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ROOTS AND CENTRALIZERS OF ANOSOV DIFFEOMORPHISMS ON TORI

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Dedicated to Leonid Shilnikov on the occasion of his 70th birthday

The centralizer of a group element g is the group of elements commuting with g. Powers and, if existent, roots of g are contained in the centralizer. We call a centralizer trivial if it consists of the integer powers of g, only.

In the group of torus homeomorphisms, we study the centralizer of torus diffeomorphisms of hyperbolic Anosov type. As a result, the centralizer can be calculated, isomorphically, in the much smaller group of affine torus automorphisms. In particular, Anosov diffeomorphisms of the 2-torus with a unique fixed point possess trivial centralizer, up to a trivial involution.

The identification of individual reactors from data on reactor cascades, for example in chemical engineering, is one source of motivation for the study of roots of diffeomorphisms. Another possible source is the study of commuting diffeomorphisms in finite-dimensional spatially or spatio-temporally chaotic systems.

Keywords: Au, pls. provide!

1. Introduction

Let G denote a group and let $g \in G$ denote any element of G. The *centralizer* $C(g)$ or, more precisely, $C_G(g)$ consists of all group elements $\gamma \in G$ commuting with g:

$$C_G(g) = \{\gamma \in G; \gamma g = g\gamma\}. \tag{1}$$

Obviously $C(g)$ is a subgroup of G. In fact, $C(g)$ is the isotropy group, or stabilizer, of g under the conjugation action of $\gamma \in G$ on itself: $\gamma g \gamma^{-1} = g$.

We call $\gamma \in G$ a *power* of $g \in G$, if $\gamma = g^n$ for some integer exponent $n \in \mathbb{Z}$. We call $\gamma \in G$ a *root* of $g \in G$, if $g = \gamma^m$ for some integer exponent $m \in \mathbb{Z}$. Clearly, powers and roots of g commute with g and hence are contained in the centralizer $C(g)$.

In the present note we compute the centralizers, and in particular, the roots of C^1 Anosov diffeomorphisms φ on the 2-torus $T^N = \mathbb{R}^N/\mathbb{Z}^N$. See [Hasselblatt & Katok, 2002] for a broad survey of Anosov maps and the contemporary theory of hyperbolic dynamics. In a C^κ genericity context, which we do not pursue here, Smale has raised the centralizer question because "*. . . it gives some focus in the dark realm, beyond hyperbolicity, where even the problems are hard to pose clearly.*" See [Smale, 1967, 1991, 2000]. More precisely, Smale asked whether centralizers of diffeomorphisms φ of compact manifolds are trivial, for generic φ. Here trivial means that the centralizer of φ consists of the integer powers of φ, only. For further discussion of this question and some more existing literature see Sec. 4.

We lift the Anosov diffeomorphism φ to a C^1 map $\overline{\varphi}$ on the covering space $\mathbb{R}^N$. Then

$$\overline{\varphi}(x + \mathbf{n}) = \overline{\varphi}(x) + A_\varphi \mathbf{n} \tag{2}$$

for all $\mathbf{n} \in \mathbb{Z}^N$. The period matrix A_φ has integer entries, determinant ± 1 by invertibility of φ, and is the map induced by φ on first homology. In other words, $A_\varphi \in L_N(\mathbb{Z})$, where the group $L_N(\mathbb{Z})$ of integer $N \times N$ matrices with integer inverse coincides with $SL_N(\mathbb{Z})$ except for admitting determinants ± 1.

Due to the hyperbolic structure of Anosov diffeomorphisms φ, their period matrix A_φ is also hyperbolic: the spectrum of A_φ is disjoint from the complex unit circle [Franks, 1970]. It has been proved by [Franks, 1970; Manning, 1974] that any C^1 Anosov diffeomorphisms φ is in fact C^0 conjugate by a C^0 homeomorphism H of T^N to its "linearization" A_φ:

$$A_\varphi = H \circ \varphi \circ H^{-1}. \tag{3}$$

It is therefore worthwhile to consider linear automorphisms B of T^N commuting with A_φ in the linear group $L_N(\mathbb{Z})$, that is $B \in C_{L_N}(A_\varphi)$. We also consider the finite fixed point set

$$\mathrm{Fix}(A_\varphi) := \{x \in T^N; Ax \equiv x (\mathrm{mod}\ \mathbb{Z})\} \tag{4}$$

as a finite Abelian group with respect to addition (mod $\mathbb{Z}$). We will then view the semidirect product

$$C_{L_N(\mathbb{Z})}(A_\varphi) \ltimes \mathrm{Fix}(A_\varphi) \tag{5}$$

as a subgroup of the semidirect product

$$AL_N(\mathbb{Z}) := L_N(\mathbb{Z}) \ltimes T^N \tag{6}$$

of affine linear torus maps. Composition of elements (B_j, τ_j) in $AL_N(\mathbb{Z})$ is defined by

$$(B_1, \tau_1)(B_2, \tau_2) := (B_1 B_2, \tau_1 + B_1 \tau_2). \tag{7}$$

We can now state our main result.

Theorem 1.1. *In the group $\mathcal{H}(T^N)$ of torus homeomorphisms, the centralizer of any C^1 Anosov diffeomorphism $\varphi: T^N \to T^N$ is isomorphic to the centralizer of its period matrix $(A_\varphi, 0)$ in the affine linear torus group $AL_N(\mathbb{Z})$. More precisely*

$$\begin{aligned} C_{\mathcal{H}(T^N)}(\varphi) &\cong C_{AL_N(\mathbb{Z})}((A_\varphi, 0)) \\ &= C_{L_N(\mathbb{Z})}(A_\varphi) \ltimes \mathrm{Fix}(A_\varphi). \end{aligned} \tag{8}$$

For centralizers in $SL_N(\mathbb{Z})$ see [Plykin, 1998]. For the 2-torus it is particularly easy to describe the centralizer $C_{L_2(\mathbb{Z})}(A_\varphi)$ in the group $L_2(\mathbb{Z})$ of integer period matrices of determinant ± 1.

Proposition 1.2. *Let $A \in L_2(\mathbb{Z})$ be hyperbolic. Then*

$$\frac{C_{L_2(\mathbb{Z})}(A)}{\{\pm id\}} \cong \mathbb{Z} \tag{9}$$

In particular, we see that the centralizer is generated by powers of a single "primitive" root of A, up to the trivial involution $-id$. Note that the centralizer in $SL_2\mathbb{Z}$ also satisfies (9), being a subgroup of the centralizer in $L_2(\mathbb{Z})$.

We now exclude nontrivial orientation preserving roots for orientation preserving Anosov diffeomorphisms φ of the 2-torus with unique fixed points. Let $S\mathcal{H}(T^2)$, in analogy to $SL_2(\mathbb{Z})$, denote the orientation preserving homeomorphisms of the 2-torus. Note $A_\psi \in SL_2(\mathbb{Z})$ for the period matrix of any $\psi \in S\mathcal{H}(T^2)$. In view of Proposition 1.2 we lift $-id$ back to $S\mathcal{H}(T^2)$, via the linearization H of φ in (3), as a unique involution

$$\kappa := H^{-1} \circ (-id) \circ H \in S\mathcal{H}(T^2). \tag{10}$$

Absence of roots of φ can then be formulated as follows.

Corollary 1.3. *Let φ be an orientation preserving C^1 Anosov diffeomorphism of the 2-torus with a unique fixed point. Let the orientation preserving torus homeomorphism ψ commute with φ. Then ψ commutes with the involution κ associated to φ, and either ψ itself or $\kappa\psi$ is an integer power of φ.*

The orientation condition on ψ is necessary for this result. Indeed the standard Anosov matrix, φ possesses a unique fixed point. But $\varphi = \psi^2$ also possesses a root: choose ψ to be the Fibonacci generator

$$\psi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

of determinant -1.

In Sec. 2 we prove Theorem 1.1. Proposition 1.2 and Corollary 1.3 are proved in Sec. 3. We conclude with some remarks including possible applications of the root problem, in Sec. 4. In particular, we contrast the Anosov case, as discussed here and elsewhere in the literature, with previous results by Belitskii and Tkachenko concerning diffeomorphisms of the interval.

2. Proof of Theorem 1.1

Conjugate group elements possess conjugate centralizers. To prove Theorem 1.1 we therefore follow [Franks, 1970; Manning, 1974] and linearize the Anosov diffeomorphism φ to be given by its integer invertible period matrix $A = A_\varphi \in L_N(\mathbb{Z})$; see (3).

A priori it is not sufficient to determine the centralizer of A in $AL_N(\mathbb{Z})$ itself. We still claim, and show below, that additional centralizing elements of A_φ in $\mathcal{H}(T^N)$ do not arise. In [Palis & Yoccoz, 1989a], a similar observation has been made under the additional assumption that ψ and φ share a common fixed point. We do not impose any such restriction: let ψ be any torus homeomorphism.

Passing to the covering space, as in (2), and skipping bars (!) we write

$$\psi(x+\mathbf{n}) = \psi(x) + B\mathbf{n} \tag{12}$$

with period matrix $B = A_\psi \in L_N(\mathbb{Z})$. In other words,

$$\psi(x) = Bx + h(x) \tag{13}$$

with a continuous function h, periodic under all lattice vectors $\mathbf{n} \in \mathbb{Z}^N$:

$$h(x+\mathbf{n}) = h(x), \tag{14}$$

for all $x \in \mathbb{R}^N$.

Now suppose the torus homeomorphism ψ commutes with the Anosov diffeomorphism $\varphi = A$. Then there exists some $\mathbf{n}_0 \in \mathbb{Z}^N$ such that

$$\psi(Ax) = A\psi(x) + \mathbf{n}_0, \tag{15}$$

for all x. In terms of h from (13) this is equivalent to the homologous equation

$$BAx + h(Ax) = ABx + Ah(x) + \mathbf{n}_0. \tag{16}$$

The periodic continuous function h is bounded. Letting $|x| \to \infty$ in (16) therefore implies commutativity

$$AB = BA \tag{17}$$

for the period matrices. Since A is hyperbolic, we can invert $(id - A)$ and define

$$\xi := (id - A)^{-1}\mathbf{n}_0 \in \mathrm{Fix}(A) \tag{18}$$

Indeed $\xi (\mathrm{mod}\ \mathbb{Z})$ defines a fixed point of $A\colon T^N \to T^N$. Returning to the covering space $\mathbb{R}^N$, we note that $\tilde{h}(x) := h(x) - \xi$ satisfies

$$\tilde{h}(Ax) = A\tilde{h}(x). \tag{19}$$

The periodic continuous function $\tilde{h}$ is bounded and the linear Anosov map A is hyperbolic. Therefore $\tilde{h} \equiv 0$ and $h(x) \equiv \xi \in \mathrm{Fix}(A)$ is a pure translation. In particular (13) implies

$$\psi = (B, \xi) \in C_{L_N(\mathbb{Z})}(A_\varphi) \ltimes \mathrm{Fix}(A_\varphi). \tag{20}$$

Conversely, any such (linearized) ψ commutes with φ. The isomorphism (8) claimed in Theorem 1.1 is simply defined by conjugation with the linearizing homeomorphism H of $\varphi = H^{-1}(A_\varphi, 0)H$.

Repeating the above arguments in the affine linear group $AL_N(\mathbb{Z})$ it is obvious that $C_{L_N(\mathbb{Z})}(A_\varphi) \ltimes \mathrm{Fix}(A_\varphi)$ is indeed the centralizer of $(A_\varphi, 0)$ in $AL_N(\mathbb{Z})$.

This proves Theorem 1.1.

3. Linear Centralizers and Unique Fixed Points

In this section we prove Proposition 1.2 and Corollary 1.3.

Proof of Proposition 1.2. Although our argument essentially follows [Plykin, 1998], where the higher-dimensional case was also addressed, we include a completely elementary proof "from scratch", at least for the case $\det A = +1$. We leave the case $\det A = -1$ as an exercise to the reader.

Let $A \in SL_2(\mathbb{Z})$ be hyperbolic. Any matrix B commuting with A shares its eigenvectors with A. Let $\lambda(B)$ denote the eigenvalue of B along the expanding eigendirection of A. Then the logarithm

$$\begin{aligned} \log\colon C_{L_2(\mathbb{Z})}(A) &\to \mathbb{R} \\ B &\mapsto \log|\lambda(B)| \end{aligned} \tag{21}$$

is a group homomorphism to the additive reals. To complete the proof it only remains to show that the image of the log homomorphism (21) is discrete, and

$$\ker \log = \{\pm id\}. \tag{22}$$

To show the image is discrete let $\tau_\pm(\lambda) := \lambda \pm 1/\lambda$. Then the eigenvalues $\lambda(B)$ satisfy

$$\tau_{\det B}(\lambda(B)) = \mathrm{tr}(B) \in \mathbb{Z}, \tag{23}$$

for any $B \in L_2(\mathbb{Z})$. Therefore $\log|\lambda(B)| \in \log|\tau_\pm^{-1}(\mathbb{Z})|$ is indeed discrete-valued.

To show the kernel is given by (22) we note that the expanding eigenvalue $\lambda(A)$ of the hyperbolic matrix $A \in SL_2(\mathbb{Z})$ is irrational. Indeed

$$\lambda(A) = \frac{1}{2}(\mathrm{tr}\, A + \sqrt{|\mathrm{tr}\, A|^2 - 4}) \notin \mathbb{Q} \tag{24}$$

because $|\mathrm{tr}\, A| \geq 3$ for real $|\lambda| \neq 1$. Because A is integer, this implies $\eta \notin \mathbb{Q}$ for the associated eigenvector $(1, \eta)$ of A. Now suppose $B \in \ker \log$. Then

$$B\begin{pmatrix}1\\ \eta\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}1\\ \eta\end{pmatrix} = \pm\begin{pmatrix}1\\ \eta\end{pmatrix}, \tag{25}$$

with integer b_{ij} and irrational η. This implies $b_{12} = b_{21} = 0$ and $b_{11} = b_{22} = \pm 1$, proving the proposition. ■

Unlike the two-dimensional result of Proposition 1.2, centralizers of hyperbolic matrices in $L_N(\mathbb{Z})$ may contain components $\mathbb{Z} \oplus \mathbb{Z}$ in dimensions $N \geq 3$; see [Plykin, 1998].

To prove Corollary 1.3 we first formulate a lemma counting fixed points of Anosov diffeomorphisms.

Lemma 3.1. *Let $A = A_\varphi \in L_N(\mathbb{Z})$ denote the period matrix of a C^1 Anosov diffeomorphism φ of T^N. Then the number of fixed points of φ is given by*

$$\# \operatorname{Fix}(\varphi) = \# \operatorname{Fix}(A) = |\det(id - A)|. \tag{26}$$

For 2-tori, $N = 2$, we have

$$\# \operatorname{Fix}(\varphi) = |\operatorname{tr} A - 2|, \tag{27}$$

in the orientation preserving case $\det A = +1$.

Proof. Part 2, (27), follows from (26) in the planar case because $\# \operatorname{Fix}(\varphi) = |p_A(1)|$ for the characteristic polynomial

$$p_A(\lambda) := \det(\lambda - A) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A. \tag{28}$$

This proves (27), up to sign. As in (24), we have $|\operatorname{tr}(A)| \geq 3$ for hyperbolic $A \in SL_2(\mathbb{Z})$. This proves the choice of sign in (27).

To count fixed points, as in (26), we note that the fixed point equation

$$(id - A)\xi = \mathbf{n}_0 \in \mathbb{Z}^N \tag{29}$$

from (18) associates to each fixed point $\xi \in T^N = \mathbb{R}^N/\mathbb{Z}^N$ a fundamental volume cell of size $|\det(id - A)|^{-1}$ in the lattice $(id - A)^{-1}\mathbb{Z}^N$. Since these volumes add up to fill the N-torus of volume 1, the Anosov diffeomorphism φ possesses $|\det(id - A)|$ fixed points. This proves (26) and the lemma. ■

As an aside we note that $\det(id - A)$ coincides with the Lefschetz number of φ and $|\det(id - A)|$ is the Nielsen number; see [Brooks *et al.*, 1975].

Proof of Corollary 1.3. Let φ be orientation preserving, C^1, and Anosov on T^2 with hyperbolic period matrix $A = A_\varphi \in SL_2(\mathbb{Z})$. By Theorem 1.1 any commuting orientation preserving homeomorphism $\varphi \in C_{\mathcal{H}(T^2)}(\varphi)$ linearizes to $(B, 0)$, where $B = A_\psi \in C_{SL_2(\mathbb{Z})}(A)$ is the period matrix of ψ. Indeed, the translation component in $\operatorname{Fix}(A_\varphi) = \{0\}$ vanishes, because φ and A_φ possess only one fixed point. It only remains to show that B, or $-B$, is a power of A. By Proposition 1.2, we have $C_{SL_2(\mathbb{Z})}(A)/\{\pm id\} \cong \mathbb{Z}$ via the log homomorphism (21). In (23)–(25) we have observed that the lower bound for nonzero $|\log|\lambda(B)||$ is attained at trace = 3. It is therefore sufficient to show

$$\operatorname{tr} A = 3 \tag{30}$$

This follows from Lemma 3.1, (27), because hyperbolicity of A implies $|\operatorname{tr} A| \geq 3$, and because φ was assumed to possess only one fixed point. This proves Corollary 1.3. ■

4. Remarks and Applications

We compare the above simple results on C^0 centralizers and roots of Anosov diffeomorphisms with some previous results in the literature. We also discuss differentiability issues for Anosov results and their connection with Belitskii linearization. We conclude by indicating how the root problem is related to an inverse problem arising in chemical engineering, and how the centralizer problem might pertain to the analysis of spatially chaotic two-dimensional patterns.

A pioneering result by Kopell [1970], asserts that generic C^1 circle diffeomorphisms φ possess trivial centralizers: only integer powers of φ commute with φ. We consider, instead, the even simpler case of an interval. Let φ be a C^2 diffeomorphism of the unit interval fixing, for simplicity, 0 and 1. Let $\mathcal{D}^2([0,1])$ denote the group of such orientation preserving diffeomorphisms. The C^0 case of orientation preserving homeomorphisms is denoted, analogously, by $\mathcal{D}^0([0,1])$. The C^κ and C^∞ cases are denoted by $\mathcal{D}^\kappa$, $\mathcal{D}^\infty$. For simplicity we also assume that Fix $(\varphi) = \{0, 1\}$ are the only fixed points of φ and are both hyperbolic.

Fix any $m \in \mathbb{N}$. Then any $\varphi \in \mathcal{D}^\infty([0,1])$ possesses a large class of roots $\psi \in \mathcal{D}^0$ which satisfy

$$\psi^m(x) = \varphi(x). \tag{31}$$

Indeed fix any $x_0 \in (0,1)$, $x_m := \varphi(x_0)$. Subdividing the interval from x_0 to x_m by a strictly monotone sequence $x_0, x_1, \ldots, x_{m-1}, x_m$, arbitrarily, we may choose any homeomorphism ψ which maps each subinterval i to subinterval $i + 1$, for $i = 0, \ldots, m - 2$. We can then use (31) to uniquely extend ψ, not only to the last subinterval $m - 1$, between x_{m-1} and x_m, but to all of (0,1), such that (31) holds. Extending ψ continuously to fix 0 and 1 then ensures $\psi \in \mathcal{H}^0$ is a root (31).

For diffeomorphisms ψ the situation changes drastically. Let $0 < a_0 := \varphi'(0) < 1$. Following

[Belitskii & Tkachenko, 2003] we may C^κ linearize $\varphi \in \mathcal{D}^\kappa$, $\kappa \geq 2$, at $x = 0$ by a diffeomorphism

$$H_0\colon [0,1) \to [0,\infty), \quad H_0(0) = 0, \tag{32}$$

such that

$$H_0 \varphi H_0^{-1}(y) = a_0 y \tag{33}$$

for all $y \geq 0$. Note that $H_0 \in C^\kappa$ is unique, up to a positive factor σ_0 multiplying H_0. At $x = 1$, $\varphi'(1) = a_1 > 1$ we linearize analogously by $\sigma_1 H_1$ such that

$$H_1\colon (0,1] \to (-\infty,0], \quad H_1(1) = 0, \tag{34}$$

$$H_1 \varphi H_1^{-1}(y) = a_1 y. \tag{35}$$

We then define the Belitskii invariant $\mu = \mu_\varphi(\tau)$ by

$$\mu_\varphi(\tau) := -\tau + \frac{1}{\log a_0} \log(H_0 \circ H_1^{-1})(-a_1^\tau). \tag{36}$$

Note that $\mu \in C^\kappa$ is unique up to constants

$$c_0 + \mu(\tau - c_1) \tag{37}$$

given by the scaling factors σ_i of the linearizations at $c_i = \log \sigma_i / \log a_i$. Moreover

$$\mu_\varphi(\tau + 1) = \mu_\varphi(\tau) \tag{38}$$

is periodic. It is therefore easy to eliminate the constants c_i, for example, by requiring $\int_0^1 \mu = 0$ and $\mu(0) = 0$. With these normalizations we easily observe

$$\mu_{\varphi^m}(\tau) = \frac{1}{m}\mu_\varphi(m\tau) \tag{39}$$

$$\mu_{\psi\varphi\psi^{-1}}(\tau) = \mu_\varphi(\tau) \tag{40}$$

$$\mu_{\varphi^{-1}}(\tau) = -\mu_\varphi(-\tau) \tag{41}$$

for all $\tau \in \mathbb{R}$, φ as above, $\psi \in \mathcal{D}^\kappa$. In particular μ is a conjugation invariant. More precisely, orientation preserving interval diffeomorphisms in $\mathcal{D}^\kappa$ are C^κ conjugate if, and only if, their invariants coincide. Reversible diffeomorphisms φ possess odd invariants μ. By (39) applied to roots $\psi^m = \varphi$, periodicity of μ_ψ immediately implies that

$$\mu_\varphi(s) = \mu_{\psi^m}(s) = \frac{1}{m}\mu_\psi(ms) \tag{42}$$

possesses a period $1/m$, rather than just 1. This shows that interval diffeomorphisms φ rarely possess roots. In fact, φ possesses an mth root in $\mathcal{D}^2$ if, and only if, $1/m$ is a (not necessarily minimal) period of μ_φ. Similarly, φ can be written as a time-1 map of a flow Φ^t on the interval $x \in [0,1]$ if, and only if, $\mu_\varphi \equiv 0$. In particular, $\mu_\varphi \equiv 0$ if, and only if, φ is conjugate to the time-1 map Φ^1 of the flow of the standard model

$$\dot{x} = x(1 - x). \tag{43}$$

See [Belitskii & Tkachenko, 2003] for further details, as well as the tutorial [Fiedler, 2005].

It is also easy to determine the centralizer $C_{\mathcal{D}^\kappa([0,1])}(\varphi)$. If $\mu_\varphi \equiv 0$, the centralizer is given by some C^κ flow conjugate to Φ^t. The conjugator itself is determined by φ and Φ^1. In particular,

$$C_{\mathcal{D}^\kappa([0,1])}(\varphi) \cong \mathbb{R}, \quad \text{for } \mu \equiv 0. \tag{44}$$

If μ_φ is nonconstant, then

$$C_{\mathcal{D}^\kappa([0,1])}(\varphi) \cong \mathbb{Z}. \tag{45}$$

A generator is given by an mth root $\psi \in C^\kappa$ of φ, where $1/m$ is the minimal period of μ_φ. The centralizer only consists of integer powers of φ if, and only if, 1 is the minimal period of μ_φ. See [Fiedler, 2005] for details. Similar arguments apply to circle diffeomorphisms and recover, after decades, Kopell's result on generic triviality of centralizers [Kopell, 1970].

In contrast to the two-dimensional Anosov case where C^0 roots are scarce and centralizers are finitely generated, with only one $\mathbb{Z}$ component, the very nonchaotic interval case possesses an abundance of C^0 roots. In class C^2 and higher, roots are scarce on the interval.

For the centralizer question in the context of C^3 Morse–Smale diffeomorphisms φ on compact manifolds see [Anderson, 1976]. See [Shilnikov *et al.*, 2001] for a background on Morse–Smale systems. Generically, centralizers are then discrete: C^0 neighborhoods of *id* in the centralizer contain *id*, only. Moreover, absence of roots is a C^∞ dense property, and hence C^κ dense for any $\kappa \geq 1$. From our view point, the above results on the interval case can be used in this context: we simply restrict φ to a one-dimensional C^2 heteroclinic orbit in the Morse–Smale system.

We now return to the Anosov case $\varphi \in C^2(T^2)$ and ask for centralizers ψ of φ in the group $\mathcal{D}^2(T^2)$ of C^2 torus diffeomorphisms. Suppose ψ is a root of $\varphi = \psi^m$. Also suppose, for simplicity, that both ψ and φ fix $x = 0$. By continuous global linearization (3) on T^2, the homoclinic points to $x = 0$ of ψ are dense in T^2 and coincide with those of φ. More precisely, the homoclinic set $W^u \cap W^s$ of the intersections of the C^2 stable and unstable manifolds W^u, W^s of $x = 0$ under φ, ψ is dense both in W^u and W^s. The Belitskii invariants $\mu_\varphi(\tau)$ and

$\mu_\psi(\tau)$ are therefore defined on a dense homoclinic subset

$$\tau \in \Gamma := \frac{1}{\log a_u} \log(-H_u(W^u \cap W^s\{0\})), \quad (46)$$

by (36). Here we replace a_0, a_1, H_0, H_1 by eigenvalues $0 < a_s < 1 < a_u$ and the Belitskii linearizations H_u, H_s on W^u, W^s, respectively. As in (38), the existence of a C^2 root $\varphi = \psi^m$ then requires μ_φ to possess period $1/m$, rather than just 1. In addition to this obstacle which, generally, excludes the presence of nontrivial C^2 roots, it now becomes a nontrivial geometric requirement that the dense domain $\Gamma \subseteq \mathbb{R}$ of definition of μ in (46), (36) has to be invariant under shift by $1/m$. Typically then, along these lines, we see how roots are excluded and centralizers reduce to mere trivial powers in the C^2 case.

For smooth Anosov diffeomorphisms φ on the N-torus, Palis and Yoccoz have in fact answered Smale's question affirmatively: for an open dense set of such φ, smooth centralizers $C(\varphi)$ are trivial, [Palis & Yoccoz, 1989a]. Generic smooth Anosov diffeomorphisms on compact manifolds, likewise, posses only trivial smooth centralizers. Moreover, for an open dense set of smooth Anosov φ, all solutions φ in $C(\varphi)$ of the generalized root equation $\varphi^m = \varphi^n$ are trivial; see [Palis & Yoccoz, 1989b]. Katok excludes such roots for sufficiently smooth φ on 2-manifolds with hyperbolic invariant measures, [Katok, 1996]. Hyperbolicity conditions which ensure discrete centralizer, generically, have been derived and surveyed by Burslem [2004]. This includes some results on trivial centralizers $C(\varphi) \cong \mathbb{R}$ for time 1 maps φ of Anosov flows. We note, however, that all these very general and beautiful results only address centralizers with high regularity. Our above results, in contrast, determine centralizers in the much larger group $\mathcal{H}(T^N)$ of torus homeomorphisms.

Even more degenerate, of course, is the rare case of global C^1 linearizability $H \in C^1$ in (3). In this case the analysis of roots and centralizes of the preceding chapters applies verbatim. This case, however, requires all Floquet exponents on the dense set of periodic orbits to coincide with the eigenvalues of the period matrix. Such a constraint imposes so much rigidity that we barely dare mention it.

In between the two extreme cases of simply heteroclinic Morse–Smale dynamics and, on the other hand, uniformly hyperbolic Anosov diffeomorphisms the pioneering work of Leonid P. Shilnihov and his school, as well as a broad international community, have partially untangled the enormously rich and complex phenomena associated with homoclinicity and its many variants. See [Shilnikov *et al.*, 1998, 2001] for a most comprehensive survey.

To be more specific, consider a planar diffeomorphism φ with hyperbolic fixed point $\varphi(0) = 0$ and everywhere transverse stable/unstable manifolds

$$W^u \pitchfork W^s. \quad (47)$$

Homoclinic orbits $x_k \in W^u \cap W^s \backslash \{0\}$ can then be classified as *1-homoclinic*, *2-homoclinic*, etc. as follows. Consider the open segment $(0, x_0)^u \subset W^u$ from 0 to x_0 in W^u and, similarly, $(x_0, 0)^s \subset W^s$ from x_0 to 0 in W^s. For x_0 to be 1-homoclinic, or of (homoclinic) type 1, we require the open segments to possess empty intersection. For 2-homoclinic orbits we require all intersection points to be of type 1, that is, 1-homoclinic. If the homoclinic point x_0 is not of any type less than n, but

$$\text{types}\left((0, x_0)^u \cap (x_0, 0)^s\right) \leq n - 1, \quad (48)$$

then we call x_0 and its orbit n-homoclinic. Applying the diffeomorphism φ, or φ^{-1}, we see that this definition does not depend on the choice of the representative x_0 of the homoclinic orbit. Denoting n-homoclinic points by Γ_n, with $\Gamma_0 := \{0\}$, we obtain a decomposition

$$W^u \cap W^s = \Gamma_0 \cup \Gamma_1 \cup \cdots. \quad (49)$$

Any homeomorphism ψ commuting with φ and fixing 0 also respects each of the n-homoclinic sets Γ_n. Indeed ψ respects W^u and W^s. Since the sets themselves are ordered canonically, along W^s or W^u, the number of $(k-)$homoclinic points between x_0 and $\varphi(x_0)$ indicates which roots of φ may be present. Since the orderings of these k-homoclinic points may differ along W^u and W^s by some permutation π we obtain further invariants which homeomorphic ψ must respect. In the differentiable case, of course, unique Belitskii linearization produces further constraints on the location of these intersections.

Even in the case of planar linear Anosov diffeomorphisms $A \in SL_2(\mathbb{Z})$ with positive irrational eigenvalues $0 < \lambda^s < 1 < \lambda^u$ it is a geometrically amusing exercise to determine all n-homoclinic orbits. The eigenvector lines W^u and W^s then divide the lattice $\mathbb{Z}^2$ into four "quadrants". Let C_1 denote the convex hull of the lattice points, separately in any of the four eigenquadrants. Let

$E_1 := \partial C_1 \cap \mathbb{Z}^2$ denote the lattice points in the boundary of C_1. (Note, in passing, how the rational slopes associated to the elements of E_1 provide the continued fraction expansions of the eigenvector slopes.) The eigenprojections of E_1 onto the eigenspaces then provide the 1-homoclinic orbits. Indeed, translate the eigenspaces to run through any element $\mathbf{n}_0$ of E_1. The resulting closed parallelogram with the original eigenspaces then does not contain any other lattice points, except 0 and $\mathbf{n}_0$. Therefore the eigenprojections of $\mathbf{n}_0$ are 1-homoclinic.

Removing the extreme points E_1 from the lattice and taking the convex hull C_2 of the remainder in each quadrant, we obtain $C_2 := \operatorname{conv}((C_1 \cap \mathbb{Z}^2)\backslash E_1)$. Let

$$E_2 := \partial C_2 \cap \mathbb{Z}^2 \tag{50}$$

Proceeding like this, we successively decompose the lattice points

$$C_1 \cap \mathbb{Z}^2 = E_1 \cup E_2 \cup \cdots \tag{51}$$

in each eigenquadrant into pairwise disjoint subsets, E_n. The n-homoclinic points Γ_n are given by the eigenprojections of the sets E_n, by construction. The recursive construction of the sets E_n can in fact be abbreviated by noting that $C_n = nC_1$. Indeed there are no lattice points left over between the boundaries of nC_1 and $(n+1)C_1$. Therefore

$$E_n = \partial(nC_1) \cap \mathbb{Z}^2 \tag{52}$$

are simply the lattice points on the scaled continued fractions boundary of the eigenvector slopes.

The rigidity results of Corollary 1.3 are then essentially a consequence of the density of the homoclinic set and its decomposition $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots$ into n-homoclinic points, which is preserved under centralizers ψ. Density does not hold, however, in the vast expanses of homoclinic phenomena covered in Shilnihov's work [Shilnikov *et al.*, 1998, 2001]. But $\psi \in C_{\mathcal{H}}(\varphi)$ or $C_{\mathcal{D}^\kappa}(\varphi)$ will preserve the nonwandering set $\Omega(\varphi)$, and will have to satisfy severely restrictive constraints there. In particular, we may speculate, the size of the centralizer could serve to measure some aspects of dynamic complexity in the nonwandering set. We have seen how these constraints depend very much, indeed, on the regularity requirements of the underlying group $G = \mathcal{H}$ or $\mathcal{D}^\kappa$ in which the centralizer is considered. In presence of C^2 differentiability, Belitskii linearization imposes severe constraints. For homeomorphisms $\mathcal{H}$ constraints can arise from density properties of

$$x \longrightarrow \boxed{\psi_1} \longrightarrow \boxed{\psi_2} \longrightarrow \cdots \longrightarrow \boxed{\psi_m} \longrightarrow \varphi(x)$$

Fig. 1. A reactor cascade of m reactors $\psi_1, \ldots, \psi_m$ converting the input x into an overall output $\varphi(x)$.

periodic, homoclinic, or other trajectories. Perhaps a combination of such monitors can be used, separately for basic sets of the nonwandering set and for their heteroclinic connections. At present, however, this direction remains largely unexplored.

Reactor cascades in chemical engineering are one applied source of our root problem $\varphi = \psi^m$. Schematically, a reactor cascade consists of m individual reactors such that, sequentially, the output of each reactor is fed forward as input into the next reactor. See Fig. 1. Let $\varphi = \varphi(x)$ denote the overall output of the last reactor in the cascade, depending on the input x to the first reactor. Let ψ_i denote the input–output map of reactor i. Then clearly

$$\varphi = \psi_m \circ \cdots \circ \psi_2 \circ \psi_1$$

is the composition of the maps ψ_i which represent the individual reactors of the cascade. In applications we may typically assume $\psi_i \in C^2$. The frequent and industrially relevant case $\psi_1 = \psi_2 = \cdots = \psi_m = \psi$ leads to our original C^2 root problem $\varphi = \psi^m$.

The task to determine the root ψ from information on the overall input–output relation φ, only, amounts to an identification of the individual input–output relation ψ without any additional costly measurements taken inside the reactor cascade.

In practice, we cannot expect the individual reactors $\psi_i \in C^1$ to be strictly identical. We are therefore also interested in quantitative monitors which measure the deviation of the reactor cascade ψ_i from the ideally identical situation $\psi_i \equiv \psi$, but depend only on φ.

In the one-dimensional case of an interval with two fixed points and $\varphi' > 0$, the deviation of the Belitskii invariant μ_φ in (36) from period $1/m$ provides such a monitor.

Even in the chaotic Anosov case, we have seen in Chapters 1–3 how to determine C^0 roots after global C^0 linearization (3) of φ. A monitor μ for C^1 roots is then also given on the dense homoclinic set $\Gamma \subseteq \mathbb{R}$ defined in (46).

We conclude with a speculation on the possible role of centralizers in the analysis of spatially chaotic patterns in dimension two or higher. We consider systems which are invariant under a lattice action, $\mathbb{Z}^2$ for simplicity.

One class of such systems are lattice dynamical system, where some nonlinear dynamics at each lattice point couples to the neighboring points, at finite or infinite range, in a translation invariant manner. Although spatio-temporal dynamics may arise, spatial chaos asks for complicated actions of lattice translations on the set of bounded stationary, that is, time-independent solutions. Elliptic partial differential equations on $\mathbb{R}^2$ with spatially doubly periodic coefficients are another class of systems which are a source of potentially interesting $\mathbb{Z}^2$ actions. A spatio-temporal example arises, even in one spatial dimension, when we consider time-periodically forced systems. One component $\mathbb{Z}$ of $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ then corresponds to a spatial shift, as before, and the other component $\mathbb{Z}$ is generated by the time period. See [Bunimovich & Sinaĭ, 1988; Mielke & Zelik, 2004] for this circle of ideas.

Decomposing $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$, we may speak of horizontal (spatial) translations $(n, 0)$ and of vertical (spatial or temporal) translations $(0, m)$. Spatial chaos expresses the phenomenon, already present in one-dimensional systems, that horizontal translations act in some complicated "chaotic" manner on the space of all bounded solutions.

We now ask whether genuinely two-dimensional chaos is possible on finite-dimensional sets of bounded solutions. For a simplistic example, we hypothesize that the set of bounded solutions constitutes some finite-dimensional differentiable torus T^N. Further, suppose that the horizontal translation (1,0) by a single lattice cell acts differentiably on T^N by an Anosov map φ.

Let ψ denote the action by vertical translation $(0,1)$ on the torus T^N of bounded solutions, and only assume continuity of ψ. Clearly our results in Secs. 1–3 then apply. These results indicate, however, how the very hyperbolicity of the horizontal shift φ imposes such strong constraints on the vertical shift ψ that both maps must be viewed as strongly dependent. For example, suppose $\psi \cong (id, \xi)$ on T^N is equivalent to a translation by a fixed point ξ of the period matrix A_φ. Then the resulting pattern will be periodic in the vertical direction and chaos is purely horizontal. If, on the other hand, $\psi \cong (A_\psi, \xi) \in C_{L_2(\mathbb{Z})}(A_\varphi) \ltimes \mathrm{Fix}(A_\varphi)$ on T^2, then there exist m, n, such that $\psi^m = \varphi^n$. The pattern can therefore be thought of as being generated by a single rational shift. We therefore hope that our elementary analysis of Anosov centralizers may contribute to an improved understanding of finite-dimensional spatial and spatio-temporal chaos, eventually.

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