Quantitative homogenization of global attractors for hyperbolic wave equations with rapidly oscillating terms

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1 Introduction

To quantify the effects of rapid spatial oscillations on the solutions of hyperbolic wave equations, we consider the following specific scalar example

$$\partial_t^2 u + \gamma^\varepsilon(x, x/\varepsilon)\partial_t u \quad = \quad \Delta u - b(x, x/\varepsilon)f(u) + g(x, x/\varepsilon), \tag{1.1}$$

$u \in \mathrm{IR}$. Here x ranges in the bounded domain $\Omega \subset \mathrm{IR}^n$, and Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

are imposed. We assume the damping coefficient γ^ε to converge rapidly of the form

$$\gamma^\varepsilon(x, z) \quad = \quad \gamma^0(x) + \varepsilon\tilde{\gamma}(x, z) \tag{1.3}$$

Our quantitive homogenization estimates for individual trajectories will include the case of negative damping. Positive damping $\gamma^0 > 0$ will be assumed only for quantitive homogenization of global attractors.
Equation (1.1) arises, for example, in the context of relativistic quantum mechanics; see [Tem88], ch. IV. 3 and the references there. Below we will impose Diophantine quasiperiodicity conditions on $b = b(x, z)$ and $g = g(x, z)$ in the rapid spatial variable $z = x/\varepsilon$. More generally, in fact, we will only require a divergence representation with respect to the fast variable z; see (1.16) – (1.21) below. In the parabolic context of reaction-diffusion systems the same divergence representations (1.16) – (1.19) have provided quantitative homogenization estimates; see [FV00].

Although $\tilde{\gamma}, b, f, g$ may also depend on ε explicitly, we suppress this dependence for notational simplicity of presentation. For well-posedness we only require

$$\begin{aligned} &b = b(x, z) \in C^0(\bar{\Omega} \times \mathrm{IR}^n) \\ &0 < \beta_1 \le b \le \beta_2 < \infty \end{aligned} \tag{1.4}$$

$$\|g(\cdot, \cdot/\varepsilon)\|_{L^2(\Omega)} \le C, \tag{1.5}$$

$$\|\gamma^\varepsilon(\cdot, \cdot/\varepsilon)\|_{L^\infty(\Omega)} \le C, \tag{1.6}$$

uniformly for $0 < \varepsilon \leq \varepsilon_0$, with generic constants C. Here $g(\cdot, \cdot/\varepsilon)$, $\gamma^\varepsilon(\cdot, \cdot/\varepsilon)$ denote the functions $x \mapsto g(x, x/\varepsilon)$, $\gamma^\varepsilon(x, x/\varepsilon)$, which are assumed to be well-defined and of class $L^2(\Omega), L^\infty(\Omega)$, respectively. The positive lower bound β_1 on b is required to fix signs for the following dissipation condition on the nonlinearity $f = f(u)$. Let F denote the primitive function of f, that is $F'(u) = f(u)$, and assume

$$F(u) \geq -\delta u^2 - C_\delta \tag{1.7}$$

for some $\delta > 0$ small enough. Specifically $\delta < \frac{1}{2}\lambda_1(\Omega)/\beta_1$ will be sufficient, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ on Ω under Dirichlet boundary conditions. Our growth condition on the C^1-nonlinearity $f = f(u)$ will be

$$|f'(u)| \leq C(1 + |u|^{p-1}), \quad \text{and} \quad 1 \leq p \leq \frac{n}{n-2} \tag{1.8}$$

in case $n = \dim x \geq 3$. Note that the limiting Sobolev exponent $p = \frac{n}{n-2}$ for the embedding $H_0^1 \hookrightarrow L^{n(p-1)} = L^{2n/(n-2)}$ is included. Here and below, polynomial growth exponents p are unconstrained for dimensions $n = 1, 2$.

Under the above conditions we obtain global solutions

$$u = u^\varepsilon(t, x), \quad t \geq 0, \tag{1.9}$$

of the Cauchy problem with prescribed initial conditions

$$\begin{aligned} u &= u_0(x) \\ u_t &= v_0(x) \end{aligned} \tag{1.10}$$

at $t = 0$; see [Tem88], [BV83, BV89] and section 2 for further details. In fact (1.1), (1.2) generate a solution semigroup $y^\varepsilon = (u^\varepsilon, v^\varepsilon) = (u^\varepsilon, u_t^\varepsilon) = S^\varepsilon(t)(u_0, v_0)$ in the Hilbert space

$$E = H_0^1(\Omega) \times L^2(\Omega). \tag{1.11}$$

In the homogenization limit $\varepsilon \searrow 0$ of rapid spatial oscillations, and under additional assumptions, we will quantitatively study convergence

$$y^\varepsilon \to y^0 \tag{1.12}$$

of these solutions in the weaker spaces

$$E_{-\alpha} := H^{1-\alpha} \times H^{-\alpha}, \tag{1.13}$$

$0 < \alpha \le 1$. We recall that $H^{-\alpha} := \mathcal{D}((-\Delta)^{-\alpha/2}), \alpha \in \mathbb{R}$, denotes the scale of graph norm spaces, associated to fractional powers of $-\Delta$ with Dirichlet boundary conditions. Note $H^0 = H = L^2(\Omega),\ H^1 = H_0^1(\Omega),\ E = E_0$, and $H^2 = H^2(\Omega) \cap H_0^1(\Omega)$, with duals H^{-1}, etc..

To establish such a convergence result we introduce the homogenized equation

$$u_{tt} + \gamma^0(x)u_t \quad = \quad \Delta u - b^0(x)f(u) - g^0(x) \tag{1.14}$$

in Ω. Here the averages b^0, g^0 are assumed to exist in the following weak sense

$$\begin{aligned} \int_\Omega \varphi(x)\gamma^\varepsilon(x, x/\varepsilon)dx \quad &\underset{\varepsilon\searrow 0}{\longrightarrow} \quad \int_\Omega \varphi(x)\gamma^0(x)dx \\ \int_\Omega \varphi(x)b(x, x/\varepsilon)dx \quad &\underset{\varepsilon\searrow 0}{\longrightarrow} \quad \int_\Omega \varphi(x)b^0(x)dx; \\ \int_\Omega \psi(x)g(x, x/\varepsilon)dx \quad &\underset{\varepsilon\searrow 0}{\longrightarrow} \quad \int_\Omega \psi(x)g^0(x)dx; \end{aligned} \tag{1.15}$$

for all $\varphi \in L^1(\Omega)$ and $\psi \in L^2(\Omega)$. We denote solutions of the homogenized problem (1.14) with the same initial conditions (1.10) by $u^0(t, x)$ and by $y^0 = (u^0, v^0) = (u^0, u_t^0) \in E$.

To obtain a *quantitative* homogenization estimate on the difference $y^\varepsilon(t) - y^0(t)$ we require the same divergence representation for b, g as in the parabolic case, see [FV00]. Specifically we require that there exist functions $G_k = G_k(x, z)$ such that

$$g(x, z) - g^0(x) \quad = \quad \sum_{i=1}^{n} \partial_{z_i} G_i(x, z) \tag{1.16}$$

holds, for $x \in \Omega \subset \mathbb{R}^n, z \in \mathbb{R}^n$. We assume bounds

$$\begin{aligned} \|G_i(\cdot, \cdot/\varepsilon)\|_{L^2(\Omega)} \quad &\le \quad C \\ \|\partial^1_{x_i} G_i(\cdot, \cdot/\varepsilon)\|_{L^{2n/(n+2)}(\Omega)} \quad &\le \quad C \end{aligned} \tag{1.17}$$

uniformly for $0 < \varepsilon \le \varepsilon_0$. Here $\partial^1_{x_i}$ denotes the partial derivative with respect to x_i, at any (x, z). Of course, all expressions are assumed to be well-defined. For b, we analogously assume a divergence representation by functions $B_i = B_i(x, z)$ such that

$$b(x, z) - b^0(x) \quad = \quad \sum_{i=1}^{n} \partial_{z_i} B_i(x, z) \tag{1.18}$$

with ε-uniform bounds

$$\begin{array}{rcl}\|B_i(\cdot,\cdot/\varepsilon)\|_{L^\infty(\Omega)} & \leq & C \\ \|\partial^1_{x_i} B_i(\cdot,\cdot/\varepsilon)\|_{L^n(\Omega)} & \leq & C.\end{array} \tag{1.19}$$

In section 3 below, we will recall sufficient conditions for such divergence representations to

hold for $b(x,z)$, $g(x,z)$ which are quasiperiodic in z, with Diophantine frequencies. See also [FV00].

For the damping coefficients $\gamma^\varepsilon(x,z), \gamma^0(x)$, finally, we require somewhat stronger conditions to hold:

$$\gamma^\varepsilon(x,z) - \gamma^0(x) = \varepsilon\tilde{\gamma}(x,z). \tag{1.20}$$

Note that, unlike for b, g, we require the perturbation to be of order ε. However, we do not require a divergence representation for $\tilde{\gamma}$. We do assume higher regularity of γ^0 instead:

$$\begin{array}{rcl}\|\gamma^0(\cdot)\|_{L^\infty} + \|\gamma^0(\cdot)\|_{W^{1,n}} & \leq & C \\ \|\tilde{\gamma}(\cdot,\cdot/\varepsilon)\|_{L^\infty} + \|\tilde{\gamma}(\cdot,\cdot/\varepsilon)\|_{W^{1,n}} & \leq & C.\end{array} \tag{1.21}$$

In particular, the second inequality strengthens our previous L^∞-bound (1.6). Together, (1.20), (1.21) can be viewed as a quantitative version of the convergence $\gamma^\epsilon \longrightarrow \gamma^0$, for $\epsilon \searrow 0$.

Theorem 1.1 *Let assumptions (1.4) – (1.8) hold. Assume weak convergence (1.15), and divergence representations (1.16) – (1.19) on the rapidly oscillating coefficients $b(x,x/\varepsilon)$, $g(x,x/\varepsilon)$, as well as convergence (1.20), (1.21) of the damping coefficient $\gamma^\varepsilon(x,x/\varepsilon)$. Consider solutions $y^\varepsilon = (u^\varepsilon(t,u), u_t^\varepsilon(t,x))$ of (1.1), (1.2) and homogenized solutions $y^0 = (u^0(t,x), u_t^0(t,x))$ of (1.14), (1.2), with the same initial condition $y_0 = (u_0(x), v_0(x)) \in E = E_0 = H_0^1(\Omega) \times L^2(\Omega)$ at time $t = 0$. Fix $\varepsilon_0 > 0$ small enough.*

Then, for any $0 < \alpha \leq 1$, there exist constants $c_0, c_1, c_2, \varrho > 0$, depending only on α, $\|y_0\|_{E_0}$, and the data, but independent of $\varepsilon \in (0, \varepsilon_0]$, such that y^ε satisfies the doubly exponential estimate

$$\|y^\varepsilon(t) - y^0(t)\|_{E_{-\alpha}} \leq \varepsilon^\alpha \exp(c_2 \exp(\varrho t) + c_1 t + c_0), \tag{1.22}$$

uniformly for all $t \geq 0$. This doubly exponential estimate also holds for damping functions $\gamma^0(x)$ which attain negative values.

If the damping coefficient $\gamma^0(x) \geq \underline{\gamma} > 0$ possesses a strictly positive lower bound, then this estimate can be sharpened to become singly exponential:

$$\|y^\varepsilon(t) - y^0(t)\|_{E_{-\alpha}} \;\leq\; C\varepsilon^\alpha \exp(\varrho t), \tag{1.23}$$

for a suitable constant C.

We address convergence of *global attractors* $\mathcal{A}^\varepsilon \to \mathcal{A}^0$ next; of course $\mathcal{A}^\varepsilon$ is again associated to global solutions $y^\varepsilon = (u^\varepsilon, u_t^\varepsilon)$ of (1.1), (1.2) and $\mathcal{A}^0$ belongs to the homogenized counterpart (1.14), (1.2). See [BV92], [Tem88] for a general background on global attractors: the minimal sets attracting all bounded sets. To assert existence and uniform boundedness, as well as relative compactness of $\bigcup_{0<\varepsilon\leq\varepsilon_0} \mathcal{A}^\varepsilon$ in $E = E_0$, we strengthen our assumptions (1.7), (1.8) on the nonlinearity $f(u)$ and its primitive $F(u)$; see [BV92], [Tem88] for details. We require that

$$f(u)u \geq -\delta u^2 - C_\delta, \tag{1.24}$$

again for some $\delta < \frac{1}{2}\lambda_1(\Omega)/\beta_1$. Moreover we further restrict our growth assumption (1.8) to be

$$|f'(u)| \leq C(1 + |u|^{p-1}), \quad \text{and } p < \frac{n}{n-2} \tag{1.25}$$

in case $n = \dim x \geq 3$. Note that we have only dropped the limiting case $p = \frac{n}{n-2}$ here. As a simplifying (but non-essential) assumption, we also impose the following Hölder condition on $f'(u)$:

$$|f'(u_1) - f'(u_2)| \leq C(1 + |u_1|^\beta + |u_2|^\beta) \cdot |u_1 - u_2|^\theta \tag{1.26}$$

holds for some positive constants C, β, θ such that $\beta + \theta < 2/(n-2)$.

We now require the damping coefficient $\gamma^0(x) > 0$ to be strictly positive. For the homogenized equation (1.14), (1.2) with global attractor $\mathcal{A}^0$, the Hamiltonian energy

$$\Phi(y^0) := \int_\Omega \left(\frac{1}{2}|\partial_t u^0|^2 + \frac{1}{2}|\nabla u^0|^2 + b^0 F(u^0) - g^0 u^0\right) dx \tag{1.27}$$

then provides a Lyapunov function. In addition we require all equilibrium solutions $(U, 0) \in E_0$ of (1.14), (1.2) to be hyperbolic. Specifically, equilibria U satisfy

$$0 = \Delta U - b^0 f(U) + g^0 \tag{1.28}$$

with Dirichlet boundary conditions. Hyperbolicity means that the linearization

$$\lambda^2 u + \lambda \gamma^0 u = \Delta u - b^0 f'(U) u \tag{1.29}$$

possesses only the trivial solution $u = 0$ in $H_0^1(\Omega)$, for $Re\lambda = 0$. It entails finiteness of the number of equilibria, a saddle-point property near equilibria U_j, the existence of finite-dimensional unstable manifolds $W_j^u \subset E_0$, and a finite Morse decomposition

$$\mathcal{A}^0 = \bigcup_j W_j^u \tag{1.30}$$

of the global attractor $\mathcal{A}^0$. As was proved in [BV89], [EFNT94] the global attractor $\mathcal{A}^0$ is then in fact exponentially attracting:

$$\mathrm{dist}_{E_0}(y^0(t), \mathcal{A}^0) \quad \leq \quad C e^{-\nu t} \tag{1.31}$$

for some positive constants C, ν which only depend on $\|y^0(0)\|_E$. For details see also section 5.

Theorem 1.2 *Let the assumptions of theorem 1.1 hold, strengthened by (1.24) – (1.26). Let the damping coefficient $\gamma^0 > 0$ be positive and independent of x. Assume hyperbolicity (1.29) of all equilibria U. Fix $\varepsilon_0 > 0$ and $0 < \alpha \leq 1$.*

Then there exists a constant $C > 0$ such that

$$\mathrm{dist}_{E_{-\alpha}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \quad := \quad \sup_{y \in \mathcal{A}^\varepsilon} \mathrm{dist}_{E_{-\alpha}}(y, \mathcal{A}^0) \leq C \varepsilon^{\alpha'} \tag{1.32}$$

holds, uniformly for $0 < \varepsilon \leq \varepsilon_0$. Here the quantitative homogenization exponent α' is related to the exponential attraction rate ν of $\mathcal{A}^0$ in (1.31) and to the exponential growth rate ϱ of the homogenization estimate (1.23) by

$$\alpha' = \alpha / (1 + \varrho / \nu). \tag{1.33}$$

Our proof of theorems 1.1 and 1.2 below will be based on an estimate of Gronwall type, which can be formulated in a quite general setting of semilinear strongly continuous semigroups. As a basis for future investigations of quantitative homogenization, for example of hyperbolic systems, we describe this abstract setting next. For a background on the theory of strongly continuous semigroups see for example [Kat66], [Paz83], [Tan79]. At the end of this introduction, after formulating our abstract result as theorem 1.3, we will outline how our specific example of the hyperbolic wave equation (1.1), (1.2) fits into this abstract framework; see also [Tem88]. For quantitative homogenization estimates of *analytic* semigroups in fractional power spaces see [FV01].

Consider a scale of Banach spaces $E_{-\alpha},\ 0 \leq \alpha \leq 1$. We assume an interpolation estimate

$$\|y\|_{-\alpha} \leq C_\alpha \|y\|_0^{1-\alpha} \|y\|_{-1}^{\alpha}. \tag{1.34}$$

in terms of the norms $\|\cdot\|_{-\alpha}$ on $E_{-\alpha}$. Let A generate a linear strongly continuous semigroup $\exp(At), t \geq 0$, on each of the spaces E_0, E_{-1}. For $\alpha = 0, 1$ we also require semigroup estimates

$$\|\exp(At)\|_{\mathcal{L}(E_{-\alpha})} \quad \leq \quad M \exp(\tilde{\varrho}_0 t) \tag{1.35}$$

for all $t \geq 0$ and suitable constants $M \geq 1, \tilde{\varrho}_0 \in \mathbb{R}$. Here $\mathcal{L}(E_{-\alpha}) = \mathcal{L}(E_{-\alpha}, E_{-\alpha})$ is the Banach space of bounded linear operators from $E_{-\alpha}$ to $E_{-\alpha}$), with the usual operator norm.

For $0 \leq \varepsilon \leq \varepsilon_0$, let $y^\varepsilon = y^\varepsilon(t) \in E_0$ denote the mild solution of the semilinear equation

$$\frac{d}{dt} y^\varepsilon \quad = \quad A y^\varepsilon + \mathcal{F}_\varepsilon(y^\varepsilon) \tag{1.36}$$

with initial condition

$$y^\varepsilon(0) \quad = \quad y_0 \in E_0. \tag{1.37}$$

By a *mild solution* we mean the solution of the integral variations-of-constants formulation

$$y^\varepsilon(t) = \exp(At) y_0 + \int_0^t \exp(A(t-\tau)) \mathcal{F}_\varepsilon(y^\varepsilon(\tau)) d\tau. \tag{1.38}$$

We assume that such mild solutions $y^\varepsilon \in C^0([0,+\infty), E_0)$ exist globally and satisfy an exponential growth estimate

$$\|y^\varepsilon(t)\|_0 \quad \leq \quad M \exp(\varrho_0 t) \tag{1.39}$$

for all $t \geq 0$, and for suitable constants $M \geq 1, \varrho_0 \in \mathrm{I\!R}$ which may depend on $\|y_0\|_0$ but not on $0 \leq \varepsilon \leq \varepsilon_0$.

We do not impose abstract conditions on the nonlinearity

$$\mathcal{F}_\varepsilon : E_0 \to E_0 \tag{1.40}$$

which guarantee the ε-uniform global exponential estimate (1.39). It allows for greater flexibility to impose (1.39) directly, to be proved for example by an energy estimate in our particular example (1.1) of hyperbolic wave equations; see section 2 below. Instead, we impose the following two crucial assumptions concerning the "homogenized" nonlinearity $\mathcal{F}_0$ and the homogenization difference $\mathcal{F}_\varepsilon - \mathcal{F}_0$, both viewed in the weaker space E_{-1} :

$$\|\mathcal{F}_0'(y)\|_{\mathcal{L}(E_{-1},E_{-1})} \quad \leq \quad C(1 + \|y\|_0^{p_1}) \tag{1.41}$$

$$\|\mathcal{F}_\varepsilon(y) - \mathcal{F}_0(y)\|_{-1} \quad \leq \quad C(1 + \|y\|_0^{p_0}) \cdot \varepsilon. \tag{1.42}$$

Here $C, p_0, p_1 \geq 1$ are suitable constants, independent of both ε and $y \in E_0$. Note that condition (1.41), if imposed on $\mathcal{F}_\varepsilon$, $\mathcal{L}(E_0, E_0)$ instead of $\mathcal{F}_0$, $\mathcal{L}(E_{-1}, E_{-1})$ implies local existence of unique mild solutions y^ε in E_0, by the usual local Lipschitz estimate. See section 2 for details.

Theorem 1.3 *Let assumptions (1.34), (1.35), (1.39) – (1.42) hold for the semilinear equations*

(1.36) and its mild solutions $y^\varepsilon(t), 0 \leq \varepsilon \leq \varepsilon_0$, with initial condition $y^\varepsilon(0) = y_0 \in E_0$.

Then there exist positive constants c_0, c_1, c_2, independent of ε, α, such that the following doubly exponential quantitative homogenization estimate holds:

$$\|y^\varepsilon(t) - y^0(t)\|_{-\alpha} \leq \varepsilon^\alpha \cdot \exp(c_2 \exp(\varrho t) + c_1 t + c_0), \tag{1.43}$$

uniformly for $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq \alpha \leq 1$. Here $\varrho := \varrho_0 p$ with $p := \max(p_0, p_1)$, as given by estimates (1.39), (1.41), (1.42). The dependencies of c_j on y_0

and α can be expressed explicitly by

$$\begin{aligned} c_0 &= (1-\alpha)\log(2M) + \log C_\alpha \\ c_1 &= (1-\alpha)\varrho_0 + \alpha(MC + \tilde{\varrho}_0) \\ c_2 &= \alpha\varrho^{-1}M^{p+1}C \end{aligned} \tag{1.44}$$

in terms of the constants $M, \tilde{\varrho}_0, \varrho_0, p_0, p_1,$ and C.

As a corollary to theorem 1.3 we note the special case $\varrho = \varrho_0 = 0$ of an estimate

$$\|y^\varepsilon(t)\|_0 \leq M(y_0), \tag{1.45}$$

for all $t \geq 0$. This includes the conservative undamped case, where solutions remain bounded in forward time, as well as the dissipative case, where solutions eventually enter a fixed large ball. The doubly exponential quantitative homogenization estimate (1.43) then simplifies to the singly exponential estimate

$$\|y^\varepsilon(t) - y^0(t)\|_{-\alpha} \leq \varepsilon^\alpha \exp(c_1 t + c_0) \tag{1.46}$$

with c_0 as in (1.44) and c_1 given explicitly by

$$c_1 = \alpha(MC(1 + M^p) + \tilde{\varrho}_0). \tag{1.47}$$

For a proof see estimates (4.22), (4.23) in section 4.

The proof of theorem 1.3 is given in section 4 below. To outline the contents of the remaining sections, and the proof of theorems 1.1, 1.2, we briefly sketch the standard semigroup formulation of the hyperbolic wave equation (1.1), (1.2). The ε-uniform exponential growth estimate (1.39) on $\|y^\varepsilon(t)\|_0$ in E_0 will then be given in section 2. The crucial estimates (1.41) and (1.42) on $\mathcal{F}_0'(y)$ and on the homogenization difference $\mathcal{F}_\varepsilon(y) - \mathcal{F}_0(y)$ in E_{-1} will be provided in section 3, for hyperbolic wave equations. Together, these estimates will prove theorem 1.1. The proof of theorem 1.2 on the quantitative homogenization of the global attractors $\mathcal{A}^\varepsilon, \mathcal{A}^0$ will be deferred to section 5.

We now recall the standard semigroup formulation (1.36) of the hyperbolic wave equation (1.1). Let $y = (u, v) \in E_{-\alpha} := H^{1-\alpha} \times H^{-\alpha}$. Note that the interpolation estimate (1.34) then follows from interpolation in the Sobolev spaces $H^\alpha = \mathcal{D}((-\Delta)^\alpha)$; see [Hen81], [Paz83], [Tan79], [Tem88]. We define

$$Ay = A(u,v) := (v, \Delta u) \tag{1.48}$$

with domain $\mathcal{D}(A) = E_{-\alpha+1}$ in $E_{-\alpha}$. Then A generates a strongly continuous semigroup on $E_{-\alpha}$. In particular the linear semigroup estimate (1.35) then holds, in fact with $M = 1$ and $\tilde{\varrho}_0 = 0$. For $\alpha = 0$ this follows because the linear semigroup $\exp(At)$ conserves the norm in E_0, which is in fact the energy norm; see [Paz83], [Tan79]. For other $\alpha \in \mathbb{R}$ this follows because $(-\Delta)^\alpha : E_0 \to E_{-\alpha}$ commutes with the semigroup $\exp(At)$ and defines a bounded linear isomorphism – which we may even consider to be isometric – between E_0 and $E_{-\alpha}$.

The nonlinearity $\mathcal{F}_\varepsilon(y)$ in (1.36) is given by

$$\mathcal{F}_\varepsilon(u,v) := (0, -b^\varepsilon f(u) + g^\varepsilon - \gamma^\varepsilon v). \tag{1.49}$$

Of course $b^\varepsilon = b(x, x/\varepsilon)$, etc.. Definitions (1.48), (1.49) clearly provide the semilinear abstract equation (1.36) as an equivalent formulation of the semilinear hyperbolic wave equation (1.1), rewritten as a first order system for $(u,v) = (u, \partial_t u)$. We will therefore first prove the slightly tricky estimates (1.41), (1.42) on the nonlinearities $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ in section 3, before we return in section 4 to the abstract semigroup homogenization theorem 1.3. It is those estimates (1.41), (1.42), where the divergence representations (1.16) – (1.21) will be used.

Once again we recall that this abstract setting will prove the doubly exponential quantitative homogenization estimate for hyperbolic wave equations, including the case where the damping coefficient $\gamma^0(x)$ does attain negative values. In the case of constant positive damping $\gamma^0 > 0$ it will also provide a quantitave homogenization estimate of the global attractor.

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2 Local and global growth estimates

In this section we recall some basic theory on local existence and uniqueness of mild solutions

$y^\varepsilon = (u^\varepsilon, \partial_t u^\varepsilon) \in E_0 = H_0^1 \times L^2$ for hyperbolic wave equations

$$\partial_t^2 u + \gamma^\varepsilon(x, x/\varepsilon)\partial_t u \quad = \quad \Delta u - b(x, x/\varepsilon) f(u) + g(x, x/\varepsilon), \tag{2.1}$$

under Dirichlet boundary conditions. See section 1 for details of notation and assumptions. In particular, we recall the semigroup formulation

$$\tfrac{d}{dt} y^\varepsilon \quad = \quad A y^\varepsilon + \mathcal{F}_\varepsilon(y^\varepsilon) \tag{2.2}$$

with initial condition $y^\varepsilon(0) = y_0 \in E_0$. We will first prove that the local Lipschitz estimate

$$\|\mathcal{F}_\epsilon'(y)\|_{\mathcal{L}(E_0)} \quad \leq \quad C(1 + \|y\|_0^{p-1}) \tag{2.3}$$

holds for (2.1). This estimate is of course crucial to local existence and uniqueness of solutions $y^\varepsilon(t)$ of (2.2); see for example [Tan79], [Paz83]. We then prove the global exponential growth estimate

$$\|y^\varepsilon(t)\|_0 \quad \leq \quad M \exp(\varrho_0 t) \tag{2.4}$$

holds for (2.1), again under the assumptions of section 1. In particular, we recall that *negative* damping values $\gamma^\epsilon(x)$ are allowed here. For *positive* damping we obtain uniform asymptotic bounds and, in fact, a global attractor.

To prove the local Lipschitz estimate (2.3), and for later reference, we first observe the following basic estimates:

$$\begin{aligned} \|u\|_{L^{2n/(n-2)}} &\leq C\|u\|_{H_0^1} \\ \|f(u)\|_{L^2} &\leq C(1 + \|u\|_{H_0^1}^p) \\ \|f'(u)\|_{L^n} &\leq C(1 + \|u\|_{H_0^1}^{p-1}). \end{aligned} \tag{2.5}$$

These estimates are immediate from the growth estimate (1.8) on $f'(u)$ together with the standard Sobolev embedding $H_0^1 \hookrightarrow L^{2n/(n-2)}$.

To prove estimate (2.3) in $E_0 = H_0^1 \times L^2$, we recall the particular form

$$\mathcal{F}_\varepsilon(u,v) \quad := \quad (0, -b^\varepsilon f(u) + g^\varepsilon - \gamma^\varepsilon v) \tag{2.6}$$

from (1.49). Therefore (2.3) will follow from the two separate, ε-uniform estimates

$$\begin{aligned} \|b^\epsilon f'(u)\tilde{u}\|_{L^2} &\leq C(1 + \|u\|_{H_0^1}^{p-1}) \cdot \|\tilde{u}\|_{H_0^1} \\ \|\gamma^\varepsilon \tilde{v}\|_{L^2} &\leq C \cdot \|\tilde{v}\|_{L^2}. \end{aligned} \tag{2.7}$$

The second of these estimates is trivial by the ε-uniform L^∞-bound (1.6) for $\gamma^\varepsilon = \gamma^\varepsilon(\cdot, \cdot/\varepsilon)$. Similarly, by the ε-uniform positivity bounds β_1, β_2 imposed on b in (1.4), the first estimate in (2.7) reduces to the simple Hölder estimate

$$\|f'(u)\tilde{u}\|_{L^2} \leq \|f'(u)\|_{L^n} \cdot \|\tilde{u}\|_{L^{2n/(n-2}}. \tag{2.8}$$

Indeed, invoking the basic estimates (2.5) proves our claim (2.7). Standard theory of strongly continuous semigroups then settles the issue of *local* existence and uniqueness of mild solutions $\gamma^\varepsilon = (u^\varepsilon, \partial_t u^\varepsilon) \in H_0^1 \times L^2 = E_0$.

We now turn to the global exponential growth estimate (2.4), which implies existence and uniqueness *globally* in time. The argument is based on an exponential Gronwall type estimate of the Hamiltonian functional

$$\Phi(u, \partial_t u) = \int_\Omega \left(\frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\bigtriangledown u|^2 + bF(u) - gu \right) dx \tag{2.9}$$

introduced in (1.27) above. (Weak) differentiation with respect to time t provides

$$\frac{d}{dt} \Phi(u, \partial_t u) = - \int_\Omega \gamma^\varepsilon |\partial_t u|^2. \tag{2.10}$$

along solutions $(u, \partial_t u)$. We write the time derivative of Φ in (2.10) for notational simplicity. More precisely, this relation should be considered in its time integrated form.

The quadratic lower estimate (1.7) on the primitive F of the nonlinearity f, on the other hand, implies

$$\Phi(u, \partial_t u) \geq \int_\Omega \frac{1}{2} |\partial_t u|^2 + \delta' |\bigtriangledown u|^2 - C \tag{2.11}$$

for some $0 < \delta' < \frac{1}{2} - \beta_1\delta/\lambda$ and some constant $C > 0$. Here we have also used the ε-uniform boundedness (1.5) of g in L^2.

Let $\underline{\gamma}$ be an essential, ε-independent lower bound of the damping coefficients $\gamma^\varepsilon(\cdot, \cdot/\varepsilon)$. We first consider the case $\underline{\gamma} \le 0$, which allows for negative damping. Then (2.10), (2.11) combined imply

$$\frac{d}{dt}(\Phi + C) \le -\underline{\gamma} \int_\Omega |\partial_t u|^2 \le -2\underline{\gamma}(\Phi + C). \tag{2.12}$$

The standard Gronwall argument therefore provides an ε-uniform exponential growth estimate

$$\Phi(y^\varepsilon(t)) + C \le M' \exp(\varrho_0 t). \tag{2.13}$$

Here $\varrho_0 := -2\underline{\gamma}$, and $M' = \Phi(y_0) + C$ depends on $\|y_0\|_{E_0}$, but not on ε. Since $\Phi + C$ in (2.11) provides a bound for the squared energy norm $\|y^\varepsilon(t)\|_0^2$, this proves the global exponential growth estimate (2.4).

It is worth mentioning the undamped Hamiltonian case $\gamma^\varepsilon \equiv 0$, for which $\underline{\gamma} = 0$ and global bounds $M = M(\|y_0\|_0)$ can be given in (2.4). In the case of x- and ε-uniformly positive damping $\gamma^\varepsilon(\cdot, \cdot/\varepsilon) \ge \underline{\gamma} > 0$, in contrast, the Hamiltonian functional Φ in fact becomes a Lyapunov function, which decreases along trajectories. This provides asymptotic bounds for Φ and implies dissipativeness in E_0, in that case; see also [BV89] and section 5.

3 Homogenization estimate

In this section we prove the two crucial homogenization estimates (1.41), (1.42), which read as follows

$$\|\mathcal{F}_0'(y)\|_{\mathcal{L}(E_{-1},E_{-1})} \le C(1 + \|y\|_0^{p_1}) \tag{3.1}$$

$$\|\mathcal{F}_\varepsilon(y) - \mathcal{F}_0(y)\|_{-1} \le C(1 + \|y\|_{-1}^{p_0}) \cdot \varepsilon. \tag{3.2}$$

We recall that these two estimates were indeed *assumed* to hold, in the semigroup formulation

$$\frac{d}{dt} y^\varepsilon = A y^\varepsilon + \mathcal{F}_\varepsilon(y^\varepsilon) \tag{3.3}$$

of the abstract homogenization result of theorem 1.3. The proof of this abstract theorem will be formulated in section 4 below. Here we only prove that the homogenization assumptions (3.1), (3.2) are indeed satisfied in the abstract setting

$$\begin{aligned} y^\varepsilon &= (u^\varepsilon, v^\varepsilon) \\ E_0 &= H_0^1 \times L^2 \\ E_{-1} &= L^2 \times H^{-1} \\ \mathcal{F}_\varepsilon(u,v) &= (0, -b^\varepsilon f(u) + g^\varepsilon - \gamma^\varepsilon v) \end{aligned} \tag{3.4}$$

of the hyperbolic wave equation (1.1), (1.2). The proof of this fact will be based on the assumptions of theorem 1.1, and in particular on the divergence representations (1.16)–(1.19). By theorem 1.3 this then proves theorem 1.1. We conclude the present section by recalling sufficient conditions for the divergence representations (1.16)–(1.19) in terms of Diophantine frequency estimates of KAM type, in the case of spatially rapidly quasiperiodic coefficients $b^\varepsilon, g^\varepsilon$.

We first prove estimate (3.1), which is an E_{-1}-variant of our previous local Lipschitz estimate (2.3), (2.7). Indeed (3.1) with $p_1 := p-1$ will follow from the two separate estimates

$$\begin{aligned} \|b^0 f'(u)\tilde{u}\|_{H^{-1}} &\leq C(1 + \|u\|_{H_0^1}^{p-1}) \cdot \|\tilde{u}\|_{L^2} \\ \|\gamma^0 \tilde{v}\|_{H^{-1}} &\leq C \cdot \|\tilde{v}\|_{H^{-1}}. \end{aligned} \tag{3.5}$$

It is sufficient to prove these estimates for smooth functions $u, \tilde{u}, \tilde{v}$. The H^{-1}-norm is then given explicitly by

$$\|\tilde{v}\|_{H^{-1}} := \sup_{\|\varphi\|_{H_0^1}=1} \int_\Omega \tilde{v}\varphi; \tag{3.6}$$

in accordance with the L^2-duality of H^{-1} and H_0^1. The second estimate in (3.5) therefore follows from the estimate

$$\begin{aligned} |\textstyle\int_\Omega \gamma^0 \tilde{v}\varphi| &\leq \|\tilde{v}\|_{H^{-1}} \cdot \|\gamma^0\varphi\|_{H_0^1} \leq \\ &\leq \|\tilde{v}\|_{H^{-1}} \cdot C(\|\gamma^0\|_{W^{1,n}} + \|\gamma^0\|_{L^\infty})\|\varphi\|_{H_0^1}, \end{aligned} \tag{3.7}$$

for smooth $\tilde{v}, \varphi$ and for γ^0 in the Sobolev space $W^{1,n} \cap L^\infty$; see (1.21).

Similarly, the first estimate in (3.2) follows from the Hölder and Sobolev estimate

$$\begin{array}{rcl} |\int_\Omega b^0 f'(u)\tilde{u}\varphi| & \leq & \|b^0\|_{L^\infty}\cdot\|f'(u)\|_{L^n}\cdot\|\tilde{u}\|_{L^2}\cdot\|\varphi\|_{L^{2n/(n-2)}} \\ & \leq & \|b^0\|_{L^\infty}\cdot C(1+\|u\|_{H_0^1}^{p-1})\cdot\|\tilde{u}\|_{L^2}\cdot\|\varphi\|_{H_0^1} \\ & \leq & C(1+\|u\|_{H_0^1}^{p-1})\cdot\|\tilde{u}\|_{L^2}\cdot\|\varphi\|_{H_0^1} \end{array} \tag{3.8}$$

with generic constants C; see also (2.5). This proves (3.5) and estimate (3.1). Similarly, estimate (3.2) with $p_0 := p$ reduces to estimating the three differences

$$\begin{array}{rcl} \|(b^\varepsilon - b^0)f(u)\|_{H^{-1}} & \leq & C(1+\|u\|_{H_0^1}^{p})\cdot\varepsilon \\ \|g^\varepsilon - g^0\|_{H^{-1}} & \leq & C\cdot\varepsilon \\ \|(\gamma^\varepsilon-\gamma^0)v\|_{H^{-1}} & \leq & C\cdot\|v\|_{H^{-1}}\cdot\varepsilon \end{array} \tag{3.9}$$

for smooth u, v and generic constants C. The last of the three differences follows easily from assumptions (1.21):

$$\begin{array}{rcl} |\int_\Omega(\gamma^\varepsilon-\gamma^0)v\varphi| & = & \varepsilon|\int_\Omega\tilde{\gamma}v\varphi| \leq \varepsilon\|v\|_{H^{-1}}\cdot\|\tilde{\gamma}\varphi\|_{H_0^1}\leq \\ & \leq & C(\|\tilde{\gamma}\|_{W^{1,n}}+\|\tilde{\gamma}\|_{L^\infty})\cdot\|v\|_{H^{-1}}\cdot\varepsilon\cdot\|\varphi\|_{H_0^1}. \end{array} \tag{3.10}$$

Here the two inequalities hold, similarly to (3.7) by assumption (1.21).

To prove the remaining two inequalities in (3.9) we use the divergence representations (1.16)–(1.19) of $b^\varepsilon - b^0$ and $g^\varepsilon - g^0$. First consider

$$g^\varepsilon - g^0 \;=\; \sum_{i=1}^n \partial_{z_i} G_i(x,z). \tag{3.11}$$

By (1.17) we can estimate the second difference in (3.9) as follows

$$\begin{array}{rcl} |\int_\Omega(g^\varepsilon-g^0)\varphi| & = & |\int_\Omega\sum_{i=1}^n\varphi\cdot\partial_{z_i}G_i(x,x/\varepsilon)| = \\ & = & |\int_\Omega\sum_{i=1}^n\varphi\cdot(\varepsilon\partial_{x_i}G_i(x,x/\varepsilon)-\varepsilon\partial_{x_i^1}G_i(x,x/\varepsilon))| \\ & \leq & \varepsilon(|\int_\Omega G\cdot\nabla_x\varphi| + \sum_{i=1}^n|\int_\Omega\partial_{x_i^1}G_i\varphi|)\leq \\ & \leq & \varepsilon\left(\|G\|_{L_2}\cdot\|\varphi\|_{H_0^1}+\sum_{i=1}^n\|\partial_{x_i^1}G_i\|_{L^{2n/(n+2)}}\cdot\|\varphi\|_{L^{2n/(n-2)}}\right) \\ & \leq & C\cdot\varepsilon\cdot\|\varphi\|_{H_0^1}. \end{array} \tag{3.12}$$

Here we have again used Hölder estimates, the Sobolev embedding $H_0^1 \hookrightarrow L^{2n/(n-2)}$, and the notations $\partial_{x^1}G(x,z)$ for partial derivatives of G with respect to the first x-components as well as $\partial_{x_i}G$ for partial derivatives of $x\mapsto G(x,x/\varepsilon)$ with respect to x_i. This proves the second inequality in (3.9).

To prove the first inequality in (3.9) we use (2.5) and the divergence representation (1.18), (1.19) of $b^\varepsilon - b^0$ to estimate quite similarly

$$
\begin{array}{rl}
& |\int_\Omega (b^\varepsilon - b^0) f(u)\varphi| = |\int_\Omega \sum_{i=1}^n \varphi \partial_{z_i} B_i(x, x/\varepsilon) f(u)| \leq \\
\leq & \varepsilon\Big(|\int_\Omega B \cdot \nabla_x (f(u)\varphi)\Big)| + \sum_{i=1}^n |\int_\Omega \partial_{x_i^1} B_i \cdot f(u)\varphi|) \leq \\
\leq & \varepsilon\Big(\|B\|_{L^\infty}(\|f'(u)\|_{L^n} \cdot \|\nabla_x u\|_{L^2} \cdot \|\varphi\|_{L^{2n/(n-2)}} + \\
+ & \|f(u)\|_{L^2} \cdot \|\nabla_x \varphi\|_{L^2}) + \\
+ & \sum_{i=1}^n \|\partial_{x_i^1} B_i\|_{L^n} \cdot \|f(u)\|_{L^2} \cdot \|\varphi\|_{L^{2n/(n-2)}}\Big) \\
\leq & C\varepsilon(1 + \|u\|_{H_0^1}^p) \cdot \|\varphi\|_{H_0^1}.
\end{array}
\tag{3.13}
$$

This completes the proof of estimates (3.9) and (3.2).

For the convenience of our readers we conclude this section with a few remarks which relate the divergence representation (1.16)–(1.19) to standard KAM Diophantine conditions, in the case of quasiperiodic coefficents $b^\varepsilon, g^\varepsilon$. See also [FV00], for further details. We only specify the Diophantine conditions for b; the omitted conditions for g are analogous. Specifically, let

$$
b(x, z) \quad = \quad \beta(x, \omega_1 z_1, \cdots, \omega_n z_n) \tag{3.14}
$$

with frequency vectors $\omega_j \in \mathbb{R}^{\ell_j}$, where β is sufficiently smooth in all its $n + \ell := n + \ell_1 + \cdots + \ell_n$ components, and 2π-periodic in all but the x-components. We then impose the standard KAM Diophantine condition

$$
|k_j \cdot \omega_j| \quad \geq \quad c|k_j|^{-(\ell_j - 1 + \delta)} \tag{3.15}
$$

for $j = 1, \cdots, n$, some constants $c, \delta > 0$ and all integer vectors $k_j \in \mathbb{Z}^{\ell_j} \backslash \{0\}$. We also recall that this condition is satisfied for a set of full Lebesgue measure in the space of frequency vectors $(\omega_1, \cdots \omega_n) \in \mathbb{R}^\ell$; see for example [Cas57]. In [FV100] an explicit divergence representation $b^\varepsilon - b^0 = \sum \partial_{z_i} B_i$ was constructed, based on the Fourier expansion

$$
\beta = \sum_k \beta_k(x) \exp(i \sum_j (k_j \omega_j) z_j) \tag{3.16}
$$

with $k = (k_1, \cdots, k_n), \quad k_j \in \mathbb{Z}^{\ell_j}$. A sufficient regularity condition for β in terms of the Fourier coefficents β_k is given by the two conditions

$$\begin{array}{rcl} \sum|\beta_k|_{L^\infty}\cdot|k_j\omega_j|^{-1} & < & \infty \\ \sum|\partial_{x^1}\beta_k|_{L^n}\cdot|k_j\omega_j|^{-1} & < & \infty. \end{array} \tag{3.17}$$

Here the sums extend over all integer vectors $k = (k_1, \cdots, k_n) \in \mathbb{Z}^\ell$ and $j = 1, \cdots, n$ for which $k_j \neq 0$. In view of the Diophantine conditions (3.15), an algebraic decay of the coefficients $|\beta_k|_{L^\infty}, \quad |\partial_{x^1}\beta_k|_{L^n}$ of order

$$(1+|k_j|)^{-(\ell_j-1+\delta)}(1+|k|)^{-\ell+\delta'} \tag{3.18}$$

for some $\delta' > 0$ is sufficient to guarantee this convergence. Of course this condition amounts to regularity conditions for the original coefficients $b(x, z)$ and $\partial_{x^1}b(x, z)$.

The conditions for the Fourier coefficients $\Gamma_k(x)$ associated to a z-quasiperiodic term $g = g(x, z)$ read

$$\begin{array}{r} \sum|\Gamma_k|_{L^2}\cdot|k_j\omega_j|^{-1}<\infty \\ \sum|\partial_x^1\Gamma_k|_{L^{2n/(n+2)}}\cdot|k_j\omega_j|^{-1}<\infty \end{array} \tag{3.19}$$

in complete analogy to (3.16), (3.17). Our algebraic decay remark (3.18) still applies.

With the above remarks we have seen how spatially quasiperiodic coefficients with KAM Diophantine frequencies provide a key example for quantitative spatial homogenization estimates of hyperbolic wave equations.

4 Proof of theorems 1.3 and 1.1

In this section we prove theorem 1.3 on quantitative homogenization of mild solutions $y^\varepsilon(t)$ of equations

$$\frac{d}{dt}y^\varepsilon = Ay^\varepsilon + \mathcal{F}_\varepsilon(y^\varepsilon), \tag{4.1}$$

for $0 \leq \varepsilon \leq \varepsilon_0$, with identical initial conditions

$$y^\varepsilon(0) = y_0 \in E_0. \tag{4.2}$$

At the end of this section, we also complete the proof of theorem 1.1.

We recall from (1.38) that mild solutions satisfy the variations-of-constants formula

$$y^\varepsilon(t) = \exp(At)y_0 + \int_0^t \exp(A(t-\tau))\mathcal{F}_\varepsilon(y^\varepsilon(\tau))d\tau. \tag{4.3}$$

To prove the doubly exponential estimate of theorem 1.3 we consider the difference

$$w(t) \;\; = \;\; y^\varepsilon(t) - y^0(t), \tag{4.4}$$

where $y^0(t) \in E_0$ satisfies the same variations-of-constants formula (4.3) with $\varepsilon = 0$. The difference $w(t)$, with $w(0) = 0$, therefore satisfies

$$w(t) = \int_0^t \exp(A(t-\tau))(\mathcal{F}_\varepsilon(y^\varepsilon(\tau)) - \mathcal{F}_0(y^0(\tau)))d\tau \tag{4.5}$$

in E_0. To derive the doubly exponential homogenization estimate (1.43) of order ε^α in the interpolation space $E_{-\alpha}$ we prove the estimates

$$\|w(t)\|_0 \;\; \leq \;\; 2M\exp(\varrho_0 t) \tag{4.6}$$

$$\|w(t)\|_{-1} \;\; \leq \;\; (\exp(\tilde{c}_2\exp(\varrho t) + \tilde{c}_1 t) - 1) \cdot \varepsilon \tag{4.7}$$

with explicit constants $\tilde{c}_1, \tilde{c}_2$ specified in (4.20) below. By interpolation (1.34) in $E_{-\alpha}$ these estimates will prove theorem 1.3, (1.43), with the explicit choices (1.44) of the coefficients c_0, c_1, c_2. Note that (4.7) is in fact slightly stronger than (1.43), where we have omitted the post-exponential term -1 which becomes irrelevant for large times.

Estimate (4.6) follows trivially from the separate exponential estimates (1.39) on $y^\varepsilon(t)$ and $y^0(t)$ in E_0. Of course this estimate provides only a bound, without any clue as to homogenization for $\varepsilon \searrow 0$.

To prove the homogenization estimate (4.7) in E_{-1} we will use a Gronwall argument for the variations-of-constants difference (4.5). To prepare this estimate, we split the $\mathcal{F}$-difference in the integrand as

$$\mathcal{F}_\varepsilon(y^\varepsilon) - \mathcal{F}_0(y^0) = (\mathcal{F}_\varepsilon(y^\varepsilon) - \mathcal{F}_0(y^\varepsilon)) + (\mathcal{F}_0(y^\varepsilon) - \mathcal{F}_0(y^0)). \tag{4.8}$$

The first difference provides a homogenization term of order ε in E_{-1}. Indeed, we have assumed

$$\|\mathcal{F}_\varepsilon(y) - \mathcal{F}_0(y)\|_{-1} \;\; \leq \;\; C(1 + \|y\|_0^{p_0})\varepsilon \tag{4.9}$$

in (1.42). Combined with the exponential estimate (1.39) on $y^\varepsilon(t)$, this provides the homogenization term

$$\|\mathcal{F}_\varepsilon(y^\varepsilon(\tau)) - \mathcal{F}_0(y^\varepsilon(\tau))\|_{-1} \;\; \leq \;\; C(1 + M^{p_0} e^{\varrho_0 p_0 \tau}) \cdot \varepsilon. \tag{4.10}$$

Similarly, the estimate

$$\|\mathcal{F}_0'(y)\|_{\mathcal{L}(E_{-1},E_{-1})} \;\; \leq \;\; C(1 + \|y\|_0^{p_1}) \tag{4.11}$$

of (1.41) provides the Gronwall term

$$\|\mathcal{F}_0(y^\varepsilon(\tau)) - \mathcal{F}_0(y^0(\tau))\|_{-1} \;\; \leq \;\; C(1 + M^{p_1} e^{\varrho_0 p_1 \tau}) \cdot \|w(\tau)\|_{-1}. \tag{4.12}$$

With the abbreviations $p = \max(p_0, p_1)$, $\varrho = \varrho_0 p$ and

$$c(\tau) \;\; := \;\; C(1 + M^p e^{\varrho \tau}) \tag{4.13}$$

the splitting (4.8) and estimates (4.10), (4.12) above add up to

$$\|\mathcal{F}_\varepsilon(y^\varepsilon(\tau)) - \mathcal{F}_0(y^0(\tau))\|_{-1} \;\; \leq \;\; c(\tau)(\|w(\tau)\|_{-1} + \varepsilon). \tag{4.14}$$

Inserting this combined estimate into the variations-of-constants difference (4.5) provides the Gronwall estimate

$$\|w(t)\|_{-1} \;\; \leq \;\; \int_0^t M e^{\tilde{\varrho}_0 (t-\tau)} c(\tau)(\|w(\tau)\|_{-1} + \varepsilon) d\tau. \tag{4.15}$$

Along the lines of the Gronwall-Lemma we can conclude the homogenization estimate

$$\|w(t)\|_{-1} \;\; \leq \;\; C(t) \cdot \varepsilon. \tag{4.16}$$

Here we have used the abbreviation

$$C(t) = \exp\left(\int_0^t (\tilde{\varrho}_0 + Mc(\tau)) d\tau \right) - 1 \tag{4.17}$$

with $c(\tau)$ defined by the exponential expression in (4.13). We have also assumed $\tilde{\varrho}_0 \geq 0$, without loss of generality.
To prove estimate (4.16), (4.17) we observe that $\xi(t) := (1 + \varepsilon^{-1}\|w(t)\|_{-1}) \exp(-\tilde{\varrho}_0 t) > 0$ satisfies $\xi(0) = 1$ and

$$\xi(t) \leq 1 + \int_0^t M c(\tau) \xi(\tau) d\tau \tag{4.18}$$

by (4.14). This implies

$$\xi(t) \leq \exp(\int_0^t Mc(\tau)d\tau),$$

by the standard Gronwall argument, and thus proves estimates (4.16), (4.17). An elementary calculation implies

$$C(t) \leq \exp(\tilde{c}_2 e^{\varrho t} + \tilde{c}_1 t) - 1 \tag{4.19}$$

with constants $\tilde{c}_1, \tilde{c}_2$ given explicitly as

$$\begin{aligned} \tilde{c}_1 &= MC + \tilde{\varrho}_0 \\ \tilde{c}_2 &= \varrho^{-1} M^{p+1} C \end{aligned} \tag{4.20}$$

if $\varrho > 0$. Combining estimates (4.16), (4.19) proves the E_{-1} homogenization estimate

$$\|w(t)\|_{-1} \leq (\exp(\tilde{c}_2 \exp(\varrho t) + \tilde{c}_1 t) - 1) \cdot \varepsilon, \tag{4.21}$$

as was claimed in (4.7) above. We explicitly mention the special conservative or dissipative case $\varrho_0 = \varrho = 0$, where $c(\tau) = C(1 + M^p)$ and (4.21) simplifies to

$$\|w(t)\|_{-1} \leq \exp(\tilde{c}_1^0 t) \cdot \varepsilon \tag{4.22}$$

with the explicit choice

$$\tilde{c}_1^0 = MC(1 + M^p) + \tilde{\varrho}_0. \tag{4.23}$$

By interpolation (1.34) in $E_{-\alpha}$, of estimates (4.6) and (4.7), as was mentioned before, this completes our proof of theorem 1.3, including corollary (1.46), (1.47).

To prove theorem 1.1, by theorem 1.3, we recall the validity of the assumptions of theorem 1.3 in the strongly continuous semigroup setting (1.48), (1.49) of the hyperbolic wave equation (1.1), (1.2). We have to check assumptions (1.34), (1.35), (1.39)–(1.42) of theorem 1.3. The interpolation estimate (1.34) holds by our choice of spaces $E_{-\alpha} = H^{1-\alpha} \times H^{-\alpha}$. For growth assumption (1.39) see section 2, (2.4). The exponential estimate (1.35) on the linear semigroup $\exp(At)$ in $\mathcal{L}(E_0)$ also follows from section 2, putting $f \equiv 0, g \equiv 0, \gamma \equiv 0$. In $\mathcal{L}(E_{-1})$ the same estimate holds, by conjugation with the isomorphism $(-\Delta)^{-1/2} : E_0 \to E_{-1}$. The homogenization estimate (1.42), together with (1.41) has also been established in section 3. Therefore theorem 1.3 indeed applies and proves the doubly exponential homogenization estimate (1.22) of theorem 1.1.

In the case $\gamma^0 \geq \underline{\gamma} > 0$ of strictly positive damping, we have claimed a sharper, singly exponential homogenization estimate (1.23). To prove this, we invoke the special case $\varrho = \varrho_0 = 0$ of theorem 1.3 as specified in estimates (1.45)–(1.47). The a priori bound $\|y^\varepsilon(t)\|_0 \leq M(y_0)$, which is assumed in (1.45), does hold for $\gamma^0 \geq \underline{\gamma} > 0$ or even for $\gamma^\varepsilon \geq 0$; see assumptions (1.21) and the remarks at the end of section 2. (Similarly, the linear semigroup estimate (1.35) holds with growth rate $\tilde{\varrho}_0 = 0$.) Therefore (1.35) implies the singly exponential estimate (1.23), which completes the proof of theorem 1.1.

5 Fractional homogenization of exponential global attractors

In this section we first prove the fractional homogenization estimate

$$\operatorname{dist}_{E_{-\alpha}}(\mathcal{A}^\varepsilon, \mathcal{A}^0) := \sup_{y \in \mathcal{A}^\varepsilon} \operatorname{dist}_{E_{-\alpha}}(y, \mathcal{A}^0) \leq C\varepsilon^{\alpha'} \tag{5.1}$$

which was claimed in theorem 1.2. The proof is based on an exponential rate of attraction for the global attractor $\mathcal{A}^0$ at $\varepsilon = 0$, as observed in (1.31) and formulated in lemma 5.1. See [HR89], [FV00] for earlier statements of this abstract principle. We then invoke earlier results by Babin and Vishik to prove the required exponential attractivity of $\mathcal{A}^0$ for the hyperbolic wave equation (1.1), (1.2) in $E_0 = H_0^1 \times L^2$. See also [EFNT94].

Our presentation of the abstract fractional homogenization estimate follows the presentation in [FV00], Lemma 4.1. For each $0 \leq \varepsilon \leq \varepsilon_0$ let $y^\varepsilon(t), t \geq 0$ denote a family of semigroups on a Banach space $E_{-\alpha}$ such that an ε-independent estimate

$$\|y^\varepsilon(t) - y^0(t)\|_{-\alpha} \leq C\varepsilon^\alpha e^{\varrho t} \tag{5.2}$$

holds, uniformly for all $0 \leq \varepsilon \leq \varepsilon_0, t \geq 0$. The constants C and ϱ are allowed to depend on α, and on the norm $\|y_0\|_0$ of the initial condition $y_0 = y^\varepsilon(0) = y^0(0)$. The space $E_{-\alpha}$ need not belong to a scale of interpolation spaces, here. But we do do assume $E_{-\alpha}$ to extend a reference subspace $E_0 \subseteq E_{-\alpha}$ with a stronger associated norm $\|y_0\|_0$ measuring the initial conditions y_0 above.

Let $\mathcal{A}^\varepsilon$, $0 \leq \varepsilon \leq \varepsilon^0$, denote a family of *negatively invariant* subsets of $E_0 \subseteq E_{-\alpha}$: for each $\tilde{y}_0 \in \mathcal{A}^\varepsilon$ and each $t \geq 0$ there exists some initial condition $y_0 \in \mathcal{A}^\varepsilon$ such that the solution y^ε in E_0 with initial condition $y^\varepsilon(0) = y_0$ satisfies $y^\varepsilon(t) = \tilde{y}_0$. We assume an exponential rate of attraction for $\mathcal{A}^0$ in $E_{-\alpha}$,

$$\mathrm{dist}_{E_{-\alpha}}(y^0(t), \mathcal{A}^0) \leq Ce^{-\nu t}, \tag{5.3}$$

uniformly for all initial conditions

$$y_0 \in \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \mathcal{A}^\varepsilon \ \subseteq \ E_0 \ \subseteq \ E_{-\alpha}. \tag{5.4}$$

In $E_{-\alpha}$, finally, we assume the fractional homogenization estimate (5.2) to hold, uniformly for all initial conditions $y_0 \in \bigcup \mathcal{A}^\varepsilon$ as in (5.4). By assumption (5.2), it is therefore sufficient to assume boundedness of $\bigcup \mathcal{A}^\varepsilon$ in the stronger reference space E_0.

Lemma 5.1 *Under the above assumptions (5.2) and (5.3), there exists an ε-independent constant $C > 0$ such that the fractional homogenization estimate (5.1) holds, with fractional order*

$$\alpha' := \alpha/(1 + \varrho/\nu). \tag{5.5}$$

The constants α, ϱ, ν were introduced in (5.2), (5.3) above.

Proof [FV00]: The proof consists of just the triangle inequality. Let $0 < \varepsilon \leq \varepsilon_0 < 1$. For any $\tilde{y}_0 \in \mathcal{A}^\varepsilon \subseteq E_0$ define $t := -\nu^{-1}\alpha' \log \varepsilon > 0$. By backwards invariance of $\mathcal{A}^\varepsilon$, choose $y_0 \in \mathcal{A}^\varepsilon$ such that $y^\varepsilon(t) = \tilde{y}_0$. Then (5.2), (5.3) and the choice of t imply

$$\begin{array}{rcl} \mathrm{dist}_{E_{-\alpha}}(\tilde{y}_0, \mathcal{A}^0) & = & \mathrm{dist}_{E_{-\alpha}}(y^\varepsilon(t), \mathcal{A}^0) \leq \\ & \leq & \|y^\varepsilon(t) - y^0(t)\|_{-\alpha} + \mathrm{dist}_{E_{-\alpha}}(y^0(t), \mathcal{A}^0) \leq \\ & \leq & C(\varepsilon^\alpha e^{\varrho t} + e^{-\nu t}) = 2C\varepsilon^{\alpha'}. \end{array} \tag{5.6}$$

This proves the fractional homogenization estimate (5.1), for a generic constant C. $\bowtie$

Let $\mathcal{A}^\varepsilon$ be the global attractor in $E_0 = H_0^1 \times L^2$ of the semigroup $S^\varepsilon(t)$ of solutions $y^\varepsilon = (u^\varepsilon, u_t^\varepsilon) \in E_0$ associated to the hyperbolic wave equation (1.1), (1.2). Similarly, $\mathcal{A}^0$ is the global attractor of the homogenized semigroup $S^0(t)$.

To prove theorem 1.2 it now remains to verify the assumptions of lemma 5.1 in our semigroup setting (1.48), (1.49) of the hyperbolic wave equation (1.1), (1.2). The fractional homogenization assumption (5.2) was proved in theorem 1.1, (1.23); see the end of section 4. The exponential attraction rate (5.3) will be established below, in fact in the stronger norm of E_0.

We summarize the necessary arguments. For earlier, technically more restrictive arguments along these lines see also [Tem88], [BV89], [EFNT94], [Rau01], [CV01]. Using the topology of $E_0 = H_0^1 \times L^2$ we observe, and partially comment below on,

(i) continuous dependence of $y^0(t; y_0)$ on $(t, y_0) \in (0, \infty) \times E_0$;

(ii) existence of the global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$;

(iii) precompactness of $\bigcup_{0 \le \varepsilon \le \varepsilon_0} \mathcal{A}^\varepsilon$ in E_0;

(iv) Lipschitz dependence in E_0,

$$\|y^1(t) - y^2(t)\|_0 \;\le\; Ce^{\tilde{\varrho} t}\|y_0^1 - y_0^2\|, \tag{5.7}$$

of solutions y^1, y^2 for $\varepsilon = 0$ with respective initial conditions y_0^1, y_0^2 in $\bigcup \mathcal{A}^\varepsilon$;

(v) continuity of the decaying Hamiltonian energy Φ in E_0, see (1.27);

(vi) finiteness and hyperbolicity of equilibria, at $\varepsilon = 0$

(vii) $C^{1+\theta}$-dependence in E_0 of the solutions $y^0(t; y_0)$, for $\varepsilon = 0$, on the initial condition y_0 at $t = 0$, near the above equilibrium set.

In [BV89], the above properties (i), (ii), (iv) – (vii) together have been shown to imply an exponential rate of attraction in E_0 of relatively compact sets

K to the global attractor $\mathcal{A}^0$ for $\varepsilon = 0$. By property (iii), proved below, $K := \bigcup \mathcal{A}^\varepsilon$ is indeed precompact. This then proves the exponential attraction property (5.3). To prove theorem 1.2 it is therefore sufficient to ensure the validity of claims (i)–(vii) above.

We first address continuous dependence of solutions $S^\varepsilon(t)y_0 = y^\varepsilon(t; y_0)$ on $(t, y_0) \in (0, \infty) \times E_0$, as claimed in (i). This is a general fact in the semigroup setting (1.48), (1.49) of our hyperbolic wave equation, and follows from local Lipschitz continuity of the abstract nonlinearity $\mathcal{F}_\varepsilon(y)$ in E_0; see (2.3) and for example [Tan79], [Paz83], [BV89, Theorem 7.1].

We address existence and precompactness of global attractors, (ii), (iii) next. The semigroups $S^\varepsilon(t)$ in E_0 possess an ε-independent absorbing set $\mathcal{B}$, which is compact in E_0, namely a closed ball of sufficiently large radius R in E_σ, for $\sigma := 1 - \frac{1}{2}(p-1)(n-2) > 0$. For $n = 3$ see [Tem88, pp. 204-206]. For $n \geq 3$ see also [Hal85], [Har85]. The proof in [Tem88] extends verbatim to rapidly oscillating coefficients $b(x, x/\varepsilon), g(x, x/\varepsilon)$, because it never uses any differentiation with respect to x. Therefore the global attractors $\mathcal{A}^\varepsilon$ exist for $0 \leq \varepsilon \leq \varepsilon_0$. Their union is precompact in E_0 being contained in the ε-independent set $\mathcal{B}$ which is compact in E_0. This proves properties (ii), (iii).

Property (iv) follows from uniform boundedness in E_0 of forward orbits starting in $\bigcup \mathcal{A}^\varepsilon$. The proof uses growth estimate (1.8) on $f'(u)$ and follows the lines of section 2.

Continuity property (v) of the Hamiltonian energy Φ follows from growth estimate (1.8), integrated twice with respect to u, and the Sobolev embedding $H_0^1 \hookrightarrow L^{2n/(n-2)}$.

Property (vi) is already assumed to hold in theorem 1.2.

The Hölder condition (1.26) on $f'(u)$ entails a corresponding Hölder property for the abstract nonlinearity $\mathcal{F}(u)$ on E_0, and therefore provides a Hölder property of the associated semigroup, as spelled out in (vii).

This completes our proof of properties (i)–(vii).

In conclusion, our proof of theorem 1.2 on quantitative homogenization of global attractors $\mathcal{A}^\varepsilon$ of damped hyperbolic wave equations (1.1), (1.2) can be summarized as follows. By [BV89], properties (i)–(vii) above imply an

ε-uniform exponential attraction rate

$$\mathrm{dist}_{H_0^1\times L^2}(S^0(t)(\bigcup_{0\leq\varepsilon\leq\varepsilon}\mathcal{A}^\varepsilon)\leq Ce^{-\nu t} \tag{5.8}$$

under the homogenized semigroup $S^0(t)$, for suitable constants $C,\nu>0$. This proves assumption (5.3) of lemma 5.1. We also recall that the fractional homogenization assumption (5.2) of lemma 5.1 follows from theorem 1.1, (1.23). In terms of $(u,u_t)=y\in E_{-\alpha}=H^{1-\alpha}\times H^{-\alpha}$ this means

$$\|u^\varepsilon(t,\cdot)-u^0(t,\cdot)\|_{H^{1-\alpha}}+\|u_t^\varepsilon(t,\cdot)-u_t^0(t,\cdot)\|_{H^{-\alpha}}\leq C\varepsilon^\alpha e^{\varrho t}, \tag{5.9}$$

for all $0\leq\alpha\leq 1$ and, again, for suitable constants $C,\alpha,\varrho>0$. Both these ingredients require positive damping. Lemma 5.1 now proves the fractional homogenization estimate

$$\mathrm{dist}_{H^{1-\alpha}\times H^{-\alpha}}(\mathcal{A}^\varepsilon,\mathcal{A}^0),\mathcal{A}^0)\leq Ce^{\alpha'} \tag{5.10}$$

for a suitable constant C and

$$\alpha':=\alpha/(1+\varrho/\nu). \tag{5.11}$$

This proves theorem 1.2, because (5.10) is equivalent to (1.32).

As a final remark we mention the open problem of quantitative homogenization in the case of limiting exponent

$$p=n/(n-2) \tag{5.12}$$

which includes the case $p=3$ of cubic f in $n=3$ space dimensions. The global attractor $\mathcal{A}^0$, as well as any single one of the attractors $\mathcal{A}^\varepsilon$, is known to be still compact in E_0, in this limiting case. See [BV89], [ACH92] and [Rau01]. Because precompactness property (iii) of

$$\bigcup_{0\leq\varepsilon\leq\varepsilon_0}\mathcal{A}^\varepsilon \tag{5.13}$$

in E_0 is not known, however, our present homogenization proof does not cover the case of limiting exponent p.

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