CHAPTER 5

Romeo and Juliet, Spontaneous Pattern Formation, and Turing's Instability

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1. Turing dreams

Crystals, snow flakes, soap bubbles, water waves, dunes, mountain valleys, pine cones, embryonic development, sunflowers, sea shells, zebra stripes, heartbeat and nerve tremors: Nearly everywhere we find ordered structures and patterns, regularities that arise as if "by themselves". This "by themselves": Does it not sound evasive – not-knowing, or not wanting to know? So we ask: How does this "by themselves", this "self-organization", work? How can shape and form bring themselves to bear, form and develop themselves from undifferentiated uniformity? How can creation so assert itself and unfold against the omnipresent powers of dissolving, of sinking back into entropic leveling and amorphous homogeneity? And so we may continue asking, astonished and perplexed.

Such questions merit a life's work and are not to be shrugged off with a lecture or a short article. Alan M. Turing (1912–1954) in his work during the year 1952 (see Reference [**11**]) developed ground-breaking insights into the problem of "self-organization" or "morphogenesis", and they still exert a lively influence. This is the same Turing who, aged 26, had laid the foundations for the modern theory of computability and the architecture of computers. During the Second World War he was deeply involved in the deciphering of the German military's secret cipher *Enigma*. But let us put all this aside, and return to his approach to morphogenesis.

Assuredly, form may be imprinted from the outside, or even be laid out in the germ, invisible to us, but nevertheless pre-formed. Soap skins for instance, stretched across wire shapes, try to minimize their energy, so that they take the wonderfully elegant forms of minimal surfaces. For the virtuoso mathematics of such optimality questions see the book of Hildebrandt and Tromba [**6**], or the article by D. Ferus in this book. But what if we do not impose any spatial structure in advance – or microscopically minimal random fluctuations, at most, of an initially homogeneous distribution? Must not such fluctuations immediately smooth out by diffusion? What would we think of a "self-organized" bath tub that un-mixed already well-tempered water "by itself": cold water to the feet and hot to the head? A crackpot, a dreamer, might fall for this – but not us! Monstrous phantom of an overheated head in urgent need of cooling . . .

Absolutely right, the bath tub does nothing like that – at least, not on realistic time scales. But Turing predicted correctly that structure can nevertheless arise "spontaneously", in a mathematically precise sense, from the interplay of spatially homogeneous reaction laws and spatially leveling diffusion. This very dream has been confirmed since by analysis, experiment and by computer-simulations. Belousov and Zhabotinski have discovered time oscillating chemical reactions – and, by the way, against the fierce resistance of those who held this to be mischief and contempt for the principles of thermodynamics. The many space-time structures that appear in such experiments substantiate Turing's dream.

Unfortunately Turing's original work required a proficiency in the analysis of ordinary and partial differential equations, that – today as then – is to be gained only through many years of serious mathematical study: proficiency in a key intellectual technology, now vanishing from among our mathematics teachers and student-teachers, or officially discouraged. In this article we shall first try to illustrate the mathematical essence of Turing's idea, requiring only the basic rules of arithmetic – and alert attention.

As a parable for the methods of mathematical dynamics we shall take the well known love story of Romeo and Juliet. Shakespeare will, I hope, forgive us for this intrusion into the love life of the immortalized couple. The claim of our essay is not literary. And in no case should the professional qualifications of marriage and partner counsellors be reduced to a mathematical diploma. But from our glimpse into questions of life and love we will learn intimately about a very rich supply of phenomena in the mathematical theory dynamical systems. At any rate, rather in this direction than the opposite.

2. Romeo and Juliet

Rather than conceiving of the love between Romeo and Juliet in words, like Shakespeare, or in music, like Prokofiev, these affairs of the heart will be vested into our arid formulae: only as a parable, as already said, for Turing's idea of morphogenesis. Be warned one last time, but urgently so, against the risks and side effects of excessive real life imitation.

So let J_n be the love that Juliet bears in her heart for Romeo on evening n. Here $n = 1$ or $n = 2$ or any other natural number: briefly, $n = 1, 2, 3 \dots$ The numerical value of J_n may be positive or negative – as the case may be. A numerical value $J_n = 0$ on our scale of love temperature will, somewhat shabbily, denote that desirable state of gentle happiness which, through the suppression of youthful enthusiasm towards ever increasing J_n, prudently avoids the catastrophe well known from literature.

If Juliet had the gift not to heed her beloved Romeo too much, her behavior might be described by the dynamics

$$J_{n+1} = J_n \tag{2.1}$$

This formula expresses that Juliet's love J_n continues to the next evening $n + 1$ with J_{n+1} exactly the same as on the previous evening. Naturally it is only an assumption that the nickname Konstanze might suit Juliet; though this was the name of the wife of Wolfgang Amadeus Mozart, who most probably did *not* follow (2.1) truly.

TABLE 2.1. The six day love cycle of Romeo and Juliet; see (2.2), (2.3).

n	1	2	3	4	5	6	7	8	9	$\cdots$
J_n	1	1	0	-1	-1	0	1	1	0	$\cdots$
R_n	0	-1	-1	0	1	1	0	-1	-1	$\cdots$

Distinct from Shakespeare's, and quite the opposite to faithful Juliet, your modern Romeo is a fickle fellow:

$$R_{n+1} = -J_n \, . \tag{2.2}$$

His love temperature R_{n+1} on evening $n+1$ thus responds to Juliet's affection $J_n > 0$ of the previous evening with a cold shoulder: "I've got *her* in the bag, time to look elsewhere", that's his creed. Only when Juliet's love cools noticeably, $J_n < 0$, he turns truly sorry: $R_{n+1} > 0$, and Juliet is again the sole queen of his roving heart.

Naturally this does not pass Juliet by unnoticed, as (2.1) would have it: we therefore replace (2.1) by

$$J_{n+1} = J_n + R_n \, . \tag{2.3}$$

This means that Romeo's affection $R_n > 0$ of the previous evening kindles the flame of Juliet's love. But a cold shoulder $R_n < 0$ understandably dampens Juliet's affection. We now combine (2.2), (2.3) in an abbreviated form:

$$z_{n+1} = Az_n \, . \tag{2.4}$$

Here the number pair $z_n = (J_n, R_n)$ describes the state of our lovers on evening n, while $Az_n = (J_n + R_n, -J_n)$ simply abbreviates the application of the right sides of (2.2) and (2.3).

Anyone who cares to may now program the recipe (2.4) and – completely independently of zodiac sign, moon phase and biorhythm – calculate the "love vector" z_n on and on, for all time.

Even old-fashioned paper and pencil are sufficient; see Table 2.1. The initial values $J_1 = 1,\ R_1 = 0$, alias $z_1 = (J_1, R_1) = (1, 0)$, are chosen arbitrarily. Substituting into (2.2), (2.3) with $n = 1$ yields $z_2 = (J_2, R_2)$. Substituting again, with $n = 2$, gives $z_3 = (J_3, R_3)$ and so on. See Figure 2.1 for a plot of the time evolution for $n = 1, 2, 3, \ldots$ A plot of the points (J_n, R_n) in the (J, R) plane is worthwhile.

Directly from Table 2.1 we observe that the love vector z_n repeats with a period of exactly six days:

$$z_{n+6} = z_n \, . \tag{2.5}$$

This happens not just for our special choice $z_1 = (1, 0)$ but for every arbitrary initial combination $z_1 = (J_1, R_1)$. Figure 2.1 clearly illustrates the time-phased oscillation between affection and antipathy of both partners. The obvious cause is Romeo's immature philandering. But Juliet could moderate these perpetually changing feelings. She might, for example, take Romeo a little less to heart: purely mathematically, replacing (2.3) with

$$z_{n+1} = J_n + 0.9 * R_n \, . \tag{2.6}$$

But we are not in therapy here.

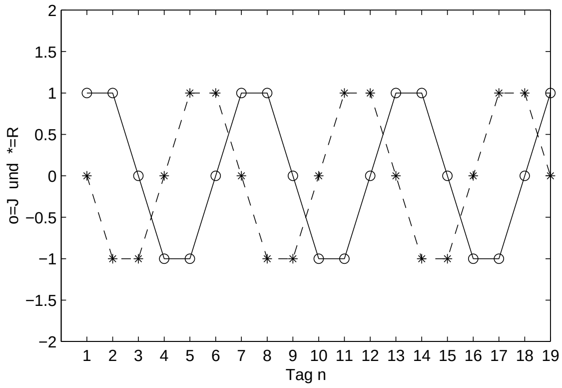

FIGURE 2.1. The six day love cycle of Romeo ($*$) and Juliet ($\circ$)

3. Roberto and Julietta

What Shakespeare did not know: Juliet has a twin sister Julietta. Monozygotic, mathematically identical twins. And Romeo likewise has a twin brother Roberto. And Roberto and Julietta are just such a romantic couple as Romeo and Juliet are.

Wherefore, such revelation? The construction is so whimsical that we would toss aside any novel based on such a premise. For our mathematical attempt, however, to understand spontaneous pattern formation and morphogenesis à la Turing, this construction fulfills an important purpose. We are interested in whether the identical relationships of the two identical romantic couples will develop in an identical way. They must, quite obviously. Or must they?

We denote by J'_n (read: J_n prime) and R'_n the love states of Julietta and Roberto on evening n, respectively. Exactly as in the previous section

$$J'_{n+1} = J'_n + R'_n \qquad R'_{n+1} = -J'_n. \tag{3.1}$$

Julietta and Juliet, resp. Roberto and Romeo, are born from the same mold, even in their relationships. Subtle differences between identical genetic influences and different imprints through environmental factors are virtually absent from our parable. For example, the vector $z'_n = (J'_n, R'_n)$ is again sentenced to a love cycle of period six:

$$z'_{n+6} = z'_n, \tag{3.2}$$

just as z_n before in (2.5), and without any hormonal basis. As in (2.4) we can abbreviate

$$z'_{n+1} = Az'_n. \tag{3.3}$$

Here the prescribed map A has no apostrophe (A'), since the z'_n indeed follow the same law A as the z_n do.

We certainly do not plan to assume that $z'_n = z_n$ for all $n = 1, 2, 3, \ldots$ For example, maybe $z'_1 \neq z_1$: the two couples may have known each other a different length of time on the reference day $n = 1$. Aha, a little tidbit of environment may creep in through this backdoor! In consequence, $z'_n \neq z_n$ for all $n = 1, 2, 3, \ldots$

Where now is morphogenesis? Both couples experience essentially the same. No surprise, since each couple remains blind and deaf to the other couple and to all the rest of the world. But soon this will change.

4. When sisters gossip . . .

We have expressed in formulae how the "love vector" z_n resp. z'_n develops, for each of the two romantic couples. Now we defer these considerations, for a moment, and examine the mutual influences of the respective sibling couples. As a simplification we first assume that only the sisters Juliet and Julietta relate their adventures, but not the brothers. The brothers will return in section 5.

How should a formula express the sisters' habit of daily gossip? We write

$$J_n = \bar{J}_n + s * (\bar{J}'_n - \bar{J}_n). \tag{4.1}$$

Here J_n again measures Juliet's love on the *evening* of day n. The new symbol $\bar{J}_n$ (read: J_n-bar) denotes Juliet's love on the *morning* of day n, after she has met Romeo. Similarly $\bar{J}'_n$ (read: J_n-bar-prime or J_n-prime-bar) measures Julietta's love temperature on *morning* n. The star, $*$, denotes conventional multiplication. The gossip susceptibility s describes how much Juliet ($\bar{J}_n$) is influenced daily by Julietta ($\bar{J}'_n$). Typically, s will be a fixed number between 0 and 1, i. e. $0 \leq s \leq 1$.

The case $s = 1$ leads to

$$J_n = \bar{J}'_n\,. \tag{4.2}$$

Perhaps Juliet thinks: "... oh, this Roberto must be just awesome, from what Julietta tells. Hm, and my Romeo is his twin brother after all. Maybe he didn't mean it yesterday, when he was so boorish to me." By the evening Juliet has fully slewed round to Julietta's line, so $J_n = \bar{J}'_n$.

The case $s = 0$ describes the other extreme:

$$J_n = \bar{J}_n. \tag{4.3}$$

Juliet does not give a damn about Julietta's gossip and sticks to her own opinion: a true Konstanze.

Values such as $s = 0.5 = 50\%$, or $30\%, 70\%$, etc. mix these two extremes to different degrees. We see that the susceptibility s simply describes the degree to which J_n responds in the direction of the difference $J'_n - J_n$.

Let us return to the general case (4.1) which covers all those hybrid forms. Since Julietta is the twin sister of Juliet she is influenced reciprocally by Juliet – by the same law. Her evening love J'_n will thus follow

$$J'_n = \bar{J}'_n + s * (\bar{J}_n - \bar{J}'_n). \tag{4.4}$$

We obtain (4.4) from (4.1) by dropping the apostrophe where there was one, and introducing one where there was none. Following the same scheme we have already obtained section 3 from section 2, for example (3.1) from (2.2), (2.3). This works, because of our twin assumption.

What has this to do with Turing? Along with Turing we are interested in the *differences* of the genetically identical twin pairs. We therefore form the difference of (4.4) and (4.1) and obtain

$$\begin{aligned} J'_n - J_n &= \bar{J}'_n + s * (\bar{J}_n - \bar{J}'_n) - (\bar{J}_n + s * (\bar{J}'_n - \bar{J}_n)) = \\ &= (1 - 2s) * (\bar{J}'_n - \bar{J}_n). \end{aligned} \tag{4.5}$$

The difference between the sisters is thus reduced by the factor $1-2s$. For $0 \leq s \leq 1$ this fraction has an absolute value between 0 and 1 (ignoring the sign). In the case $s = 0$ nothing changes, because both sisters are Konstanzes. In the case $s = 1$ the rôles of the sisters reverse. The middle course $s = 0.5 = 50\%$ leads to perfect and immediate harmony $J'_n - J_n = 0$ (of the sisters) and hence

$$J'_n = J_n = \frac{1}{2}(\bar{J}'_n + \bar{J}_n)\,. \tag{4.6}$$

The women cleverly agree on the arithmetic mean and so sort out their sibling differences. The compensating leveling influence of gossiping, setting a trend for each $0 < s < 1$, is most pronounced here.

We now have both ingredients ready to realize Turing's idea: the dynamics proper of each romantic couple, and the compensating leveling function of the gossiping sisters. Let us combine the two effects. As in section 2 the nightly encounters of Romeo and Juliet lead to

$$\begin{aligned} \bar{J}_{n+1} &= J_n + R_n \\ \bar{R}_{n+1} &= -J_n. \end{aligned} \tag{4.7}$$

Here we have transcribed (2.2), (2.3) and replaced (J_{n+1}, R_{n+1}) by $(\bar{J}_{n+1}, \bar{R}_{n+1})$: at the end of the love night we have arrived at the *morning* of day $n+1$, and with the inquisitive sisters we await the latest news. Correspondingly, for Roberto and Julietta,

$$\begin{aligned} \bar{J}'_{n+1} &= J'_n + R'_n \\ \bar{R}'_{n+1} &= -J'_n. \end{aligned} \tag{4.8}$$

On day $n+1$, for Romeo, $R_{n+1} = \bar{R}_{n+1}$; he remains uninfluenced by his brother Roberto until the next evening. For Juliet we substitute the gossip dynamics (4.1) – of course with $n+1$ instead of n, since we are already in day $n+1$:

$$J_{n+1} = \bar{J}_{n+1} + s * (\bar{J}'_{n+1} - \bar{J}_{n+1}). \tag{4.9}$$

Substituting (4.7), (4.8) into (4.9) yields (with a little pencil and paper)

$$\begin{aligned} J_{n+1} &= J_n + R_n + s * (J'_n - J_n) + s * (R'_n - R_n) \\ R_{n+1} &= \bar{R}_{n+1} = -J_n \end{aligned} \tag{4.10}$$

for Romeo and Juliet on the next evening $n+1$.

Our apostrophe trick from (4.4) yields the corresponding equations for Roberto and Julietta:

$$\begin{aligned} J'_{n+1} &= J'_n + R'_n + s * (J_n - J'_n) + s * (R_n - R'_n) \\ R'_{n+1} &= -J'_n. \end{aligned} \tag{4.11}$$

The casting of our four actor drama is now almost complete, but somewhat confusing. We simplify in terms of the arithmetic means J_n^+, R_n^+ and differences J_n^-, R_n^-, defined as follows:

$$\begin{aligned} J_n^{\pm} &= \frac{1}{2}(J_n \pm J'_n) \\ R_n^{\pm} &= \frac{1}{2}(R_n \pm R'_n). \end{aligned} \tag{4.12}$$

The symbols $\pm$ can be read either as $+$ or as $-$, but consistently in all places. Now, along with Turing, we are most interested in the differences J_n^-, R_n^- of the genetically identical sibling pairs. For example, (4.5) now reads

$$J_n^- = (1-2s) * \bar{J}_n^-. \tag{4.13}$$

On adding the equations (4.10), (4.11) we obtain

$$\begin{aligned} J_{n+1}^+ &= J_n^+ + R_n^+ \\ R_{n+1}^+ &= -J_n^+ \end{aligned} \tag{4.14}$$

and, by fearless subtraction,

$$\begin{aligned} J_{n+1}^- &= (1-2s) * (J_n^- + R_n^-) \\ R_{n+1}^- &= -J_n^-. \end{aligned} \tag{4.15}$$

Still four equations, like (4.10) and (4.11), but now neatly uncoupled. For example (4.14): These are exactly the same equations as (2.2), (2.3), discussed in section 2, just featuring other letters: J_n^+, R_n^+ in place of J_n, R_n. Consequently the mean values $z_n^+ = (J_n^+, R_n^+)$ in our total system including the gossiping sisters also oscillate lustily with love period 6:

$$z_{n+6}^+ = z_n^+. \tag{4.16}$$

And how about the differences $z_n^- = (R_n^-, J_n^-)$? Clearly $z_1^- = (R_1^-, J_1^-) = (0,0)$ leads, by (4.15), successively to

$$z_{n+1}^- = z_n^- = \ldots = z_2^- = z_1^- = (0,0). \tag{4.17}$$

We simply have two identical copies of the same lovers and the consistently unanimous sisters might as well have spared their gossiping s.

Figure 4.1 illustrates the effect of the gossiping sisters for initial values $z_1^- = (J_1^-, R_1^-) = (1,0)$ and different values of the gossip susceptibility s. For $0 < s \leq 0.5 = 50\%$ we observe that the increasing gossip susceptibility of the sisters increasingly dampens the differences $z_n^- = (J_n^-, R_n^-)$ towards zero. This holds for the brothers R_n^- too, although they have never once talked to each other about their sweethearts, cool as they are. When $50\% < s < 75\%$ we see two-day oscillations at first, because the sisters gossip too much. But still these oscillations dampen out and the two couples synchronize at last. Readers with some amatory experience will recognize this effect, won't they?

Gossip addicted sisters with $s > 75\%$, however, head directly into catastrophe. Both couples experience an increasing, ever worsening, Up and Down of love-hate cycles, affection and dislike, that may eventually tragically destroy both relationships. The vicious cycle grows, independently of how minute the differences z_1^- might have been initially, and finally overwhelms the 6-day cycle of the mean values z_n^+. Juliet and Julietta are consistently at odds in their feelings, and so are Romeo and Roberto. The effects on such different characters as Romeo and Juliet are remarkable: Both experience an ever increasing affection, or dislike, for each other – despite substantially different "love strategies" – in complete synchrony.

Thus Turing's instability dreadfully affects our two paradigmatic couples of loving sweethearts. Despite the virtually identical initial conditions of the identical pairs, and even discounting the leveling influence of the sisterly gossip, the differences z_n^- finally turn catastrophic.

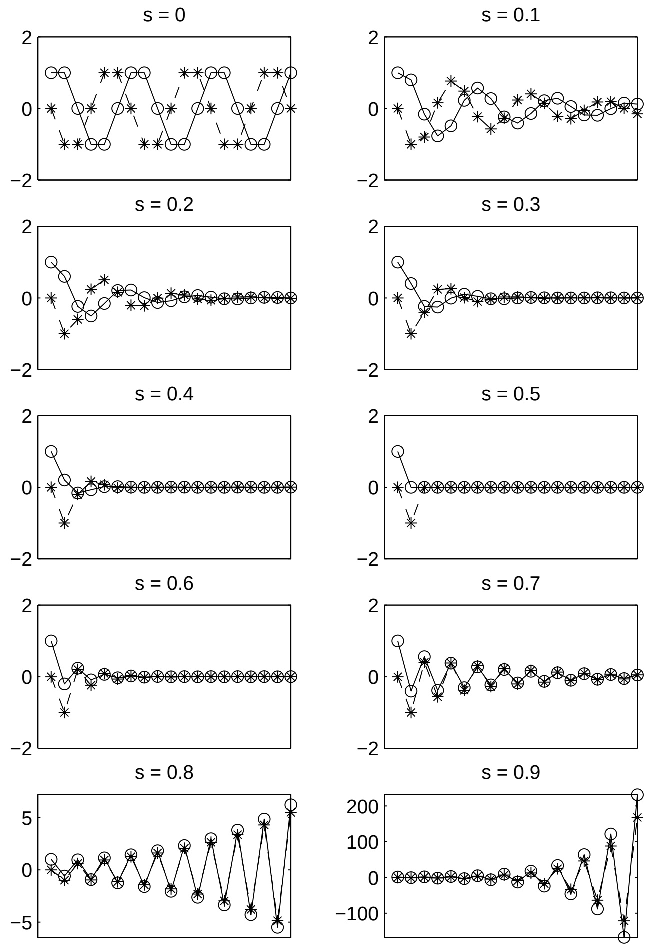

FIGURE 4.1. Gossiping sisters lead to Turing instability; $\circ = J_n^-, * = R_n^-$

And the moral? Gossip a little, ye ladies, but not too much. The border lies at 75%! Oh, well – in our model ...

5. ... and brothers brag

The Turing instability of the two romantic couples can equally well be caused by the brothers Romeo, R_n, and Roberto, R'_n. We simply have to transfer the discussion of the previous sections regarding the sisters, J_n and J'_n, to the brothers. Experienced in modeling issues of love, as we have since become, we can now abbreviate the interpretation of the individual steps.

Analogously to (4.1), (4.9), on day $n+1$ Romeo's evening love R_{n+1} for Juliet is given by

$$R_{n+1} = \bar{R}_{n+1} + p * (\bar{R}'_{n+1} - \bar{R}_{n+1}), \tag{5.1}$$

after his twin brother Roberto, $\bar{R}'_{n+1}$, during bright daylight, has sufficiently enflamed or chilled him with his night-time stories of Julietta. The brag parameter p

is chosen fixed again, with $0 \leq p \leq 1$. In section 4, for example, we have chosen $p = 0$. Conversely, then, we of course have

$$R'_{n+1} = \bar{R}'_{n+1} + p * (\bar{R}_{n+1} - \bar{R}'_{n+1}), \tag{5.2}$$

analogously to (4.4). This time the sisters remain stoically unimpressed:

$$J_{n+1} = \bar{J}_{n+1}, \qquad J'_{n+1} = \bar{J}'_{n+1}. \tag{5.3}$$

This corresponds to the choice $s = 0$ in the previous section.

The total system for the two romantic couples, this time, reads

$$\begin{aligned} J_{n+1} &= J_n + R_n \\ R_{n+1} &= -J_n \; - p * (J'_n - J_n) \end{aligned} \tag{5.4}$$

$$\begin{aligned} J'_{n+1} &= J'_n + R'_n \\ R'_{n+1} &= -J'_n \; - p * (J_n - J'_n) \end{aligned} \tag{5.5}$$

in analogy to (4.10), (4.11). In terms of the variables $J_n^{\pm}, R_n^{\pm}$ from (4.12) which express the sibling mean values (+) resp. differences (−), our system is

$$\begin{aligned} J^+_{n+1} &= J^+_n + R^+_n \\ R^+_{n+1} &= -J^+_n \end{aligned} \tag{5.6}$$

$$\begin{aligned} J^-_{n+1} &= J^-_n + R^-_n \\ R^-_{n+1} &= -(1-2p) * J^-_n \,. \end{aligned} \tag{5.7}$$

This resembles (4.14), (4.15), but not quite. Certainly the mean values $z_n^+ = (J_n^+, R_n^+)$ again follow the path of a single romantic couple with a six day love cycle, $z_{n+6} = z_n$, unperturbed by the brothers' bragging p.

Figure 5.1 illustrates the effect of the brothers' bragging for initial values $z_1^- = (J_1^-, R_1^-) = (1, 0)$ and various values of the brag parameter p. For $0 < p \leq 50\%$ we concede the soothing influence of bragging among brothers, which, in its own way, serves to exchange information and to balance tempers, much as the sisters' gossip did.

Mathematically there is no difference between (4.1), (4.4) on the one hand and (5.1), (5.2) on the other: We have only replaced the letter J by R, and s by p. The effects on the total system, however, are different, caused by the fundamentally different attitudes of the twin brothers.

Indeed, for $p = 48\%$ we notice a clearly decelerated stabilization. Already for a brag parameter $p > 50\%$ the differences $z_n^- = (J_n^-, R_n^-)$ grow ever faster, again directly into catastrophe. Again the catastrophic differences z_n^- between the siblings triumph over the 6-day-cycle of the arithmetic means $z_n^+ = (J_n^+, R_n^+)$. The progression is quite different, however, from when the sisters gossip. Take Juliet, J_n, for example. Since

$$J_n = \frac{1}{2}(J_n^+ + J_n^-)$$

she is wafted into an ever more ecstatic state of bliss, just as in Shakespeare. Romeo follows her according to

$$R_n = \frac{1}{2}(R_n^+ + R_n^-)\,,$$

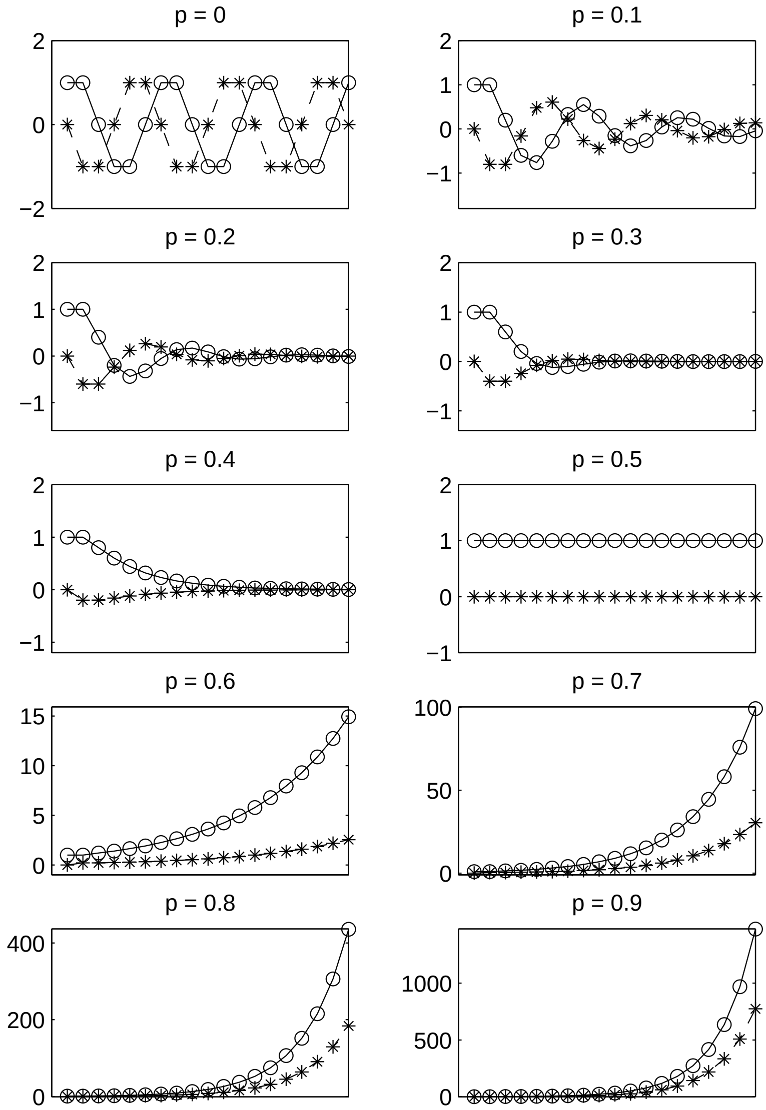

FIGURE 5.1. Bragging brothers induce Turing instability; $\circ = J_n^-$, $* = R_n^-$.

even if at a measured distance. For Julietta, however, things look rather different. According to

$$J_n' = \frac{1}{2}(J_n^+ - J_n^-)$$

the unbounded increase of the values J_n^- forces her into the repulsive region of increasing negativity. Again Roberto mimics her disdain:

$$R_n' = \frac{1}{2}(R_n^+ - R_n^-),$$

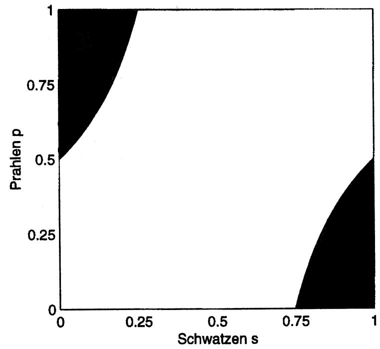

FIGURE 6.1. Domain of Turing instability (black), when sisters gossip (s) and brothers brag (p).

if hesitantly. Thus while Romeo and Juliet might be heading towards their immortal literary catastrophe of unimpeded love, Roberto and Julietta just as quickly settle on divorce, purportedly due to irreconcilable differences, and despite mathematically exactly equal genetic dispositions. But in reality because of Turing's instability.

And the moral? Brag a little, ye merry gentlemen, but not too much. The critical border lies at 50% – and that is *less* than the critical 75% for women! Oh, well – in our model ...

6. Turing's theorem

The preceding sections have illustrated Turing's idea on instability in terms of a single parable. The idea, however, reaches far beyond our simple example and our more or less amatory interpretations. Turing's result of 1952 [**11**] can be summarized approximately as follows.

Theorem. *Identical individually stable systems can become* destabilized *by interactions which, in themselves, would appear to stabilize.*

Our formulation may not be fully rigorous, but it captures the essence. The "systems" are, in our parable, the two couples, identical as monozygotic twins can ever be. The interactions in section 4 are the gossiping susceptibility s of the sisters, resp. the brag parameter p of the brothers in section 5. In themselves, s and p are stabilizing, as long as $0 < s < 1, 0 < p < 1$, since they reduce the differences by a daily factor $1 - 2s$ resp. $1 - 2p$. Turing instability of the total system nevertheless strikes at $s = 75\%$, resp. $p = 50\%$.

We can also study, without much effort, the hybrid forms that arise from a more or less intense daily exchange between *both* sibling pairs; cf. Figure 6.1. The corresponding equations are

$$
\begin{aligned}
J_{n+1} &= J_n + R_n + s * (J'_n - J_n) + s * (R'_n - R_n) \\
R_{n+1} &= -J_n - p * (J'_n - J_n) \,.
\end{aligned}
\tag{6.1}
$$

with corresponding equations for J'_{n+1} and R'_{n+1}; see (4.10) and (5.4). For the mean values J^+_n, R^+_n we find clear 6-day-cycles as in sections 2, 4 and 5. The differences

J_n^-, R_n^- obey

$$(6.2) \qquad \begin{aligned} J_{n+1}^- &= (1-2s) * (J_n^- + R_n^-) \\ R_{n+1}^- &= -(1-2p) * J_n^- . \end{aligned}$$

This time (4.15) resp. (5.7) appear as the special cases $p = 0$ resp. $s = 0$. Figure 6.1 sketches how the behavior patterns of the differences J_n^-, R_n^- of section 4, resp. section 5, correspond to the points of the s-axis $p = 0$ (resp. of the p-axis $s = 0$) and carry over to the hybrid cases. For the interpretation of the different monotone, oscillatory and alternating regions see Figures 4.1 and 5.1.

A small geometrical, one might say "gender specific", difference is perhaps worth noting. Each horizontal line (except $p = 50\%$) intersects the Turing instability region – but no vertical line with $25\% < s < 75\%$. Thus the brother-siblings alone, as well as the sister-siblings, can affect the total system through a clever *choice* of their respective parameters p resp. s. Moreover, the sisters, by an unclever choice, too large or too small, of their gossip susceptibility s, can maneuver both relationships into instability, provided $p \neq 50\%$: femmes fatales. Conversely, smart sisters have it in their power to stabilize both relationships beneficially, through the choice $25\% < s < 75\%$, a healthy – but not exaggerated – gossipiness. The philandering brothers may then, with $0 < p < 100\%$, brag as they please: The smart sisters have indestructibly stabilized both their relationships.

Every theorem requires a proof, as everyone has known since Euclid's geometry at least – otherwise it has no claim to mathematical validity. We have not proved Turing's theorem here – we have phrased it too vaguely, and within the amatory confines of this article, as already indicated, we lack the mathematical tools, by far. For an indication of the mathematical framework, though not the proof, we refer to section 7. Turing's groundbreaking achievement lay, however, less in the computations involved in proving Theorem 6.1. The breakthrough of this genius of the century lay much more in the profound insight that this mechanism of instability is, paradoxically, mathematically at all possible, and might serve to elucidate pattern formation and morphogenesis – against all entropically leveling forces such as gossiping, bragging, etc.

Tragically, the year 1952 marked not only this last scientific triumph in the life of Alan Turing, but also the year of his social ostracism. See for example the brief account in [**7**]. He was arrested for "gross indecency" as a homosexual, and brought to trial. Sentenced to psychoanalysis and hormone "treatment", he committed suicide in 1954.

7. Mathematical summary

For the more mathematically inclined reader we give a short synopsis of the mechanism that underlies Turing's instability and the adaptation sketched above. The mathematically less ambitious may safely skip this section for now.

A linear iteration

$$(7.1) \qquad \begin{aligned} \bar{z}_{n+1} &= A z_n \\ \bar{z}'_{n+1} &= A z'_n, \end{aligned} \qquad A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

for $z_n \in \mathbb{R}^2$ was introduced in sections 2 and 3. The components of $z_n = (J_n, R_n)$ were called Juliet and Romeo from then on. The matrix A has eigenvalues $\exp(\pm i\pi/3)$, which are sixth roots of unity. Hence the period six of all z_n.

Gossiping, resp. bragging, follows the equations

$$\begin{aligned} z_{n+1} &= \bar{z}_{n+1} + D(\bar{z}'_{n+1} - \bar{z}_{n+1}) \\ z'_{n+1} &= \bar{z}'_{n+1} + D(\bar{z}_{n+1} - \bar{z}'_{n+1}), \end{aligned} \qquad D = \begin{pmatrix} s & 0 \\ 0 & p \end{pmatrix}. \tag{7.2}$$

Together, (7.1) and (7.2) imply

$$\begin{aligned} z_{n+1} &= Az_n + DA(z'_n - z_n), \\ z'_{n+1} &= Az'_n + DA(z_n - z'_n)\,. \end{aligned} \tag{7.3}$$

The manifest linear $\mathbb{Z}_2$-symmetry of the linear "twins"-iteration of $(z_n, z'_n) \in \mathbb{R}^4$ decomposes into representations $\pm\mathrm{id}$ via the coordinates

$$z_n^{\pm} := \frac{1}{2}(z_n \pm z'_n). \tag{7.4}$$

Thus (7.3) uncouples to

$$\begin{aligned} z_{n+1}^{+} &= Az_n^{+} \\ z_{n+1}^{-} &= (\mathrm{id} - 2D)Az_n^{-}\,. \end{aligned} \tag{7.5}$$

The homogeneous part z_n^+ knows nothing of D. The difference part z_{n+1}^- – this is the essence of Turing's idea – can be unstable, even though A itself and $(\mathrm{id} - 2D)$ are each stable.

It is well known that the stability of a matrix iteration (with simple eigenvalues) is determined by the spectrum of the matrix. Stability then means that all eigenvalues lie in the (closed) complex unit disk in $\mathbb{C}$. Instability sets in as soon as at least one eigenvalue lies outside the unit disk. From the stability of A and $(\mathrm{id} - 2D)$ it does *not* follow that the product of the two matrices is stable. This is the very essence of Turing's idea. In section 4 instability was generated by an eigenvalue $\mu < -1$ of the matrix product, and in section 5 by $\mu > 1$.

This idea is of course not confined to our special choices of A, D – nor to our wrapping parable – nor to $z_n \in \mathbb{R}^2$. Nonlinear systems can be described as well, and globally, together with spatio-temporal pattern forming bifurcations which arise from the germ of a Turing instability; see, for example, the author's work [**1**] mentioned in the list of references.

Turing was actually interested in reaction-diffusion equations of the form

$$\partial_t z = D\Delta_x z + Az \tag{7.6}$$

Here $z = (z_1, \ldots, z_n) \in \mathbb{R}^n, x \in \Omega \subset \mathbb{R}^N$, Δ_x is the Laplace operator with appropriate boundary conditions on Ω, and $z = z(t, x)$ is the pattern forming solution in time t and space x that we seek. The diffusion matrix D is positive diagonal, as in our example, and A is an almost arbitrary $n \times n$-matrix which describes the linearization of some "reactions". For $A = 0$ this provides the entropically leveling influence of the uncoupled heat equation for the components $z_1, \ldots, z_n$. Stability is also assumed for $D = 0$:

$$\mathrm{Re\ spec}\, A < 0\,; \tag{7.7}$$

the eigenvalues of A should have negative real part. Now let $\lambda > 0$ be an eigenvalue of $-\Delta_x$ on Ω, corresponding to a "resonance" of the region Ω. It does *not* follow that the stability condition

$$\operatorname{Re}\operatorname{spec}\,(-\lambda D + A) \leq 0 \tag{7.8}$$

holds. In particular the interaction of the reaction matrix A with the diffusion matrix $D \neq \mathrm{id}$ may lead to instability of the individual z-components, that is, to Turing instability.

In our essay this result has been reduced to its simplest possible form: A region Ω = two points, the two sibling pairs, and discrete time. In this way we have circumvented even that last vestige of differential calculus which is genuinely innate to any culture of the continuous.

8. Outlook

Murray's eminent book [**10**] teems with inspiring illustrations of conjectured effects of Turing instability in animals: see Figure 8.1. The mathematician does

FIGURE 8.1. A Turing goat: courtesy of [**10**].

not flinch at renaming Romeo/Juliet (resp. Roberto/Julietta) to become the rear- (resp. front-) ends of a goat – instead of J_n and R_n one can employ other letters and circumstances without further ado. The modeling can then describe completely different interdependencies than the chemistry of love affinity: The abstract possibility of a Turing instability persists. The instability of section 5, where Romeo/Juliet float away into their realm of bliss, while Roberto/Julietta plunge into the perdition of divorce, expresses itself less dramatically, but all the same visibly, in a white rear- (resp. black front-) end.

carbon monoxide oxidation on Pt; [Ertl et al](1998)

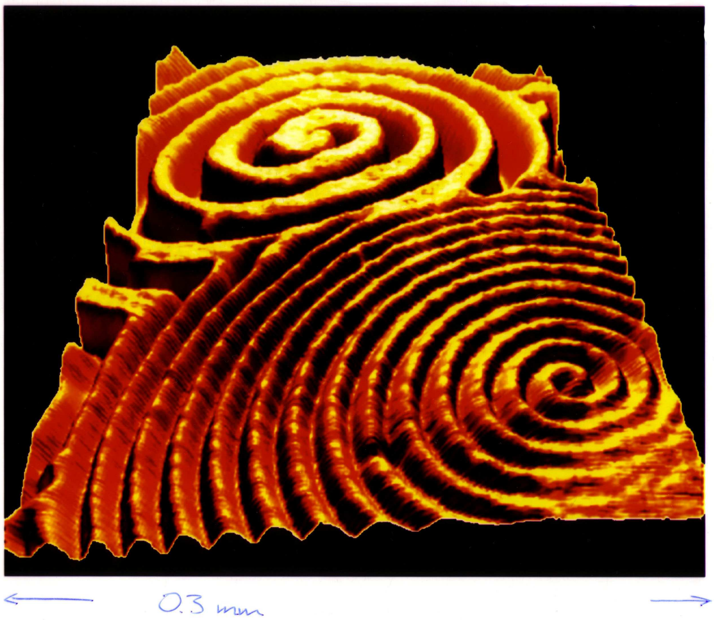

Figure 8.2. Snapshot of rotating oxidation pattern on a platinum catalyst, resolved by photo electron emission microscopy. Object size 300μm (Fritz-Haber-Institut der Max-Planck-Gesellschaft, Berlin)

Whimsical? The exact molecular biological basis is being contested. Solid molecular evidence is required, for reactions and interactions, to substantiate Turing's far-reaching dream on morphogenesis, way beyond mere mathematics. Lifetime achievements of Christiane Nüsslein-Volhard on drosophila, and Chica Schaller on hydra, should at least be mentioned here. The chemistry of time oscillating reactions and catalysis on platinum surfaces also provides well understood and modeled patterns, very close to Turing's ideas, and even temporally variable; see Figure 8.2. For comprehensive ongoing research material, above all from physics and chemistry, see the work of Gerhard Ertl and his group, for example, and also the series published by Hermann Haken [**5**], and the surveys and collections in [**2**], [**3**], [**4**].

Meinhardt and Gierer have carried out computer simulations, over many years, in the framework of *activator inhibitor systems*. On the one hand these seek to simulate biological context, and on the other hand they demonstrate Turing's idea impressively. Meinhardt's wonderful book [**9**] abounds with computer simulations of real patterns on sea shells, that reflect the temporal development $z(t, x)$ of concentrations in nonlinear reaction-diffusion systems with Turing instabilities; see Figure 8.3. The book is prefaced with a quote from "Doctor Faustus" by Thomas Mann [**8**]. The father, Jonathan, of Adrian Leverkühn who later becomes a composer and signs a contract with the Devil, ponders the diversity of patterns in his snail collection "to speculate the elements":

FIGURE 8.3. Photograph (left) and mathematical simulation (right) of *Oliva pophyria*; see [**9**]

> "It has turned out to be impossible to get at the meaning of these marks. Unfortunately, my dears, such is the case. They refuse themselves to our understanding, and will, painfully enough, continue to do so. But when I say refuse, that is merely the negative of reveal – and that Nature painted these ciphers, to which we lack the key, merely for ornament on the shell of her creature, nobody can persuade me. Ornament and meaning always run along each other; the old writings too served for both ornament and communication. Nobody can tell me that there is nothing communicated here. That it is an inaccessible communication, to plunge into this contradiction, is also a pleasure."

And Thomas Mann himself, alias Serenus Zeitblom, muses on:

> Did he think, if it were really a case of secret writing, that Nature must command a language born and organized out of her own self? For what man-invented one should she choose, to express herself in?

Only five years after the appearance of these lines Alan Turing may have unlocked this cipher of self-organization. Maybe, and certainly only partially so – but in the universal language of mathematics.

Acknowledgement. The author is much indebted to Nitsan Ben-Gal for enhancing this translation to a level which surpasses the German original.

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