for $0 \leq t \leq T(u_0)$. *As before,* γ *is the regularized Hölder exponent with respect to* ε, *as was specified in assumption* (2.14).

Proof. The proof is similar to, but simpler than, the invariant manifold proofs given above. The fixed point form for mild and strong solutions of (2.1) is

$$u^\varepsilon(t) = e^{At} u_0 + \int_0^t e^{A(t-\tau)} F^\varepsilon(u^\varepsilon(\tau))\, d\tau, \tag{2.47}$$

by the variation-of-constants formula. The proof then proceeds as before, replacing the contraction constant κ in (2.30) by

$$\kappa := \eta M \int_0^T \tau^{-(\alpha+\beta)} d\tau = \frac{\eta M}{1-(\alpha+\beta)} T^{1-(\alpha+\beta)}. \tag{2.48}$$

Here we have assumed the semigroup $\exp(At)$ to be bounded by M, without loss of generality. Indeed we may shift the spectrum of A by subtracting a multiple of identity, adding this multiple to the nonlinearity F, instead. Clearly $\kappa < 1$ becomes a contraction for small $T > 0$. Time stepping with respect to T then proves the corollary. $\square$

Conversely, Corollary 2.4 is a viable approach to proving invariant manifold theorems. We chose to prove the slightly more involved invariant manifold theorem directly, to avoid excessive hand waving.

We conclude this section by describing the necessary cut-off modifications for local C^1 versions of the invariant manifold results stated in Corollaries 2.2 and 2.3. Specifically, we replace assumptions (2.14) and (2.42) as follows. Let $F^\varepsilon \in C^1(X^\alpha, X^{-\beta})$ be such that

$$\begin{aligned} F^\varepsilon(0) = 0, \quad D_u F^0(0) = 0 \\ \|F^\varepsilon(u) - F^0(u)\|_{\alpha,-\beta} \leq C\varepsilon^\gamma \\ \|D_u F^\varepsilon(u) - D_u F^0(u)\|_{\alpha,-\beta} \leq C\varepsilon^\gamma \end{aligned} \tag{2.49}$$

for $0 \leq \varepsilon < \varepsilon_0$, all $\|u\|_\alpha \leq \delta_0$ and suitable positive constants C, γ. We also fix a smooth scalar cut-off function $\chi \in C^\infty([0,\infty),[0,1])$, identically 1 for arguments in $[0,1]$ and identically 0 for arguments above 2. Then χ defines a cut-off $\tilde{F}^\varepsilon$ of F^ε via

$$\tilde{F}^\varepsilon(u) := \chi(\|u\|_\alpha/\delta) \cdot F^\varepsilon(u). \tag{2.50}$$

Finally, we require differentiability of the norm $\|\cdot\|_\alpha$ on X^α. This holds, for example, for spaces X^α based on L^p spaces X with $1 < p < \infty$, $2\alpha > n/p$.

Corollary 2.5. *Consider the cut-off $\tilde{F}^\varepsilon$ of a nonlinearity F^ε satisfying* (2.49) *and* (2.50) *above. Then for any $\eta > 0$ there exists $\delta > 0$ and $\varepsilon_0 > 0$ such that the cut-off nonlinearity $\tilde{F}^\varepsilon$ satisfies assumptions* (2.14) *and* (2.42), *globally for all $u_0 \in X^\alpha$ and all $0 \leq \varepsilon < \varepsilon_0$. In particular, Corollaries* 2.2 *and* 2.3 *hold true for $\tilde{F}^\varepsilon$, giving rise to local stable and unstable manifolds of the trivial equilibrium $u = 0$ of* (2.1) *with the original nonlinearity F^ε.*

Proof. We have to prove (2.14) and (2.42) for prescribed small $\eta > 0$ and all $u \in X^\alpha$. Since $\tilde{F}^\varepsilon(u) \equiv 0$ for $\|u\|_\alpha \geq 2\delta$, it is sufficient to consider $\|u\|_\alpha < 2\delta$. We only address the derivative estimates in (2.42), the remaining claims being obvious. In other words, we have to prove that

$$\|D_u \tilde{F}^\varepsilon(u)\|_{\alpha,-\beta} \leq \eta \tag{2.51}$$

$$\|D_u \tilde{F}^\varepsilon(u) - D_u \tilde{F}^0(u)\|_{\alpha,-\beta} \leq C\varepsilon^\gamma \tag{2.52}$$

for $\|u\|_\alpha < 2\delta$ and suitably chosen δ.

We prove (2.51) first. By the product rule,

$$D_u\tilde{F}^\varepsilon(u) = \chi'(\|u\|_\alpha/\delta)\cdot\delta^{-1}D_u\|u\|_\alpha\cdot F^\varepsilon(u) + \chi(\|u\|_\alpha/\delta)\cdot D_uF^\varepsilon(u). \tag{2.53}$$

The second term is small, for δ small and $0 \le \varepsilon < \varepsilon_0$. Indeed, χ is bounded and

$$\begin{aligned}\|D_uF^\varepsilon(u)\|_{\alpha,-\beta} &\le \|D_uF^\varepsilon(u) - D_uF^0(u)\|_{-\beta} + \|D_uF^0(u) - D_uF^0(0)\|_{-\beta}\\ &\le C\varepsilon^\gamma + o(1)\end{aligned} \tag{2.54}$$

is arbitrarily small for ε_0, δ small enough. Here we have used assumption (2.49).

The first term in (2.53) features uniformly bounded χ' and $D_u\|u\|_\alpha$, for $\|u\|_\alpha < 2\delta$. Concerning the $X^{-\beta}$ norm of the remaining factor $\delta^{-1}F^\varepsilon(u)$, we use $F^\varepsilon(0) = 0$ and estimate

$$\begin{aligned}\delta^{-1}\|F^\varepsilon(u)\|_{-\beta} &= \delta^{-1}\|F^\varepsilon(u) - F^\varepsilon(0)\|_{-\beta}\\ &\le \delta^{-1}\cdot 2\delta\cdot \sup_{\|u\|_\alpha\le 2\delta}\|D_uF^\varepsilon(u)\|_{\alpha,-\beta},\end{aligned} \tag{2.55}$$

which is small by the above uniform estimate (2.54). This proves estimate (2.51).

To prove (2.52), we expand $D_u\tilde{F}^\varepsilon(u)$ and $D_u\tilde{F}^0(0)$ as in (2.53) above. By assumption (2.49) and boundedness of $\chi, \chi', D_u\|u\|_\alpha$ it is then sufficient to estimate

$$\begin{aligned}&\delta^{-1}\|F^\varepsilon(u) - F^0(u)\|_{-\beta} = \delta^{-1}\|(F^\varepsilon(u) - F^\varepsilon(0)) - (F^0(u) - F^0(0))\|_{-\beta}\\ &= \delta^{-1}\Big\|\int_0^1 (D_uF^\varepsilon(\theta u) - D_uF^0(\theta u))\,d\theta\cdot u\Big\|_{-\beta}\\ &\le \int_0^1 \|D_uF^\varepsilon(\theta u) - D_uF^0(\theta u)\|_{\alpha,-\beta}\,d\theta\cdot\delta^{-1}\|u\|_\alpha \le 2C\varepsilon^\gamma.\end{aligned} \tag{2.56}$$

This proves (2.52), for generic constants C, and the corollary. □

3. Diophantine estimates

The crucial assumptions for the construction of local invariant manifolds, in the preceding section, are the estimates (2.14), (2.42) and (2.49) in the fractional power spaces X^α and $X^{-\beta}$ introduced in (2.5) and (2.6). As a bridge to our motivating example (1.1), (1.4), we now provide related estimates for sufficiently smooth functions $b^\varepsilon(x) := B(x, \omega x/\varepsilon)$ which are quasiperiodic in the rescaled variable x/ε and satisfy a Diophantine condition for the frequency matrix ω. Specifically, we assume

$$B : T^n \times T^N \to \mathbb{R}, \quad (x, y) \mapsto B(x, y) \tag{3.1}$$

is 2π-periodic in each component of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$. In Fourier series,

$$B(x, y) = \sum_{j \in \mathbb{Z}^N} B_j(y) \exp(ij^T x), \quad B_j(y) = \sum_{k \in \mathbb{Z}^n} B_{jk} \exp(ik^T y). \tag{3.2}$$

For regularity we assume

$$\|B\|_{\alpha,s} := \Big(\sum_{\substack{j \in \mathbb{R}^n \\ k \in \mathbb{R}^N}} (1 + j^2)^{2\alpha} (1 + k^2)^s |B_{jk}|^2 \Big)^{1/2} < \infty \tag{3.3}$$

where we abbreviate $j^2 = j^T j$ and $k^2 = k^T k$. Thinking of y-averages as having been subtracted, we require zero y-average

$$\begin{aligned} (2\pi)^{-N} \textstyle\int_{T^N} B(x, y)\, dy = 0, &\quad \text{for all } x, \text{ alias} \\ B_{j0} = 0, &\quad \text{for all } j \in \mathbb{Z}^n. \end{aligned} \tag{3.4}$$

For the $N \times n$ frequency matrix ω we use the notation $\omega = (\omega_1, \ldots, \omega_n)$ with columns $\omega_\varrho \in \mathbb{R}^N$.

The main result of this section is

Proposition 3.1. *Let $B : T^n \times T^N \to \mathbb{R}$ satisfy assumptions* (3.2)–(3.4) *above. Assume the frequency matrix ω satisfies the Diophantine condition*

$$\min_{\varrho=1,\ldots,n} |k^T \omega_\varrho| > c|k|^{-(N-1)-\vartheta} \tag{3.5}$$

for some $c, \vartheta > 0$ and all $k \in \mathbb{Z}^N \backslash \{0\}$. Fix α, β, and s such that

$$0 < \beta \leq 1/2; \quad \beta < \alpha - n/4; \quad s > 2(N - 1 + \vartheta)\beta + N/2. \tag{3.6}$$

Then $b^\varepsilon(x) := B(x, \omega x/\varepsilon)$, for $0 \leq x_\varrho < 2\pi$, satisfies

$$\|(-A_0)^{-\beta} b^\varepsilon\|_{L^2(T^n)} \leq C \|B\|_{\alpha,s}\, \varepsilon^{2\beta}, \tag{3.7}$$

where $A_0 + id = \Delta$ denotes the Laplacian (*with periodic boundary conditions*) *on the standard n-torus $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. The constant $C = C(n, N, c, \vartheta, \alpha, \beta, s)$ is independent of B and $\varepsilon > 0$.*

We repeat that the set of frequencies ω_ϱ satisfying the Diophantine condition (3.5) is a set of full Lebesgue measure in $\mathbb{R}^{Nn}$; see [5]. The proof of Proposition 3.1 will be prepared by Lemmas 3.2–3.4. We first consider the special case where $B(x, y) = B_0(y)$ is independent of x and $b^\varepsilon(x) = B_0(\omega x/\varepsilon)$ is just a rescaled quasiperiodic function.

Lemma 3.2. *If $B(x,y) = B_0(y)$ is independent of x, then Proposition* 3.1 *holds for any $\beta \in (0, 1/2]$, with*

$$\|B\|_{\alpha,s} = \|B_0\|_{H^s} := \Big(\sum_{k \in \mathbb{Z}^N} (1+k^2)^s |B_{0k}|^2 \Big)^{1/2}. \tag{3.8}$$

We prepare for the proof of Lemma 3.2 with an elementary estimate.

Lemma 3.3. *Let $\mu, \eta \in \mathbb{R}$ and $p \geq 1$. Then*

$$(1+\eta^2)^{1/p}(1+(\mu-\eta)^2) \geq (1+(\mu/2)^2)^{1/p}(1+\min(\eta^2,(\mu-\eta)^2)). \tag{3.9}$$

Proof. Fix μ and η. Since the quotient $((1+\eta^2)/(1+(\mu/2)^2))^{1/p}$ is a monotone function of $p \in [1,\infty)$, it is sufficient to prove (3.9) for the extreme cases $p = 1$ and $p = \infty$.

The case $p = \infty$ is trivial. The case $p = 1$ follows from the fact that

$$(\mu/2)^2 \leq \max(\eta^2, (\mu-\eta)^2),$$

as is to be expected for the arithmetic mean $\mu/2$ of η and $\mu - \eta$. This proves Lemma 3.3. ☐

Proof of Lemma 3.2. We expand

$$B(x,y) = B_0(y) = \sum_{k \in \mathbb{Z}^N} B_{0k}(y) \exp(ik^T y).$$

Recall the y-average $B_{00} = 0$, by assumption (3.4). Restricting attention to a single Fourier term, we will prove an estimate

$$\|(-A_0)^{-\beta} \exp(\varepsilon^{-1} ik^T\omega\cdot)\|^2_{L^2(\Omega)} \leq C(1+k^2)^{4\beta(N+\vartheta)}\varepsilon^{4\beta}, \tag{3.10}$$

uniformly in $k \in \mathbb{Z}^N \setminus \{0\}$ and $\varepsilon > 0$. Summing over k will then prove the lemma.

To prove (3.10), we first compute the j-th Fourier coefficient $e^{\varepsilon,-\beta}_{k,j}$, $j \in \mathbb{Z}^n$, of the fractional power. Since $-A_0 = -\Delta + id$ acts on the j-th Fourier mode of T^n as multiplication by $1 + j^2$, we obtain

$$\begin{aligned} e^{\varepsilon,-\beta}_{k,j} &:= ((-A_0)^{-\beta}\exp(\varepsilon^{-1} ik^T\omega\cdot))^\wedge_j = (1+j^2)^{-\beta}(\exp(\varepsilon^{-1}ik^T\omega\cdot))^\wedge_j \\ &= (1+j^2)^{-\beta}(2\pi)^{-n}\int_{T^n} \exp(i(-j^T + k^T\omega/\varepsilon)x)\,dx \qquad (3.11) \\ &= (1+j^2)^{-\beta}\prod_{\rho=1}^{n}\Big(\frac{\sin \pi(-j_\varrho + k^T\omega_\rho/\varepsilon)}{\pi(-j_\rho + k^T\omega_\rho/\varepsilon)}\exp(\pi i(-j_\rho + k^T\omega_\rho/\varepsilon))\Big). \end{aligned}$$

Here j_ρ denote the components of $j \in \mathbb{Z}^n$, and $\omega_\rho \in \mathbb{R}^N$ is the ρ-th column of the $N \times n$ frequency matrix ω.

We abbreviate $\mu_\rho := k^T\omega_\varrho/\varepsilon$ and use $\sin^2 \pi x \le \pi^2 x^2/(1+x^2)$ to estimate

$$|e_{k,j}^{\varepsilon,-\beta}|^2 \le (1+j^2)^{-2\beta} \prod_{\rho=1}^{n} (1+(\mu_\rho - j_\rho)^2)^{-1}. \tag{3.12}$$

Using (3.12) and summing over $j \in \mathbb{Z}^n$, we obtain an estimate of the L^2-norm

$$\|(-A_0)^{-\beta} \exp(\varepsilon^{-1} i k^T \omega \cdot)\|_{L^2}^2 = \sum_{j\in\mathbb{Z}^n} |e_{k,j}^{\varepsilon,-\beta}|^2 \tag{3.13}$$

for $0 < \beta \le 1/2$ as follows. We regroup the factors in (3.12) and apply Hölder's inequality with $p = 1/2\beta,\ p' = 1/(1-2\beta)$ to obtain

$$\begin{aligned}
&\|(-A_0)^{-\beta} \exp(\varepsilon^{-1} i k^T \omega \cdot)\|_{L^2}^2 \le \\
&\sum_{j\in\mathbb{Z}^n} \Big(\prod_{\rho=1}^{n} (1+j_\rho^2)^{\frac{1}{n}} (1+(\mu_\rho - j_\rho)^2)\Big)^{-2\beta} \Big(\prod_{\rho=1}^{n} (1+(\mu_\rho - j_\rho)^2)\Big)^{-(1-2\beta)} \le \\
&\Big(\sum_{j\in\mathbb{Z}^n} \prod_{\rho=1}^{n} (1+j_\rho^2)^{\frac{-1}{n}} (1+(\mu_\rho - j_\rho)^2)^{-1}\Big)^{2\beta} \Big(\sum_{j\in\mathbb{Z}^n} \prod_{\rho=1}^{n} (1+(\mu_\rho - j_\rho)^2)^{-1}\Big)^{1-2\beta}.
\end{aligned} \tag{3.14}$$

A majorant sum for the second factor is in fact given by $C_1^{(1-2\beta)n}$ with

$$C_1 := \max_{\mu_0\in\mathbb{R}} \sum_{j_0\in\mathbb{Z}} (1+(\mu_0 - j_0)^2)^{-1} < \infty. \tag{3.15}$$

Note that the sum is convergent, because it runs over the one-dimensional "lattice" $j_0 \in \mathbb{Z}$, only. Also, the estimate is uniform with respect to $\mu_0 \in \mathbb{R}$, by continuous dependence and 1-periodicity in μ_0.

The first factor in (3.14) will provide the $\varepsilon^{2\beta}$ estimate. We apply Lemma 3.3 and (3.9), with $p = n,\ \mu = \mu_\rho$, and $\eta = j_\rho$, and obtain

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^n} \prod_{\rho=1}^{n} (1+j_\rho^2)^{-1/n} (1+(\mu_\rho - j_\rho)^2)^{-1} \\
&\le \sum_{j\in\mathbb{Z}^n} \prod_{\rho=1}^{n} (1+(\mu_\rho/2)^2)^{-1/n} (1+\min(j_\rho^2, (\mu_\rho - j_\rho)^2))^{-1} \\
&= \prod_{\rho=1}^{n} \Big(\sum_{j_0\in\mathbb{Z}} (1+(\mu_\rho/2)^2)^{-1/n} (1+\min(j_0^2, (\mu_\rho - j_0)^2))^{-1}\Big)
\end{aligned} \tag{3.16}$$

$$\leq \Big(\prod_{\rho=1}^{n} (2/\mu_\rho)^2 \Big)^{1/n} \cdot \prod_{\rho=1}^{n} \Big(\frac{1}{2} \sum_{j_0 \in \mathbb{Z}} \Big(\frac{1}{1+j_0^2} + \frac{1}{1+(\mu_\rho - j_0)^2} \Big) \Big)$$

$$\leq \frac{1}{n} \Big(\sum_{\rho=1}^{n} 4/\mu_\rho^2 \Big) \cdot C_1^n = \frac{4}{n} C_1^n \cdot \Big(\sum_{\rho=1}^{n} (k^T \omega_\rho)^{-2} \Big) \varepsilon^2.$$

Here C_1 is the same constant as in (3.15), and we have restored $k^T\omega_\rho/\varepsilon$ for μ_ρ. Combining estimates (3.14)–(3.16) with the *Diophantine condition* (3.5) for $k \in \mathbb{Z}^N \setminus \{0\}$ we obtain

$$\begin{aligned} \|(-A_0)^{-\beta} \exp(\varepsilon^{-1} i k^T \omega \cdot)\|^2_{L^2(T^n)} &\leq C \Big(\sum_{\rho=1}^{n} (k^T \omega_\rho)^{-2} \Big)^{2\beta} \varepsilon^{4\beta} \\ &\leq C|k|^{4(N-1+\vartheta)\beta} \varepsilon^{4\beta} \end{aligned} \tag{3.17}$$

with generic constants $C > 0$.

We sum over $k \in \mathbb{Z}^N \setminus \{0\}$ next. By Fourier expansion of $B(x, y) = B_0(y) = \sum B_{0k} \exp(ik^T y)$ and $b^\varepsilon(x) = B_0(x/\varepsilon)$, we have

$$\begin{aligned} \|(-A_0)^{-\beta} b^\varepsilon\|_{L^2(T^n)} &\leq \sum_{k \in \mathbb{Z}^N \setminus \{0\}} |B_{0k}| \cdot \|(-A_0)^{-\beta} \exp(\varepsilon^{-1} i k^T \omega \cdot)\|_{L^2(T^n)} \\ &\leq C \sum_k |B_{0k}| \cdot (1 + |k|^2)^{2(N-1+\vartheta)\beta} \varepsilon^{2\beta} \qquad (3.18) \\ &\leq C \Big(\sum_k |B_{0k}|^2 (1+|k|^2)^s \Big)^{1/2} \Big(\sum_k (1+|k|^2)^{2(N-1+\vartheta)\beta - s} \Big)^{1/2} \varepsilon^{2\beta} \end{aligned}$$

for generic constants C, by Cauchy–Schwarz. The first factor equals $\|B_0\|_{H^s}$. The second sum is bounded because we have assumed sufficient B-regularity in Proposition 3.1:

$$s > 2(N - 1 + \vartheta)\beta + N/2. \tag{3.19}$$

This proves the lemma. □

We have now proved Proposition 3.1 for the special case $B(x, y) = B_0(y)$. Next we aim at the special case

$$B(x, y) = e_j(x) B_j(y), \tag{3.20}$$

where we abbreviate $e_j(x) = \exp(ij^T x)$ and $j \in \mathbb{Z}^n \setminus \{0\}$ is fixed.

Lemma 3.4. *Let $B(x, y) = e_j(x)B_j(y)$ consist of only the j-th Fourier component, $j \in \mathbb{Z}^n$. As before, let $b_j^\varepsilon(x) := B(x, \omega x/\varepsilon)$. Then Proposition 3.1 holds for any $0 < \beta \leq 1/2$, with*

$$\|(-A_0)^{-\beta} b_j^\varepsilon\|_{L^2(T^n)} \ \leq \ C \|e_j\|_\beta \|B_j\|_{H^s} \ \varepsilon^{2\beta} \tag{3.21}$$

and $\|B_j\|_{H^s}$ *as defined in* (3.8). *The constant* C *does not depend on* j *or* ε.

Proof. Using the special case $j = 0$ of Lemma 3.2, we obtain

$$\begin{aligned}
&\|(-A_0)^{-\beta} b_j^\varepsilon\|_{L^2(T^n)} = \|(-A_0)^{-\beta}(e_j(\cdot)B_j(\varepsilon^{-1}\omega\cdot))\|_{L^2(T^n)} \\
&= \|(-A_0)^{-\beta} e_j(\cdot)(-A_0)^{\beta}(-A_0)^{-\beta} B_j(\varepsilon^{-1}\omega\cdot)\|_{L^2(T^n)} \\
&\leq |\|(-A_0)^{-\beta} e_j(-A_0)^{\beta}\|| \cdot \|(-A_0)^{-\beta} B_j(\varepsilon^{-1}\omega\cdot)\|_{L^2(T^n)} \qquad (3.22)\\
&\leq |\|(-A_0)^{-\beta} e_j(-A_0)^{\beta}\|| \cdot C\|B_j\|_{H^s}\, \varepsilon^{2\beta}.
\end{aligned}$$

It is therefore sufficient to estimate the L^2 operator norm of the operator $(-A_0)^{-\beta} e_j(-A_0)^{\beta}$ by $C\|e_j\|_\beta$; note that e_j denotes the multiplication operator here.

Let $\varphi(x) = \sum_{m\in\mathbb{Z}^n} \varphi_m \exp(im^T x) \in X^\beta$. We have to prove an estimate

$$\|(-A_0)^{-\beta} e_j(-A_0)^{\beta}\varphi\|_{L^2(T^n)} \leq C\|e_j\|_\beta\, \|\varphi\|_{L^2(T^n)}. \qquad (3.23)$$

This is sufficient because X^β is dense in $X = L^2(T^n)$. It is elementary to see that

$$0 < q_{mj} := \frac{1 + (m-j)^2}{(1+m^2)(1+j^2)} \leq 2 \qquad (3.24)$$

holds for all $m, j \in \mathbb{Z}^n$. Therefore,

$$\begin{aligned}
&\|(-A_0)^{-\beta} e_j(-A_0)^{\beta}\varphi\|_{L^2(T^n)} \qquad (3.25)\\
&= \|\left((1+m^2)^{-\beta}(1+(m-j)^2)^{\beta}\varphi_{m-j}\right)_{m\in\mathbb{Z}^n}\|_{\ell^2} \\
&= \|\left((1+j^2)^{\beta} q_{mj}^{\beta}\, \varphi_{m-j}\right)_m\|_{\ell^2} \qquad (3.26)\\
&\leq 2^\beta(1+j^2)^\beta\|(\varphi_{m-j})_m\|_{\ell^2} = 2^\beta\|e_j\|_\beta \cdot \|\varphi\|_{L^2}.
\end{aligned}$$

This proves (3.23) and the lemma.

Proof of Proposition 3.1. In view of Lemma 3.4 it remains only to consider the Fourier decomposition

$$B(x,y) = \sum_{j\in\mathbb{Z}^n} B_j(y)e_j(x). \qquad (3.27)$$

With the notation $b_j(x) := B_j(x/\varepsilon)e_j(x)$, estimate (3.21) implies

$$\begin{aligned}
&\|(-A_0)^{-\beta} b^\varepsilon\|_{L^2(T^n)} \leq \sum_{j\in\mathbb{Z}_n} \|(-A_0)^{-\beta} b_j^\varepsilon\|_{L^2(T^n)} \leq C\varepsilon^{2\beta} \sum_j \left(\|e_j\|_\beta \cdot \|B_j\|_{H^s}\right) \\
&\leq C\varepsilon^{2\beta} \sum_j \left((1+j^2)^{\beta-\alpha} \cdot \left((1+j^2)^\alpha \|B_j\|_{H^s}\right)\right) \qquad (3.28)
\end{aligned}$$

$$\leq C\varepsilon^{2\beta}\Big(\sum_j (1+j^2)^{2(\beta-\alpha)}\Big)^{1/2}\Big(\sum_j (1+j^2)^{2\alpha}\|B_j\|^2_{H^s}\Big)^{1/2} \leq C\|B\|_{\alpha,s}\varepsilon^{2\beta}.$$

Here we have again used generic constants C, the Cauchy–Schwarz inequality, and summability of $(1+j^2)^{2(\beta-\alpha)}$ for $\alpha > \beta + n/4$. This completes the proof of Proposition 3.1. □

4. Example: Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1, 1.2 on quantitative homogenization for the reaction–diffusion equation

$$u_t = \Delta u + f^\varepsilon(x, \omega x/\varepsilon, u). \tag{4.1}$$

In view of Theorem 2.1 and Corollaries 2.2–2.5 on fractional homogenization of invariant manifolds, it remains only to check assumptions (2.49) for

$$(F^\varepsilon(u))(x) := f^\varepsilon(x, \omega x/\varepsilon, u(x)) - f^0_u(x,0)u(x). \tag{4.2}$$

Here we have subtracted the linearization $f^0_u(x,0)$ at $u=0$ of the spatial y average $f^0(x,u)$ of $f^\varepsilon(x,y,u)$ over $y \in T^N$ at $\varepsilon = 0$. We recall $f^0(x,0)=0$, by assumption (1.10). Moreover, the linearization $f^0_u(x,0) = a^0_1(x)$ is subsumed in the sectorial operator A; see (1.10) again. With the obvious notation $F^0(u) := f^0(x,u) - f^0_u(x,0)u$ we therefore have achieved $F^0(0) = 0$ and $D_uF^0(0) = 0$, as has been anticipated in (2.13) and (2.14).

Specifically, we have to show $F^\varepsilon \in C^1(X^\alpha, X^{-\beta})$ and local fractional order convergence of $F^\varepsilon(u), D_uF^\varepsilon(u)$, for $\varepsilon \searrow 0$. Since f^ε is polynomial in u, the necessary fractional estimates are completely analogous for $F^\varepsilon(u)$ and $D_uF^\varepsilon(u)$. We therefore address only the estimate

$$\|F^\varepsilon(u) - F^0(u)\|_{-\beta} \leq C\varepsilon^\gamma, \tag{4.3}$$

for $0 \leq \varepsilon < \varepsilon_0$ and $\|u\|_\alpha \leq 2\delta_0$.

Because $f^\varepsilon(x,y,u)$ are smooth functions, polynomial in u, and because $X^\alpha \hookrightarrow C^0(T^n)$ is an algebra for $\alpha > \frac{n}{4}$, we obviously have $F^\varepsilon \in C^1(X^\alpha, X^{-\beta})$, for all $\beta \geq 0$ and

$$\alpha > n/4. \tag{4.4}$$

In view of assumption (2.11), $\alpha+\beta < 1$, this limits our example to dimensions

$$n \leq 3. \tag{4.5}$$

To prove (4.3) we choose

$$0 < 2\beta = \gamma < \gamma^*(n) := 1 - n/4, \qquad \frac{n}{4} < \alpha < 1-\beta. \tag{4.6}$$

Inserting (4.2) into (4.3), we estimate

$$\|F^\varepsilon(u) - F^0(u)\|_{-\beta} = \|f^\varepsilon(\cdot, \varepsilon^{-1}\omega\cdot, u) - f^0(\cdot, u)\|_{-\beta} \tag{4.7}$$
$$\leq \sum_{m=1}^{\alpha} \|(a_m^\varepsilon(\cdot, \varepsilon^{-1}\omega\cdot) - a_m^0(\cdot)) \cdot u^m\|_{-\beta} = \sum_{m=1}^{\alpha} \|b_m(\cdot)\|_{-\beta}$$

if we define $b_m(x) := B_m(x, \omega x/\varepsilon)$, where

$$B_m(x, y) := (a_m^\varepsilon(x, y) - a_m^0(x))u^m(x). \tag{4.8}$$

Invoking the fractional-order homogenization estimate (3.7) of Proposition 3.1, we only have to show that

$$\|B_m\|_{\alpha,s} \leq C \tag{4.9}$$

is uniformly bounded for $\|u\|_\alpha \leq 2\delta_0$, our choice (4.6) of α, and some

$$s > 2(N - 1 + \vartheta)\beta + N/2. \tag{4.10}$$

Recalling the definition (3.3) of $\|B\|_{\alpha,s}$, we choose s to be an even integer, $A_0 := \Delta_x - id$ and observe

$$\|B_m\|_{\alpha,s} = \|(-A_0)^\alpha(-\Delta_y + id)^{s/2} B_m(x, y)\|_{L^2(T^n \times T^N)}. \tag{4.11}$$

Since a_m^ε are assumed to be smooth, all y-derivatives of B_m take the same form as (4.8):

$$D_y^r B_m(x, y) = \tilde{a}_{m,r}^\varepsilon(x, y) u^m(x) \tag{4.12}$$

with smooth coefficients $\tilde{a}_{m,r}^\varepsilon$ of zero y-average, $r = 0, \ldots, s$. It is therefore sufficient to show bounds

$$\|(-A_0)^\alpha(\tilde{a}_{m,r}^\varepsilon(x, y)u^m(x))\|_{L^2(T^n)} \leq C, \tag{4.13}$$

uniformly in y. Such a bound follows because all $\tilde{a}_{m,r}^\varepsilon$ are smooth, $u \in X^\alpha$, and X^α is an algebra for $\alpha > n/4$; see [16]. As was pointed out in our discussion of smoothness assumption (1.8), estimates (4.10), (4.12), and (4.13) specify the smoothness assumptions on the coefficients $a_m^\varepsilon(x, y)$. Specifically the required x smoothness is

$$\|(-A_0)^\alpha \tilde{a}_{m,r}^\varepsilon(x, y)\|_{L^2(T^n)} \leq C \tag{4.14}$$

for $m = 0, \ldots, d,\ \ 0 < \varepsilon < \varepsilon_0,\ \ r = 0, \ldots, s$.

By Corollary 2.4 and Proposition 3.1, this proves Theorem 1.1. By Corollaries 2.2, 2.3, 2.5 and Proposition 3.1, this also proves Theorem 1.2.

5. EXAMPLE: THE NAVIER-STOKES EQUATION

In this section we apply our main abstract result, Theorem 2.1, to derive a quantitative homogenization result for invariant manifolds of the incompressible Navier–Stokes equation, both in dimensions 2 and 3. Specifically we consider the system

$$\begin{array}{rcl} u_t & = & \nu\Delta u - (u\cdot\nabla)u + b^\varepsilon(x) - \int_{T^n} b^\varepsilon(x)\,dx + \nabla p \\ 0 & = & \nabla\cdot u \end{array} \tag{5.1}$$

for $u \in \mathbb{R}^n$, $n = 2$ or 3, with periodic boundary conditions $x \in T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. We require the usual mean velocity condition

$$\int_{T^n} u\;dx = 0 \tag{5.2}$$

which eliminates uniform drift. The time-independent external force $b^\varepsilon(x) - \int_{T^n} b^\varepsilon$ also satisfies (5.2), to eliminate mean acceleration. As in Section 2, we assume

$$b^\varepsilon(x) = B(x, \omega x/\varepsilon) \tag{5.3}$$

to be quasiperiodic, due to periodicity of B in the rescaled variable $y = \omega x/\varepsilon$ with $N\times n$ frequency matrix $\omega = (\omega_1, \ldots, \omega_n)$. For regularity of B we assume

$$\|B\|_{\sigma/2,s'} \le C \tag{5.4}$$

to be bounded, where the respective regularities σ, s' in the slow, rapid variables x, y will be chosen below. See (3.3) for our definition of the norm $\|\cdot\|_{\alpha,s}$. Following Proposition 3.1, (3.5), we also assume the Diophantine condition

$$\min_{\varrho=1,\ldots,n} |k^T\omega_\varrho| > c|k|^{-(N-1)-\vartheta} \tag{5.5}$$

for some $c, \vartheta > 0$ and all $k \in \mathbb{Z}^N \setminus \{0\}$.

The formally homogenized Navier–Stokes equation, for $\varepsilon = 0$, reads

$$\begin{aligned} u_t &= \nu\Delta u - (u\cdot\nabla)u + b^0(x) - \int_{T^n} b^0 + \nabla p \\ 0 &= (\nabla\cdot u) \end{aligned} \tag{5.6}$$

with the spatial average

$$b^0(x) := (2\pi)^{-N}\int_{T^N} B(x,y)\,dy. \tag{5.7}$$

For $b^0 \equiv 0$, the dynamics of (5.6) are given by a globally attracting trivial equilibrium $u = U^0 \equiv 0$. Existence and smoothness problems about equilibria of the stationary Navier–Stokes equation are studied in [11], [13], and [20], for example.

We assume that for the forcing term $b^0(x) - \int_{T^n} b^0$ there exists a hyperbolic equilibrium $U^0(x)$ of the averaged system (5.6), and $U^0 \in H^2(T^n)$. For rectangle domains ($n = 2$) with large aspect ratio and periodic boundary conditions, equilibria with arbitrarily high unstable dimension do in fact exist; see [15], [23], and [4].

For a more precise formulation, it is useful to rewrite the Navier–Stokes system (5.1) as an abstract semigroup amenable to the setting of Section 2. See for example [9] for a background. We first orthogonally decompose

$$X = L^2(T^n) \cap \left\{ \int_{T^n} u \, dx = 0 \right\} = H_s \oplus H_p \tag{5.8}$$

into the L^2 closures H_s of divergence-free velocity fields and H_p of gradient velocity fields. Let P denote a projection onto H_s. Since the Laplacian commutes with this decomposition, we only have to solve the projected system

$$u_t = \nu \Delta u - P(u \cdot \bigtriangledown)u + P\Big(b^\varepsilon - \int_{T^n} b^\varepsilon dx\Big) \tag{5.9}$$

for $u \in H_s$. We abbreviate $A := \nu\Delta|_{H_s},\ N(u) := -P(u \cdot \bigtriangledown)u,\ g^\varepsilon = P(b^\varepsilon - \int_{T^n} b^\varepsilon dx)$ and rewrite (5.9) as an abstract equation

$$u_t = Au + N(u) + g^\varepsilon. \tag{5.10}$$

The self-adjoint invertible Stokes operator A possesses compact resolvent and defines the scale $X_s^\alpha, \alpha \in \mathbb{R}$, of fractional power spaces as outlined in Section 2; for example $X_s^0 = H_s$. As before $\|\cdot\|_\alpha$ denote the associated graph norms, which can be identified with the restriction to H_s of the Fourier norms used in Section 1:

$$\|u\|_\alpha = \|u\|_{H^{2\alpha}(T^n)}, \quad X^\alpha = H_s^{2\alpha}. \tag{5.11}$$

We summarize our assumptions next. We consider the incompressible Navier –Stokes system (5.1) on $x \in T^n$, for $n = 2$ or 3, and with rapid quasiperiodic forcing b^ε as in (5.3). Assume the Diophantine frequency condition (5.5) holds. Concerning the (formally) homogenized system (5.6), (5.7), we assume existence of a hyperbolic smooth equilibrium solution $u = U^0(x)$. In other words, the imaginary axis belongs to the resolvent set of the (compact) resolvent of the linearization of (5.6), (5.7) at U^0. We now choose real values σ, γ in the triangle

$$0 < \gamma < \min(\sigma - \frac{n}{2}, 2 - \sigma), \tag{5.12}$$

as in Theorem 1.1 and (1.14). Finally, as in assumption (4.10) of Section 4, let

$$s' > 2(N-1+\vartheta)\beta + N/2 \tag{5.13}$$

be the smoothness required for the quasiperiodic dependence of the external forces b^ε; see (5.3) and (5.4).

Theorem 5.1. *Let assumptions* (5.3)–(5.5), (5.12), *and* (5.13) *hold, as summarized above. Let* U^0 *denote a hyperbolic equilibrium of the homogenized Navier–Stokes system* (5.2), (5.6), (5.7), *in dimension* $n = 2$ *or* 3.

Then there exist C *and* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the Navier-Stokes system* (5.1) *possesses an equilibrium* $U^\varepsilon \in H_s^\sigma$ *with*

$$\|U^\varepsilon - U^0\|_{H_s^\sigma} \le C\varepsilon^\gamma. \tag{5.14}$$

Moreover, U^ε *is hyperbolic with the same unstable dimension as* U^0*, and the local stable and unstable manifolds* W_ε^s *and* W_ε^u *of class* C^1 *converge in* H_s^σ *to their homogenized counterparts* W_0^s, W_0^u *with that same fractional order* ε^γ*. See Theorem* 2.1 *and Corollaries* 2.2, 2.3, *and* 2.5 *for a complete technical formulation.*

We prepare for the proof of Theorem 5.1 with a technical observation for the quadratic nonlinearity $N(u) = P(u\cdot\triangledown)u$.

Lemma 5.2. *Let* $\frac{n}{4} < \alpha < 1$ *with* $n = 2$ *or* 3*. Then the quadratic term*

$$\begin{array}{rcl} N : X^\alpha & \to & X^0 := X = H_s(T^n) \\ u & \mapsto & N(u) \end{array} \tag{5.15}$$

is twice continuously differentiable with constant second derivative. In particular

$$\|N'(u)\|_{\alpha,0} \le C \cdot \|u\|_\alpha \tag{5.16}$$

holds for the operator norm $\|\cdot\|_{\alpha,0}$ *from* X^α *to* $X^0 = X$.

At the hyperbolic equilibrium U^0 *of the homogenized Navier–Stokes system* (5.6), (5.7), *the linearization*

$$A_0 = A + N'(U^0): \quad X^\alpha \to X^{\alpha-1} \tag{5.17}$$

is bounded with bounded inverse.

Proof of Lemma 5.2. Since $\alpha > 1/2$ we have the continuous embedding $X^\alpha \hookrightarrow H_s^1(T^n)$. Since $\alpha > n/4$, we also have $X^\alpha \hookrightarrow C^0(T^n) \hookrightarrow L^\infty(T^n)$. Therefore the map $(u_1, u_2) \mapsto P(u_1 \cdot \triangledown)u_2$ is bounded and bilinear from $X^\alpha \times X^\alpha$ to $X = H_s$. This proves claims (5.15) and (5.16).

The remaining claim (5.17) holds for $\alpha = 1$, by our hyperbolicity assumption. Moreover, (5.17) holds for any α, by definition, if we replace the linearization $A_0 = A + N'(U^0)$ by the unperturbed invertible Stokes operator A itself. The perturbation $N'(U^0) : X^\alpha \to X^0 \hookrightarrow X^{\alpha-1}$ is bounded and, in fact, compact for $n/4 < \alpha < 1$. Therefore A_0 in (5.17) is bounded and, in fact, Fredholm of Fredholm index zero. To show bounded invertibility of A_0 in (5.17), not only for $\alpha = 1$ but likewise for $n/4 < \alpha < 1$, it is therefore sufficient to show injectivity of A_0. But injectivity of A_0 for $n/4 < \alpha < 1$ is clearly inherited from injectivity of A_0 for $\alpha = 1$, because any kernel element $A_0 u = 0$ in X^α satisfies $u = -A^{-1}N'(U^0)u \in X^1$. This proves the lemma. $\square$

Proof of Theorem 5.1. Let $\alpha := \sigma/2$, $\beta := \gamma/2$ so that $X^\alpha = H_s^\sigma$ and

$$\alpha > \beta + n/4, \quad 0 < \beta < \frac{1}{2}, \quad \alpha + \beta < 1. \tag{5.18}$$

Here we recall $n = 2$ or 3.

We first prove the existence of equilibria U^ε near U^0, together with a quantitative estimate

$$\|U^\varepsilon - U^0\|_{1-\beta} \leq C\varepsilon^{2\beta}. \tag{5.19}$$

We then subtract $U^\varepsilon \in X^{1-\beta} \hookrightarrow X^\alpha$ from the solutions u, defining

$$v := u - U^\varepsilon. \tag{5.20}$$

The corresponding semigroup equation for v reads

$$v_t = A_0 v + F^\varepsilon(v) \tag{5.21}$$

with $A_0 = A + N'(U^0)$ and the nonlinearity

$$F^\varepsilon(v) := N(U^\varepsilon + v) - N(U^\varepsilon) - N'(U^0)v. \tag{5.22}$$

In view of Theorem 2.1 and Corollary 2.5 we only have to check assumptions (2.10), (2.11), (2.14), and (2.49), for F^ε. Assumption (2.10) holds because $N : X^\alpha \to X \hookrightarrow X^{-\beta}$ is in fact continuously differentiable by Lemma 5.2. Assumption (2.11), $\alpha + \beta < 1$, holds by (5.18). Obviously $F^\varepsilon(0) = 0$, as was required in (2.14). To complete our proof of (2.14), and (2.49), it is therefore sufficient to show the estimates

$$\|F^\varepsilon(v) - F^0(v)\|_{-\beta} \leq C\varepsilon^{2\beta} \tag{5.23}$$

$$\|D_v F^\varepsilon(v) - D_v F^0(v)\|_{\alpha,-\beta} \leq C\varepsilon^{2\beta}, \tag{5.24}$$

locally for small $\|v\|_\alpha$, together with the $\varepsilon^{2\beta}$ estimate (5.19) on the equilibria U^ε. To show estimate (5.19) we use the implicit function theorem to solve

$$\Phi(U,g) := AU + N(U) + g = 0 \tag{5.25}$$

for U near the given hyperbolic solution

$$U = U^0, \quad g = g^0 = Pb^0. \tag{5.26}$$

By Lemma 5.2, (5.17), the map $\Phi : X^{1-\beta} \times X^{-\beta} \to X^{-\beta}$ is twice continuously differentiable with invertible linearization

$$A + N'(U^0): \quad X^{1-\beta} \to X^{-\beta}. \tag{5.27}$$

Indeed $1-\beta < 1$ and $1-\beta > 1-\alpha+n/4 > n/4$, by (5.18). The implicit function theorem therefore provides a local solution $U = U(g)$ near $U^0 = U(g^0)$. Defining $U^\varepsilon := U(g^\varepsilon)$, and invoking estimate (3.7) of Proposition 3.1, we obtain

$$\begin{aligned} \|U^\varepsilon - U^0\|_{1-\beta} &\leq C \cdot \|g^\varepsilon - g^0\|_{-\beta} \\ &\leq C\|P(b^\varepsilon - b^0)\|_{-\beta} + C\|P\int_{T^n}(b^\varepsilon - b^0)\|_{-\beta} \\ &\leq C\|b^\varepsilon - b^0\|_{-\beta} \leq C\|B\|_{\alpha,s'}\varepsilon^{2\beta}. \end{aligned} \tag{5.28}$$

Here C denotes generic constants and s' is the high regularity of the quasi-periodic dependence as required in (5.13) above. We also recall $\alpha = \sigma/2$, so that $\|B\|_{\alpha,s'}$ is in fact bounded, by assumption (5.4).

Because of the continuous embedding $X^{1-\beta} \hookrightarrow X^\alpha$, induced by $\alpha+\beta<1$, this proves the quantitative homogenization estimate (5.19).

To show the local estimate (5.23), we compute

$$\begin{aligned} \|F^\varepsilon(v) - F^0(v)\|_{-\beta} &\leq \|N(U^\varepsilon + v) - N(U^0+v)\|_{-\beta} + \|N(U^\varepsilon) - N(U^0)\|_{-\beta} \\ &\leq 2\Big(\sup_{\|u\|_\alpha \leq \|U^0\|_\alpha + 1} \|N'(u)\|_{\alpha,-\beta}\Big) \cdot \|U^\varepsilon - U^0\|_\alpha \\ &\leq 2\Big(\sup_{\|u\|_\alpha \leq \|U^0\|_\alpha + 1} \|N'(u)\|_{\alpha,0}\Big) \cdot \|U^\varepsilon - U^0\|_{1-\beta} \leq C\varepsilon^{2\beta}. \end{aligned} \tag{5.29}$$

Here we have used the $\varepsilon^{2\beta}$ estimate (5.19) for $U^\varepsilon - U^0$ and Lemma 5.2, (5.16). To show the local estimate (5.24), we compute

$$\begin{aligned} \|D_vF^\varepsilon(v) - D_vF^0(v)\|_{\alpha,-\beta} &= \|D_vN(U^\varepsilon+v) - D_vN(U^0+v)\|_{\alpha,-\beta} \\ &\leq \|N''\|_{\alpha,-\beta} \cdot \|U^\varepsilon - U^0\|_\alpha \leq C\varepsilon^{2\beta}. \end{aligned} \tag{5.30}$$

Here we have used (5.19) together with Lemma 5.2 and the fact that $N''(u) = N''$ is independent of u. Recalling $\beta := \gamma/2$, this proves Theorem 5.1. □

Corollary 5.3. *Let assumptions* (5.3)–(5.5), (5.12), *and* (5.13) *hold in dimension* $n = 2$ *or* 3, *as in Theorem* 5.1. *Consider any initial condition* $u_0 \in H_s^\sigma(T^n)$ *and any finite time* $T(u_0) > 0$ *such that the corresponding solution* $u^0(t)$ *of the homogenized Navier–Stokes system* (5.2), (5.6), (5.7) *exists for* $0 \le t \le T(u^0)$.

Then there exists $\varepsilon_0 > 0$ *and a constant* $C > 0$ *such that the solution* $u^\varepsilon(t)$ *of* (5.1), (5.3) *with the same initial condition* $u^\varepsilon(0) = u_0$ *exists for* $0 \le t \le T(u^0)$ *and satisfies the quantitative homogenization estimate*

$$\|u^\varepsilon(t) - u^0(t)\|_{H_s^\sigma(T^n)} \le C\varepsilon^\gamma, \tag{5.31}$$

uniformly for $0 \le t \le T(u^0)$ *and* $0 < \varepsilon < \varepsilon_0$.

Proof. The proof follows from Corollary 2.4 together with the estimates given in the proof of Theorem 5.1 as based on Lemma 5.2. □

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