Mathematical models of flageolet harmonics on stringed instruments

– dedicated to Klaus Böhmer and the muses –

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Abstract

Flageolet is a common technique to elicit harmonics on stringed instruments like guitars, pianos, and the violin family: the bowed or plucked string is subdivided by a slight touch of the finger. The paper discusses appropriate linear wave equations which model the flageolet phenomenon. The standard second order wave equation fails, because the resulting Dirichlet boundary condition at the finger uncouples the two parts of the string and produces tones different from the flageolet. We include and discuss fourth order corrections, due to string stiffness, as a possible source for the flageolet phenomenon.

1 Introduction

All music instruments cultivate harmonics or "overtones", which are supposed to determine the "color" quality of the sounds produced. See for example [FleRos10] for an encyclopedic overview from a physics perspective, and also the deeply inspiring Helmholtz classic [Hel1863] of 1863. For centuries organ pipes have evolved into a broadly colored variety of registers, including the ever elusive *vox humana*. Drums, bells, and gongs are judiciously hit at various points; lip tension and/or air jets delicately control the sound of wind instruments; and the complications of carefully playing plucked or bowed string instruments like guitars or the violin family are legendary. Harmonic or "overtone" singing of Mongolian style is a striking example of human vocal control skill and art.

Mathematically speaking, the "color" of harmonics or "overtones" is supposedly represented by amplitudes of the Fourier decomposition of the produced sound. This amounts to a discussion of the spectral decomposition of the wave operator, in the linear case – be that spectrum harmonic or otherwise. A tacit assumption here, which we only adopt reluctantly and with simplicity as our only excuse, is the infinite periodic time extension of the sound itself. In particular we largely ignore the musically important questions of sound initiation, termination, hull curve and modulation.

Flageolet is a technique of playing stringed instruments such that specific harmonics are accessed directly, as principal or ground tones. The finger touches the string slightly, at a rational fraction $0 < p/q < 1$ of the total length; see figure 1. The slight touch prevents string motion at the touching point, but does not clamp the string. The result is a sound of

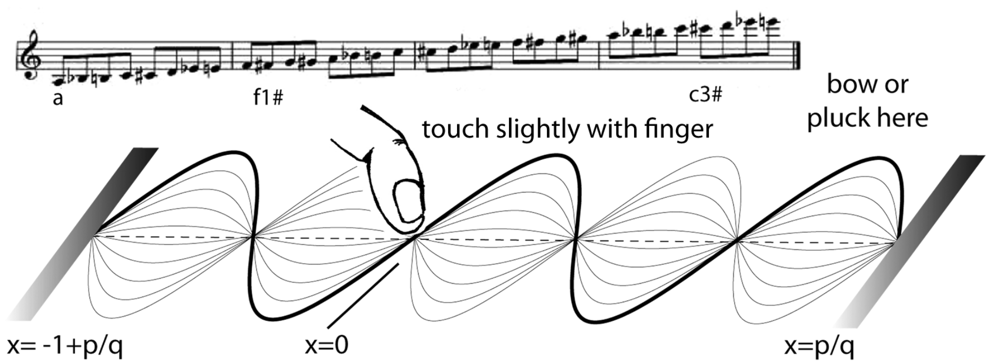

Figure 1: *Flageolet playing technique by slight finger touch. The audible frequency amounts to a half wave length of* $\gcd(p, q-p)/q$*. Illustrated is the case* $p = 3$*,* $q = 5$ *which results in the fifth harmonic c3# of a. Pressing firmly, instead, results in f1#.*

half wave length given by

$$\gcd(p, q-p)/q = \gcd(p, q)/q\,, \tag{1.1}$$

where gcd denotes the greatest common divisor. This flageolet phenomenon occurs for bowed strings on all instruments of the violin family. It has been used systematically to access high ranging notes, at least ever since Paganini. See also Sarasate's celebrated Carmen fantasy [Sar1883] for violin, [Rav20] for cello, or the priceless "Mistfinken" movement for two violins [Hin23]. Typically $q \leq 7$ and $q \leq 16$ are accessible with reasonable reliability on a violin and a cello, respectively. Another common use is for tuning strings by comparing plucked flageolet tones, e.g. on guitars. Flageolet sounds can also be produced on a (grand) piano, touching strings lightly at p/q fractions either before or after hitting the respective key. We subsume this piano case of a string hit by a hammer under the notion of plucking. As a result, in all cases, one obtains an eerie sound of a wind instrument like, "flautando" quality. Hence the name flageolet which also designates a French woodwind instrument of the flute family.

Fixing the string at the same point $x = 0$, in contrast, e.g. by firm pressure of the finger, the bowed part of length p/q will equal the half wave length. In the example of figure 1, where $p = 3$, $q = 5$, this will be one octave and a pure fifth below the Flageolet tone. Specifically, on a cello A-string of frequency $a = 220$ Hz our example produces a flageolet of $c3\# \approx 5 \cdot a = 1100$ Hz, while a firm push produces a regular tone at

$f1\# \approx \frac{5}{3}a \approx 367$ Hz.

We conclude this review of common musical practice with a summary of properties of the flageolet technique which are well-known to practical musicians after only a few years of practice, even on an amateur level:

(f.i) flageolet (1.1) works, stably and reliably, both for plucked and bowed strings;

(f.ii) flageolet is produced with the finger preventing the motion of the string by a slight touch, but without damping the string;

(f.iii) flageolet can be produced down to very low vibration amplitudes;

(f.iv) bowed flageolet can be sustained indefinitely;

(f.v) plucked flageolet works with the finger touching and resting on the string either after or before plucking the string.

So far for ubiquitously well-known technique and practice. From a mathematical view point, however, some difficulties arise. We highlight these difficulties next, with a critique of the two main modelling attempts found in folklore explanation and the mathematical literature. We then propose a more detailed model which suggests string stiffness – as opposed to mere elasticity – as a primary source of the flageolet phenomenon. For simplicity, and based on low amplitude "ppp" flageolet property (f.iii), we will restrict ourselves to linear models in this paper. With all due apologies now, this part is going to increasingly address amateur and professional mathematicians, rather than amateur and professional musicians. For further reading specific to the violin see for example [Kol03], [Cre84]. For mathematical prerequisites see for example [GilTru83] or [Ev10].

The standard mathematical model for the vibrating string goes back to Daniel Bernoulli, Euler, d'Alembert, Lagrange, Helmholtz, and others. For a more contemporary account see for example [FleRos10]. Not surprisingly, that standard model is called the *string equation*:

$$u_{tt} - u_{xx} = 0 \qquad \text{for } -\ell_- < x < \ell_+ \tag{1.2}$$

$$u = 0 \qquad \text{for } x = \pm\ell_\pm \tag{1.3}$$

Here $\ell_\pm > 0$ indicate the string interval $x \in [-\ell_-, \ell_+]$ of nomalized length $\ell = \ell_- + \ell_+ = 1$. Subscripts t, x indicate partial derivatives with respect to time t and position x on the vibrating string.

Flageolet is supposed to be achieved by the finger touching the string slightly at $x = 0$. The folklore explanation for the resulting flageolet sound goes as follows. The slight touch eliminates all spatial eigenfunction components $a_k \varphi_k$ resulting from eigenfunctions φ_k with $\varphi_k(0) \neq 0$ at the touching point. Since

$$\varphi_k(x) = \sin\Big(\frac{k\pi}{\ell_- + \ell_+}(x + \ell_-)\Big) \tag{1.4}$$

this implies vanishing amplitudes $a_k = 0$, for all $k \in \mathbb{N}$, unless

$$\frac{p}{q} = \frac{\ell_-}{\ell_- + \ell_+} \in \frac{1}{k}\mathbb{Z} \subseteq \mathbb{Q} \tag{1.5}$$

is at a rational partition p/q with $\gcd(p, q)/q = 1/k$.

This argument pretends to invoke flageolet property (f.ii) which requires an interior condition

$$u = 0 \qquad \text{for } x = 0 \tag{1.6}$$

in addition to the Dirichlet boundary conditions (1.3). However, condition (1.6) decouples the string equation (1.2) into two independent pieces, one on the left interval $I_- : -\ell_- < x < 0$ and another one on $I_+ : 0 < x < \ell_+$. On each subinterval, separately, we obtain a well-posed Cauchy initial-value problem which does not see the solution on the other subinterval at all. The resulting Dirichlet problem cannot accommodate any coupling between the intervals $I_\pm$. The standard setting $(u, u_t) \in H^1 \times L^2$, for example, does not allow for coupling via $u_x(0^\pm)$. Specifying $u_x(0)$ would overdetermine the Dirichlet problem. The practical flageolet condition (f.ii) becomes incomprehensible, in fact: what is it even supposed to mean, to enforce $u = 0$ by a slight touch, but without damping?

Motivated by such difficulties Bamberger, Rauch and Taylor proposed a mathematical extremely elegant and interesting solution to the flageolet problem; see [BRT82]. They postulated a linear damping term

$$u_{tt} + a(x)u_t - u_{xx} = 0 \tag{1.7}$$

to replace the standard string equation (1.2). Here $a(x) \geq 0$ of compact support around the touching point $x = 0$ is supposed to model the slight touch by the finger. In the limit $a \to a_0\delta_0$ of damping coefficients approximating the Dirac distribution δ_0 at $x = 0$ with weight a_0 they suggest and analyze the following limiting problem:

$$u_{tt} - u_{xx} = 0 \tag{1.8}$$

on the two intervals $I_\pm$ with Dirichlet boundary conditions $u = 0$ at $x = \pm\ell_\pm$ and with the jump conditions (alias transmission conditions)

$$[u_t]_0 = 0, \quad a_0 u_t = [u_x]_0 \tag{1.9}$$

at the touch point. Here and below

$$[\varphi]_0 := \varphi(0^+) - \varphi(0^-) \tag{1.10}$$

denotes the jump at $x = 0$ of the one-sided limits $\varphi(0^\pm) := \lim_{\pm x \searrow 0} \varphi(x)$.

Beautiful as their mathematical treatment certainly is, it does not withstand scrutiny by musical practice. Quoting from their own analysis in slightly adapted notation:

"... When playing harmonics, a musician removes his finger from the string after a short time. In view of the fact that for a friction $a(x)$ spread over a finite interval all solutions tend to zero, this seems wise. Presumably what is happening is that the rate of decay is much slower for the modes vanishing at 0 (a fact that is rigorously true in the limit $a(x) \to a_0\delta_0$); thus the musician leaves his finger in contact with the string long enough to damp the components in $M^\perp$, but not those in M. As this description indicates, the playing of harmonics is a delicate matter, a fact that can easily be verified by anyone inexperienced in the art.

There is an additional sensitivity to the artistry of the player, clearly indicated in Theorem 6. The object in playing harmonics is to obtain as rapid decay as possible on $M^\perp$. ..."

Here M denotes the span of the eigenfunctions of the standard wave operator (1.2), (1.3) which vanish at $x = 0$.

Admittedly, this explanation has practical merit if we *assume* the finger touches the string briefly, after plucking, and then is being removed again from the string. A damping argument is then sufficient to explain the prevalence of eigenfunctions in M, which vanish at $x = 0$. Why the damping should be proportional to the string velocity u_t relative to the resting finger at position $x = 0$, rather than of highly nonlinear slip-stick type as we will discuss for bowed strings in section 5, remains yet another matter of doubt. The above conclusion clearly contradicts common flageolet practice (f.iv) on bowed string instruments, anyway. Indeed, removing the finger from the string "after a short time" terminates the flageolet sound immediately. Even for plucked strings, which the authors probably had in mind, their observation contradicts flageolet property (f.v) which allows the finger to rest on the plucked string indefinitely, after plucking.

To explain sustainable flageolet under modest demands on the skills of the player, we suggest to consider stiff and damped strings which satisfy the PDE

$$u_{tt} + \varepsilon\beta u_{txxxx} + \varepsilon^2 u_{xxxx} - \alpha u_{txx} - u_{xx} = 0 \tag{1.11}$$

on the two open intervals $I_- := \{-\ell_- < x < 0\}$ and $I_+ := \{0 < x < \ell_+\}$ of joined length $\ell_- + \ell_+ = 1$. At $x = \pm\ell_\pm$ we impose standard *clamped boundary conditions*

$$u = u_x = 0 \tag{1.12}$$

on the above fourth order damped wave equation. At the delicate point $x = 0$ where the flageolet finger touches slightly, we impose the *node condition*

$$u = 0 \tag{1.13}$$

together with the continuity and matching conditions

$$[u_x]_0 = [\frac{1}{\varepsilon}\beta u_{txx} + u_{xx}]_0 = 0. \tag{1.14}$$

In the clamped case, when the finger firmly pushes down on the string, the matching conditions (1.14) would be replaced by $u_x = 0$ at $x = 0$, so that the clamped boundary condition (1.12) holds at $x = 0$ and at $x = \pm\ell_\pm$ alike. This decouples the damped stiff string completely into the two independent intervals $-\ell_- < x < 0$, with the string at rest, and $0 < x < \ell_+$, the plucked or bowed part. This is the ordinary case of non-flageolet sounds.

In section 2 we will relate the *flageolet equations* (1.11) – (1.14) to considerations of the elastic energy E_1 and the bending or stiffness energy

E_2 of the string, given by

$$E_1 := \tfrac{1}{2} \int_{-\ell_-}^{\ell_+} u_x^2 dx\,; \tag{1.15}$$

$$E_2 := \tfrac{1}{2}\varepsilon^2 \int_{-\ell_-}^{\ell_+} (u_{xx})^2 dx\,. \tag{1.16}$$

Indeed it is the elastic energy E_1 of Hooke's law which defines the basic frequency, say, of a vibrating plucked or bowed A-string. We have normalized this frequency to $\omega = \pi$ on our string of unit length, even though it corresponds to the Hooke elasticity tension parameter which we adjust when tuning the string. The standard string model (1.2), (1.3) is based on the elastic energy E_1, alone. We suggest to incorporate the bending energy E_2, responsible for the "singing saw", to model the flageolet phenomenon more accurately.

In section 3 we qualitatively estimate the stiffness parameter $\varepsilon > 0$ and the damping parameters $\alpha, \beta > 0$ for standard violin strings. We are only interested in orders of magnitude here, since precise experimental values will vary between manufacturers and string types. We therefore base our estimates on rather crude measurements which are easily repeated by anyone in possession of a string instrument. Our qualitative results on the feasibility of the flageolet technique should not depend on the precise numerical values, anyway.

Section 4 addresses the eigenvalue spectrum and the associated eigenfunctions of the flageolet wave operator (1.11) – (1.14) in the undamped case. See theorem 4.1 which supports the flageolet rule (1.1). It turns out that the resonant flageolet (1.1) is characterized by double eigenvalues. At nonresonance, both intervals before and after the string "survive" as simple eigenvalues, for small $\varepsilon > 0$. We conclude with a discussion of damping, as a possible selection mechanism, and nonlinear issues in section 5.

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2 Energy of damped stiff strings

In this section we relate the flageolet string equations (1.11) – (1.14) to general energy considerations for second order damped Hamiltonian systems. The Hamiltonian energy approach also suggests an appropriate weak formulation and the indicated flageolet matching conditions (1.14) at $x = 0$. The spectral analysis of section 4 will in fact reflect the strongly continuous semigroup of our particular damping in (stiff) strings.

In the linear case the quadratic Hamiltonian energy of a spatially homogeneous stiff elastic string is given by

$$H = \tfrac{1}{2} \int_{-\ell_-}^{\ell_+} (u_t^2 + u_x^2 + \varepsilon^2 (u_{xx})^2) dx. \tag{2.1}$$

The first summand provides the kinetic energy $E_0 = \frac{1}{2} \int u_t^2$; the other terms E_1 and E_2 account for elasticity and bending stiffness; see (1.15), (1.16). Indeed geometric nonlinearities in the arc length and curvature terms of the string disappear, at this linear level.

To illustrate our approach to damping of second order Hamiltonian systems in the simplest ODE case we briefly digress to consider finite-dimensional Hamiltonians of the form

$$H = \tfrac{1}{2} \dot{u}^T \dot{u} + E(u) \tag{2.2}$$

for $u \in \mathbb{R}^N$ and a C^2-differentiable scalar potential $E : \mathbb{R}^N \to \mathbb{R}$. To introduce damping in the resulting second order system $\ddot{u} + E'(u) = 0$, with the gradient force field $-E'(u)$ of the potential $E(u)$ under the standard scalar product and symplectic form, we consider

$$\ddot{u} + a''(u)\dot{u} + E'(u) = 0. \tag{2.3}$$

Here $a : \mathbb{R}^N \to \mathbb{R}$ is a scalar damping potential of differentiability class C^3. We assume convexity, i.e. the symmetric square Hessian matrix $a''(u)$

is strictly positive definite:

$$a''(u) > 0 \tag{2.4}$$

for all u. Then the Hamiltonian H becomes a Lyapunov function, i.e.

$$\dot{H} = -a''(u)[\dot{u}]^2 < 0 \tag{2.5}$$

is strictly negative except at equilibria. For linear second order equations (2.3), of course $a(u)$ and $E(u)$ must be quadratic forms and $a''(u) = a''$ becomes independent of u itself.

Returning to the Hamiltonian (2.1) of the stiff elastic string we have the potential energy

$$E(u) = E_1(u) + E_2(u) = \tfrac{1}{2}\int u_x^2 + \tfrac{1}{2}\varepsilon^2 \int (u_{xx})^2. \tag{2.6}$$

For the quadratic damping potential $a(u)$ we rely on the very same constituents E_1 and E_2 of elastic and bending energy, albeit with different factors $\alpha_1, \alpha_2 > 0$ of proportionality to account for the different physics of tension and bending damping on the atomic level,

$$a(u) := \varepsilon\alpha E_(u) + \varepsilon\beta E_2(u). \tag{2.7}$$

Since E_1, E_2 and $a(u) = a[u,u]$ are symmetric quadratic forms this leads to the second order equation

$$u_{tt} + 2a[u_t, \cdot] + E'(u) = 0. \tag{2.8}$$

To derive the flageolet equations (1.11) – (1.14) from (2.8) we work in the energy space $H < \infty$ of bounded total energy. In other words we consider $v = u_t$ in the space L^2 of Lebesgue square integrable functions, and u in the Sobolev space H_0^2 of square integrable second weak derivatives $u_{xx} \in L^2$ for which u and u_x vanish at the boundary. This already accounts for the clamped boundary condition $u = u_x = 0$ at $x = \pm\ell_\pm$, as required in (1.12). Indeed the string extends in a straight line into the pegbox and beyond the bridge. Any $u_x \neq 0$ at $x = \pm\ell_\pm$ would therefore produce infinite bending energy E_2. Obviously the geometry itself fixes $u = 0$ there. For simplicity, here we suppress the critical energy transfer to the resonance body of the instrument through vibrations of the bridge itself at $x = \ell_+$, [Cre84].

Similarly the flageolet continuity condition $[u_x]_0 = 0$ of (1.4) at the flageolet point $x = 0$ follows from $u \in H_0^2 \hookrightarrow C^1$, e.g. by Sobolev embedding.

Working in the closed linear subspace $u \in H_0^2 \cap \{u(0) = 0\}$ we can easily accommodate condition (1.13), again by Sobolev embedding.

To derive the PDE (1.11) itself, as well as the remaining flageolet matching condition

$$[\frac{1}{\varepsilon}\beta u_{txx} + u_{xx}]_0 = 0 \tag{2.9}$$

of (1.14) at $x = 0$, we test our second order Euler-Lagrange equation (2.8) with $\varphi \in H_0^2 \cap \{\varphi(0) = 0\}$ and integrate by parts:

$$\begin{aligned} 0 &= \int (u_{tt}\varphi + \varepsilon\alpha u_{tx}\varphi_x + \varepsilon\beta u_{txx}\varphi_{xx} + u_x\varphi_x + \varepsilon^2 u_{xx}\varphi_{xx}) \\ &= \int (u_{tt} - \varepsilon\alpha u_{txx} + \varepsilon\beta u_{txxxx} - u_{xx} + \varepsilon^2 u_{xxxx})\varphi \\ &\quad + [\varepsilon\alpha u_{tx}\varphi + \varepsilon\beta(u_{txx}\varphi_x - u_{txxx}\varphi) + u_x\varphi + \\ &\quad + \varepsilon^2(u_{xx}\varphi_x - u_{xxx}\varphi)]_{\pm\ell_\pm,0} \end{aligned} \tag{2.10}$$

Here we have assumed sufficient piecewise regularity of u for all terms to be defined. Pieces refer to the open subintervals

$$I_- := (-\ell_-, 0) \quad I_+ := (0, \ell_+), \quad \text{of } I := (-\ell_-, \ell_+). \tag{2.11}$$

The jump term $[\cdot]_{\pm\ell_\pm,0}$ abbreviates the sum $[\cdot]_{-\ell_-}^0 + [\cdot]_0^{\ell_+}$. Choosing φ with compact support in $I_- \cup I_+ = I \setminus \{0\}$ we obtain PDE (1.11) from the vanishing second line of (2.10). For suitable $\varphi \in H_0^2 \cap \{\varphi(0) = 0\}$ equation (2.10) moreover implies

$$0 = [\varepsilon^2(\frac{\beta}{\varepsilon}u_{txx} + u_{xx})\varphi_x]_0 = \varepsilon^2\varphi_x(0)[\frac{\beta}{\varepsilon}u_{txx} + u_{xx}]_0 \tag{2.12}$$

For $\varphi_x(0) \neq 0$ this provides the remaining matching condition (2.9) at the flageolet point $x = 0$.

Substituting $\varphi = u_t$ in (2.12), we of course recover the energy decay (2.5) of the Hamiltonian H from (2.1). Indeed any violation of the matching condition (2.9) would enable us to construct solutions for which energy increases, at least initially. Notably $\beta/\varepsilon\, u_{txx} + u_{xx} = 0$ is the appropriate boundary condition for a beam or stiff string which is hinged at that boundary. For the stiff string in flageolet, (2.9) states that the “hinged” boundary conditions of u_{xx} match at $x = 0$, in addition to the slope u_x.

3 String data

We do not seek best numerical fits of the stiffness parameter ε^2 and the damping coefficients α, β of tension and bending in our flageolet string equation (1.11) – (1.14). Such data would vary noticeably across the large family of stringed instruments, and even between different manufacturers and types of strings on the same instrument. For example, metal, nylon, and gut of various animals are used today – with varnished gut the preferred material well into the 20th century and certainly with historically inclined ensembles today. For the preparation of gut strings see for example [Moz1787]. The noticeable stiffness even of gut strings came as a surprise to the author.

In view of such broad variability, we therefore describe a few elementary observations which allow us, and anyone, to estimate the parameters at least qualitatively. Data below are for standard Pirastro Eudoxa violin strings.

In PDE (1.11) we have normalized parameters such that the standard frequency f_1 of the primary eigenmode is achieved by the standard wave equation $u_{tt} - u_{xx} = 0$ at a length of 33 cm of the mounted string. For simplicity we ignore the slight frequency shifts by stiffness and damping. To estimate the *stiffness parameter* ε^2 we clamp an unmounted violin string at the same length of 33 cm and let the other end swing freely, in absence of tension and by bending only. The result is a much lower bending frequency f_2. In terms of the primary (first) eigenvalues

$$
\begin{aligned}
-\lambda_{10}\varphi &= \varphi_{xx} \\
\lambda_{20}\psi &= \psi_{xxxx}
\end{aligned}
\tag{3.1}
$$

under Dirichlet and clamped-open boundary conditions, respectively, this implies

$$
f_2^2/f_1^2 = \varepsilon^2 \lambda_{20}/\lambda_{10}\,. \tag{3.2}
$$

Plugging in $\lambda_{10} = \pi^2$ and $\lambda_{20} = (1.875\ldots)^4$ we obtain the small coefficient

$$
\varepsilon = \frac{\pi f_2}{1.875^2 f_1} = 0.003\ldots 0.007\,. \tag{3.3}
$$

This will justify a singular perturbation and boundary layer approach in section 4.

Estimates of damping are virtually absent in the literature; at best vague references to the presence of damping are made or damping is introduced based on the resonant eigenmodes in an ad-hoc manner and separately for each mode.

To estimate the elastic damping α we use the solution $\exp(-\frac{1}{2}\alpha t + i\Omega t)$ of the damped wave equation $u_{tt} - \alpha u_{txx} - u_{xx} = 0$. For small α we again ignore the frequency shift and replace $\Omega = \sqrt{1-\alpha^2/4} \approx 1$. Then an amplitude decay to $1/e$ occurs at time $t = 2/\alpha$. With typical pizzicato decay times of plucked violin strings we obtain

$$\alpha = 0.0015 \ldots 0.003\,. \tag{3.4}$$

Similarly suppose we count approximately k period cycles for the $1/e$ decay of the slow bending oscillation. Then $\beta\lambda_{20}^{1/2} = 1/(k\pi)$ provides a range of

$$\beta = 0.006 \ldots 0.01 \tag{3.5}$$

for the coefficient β of damping by bending. Note that ε does not appear explicitly in this scaling of β.

4 Flageolet undamped

In this section we consider the spectrum of the undamped flageolet equation

$$u_{tt} + \varepsilon^2 u_{xxxx} - u_{xx} = 0\,, \tag{4.1}$$

i.e. the stiff string equation (1.11) with zero damping $\alpha = \beta = 0$. We obtain the pure point spectrum $\lambda = \pm i\Omega$ for the Hamiltonian system (4.1) via the exponential dispersion Ansatz $u = \exp(\lambda t + \mu x)$ in (4.1) with dispersion relation

$$-\Omega^2 + \varepsilon^2\mu^4 - \mu^2 = 0\,. \tag{4.2}$$

We will expand the spectrum and the spatial eigenfunctions, uniformly for bounded frequencies Ω, with respect to small $\varepsilon > 0$.

From (4.2) we immediately obtain the two pairs of spatial modes

$$\begin{aligned} \pm\mu &= \varepsilon^{-1}\sigma = \varepsilon^{-1}(1 + \tfrac{1}{2}\varepsilon^2\Omega^2 + \ldots), \text{ or} \\ \pm\mu &= i\omega = i(\Omega - \tfrac{1}{2}\varepsilon^2\Omega^2 + \ldots) \end{aligned} \tag{4.3}$$

with omitted terms of ε-order raised by two or more. In view of the clamped boundary conditions (1.12) we can therefore write the eigenfunction $u(t,x) = \exp(\lambda t)\xi_\pm(x)$ on the two subintervals $x \in I_\pm$ in the form

$$\varphi_\pm := a_\pm \boldsymbol{s}_\pm + b_\pm \boldsymbol{c}_\pm + \eta_\pm \boldsymbol{e}_\pm + \zeta_\pm \boldsymbol{e}_0 \tag{4.4}$$

with the abbreviations

$$\begin{aligned}
\boldsymbol{s}_\pm &= \sin \omega\xi_\pm && := \tfrac{1}{2i}(e^{i\omega\xi_\pm} - e^{-i\omega\xi_\pm}) \\
\boldsymbol{c}_\pm &= \cos \omega\xi_\pm && := \tfrac{1}{2}(e^{i\omega\xi_\pm} + e^{-i\omega\xi_\pm}) \\
\boldsymbol{e}_\pm &= \exp(-\sigma\xi_\pm/\varepsilon) && \\
\boldsymbol{e}_0 &= \exp(-\sigma|x|/\varepsilon) && \\
\xi_\pm &= \ell_\pm \mp x &&
\end{aligned} \tag{4.5}$$

and eight real valued coefficients $a_\pm, b_\pm, \eta_\pm, \zeta_\pm$ on the subintervals $I_\pm$: $0 < \xi_\pm \leq \ell_\pm$.

By dispersion relation (4.2) this solves the PDE (4.1) on $I_\pm$. Note how the sine terms represent the uncoupled wave equation (1.2), (1.3) of the formal limit $\varepsilon = 0$. The exponential terms indicate singular boundary layers, $\boldsymbol{e}_\pm$ at the interval ends $x = \pm\ell_\pm$ alias $\xi_\pm = 0$ and $\boldsymbol{e}_0$ at the flageolet point $x = 0$, which are due to the stiffness correction by $\varepsilon^2 u_{xxxx}$.

It is straightforward to successively determine expansions for the eight coefficients from the boundary conditions (1.12) – (1.14) , which read

$$\begin{aligned}
b_\pm + \eta_\pm &= 0; \\
\delta a_\pm - \eta_\pm &= 0;
\end{aligned} \tag{4.6}$$

$$s_\pm a_\pm + c_\pm b_\pm + \zeta_\pm = 0; \tag{4.7}$$

$$\begin{aligned}
\delta c_- a_- - \delta s_- b_- + \zeta_- &= -\delta c_+ a_+ + \delta s_+ b_+ - \zeta_+; \\
-\delta^2 s_- a_- - \delta^2 c_- b_- + \zeta_- &= -\delta^2 s_+ a_+ - \delta^2 c_+ b_+ + \zeta_+;
\end{aligned} \tag{4.8}$$

in truncated form, respectively. Here we abbreviate

$$\delta := \frac{\omega}{\sigma}\varepsilon \tag{4.9}$$

and write $s_\pm := \boldsymbol{s}_\pm(\omega\ell_\pm)$ etc. It is now straightforward to obtain the truncated expansions

$$\begin{aligned} \eta_\pm &= \delta a_\pm \\ b_\pm &= -\delta a_\pm \end{aligned} \tag{4.10}$$

from (4.6), and similarly from (4.7)

$$\zeta_\pm = (\delta c_\pm - s_\pm)a_\pm\,. \tag{4.11}$$

Inserting (4.10), (4.11) into (4.8) in search of a nontrivial solution $a_\pm$ we only require vanishing determinant of the remaining 2×2 coefficient matrix for $a_\pm$:

$$\begin{pmatrix} (1-\delta^2)s_- - 2\delta c_- & (1-\delta^2)s_+ - 2\delta c_+ \\ -s_- + \delta c_- & s_+ - \delta c_+ \end{pmatrix} \begin{pmatrix} a_- \\ a_+ \end{pmatrix} = 0 \tag{4.12}$$

Barehandedly we obtain the spectral conditon

$$s_+ s_- = \tfrac{3}{2}(s_+ c_- + c_+ s_-)\delta - 2c_- c_+ \delta^2 + \ldots \tag{4.13}$$

with omitted terms of order δ^3 or higher. In other words the spatial modes satisfy

$$\sin(\omega\ell_+)\sin(\omega\ell_-) = \tfrac{3}{2}\varepsilon\omega\sin(\omega(\ell_+ + \ell_-)) + \mathcal{O}(\varepsilon^2) \tag{4.14}$$

Theorem 4.1. *The frequency spectrum $\Omega = \omega + \ldots$ of the flageolet string without damping and with near-vanishing stiffness $\varepsilon^2 > 0$ is given by the two series*

$$\omega_k^\pm = k\pi/\ell_\pm + \ldots \tag{4.15}$$

of spatial modes $k \in \mathbb{N}$, where $\ell_\pm$ with $\ell_- + \ell_+ = 1$ are the lengths of the subintervals $I_\pm$ separated by the flageolet point $x = 0$.
For sufficiently small $0 < \varepsilon < \varepsilon(k)$, depending on the wave number k, these are algebraically simple eigenvalues, except when the flageolet resonance condition

$$\ell_- = 1 - \frac{p}{q}, \qquad \ell_+ = \frac{p}{q} \tag{4.16}$$

holds. In that latter resonant case we formally obtain an algebraically double and geometrically simple eigenvalue at

$$\ell_-/k_- = \pi/\omega_{k_-}^- = \pi/\omega_{k_+}^+ = \ell_+/k_+ \tag{4.17}$$

In the resonant case, only, the identical half wave length $\ell_\pm/k_\pm$ is a common divisor of both subintervals $\ell_\pm$. The associated sound is a harmonic of order $q/\gcd(p,q)$ to the fundamental frequency of the string, as claimed in (1.1).

Proof.
By the above discussion, the proof reduces to the spectral condition (4.14), for simple eigenvalues. For $\varepsilon = 0$ we obtain either $\sin(\omega\ell_+) = 0$ or $\sin(\omega\ell_-) = 0$ but not both. The case of a double eigenvalue, in the formal limit $\varepsilon \to 0$, corresponds to both sine-terms vanishing simultaneously. Geometric simplicity of the kernel follows because the reduced 2×2 matrix for $a_\pm$ never vanishes identically, except for $\varepsilon = 0$. This proves the theorem. □

In the near-resonant case

$$\ell_+/k_+ = \ell_-/k_- + \ldots$$

we may consider the splitting point $x = 0$ to be located at a simple zero of an eigenfuction $\varphi_k(x)$ of the undamped wave equation (4.1) on the total interval $I = (-\ell, \ell_+)$, with clamped boundary (1.12). Then the flageolet condition (1.13) is satisfied by definition and the continuity and jump conditions (1.14) are automatically satisfied by smoothness of $\varphi(x)$. This is the flageolet profile which we actually see, even with damping. As an eigenfunction to a simple eigenvalue, on all I, it indeed persists for small damping α, β.

5 Discussion

We have presented a model of a clamped stiff string for the flageolet resonance phenomenon of stringed instruments. Stiff string models by themselves are not new: it is our point here, that stiffness provides a crucially important coupling between the intervals I_+ and I_- which is necessary to appropriately model the flageolet phenomenon. Mentioning some of the literature along our way we also discuss the relevance of damping. Indeed the simple eigenvalues of ill-placed nonresonant flageolet are not heard, in practice, as opposed to the double eigenvalues of resonant flageolet harmonics. We model this suppression qualitatively by nonresonant oscillators which are coupled only via damping terms. Complementing our discussion of linear effects we also mention some spectral issues. We conclude with a brief outlook on nonlinearity.

Stiffness originally appears in beam models attributed to Jacob and Daniel Bernoulli as well as Euler, in the static case, and to Rayleigh [SR1896] in the dynamic case; see [HBW99] for a survey of standard beam theories. The combination

$$u_{tt} + \varepsilon^2 u_{xxxx} - u_{xx} = 0 \tag{5.1}$$

with the string equation appears in [Mor36] for example. Under clamped boundary conditions $u = u_x = 0$ on an interval of length π this leads to an inharmonic spread of the k-th harmonics $\Omega_k = k(1 + \frac{1}{2}\varepsilon^2 k^2 + \ldots)$. See for example [Bry09] for a nicely commented version of a 1961 letter by tone-deaf Richard Feynman to his piano-tuner. Feynman attributes the common tuning skill of "spreading the octaves" to a desire for resonance of harmonics in spite of that frequency spread. Numerical values of $\varepsilon^2 \sim 0.0004$ for low piano strings [RLV07] are in reasonable agreement with our primitive bending experiment; see (3.3). For stiff violin strings see also [Cre84].

Damping is rarely considered in the literature, and mostly in an ad hoc manner. See for example [ChaAsk95], where damping terms like u_t and u_{ttt} are considered. Fitting of data, with separate damping coefficients $+\alpha_k \dot{z}_k$ for each mode z_k, also seems common in physical modeling. Instead, we tried to model damping mechanisms which are simply proportional to the type of stored potential energy; see section 2.

In our specific case of the damped stiff string equation

$$u_{tt} + \varepsilon\beta u_{txxxx} - \alpha u_{txx} + \varepsilon^2 u_{xxxx} - u_{xx} = 0 \tag{5.2}$$

the *effects of damping* are quite dramatic, for $\varepsilon \searrow 0$. We describe some of these effects next in the scaling $\alpha = \varepsilon\widetilde{\alpha}$ of small elastic damping α. The scaling is justified by our quantitative estimates (3.3), (3.4).

Consider the parts $\varphi_\pm$ of eigenfunctions φ on the subintervals $I_\pm$,

$$\varphi_\pm = a_\pm \boldsymbol{s}_\pm + b_\pm \boldsymbol{c}_\pm + \eta_\pm \boldsymbol{e}_\pm + \zeta_\pm \boldsymbol{e}_0. \tag{5.3}$$

An analysis in the spirit of section 4, for the nonresonant flageolet case, provides oscillatory exponential boundary layers of width $\sqrt{\varepsilon}$, rather than monotone layers of width ε. The bending and elastic energy content of these layers is of order $\sqrt{\varepsilon}$, compared to an energy content of order ε in the dominant $\boldsymbol{s}_+$ component. The boundary layers which mediate the coupling between the subintervals I_+ and I_- are of equal strength on either side of the flageolet point $x = 0$. Aside from that layer the remaining components of the φ_- part of the eigenfunction φ on the interval $I_- = (-\ell_-, 0)$ are again of order $\sqrt{\varepsilon}$ smaller than their counterparts on I_+. Alternatively we can reverse the roles of $I_\pm$ by reflection in x. Note how the presence of boundary layers on either side of $x = 0$ implies non-orthogonality of the respective eigenfunctions.

As a somewhat simplistic model let us consider two second order damped oscillators $z_\pm$ which are coupled through damping terms

$$\ddot{z}_\pm + \varepsilon\beta(\dot{z}_\pm - \dot{z}_\mp) + \varepsilon\widetilde{\alpha}\dot{z}_\pm + \omega_\pm^2 z_\pm = 0. \tag{5.4}$$

For $\omega_+ = \omega_- = \omega$ we obtain a damping rate $\varepsilon\widetilde{\alpha}$ for the sum $z_+ + z_-$ and $\varepsilon(\widetilde{\alpha} + \beta)$ for the difference $z_+ - z_-$. For $\omega_\pm$ which differ from each other sufficiently, in contrast, periodic input will find at most one of the oscillators at resonance, again with an effective damping rate $\varepsilon(\widetilde{\alpha} + \beta)$. For coupling damping β large compared to the individual damping $\widetilde{\alpha}$ this amounts to a sizable advantage of β/α for the synchrony resonance of $\omega_+ = \omega_-$. This mimics a flageolet harmonic on the resonant subintervals $x \in I_\pm$ of the string.

A somewhat simpler, but less specific, argument to understand the strong damping of non-flageolet strings just relies on the characterization of the flageolet configuration dy double eigenvalues, in the undamped case, which we derived in section 4. Power spectra at near resonant frequencies $\omega \approx \omega_0$ are well approximated by suitably scaled multiples of

$$((\omega - \omega_0)^2 + \gamma^2\omega_0^2)^{-1} \tag{5.5}$$

in presence of small damping $\gamma \sim \varepsilon(\widetilde{\alpha} + \beta)$. This is the case of resonance by a simple eigenvalue $\lambda = i\omega_0$ which is usually encountered for the damped pendulum. At a double eigenvalue, however, as encountered in the flageolet situation, the analogous approximation of the resonant power spectrum will read

$$((\omega - \omega_0)^4 + \gamma^2\omega_0^2)^{-1}. \tag{5.6}$$

In competition with simple resonances (5.5), the double resonance (5.6) represents a much larger part of the total available energy of the string vibration. This accounts for the suppression of the non-flageolet frequencies associated to the subintervals $I_\pm$ alone, in the flageolet case.

In contrast, consider the case that the finger touches the string slightly, but at an “irrational”, “non-flageolet” position. Then we only have to remember density of rational numbers to understand that the frequencies of $I_\pm$, separately, are not heard. Indeed the double eigenvalues then occur at string divisions p/q, in figure 1, which are too high to be reliably excited by the bow: $p > 8$ or 17, in our violin or cello example. Alternatively, these "flageolets out of reach" can be viewed as consuming the available input energy in a non-sustainable frequency range.

In practice matters are not quite as simple. In fact, practically the same large damping which couples the boundary layers at the flageolet point $x = 0$ is also present at the end points $x = \pm\ell_\pm$ of the string. Therefore it is not an easy task, at least on the violin, to obtain a higher flageolet harmonic with $\ell_+ \neq \ell_-$ by first placing the finger in the appropriate position, at rest, and then plucking the string. Indeed the basic frequency of I_+ remains present, even under slight touches. On low piano strings, higher flageolet harmonics can be excited in this manner with sufficiently strong key strokes. The comparative ease by which higher flageolet harmonics are elicited on the bowed string is quite remarkable in this context. We repeat that we place the finger onto the string first, for these experiments, before plucking. Placing the finger after the string has been plucked just produces the intersection of the eigenspaces and spectra of the full string with the flageolet string, and therefore cannot select anything but harmonics. This technique is common when tuning guitars.

The boundary layers discussed above are also important for sound transmission from the vibrating string to the body of the instrument, at the bridge $x = \ell_+$. See [Cre84] and for an interesting discussion including active control of the linear frequency response also [BouBes08]. Of course the presence of the elastic bridge saps energy from the boundary layer, effectively changing the boundary condition, and affects the above considerations on damping.

As a curiosity we note that violin strings tend to tear at the far peg box end $x = -\ell_-$ for violins, even at rest when the strings are neither being tuned nor played. Again the bending of the clamped string into the peg box at that end is a delicate matter of design of the topnut just below $x = -\ell$ and further testifies to the relevance of string stiffness.

In reality, the string itself and the bow are main sources of *nonlinearities*. We ignore any sophisticated structure of the string itself, like its core and one or several layers of spun windings of different materials, which are designed to enhance mass density without increasing stiffness. Even for perfectly homogeneous strings a large variety of geometric nonlinearities enter. See for example [Nar67] who admits two transverse directions but obtains nonlocal quadratic terms even in the planar case and in absence of stiffness. See [BajJoh92], [MoTu04] for a comparison of the Narasimha model with experiments, and for some hints at the role of damping. Stiffness was added by [Wat92] albeit with a non-physical (air) damping term u_t.

One nonlinear approach to the flageolet problem, even for linear strings, consists in the temptation to view the flageolet condition at $x = 0$ as an *obstacle problem* $u \geq 0$. Mathematical appealing as it may be: this is inappropiate in the present context. Indeed the flageolet finger touches the string slightly, vertically from above. It does not unilaterally obstruct movement in the horizontal plane of the string vibration. In experiments with a one-sided metal obstacle we were not able to elicit flageolet phenomena from the string.

The *bow*, possibly of Mongolian origin before the 10th century, is the most striking source of nonlinearity in string instruments. Horse hair and rosin make for interaction with the string, based on slip-stick friction. Helmholtz was the first to discover the saw-tooth response of the bowed string in 1863. Under a microscope, he observed the resulting resonant Lissajous figures of a point marked on the transversely bowed string. The microscope lens was vibrating longitudinally, along the string direction, via a mounting on a tuning fork which was kept in resonant electromagnetic excitation. See [Hel1863] for this brilliant example of a beautiful experiment presented in accomplished style. A more recent survey is [WoGa04].

At present we can only speculate how the Helmholtz kink of the sawtooth response may relate to the flageolet harmonics on bowed strings. The kink is generated when the string tears off the sticky bow and enters slip motion. Originally this should occur at a frequency ω_+ which is the resonance of the bowed interval I_+. While the kink itself is rounded off by string stiffness, the slope continuity $[u_x]_0 = 0$ at $x = 0$ transmits a sawtooth-like excitation to the inert I_- part behind the finger. The flageolet conditions (1.13), (1.14) should allow the rounded kink to pass on into the I_- interval behind the slightly touching finger. This amounts to a sawtooth-like excitation of I_- through the flageolet boundary. With all integer harmonics $k_+\omega_+$ of I_+ present in this excitation, resonance of I_- occurs iff $k_+\omega_+ = k_-\omega_-$. This is the flageolet resonance condition (4.17) of theorem 4.1. In the nonresonant case, however, all energy transmitted through the flageolet boundary will return to I_+ at inappropriate nonresonant times, possibly to disrupt periodic slipstick excitation by the bow.

In conclusion, a comprehensive answer to the flageolet question on bowed string instruments, deceptively linear and elementary as it appeared at first, may well involve genuinely nonlinear aspects. It remains to humbly respect and deeply admire, now as then, the degree of practical skill and insightful sophistication of those who dedicated and dedicate their lives to the century old development and art of our music instruments.

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