Global dynamics of blow-up profiles in one-dimensional reaction diffusion equations

Dedicated to Palo Brunovský on the occasion of his birthday

Bernold Fiedler*, Hiroshi Matano**

*Division of Applied Mathematics

Brown University, Box F

Providence, R.I. 02912, USA

**Graduate School of Mathematical Sciences

University of Tokyo

Komaba, Tokyo 153-8914, JAPAN

version of June 6, 2007

Abstract

We consider reaction diffusion equations of the prototype form

$$u_t = u_{xx} + \lambda u + |u|^{p-1}u$$

on the interval $0 < x < \pi$, with $p > 1$ and $\lambda > m^2$. We study the global blow-up dynamics in the m-dimensional fast unstable manifold of the trivial equilibrium $u \equiv 0$. In particular, sign-changing solutions are included.

Specifically, we find initial conditions such that the blow-up profile $u(t,x)$ at blow-up time $t = T$ possesses $m+1$ intervals of strict monotonicity with prescribed extremal values $u_1, ..., u_m$. Since $u_k = \pm\infty$ at blow-up time $t = T$, for some k, this exhausts the dimensional possibilities of trajectories in the m-dimensional fast unstable manifold.

Alternatively, we can prescribe the locations $x = x_1, ..., x_m$ of the extrema, at blow-up time, up to a one-dimensional constraint.

The proofs are based on an elementary Brouwer degree argument for maps which encode the shapes of solution profiles via their extremal values and extremal locations, respectively. Even in the linear case, such an "interpolation of shape" was not known to us. Our blow-up result generalizes earlier work by [CM89] and [Mer92] on multi-point blow-up for positive solutions, which were not constrained to possess global extensions for all negative times. Our results are based on continuity of the blow-up time, as proved by [Mer92], [Qui03], and on a refined variant of Merle's continuity of the blow-up profile, as addressed in the companion paper [MF07].

1 Introduction

For nonlinearities $f = f(u)$ which grow superlinearly, blow-up solutions $u = u(t, x)$ are known to exist for the reaction diffusion equation

$$u_t = u_{xx} + f(u). \tag{1.1}$$

See for example [SGKM95] and the extensive bibliography there. To be specific, we consider the interval $x \in (0, \pi)$ with Dirichlet boundary conditions

$$u = 0 \text{ at } x = 0, \pi. \tag{1.2}$$

For the nonlinearity $f \in C^2$ we require

$$f(0) = 0, \quad f'(0) > m^2 \tag{1.3}$$

for some positive integer m. This assumption implies that the trivial equilibrium $u \equiv 0$ of (1.1), (1.2) possesses an unstable manifold $W^u(0)$ of dimension at least m. The ordinary differential equation for equilibria, $0 = u_{xx} + f(u)$, is integrable. We consider the "hard spring" case where

$$f(u)/|u| \text{ is nondecreasing,} \tag{1.4}$$

see [Ura67], [BR78]. The period of the closed orbits surrounding the only fixed point $u \equiv 0$ in the (u, u_x)-plane decreases strictly monotonically with amplitude. We finally require f to grow at least like $|u|^{p-1}u$ for $u \to \pm\infty$ and for some $p > 1$. More precisely, following [MF07] we require

$$f(u)/|u|^{p'} \text{ is nondecreasing for } |u| > c_1, \tag{1.5}$$

for some $p' > 1$ and some large constant c_1. The prototype for our assumptions (1.3)–(1.5) is given by

$$f(u) = \lambda u + |u|^{p-1}u, \qquad \text{with } \lambda > m^2. \tag{1.6}$$

In this setting, we will consider initial conditions $u(0,x) = u^0(x) \not\equiv 0$ in the unstable manifold $W^u(0)$ of the trivial equilibrium. All solutions in $W^u(0)\backslash\{0\}$ become unbounded at some finite positive *blow-up time* $t = T(u^0) > 0$; see proposition 2.3 below. In contrast to forward blow-up, these solutions possess a bounded global *backwards* time extension, and thus exist for

$$-\infty < t < T(u^0). \tag{1.7}$$

Such solutions are sometimes called *ancient*. Note that $u^0 \in W^u(0)$ implies $u(t,\cdot) \to 0$ for $t \to -\infty$, by definition. See (1.11) below for an appropriate phase space setting of this convergence.

Our main results, theorems 1.1 and 1.2 below, construct such ancient solutions $u(t,\cdot)$ with prescribed "shape" at blow-up time $t = T(u^0)$. More precisely, at $t = T(u^0)$ the x-profiles $u(t,\cdot)$ will be piecewise strictly monotone with local extrema $u_1, ..., u_m \in \mathbb{R} \cup \{\pm\infty\}$ at successive locations $0 < x_1 \leq ... \leq x_m < \pi$. Note that $u_j \in \{\pm\infty\}$ for at least one j, because $t = T(u^0)$ is the blow-up time. In terms of the x-profile

$$x \mapsto u(T(u^0), x), \tag{1.8}$$

at blow-up, we may view $0 < x_1 < ... < x_m < \pi$ as *critical points* and $u_1, ..., u_m$ as the associated *critical values*. In theorem 1.1 below, we will see how those critical *values* can be prescribed arbitrarily – up to the obvious order constraint

$$0 < u_1 \geq u_2 \leq u_3 \geq \leq \dots \quad \text{or} \tag{1.9}$$
$$0 > u_1 \leq u_2 \geq u_3 \leq \geq \dots$$

Note that equalities in (1.9) indicate blow-up profiles with less than $m+1$ intervals of strict monotonicity.

In theorem 1.2, we find ancient solutions $u(t,x)$ with prescribed locations $x_1, ..., x_m$ of the critical *points*, up to a constraint of codimension one. In this case, we do not know at which x_j blow-up does occur. For a precise formulation of the codimension one constraint on $x_1, ..., x_m$ see (1.24), (1.25) below. Roughly speaking the codimension one constraint determines the total length of intervals of increase versus decrease in the blow-up profile. Apart from this constraint the locations of the critical points $x_1, ..., x_m$ can be prescribed arbitrarily.

In a celebrated paper, Merle has constructed positive initial conditions u^0 with *arbitrarily* prescribed blow-up locations for $f(u) = |u|^{p-1}u$; see [Mer92]. So, what is new and different in the present paper, besides a slightly modified nonlinearity?

We allow solutions $u(t, \cdot)$ to change sign, and we consider more general classes of nonlinearities $f(u)$. But more importantly, our main goal is not to construct one or the other example of a particular blow-up profile. Instead we attempt to qualitatively characterize *all* solutions in the unstable manifold $W^u(0)$, globally, in terms of their blow-up shape. In particular, the construction of blow-up profiles is achieved under the restriction of global backward existence for $t \in (-\infty, T(u^0))$, as was explained in (1.7).

Phrased differently, we require our blow-up solutions $u(t,x)$ to possess at most m local extrema throughout $t \in (-\infty, T(u^0))$. This requirement restricts u^0 to lie in the m-dimensional fast unstable manifold W_m of zero; see proposition 2.3 below. Theorems 1.1, 1.2 then essentially exhaust the dimensional freedom provided by W_m. Differently from [Mer92], we keep track of all extrema of the profile $u(T(u^0), \cdot)$ rather than just those extremal values $\pm\infty$ which blow up.

The questions which we address, therefore, are of a more global nature than the analysis of blow-up profiles and blow-up rates near blow-up. Accordingly, we use elementary but global topological techniques involving Brouwer degree.

Before we formulate our main results, we fix our semigroup setting for the reaction diffusion equation (1.1), (1.2). We then explain the variational and the Sturm structure of the semiflow, and recall some known facts on single point blow-up.

The solutions $u(t, \cdot)$ of our reaction diffusion equation (1.1), (1.2) generate an analytic semigroup

$$(t, u_0) \mapsto u(t, \cdot) \tag{1.10}$$

on the Sobolev space

$$u(t, \cdot) \in X := H_0^1. \tag{1.11}$$

of profiles in H^1 satisfying Dirichlet boundary conditions (1.2). The semigroup is only local in time t, allowing for finite time blow-up. By the regularizing property of the parabolic equation (1.1), (1.2), solution profiles (1.10) are in fact continuous from $(t, u^0) \in (0, T(u^0)) \times X$ to C^2. See [Hen81], [Paz83] for details. Backward extensions in the sense of (1.7) are in fact unique in this setting; see for example [Fri64].

The spatially one-dimensional, scalar reaction diffusion equation (1.1), (1.2) possesses two additional remarkable features which we explain next: a variational structure and the Sturm property of its solutions. The *variational structure*, which also holds in more than one space dimension, is given by the *continuous Lyapunov functional*

$$V(u(t,\cdot)) := \int_0^\pi (\frac{1}{2}u_x^2 - F(u))dx. \tag{1.12}$$

Here and below F denotes the primitive of the nonlinearity $f = F_u$. Differentiating (1.12) with respect to time $t > 0$ along solutions $u(t,\cdot)$, we indeed obtain

$$\frac{d}{dt}V(u(t,\cdot)) = -\int_0^\pi u_t^2 dx. \tag{1.13}$$

This clearly identifies $V(u(t,\cdot))$ as a Lyapunov functional which decreases strictly outside of equilibrium solutions. Our growth assumption (1.5), together with the L^2 variational structure (1.12), (1.13), in fact implies a *blow-up dichotomy* for any maximal forward solution $u(t,\cdot) \in X$, $0 \le t < T(u^0)$:

- either, $T(u^0) = +\infty$ and the solution $u(t,\cdot)$ converges to equilibrium for $t \to +\infty$, staying bounded in particular, or else
- $T(u^0) < +\infty$ is finite and H^1 blow-up occurs at time $t = T(u^0)$.

See proposition 2.1 for a brief review of this fact. As we will recall there, blow-up in the H^1-norm is equivalent to blow-up in the L^∞-norm. For related observations see [CL84].

The second remarkable feature of our reaction diffusion equation (1.1), (1.2) amounts to a *discrete-valued Lyapunov function* for differences of solutions and is available only in one space dimension: the *Sturm structure.* Following [Mat82], we define the *lap number* $\ell(u(t,\cdot)) \leq \infty$ as the minimal number of intervals partitioning $x \in [0,\pi]$ such that $x \mapsto u(t,x)$ is strictly monotone on each interval. In particular, $\ell(u(t,\cdot)) = \infty$ if $u(t,\cdot)$ is not piecewise monotone. Counting strict interior minima and maxima by the minmax number μ and strict interior sign changes by the zero number z, we see that

$$\mu(u(t,\cdot)) = z(u_x(t,\cdot)) = \ell(u(t,\cdot)) - 1 \tag{1.14}$$

for $t > 0$.

For $f = f(u)$ independent of x, the lap-number ℓ and the minmax number μ are both nonincreasing along solutions,

$$t \mapsto \mu(u(t,\cdot)) \qquad \text{decreasing.} \tag{1.15}$$

This crucial fact was proved and successfully applied to semilinear parabolic equations by [Mat82]. A variant for linear autonomous equations goes back as far as [Stu36]. From [Ang88] it follows that, in fact, μ is finite for positive t. Moreover μ drops *strictly* at any $t = t_0 > 0$, such that $u_x(t_0,\cdot)$ possesses a multiple zero at any interior point $x_0 \in (0,\pi)$ or becomes zero at the boundary $x_0 \in \{0,\pi\}$. More generally the same strict dropping occurs at $t = t_0$ for

$$t \mapsto z(u^1(t,\cdot) - u^2(t,\cdot)), \tag{1.16}$$

whenever a multiple zeros x_0 arises for the difference $u^1(t_0,\cdot) - u^2(t_0,\cdot)$ of any two nonidentical solutions.

This *Sturm property* and its variants impose very strong restrictions on the global dynamics of our reaction diffusion equation. For a detailed analysis of the dissipative case with general nonlinearities $f = f(x, u, u_x)$, for example, see [FR00] and the references there. In a blow-up context, the Sturm property has been exploited systematically by [Gal04] to study specific blow-up properties of single solutions to very general types of nonlinear parabolic equations. Suffice it here to emphasize that the Sturm property is much stronger and finer than the usual comparison principles, albeit restricted to a single space dimension or radially symmetric solutions.

Aiming for blow-up solutions we recall that solutions $u(t, \cdot)$ can become unbounded in $X = H_0^1$ at some finite time $T(u_0)$, only if they become unbounded in L^∞, by the regularizing property of parabolic equations. More specifically, *finite point blow-up* was proved by [CM89]. This means that a limiting *blow-up profile*

$$u(T(u^0), \cdot) \tag{1.17}$$

exists, with values $\pm\infty$ at most at finitely many interior points $x_j \in (0, \pi)$. Moreover, $u(T(u^0), \cdot) \in C^2$ is smooth outside the finite blow-up set.

With these preparations we can now state our first main result. For $1 \leq m^2 < f'(0)$ we denote by W_m the set of those initial conditions $u^0 = u(0, \cdot) \in X$ which possess a solution $u(t, \cdot)$ for $-\infty < t < T(u^0)$, uniformly bounded for $t \to -\infty$ and such that

$$\mu(u(t, \cdot)) \leq m \tag{1.18}$$

for all t. In other words, the ancient solution u^0 extends backwards, boundedly, with at most m local maxima/minima in the x-profiles $u(t, \cdot)$. As we will see in proposition 2.3 below, W_m coincides with the m-dimensional strong unstable manifold of the trivial equilibrium. As in (1.8), (1.9) above, we can associate critical points $0 = x_0 \leq x_1 \leq ... \leq x_m \leq x_{m+1} = \pi$, a sign $\iota \in \{\pm 1\}$, and critical values $u_1, ..., u_m \in \mathbb{R} \cup \{\pm\infty\}$ to the blow-up profile $u(T(u^0), \cdot)$ such that

$$\begin{aligned} u_j &= u(T(u^0), x_j) \\ 0 &= u_0 = u_{m+1} \quad (1.19) \\ \iota \cdot (-1)^{-j} u(T(u^0), \cdot) & \quad \text{increases strictly on} [x_j, x_{j+1}], \end{aligned}$$

for all j. In particular, the profiles $u(t, \cdot)$ are piecewise strictly monotone with lap number at most $m + 1$. For a first reading of the following theorems it may be useful to just consider the case where $u_j \neq u_{j+1}$ for $j = 0, ..., m$, and therefore $x_j < x_{j+1}$ are all different. We will comment later on the limiting cases where some of these values coincide.

Theorem 1.1 *Let m be a positive integer. Assume $f \in C^2$ satisfies $f(0) = 0$, $f'(0) > m^2$, and grows superlinearly at least like u^p for some $p > 1$, in the sense of assumptions (1.4), (1.5). Choose values $u'_1, \ldots, u'_m \in \mathbb{R} \cup \{\pm\infty\}$, $u'_0 = u'_{m+1} := 0$, such that $u'_j = \pm\infty$ for at least one j. Moreover let*

$$\iota'(-1)^j (u'_{j+1} - u'_j) \geq 0 \quad (1.20)$$

for some fixed sign $\iota' \in \{\pm 1\}$ and all $j = 0, ..., m$.

Then there exists an initial condition u^0 with associated ancient solution $u(t, \cdot)$ for $-\infty < t < T(u^0) < +\infty$, which converges to the trivial equilibrium $u \equiv 0$ for $t \to -\infty$ and blows up at $t = T(u^0)$ with piecewise stricly monotone *blow-up profile $u(T(u^0),\cdot)$ as follows. The* critical values $u_1,\dots,u_m$ *in the sense of (1.19) are as was prescribed:*

$$u_j = u'_j, \tag{1.21}$$

for all $j=1,...,m$. Moreover

$$\mu(u(t,\cdot)) = m \tag{1.22}$$

for all $-\infty < t < T(u^0)$.

The theorem allows us to choose some adjacent critical values u'_j, u'_{j+1} to be equal, at blow-up time $T(u^0)$. By (1.22) such a situation can occur if, and only if, the adjacent critical values $u_j(t), u_{j+1}(t)$ as well as their associated critical points $x_j(t), x_{j+1}(t)$ of the solution profile $u(t,\cdot)$ have just merged at $t=T(u^0)$. We may therefore view $u'_j = u'_{j+1}$ as a degenerate critical value of the blow-up profile. We will explain in section 5 below how further information on the evolution to blow-up can be encoded by prescribing coinciding x-adjacent values u'_j.

Theorem 1.1 loses track of the locations $x_1,...,x_m$ of the critical points at blow-up. Adhering to the notation (1.19), for a remedy we define the lengths

$$\begin{aligned} \ell_j &:= x_{j+1} - x_j \geq 0 \\ \ell_{\text{even}} &:= \ell_0 + \ell_2 + ... \\ \ell_{\text{odd}} &:= \ell_1 + \ell_3 + ... \end{aligned} \tag{1.23}$$

of the monotonicity intervals, for $j=0,...,m$. Note that $\iota u(T(u^0),\cdot)$ increases strictly, on the even numbered intervals, while decreasing strictly on odd intervals, for some suitably fixed sign $\iota = \pm 1$.

Theorem 1.2 *Under the assumptions of theorem 1.1 choose* $\sigma_0, ..., \sigma_m \geq 0$ *with normalized even/odd sums*

$$\begin{aligned} \sigma_0 + \sigma_2 + ... &= 1 \\ \sigma_1 + \sigma_3 + ... &= 1 \end{aligned} \tag{1.24}$$

and pick a sign $\iota' \in \{\pm 1\}$.

Then there exists an initial condition u^0 *with associated ancient solution* $u(t, \cdot)$, $-\infty < t < T(u^0) < +\infty$, *which converges to* $u \equiv 0$ *for* $t \to -\infty$ *and blows up at* $t = T(u^0)$ *with piecewise strictly monotone blow-up profile* $u(T(u^0), \cdot)$ *as follows. The interval lengths* $\ell_j = x_{j+1} - x_j$ *of the partition* $0 = x_0 \leq x_1 \leq ... \leq x_m \leq x_{m+1} = \pi$ *by the critical points of the blow-up profile in the sense of (1.19), (1.23) satisfy*

$$\ell_j = \begin{cases} (\pi - \vartheta)\sigma_j & \text{for } j \text{ odd} \\ \vartheta \sigma_j & \text{for } j \text{ even} \end{cases} \tag{1.25}$$

for all $j = 0, ..., m$ *and some* $\vartheta = \vartheta(u^0) \in (0, \pi)$. *Moreover* $\iota = \iota'$, *that is*

$$\iota'(-1)^j (u_{j+1} - u_j) \geq 0 \tag{1.26}$$

for $j = 0, ..., m$ *and with* $u_0 = u_{m+1} := 0$.

Keeping in mind that $u^0 \in W_m$ and $\dim W_m = m$, by proposition 2.3 below, only $m-1$ of the quantities $x_1, ..., x_m$ can be prescribed. This fact is reflected by the appearance of

$$\vartheta = \ell_{\text{even}} = \pi - \ell_{\text{odd}} \tag{1.27}$$

which cannot be prescribed at blow-up time $T(u^0)$, for $u^0 \in W_m$. Only the scaled partitioning of $\ell_{\text{even}}, \ell_{\text{odd}}$ into increasing/decreasing subintervals, respectively, can be adjusted arbitrarily by $m - 1$ independent quantities $\sigma_0, ..., \sigma_m$ with normalized even/odd sums (1.24). Although ϑ then determines all locations x_j of critical points, in theorem 1.2, we lose track of the information on the critical values u_j. We do not even know which u_j has become infinite. Such information is only available in theorem 1.1.

The remaining sections are organized as follows. In section 2 we collect some background on the dichotomy of global boundedness versus finite time blow-up, and some continuity results on finite time blow-up. Sections 3 and 4 are devoted to the proofs of theorems 1.1, 1.2, respectively. After a proper set-up which relies crucially on the Sturm feature of the discrete Lyapunov function μ, both proofs reduce to elementary applications of Brouwer degree on hemispheres. We conclude, in section 5, with a brief discussion of these result.

Acknowledgement. We are indebted to Sigurd Angenent for helpful hints concerning the relation between blow-up and variational structure, to Pavol Quittner for clarifying the continuity property of blow-up time, to Marc Georgi and Felix Schulze for several insightful remarks, and to the referee for indicating a glitch in proposition 2.1. The authors are also grateful for mutual and repeated hospitality of their institutions.

This work was supported by the Deutsche Forschungsgemeinschaft, and the Japan Society for the Promotion of Sciences.

2 Background

In this section we collect some facts on finite point blow-up. In proposition 2.1, we address the dichotomy between global boundedness of solutions $u(t, \cdot)$, for $t \nearrow +\infty$, and finite time blow-up, for $t \nearrow T(u^0) < +\infty$. Proposition 2.2 states the continuity of the blow-up time $T(u^0)$ and of the blow-up profile $u(T(u^0), \cdot)$ with respect to both the initial condition $u^0 \in H_0^1$ and the nonlinearity f. We use [Qui03] for these results, which are proved in detail in [MF07]. Proposition 2.3 summarizes existence, uniqueness, continuous dependence, and behavior of lap numbers and zero numbers on fast unstable manifolds.

Proposition 2.1 *Let $f = f(u) \in C^2$ satisfy the p-growth assumption (1.5), for some $p > 1$, but not necessarily (1.3), (1.4).*

Then any solution $u(t, \cdot)$ of (1.1), (1.2) with initial condition $u^0 \in H_0^1$ satisfies the following dichotomy on its maximal positive interval of existence $0 \leq t < T(u^0)$:

(i) either $T(u^0) < +\infty$ and the solution blows up at finitely many points, see (1.17);

(ii) or else $T(u^0) = +\infty$, with $u(t, \cdot)$ remaining uniformly bounded and converging to some equilibrium in H_0^1.

In the blow-up case (i), the Lyapunov functional V also blows up:

$$\lim_{t \to T(u^0)} V(u(t, \cdot)) = -\infty. \tag{2.1}$$

Proof: The H_0^1 dichotomy (i)–(ii), as well as blow-up (2.1) of the Lyapunov functional V, is proved in [Qui03] under a slightly less restrictive variant of growth assumption (1.5). See also [MF07]. The fact that blow-up occurs at finitely many points is proved in [CM89]. $\bowtie$

To formulate continuous dependence of the blow-up time $T = T(u^0, f)$ and the blow-up profile $u(T, \cdot)$ on the initial condition $u^0 \in H_0^1$ and on the nonlinearity f, we fix the following spaces. Let $\mathcal{F}$ denote the set of all C^2-functions f which satisfy the growth condition (1.5). For a topology on $\mathcal{F}$ we choose locally uniform C^2 convergence C^2_{loc}. Consider the blow-up space

$$\mathcal{B} := \{(u^0, f) \in H_0^1 \times \mathcal{F};\ T(u^0, f) < \infty\}. \tag{2.2}$$

Proposition 2.2 *The blow-up space $\mathcal{B}$ is open in the topology of $H_0^1 \times \mathcal{F}$. The blow-up time*

$$\begin{array}{rccl} T: & \mathcal{B} & \to & (0, \infty) \\ & (u^0, f) & \mapsto & T(u^0, f) \end{array} \tag{2.3}$$

is continuous. Similarly, the compactified solution profile $\arctan u(t, \cdot) \in C^0([0, \pi], \mathbb{R})$ *depends continuously on* $(u^0, f) \in \mathcal{B}$.

Proof: The proof of this proposition is the main contents of the independent paper [MF07]. Continuity of the blow-up time was first proved in [Mer92] for the case $f(u) = u^p$ by using an energy method, and later in [Qui03] for a more general class of f by using bootstrap arguments. Continuity of the compactified blow-up profile, a substantial refinement of the finite point blow-up result in [CM89], is proved in [MF07]. It is here that the more restrictive growth assumption (1.5) is used. Note that a slightly weaker version of the result on profile continuity can already be found in [Mer92]. $\bowtie$

Continuity proposition 2.2 is formulated in complete generality, for reference. Below we use continuity of T with respect to u^0, only. A homotopy of the nonlinearity f will only be performed locally, near the trivial equilibrium $u \equiv 0$, where blow-up is not an issue.

We now discuss the fast unstable manifold $W_m \subseteq W^u(0)$. Recall the eigenvalues

$$\lambda_k := f'(0) - k^2 \tag{2.4}$$

$k = 1, 2, ...$, with eigenfunction $\sin(kx)$ of the linearization of (1.1), (1.2) at $u \equiv 0$.

Proposition 2.3 *Let $f = f(u) \in C^2$ satisfy assumption (1.3), that is, $f(0) = 0$ and $\lambda_m > 0$.*

Then there exists a submanifold W_m of class C^1 in the unstable manifold $W^u(0)$ with the following properties

(i) $E_m := T_0 W_m = \mathrm{span}\{\sin x, ..., \sin\,(mx)\}$*;*

(ii) $u^0 \in W_m$ *if, and only if,*

$$\lim_{t \to -\infty} u(t, \cdot) e^{(\lambda_m - \eta)t} = 0,$$

for any $\eta > 0$*;*

(iii)

$$\lim_{t \to -\infty} u(t, \cdot)/||u(t, \cdot)||_{H^1} \; \in E_m$$

is an eigenfunction, for $u^0 \in W_m \setminus \{0\}$*;*

(iv)

$$\mu(u^0) \leq m, \;\; z(u^1 - u^2) \leq m - 1$$

for all $u^0, u^1, u^2 \in W_m$*;*

(v) the L^2*-orthogonal eigenprojection*

$$P : W_m \to E_m$$

is injective;

(vi) for small $\rho > 0$*, the local inverse*

$$W_m^{loc} := P^{-1}\{e \in E_m; \;\; ||e_m|| < \rho\}$$

depends continuously on $f \in C^2$*.*

If f satisfies the "hard spring" assumption (1.4), in addition, then the closure $\bar{W}_m$ *of* $W_m \subseteq X$ *coincides with* W_m *and is an embedded manifold without boundary, diffeomorphic to* $\mathbb{R}^m$*. If moreover p-growth assumption (1.5) holds, then any solution* $u(t, \cdot)$ *associated to* $u^0 \in W_m \setminus \{0\}$ *blows up in finite time.*

Proof: For a detailed proof of properties (i)-(iii), (vi) in an abstract semigroup setting see for example [BF86]. Properties (iv) follow from (iii) and the monotonicity properties (1.15), (1.16). Indeed, (iv) holds true for $t \to -\infty$, where u^0, u^1, u^2 become elements of the Sturm-Liouville eigenspace E_m.

One minor technical point here is the evaluation of $\mu = z(u_x)$, which involves pointwise values for u_x. Although such pointwise values are not available in H_0^1, directly, the smoothing property of the parabolic equation guarantees their existence and continuity. Moreover $\mu \leq m$ follows from (iii) because the norms in H^1 and in H^2 are equivalent on the finite-dimensional local unstable manifold W^u.

Property (v) follows from (iv), indirectly. Suppose $u^1, u^2 \in W_m$, although not identical, possess the same projection, $P(u^1 - u^2) = 0$. Then

$$(2.5) \qquad 0 \neq u^1 - u^2 \in E_m^{\perp} = \text{span}\{\sin(m+1)x, ...\}.$$

As already known to [Stu36], this implies $z(u^1 - u^2) \geq m$, which contradicts (iv). These observations prove (i)-(vi).

In the "hard spring" case (1.4) all nontrivial periodic orbits in the phase plane (u, u_x) of $0 = u_{xx} + f(u)$ are periodic and cross the u_x-axis $\{u = 0\}$ twice, say at $u_x = \pm a$. For each $a > 0$ let $\Theta_\pm(a)$ denote the return times in the right and left half plane, respectively. More precisely, $\Theta_+(a)$ and $\Theta_-(a)$ denote the travel time from $(0, a)$ to $(0, -a)$, and from $(0, -a)$ to $(0, a)$, respectively. Then the minimal period $\Theta(a) = \Theta_+(a) + \Theta_-(a)$ decreases strictly with $a > 0$, as do the return times $\Theta_\pm(a)$, separately. See [Ura67] and also [Sch90]. Note that

$$(2.6) \qquad \Theta_\pm \leq \lim_{a \to 0} \Theta_\pm(a) = \frac{1}{2} \frac{2\pi}{\sqrt{f'(0)}} < \frac{\pi}{m},$$

by assumption (1.3). Therefore $\mu > m$, for any nontrivial equilibrium. In particular, the closure $\bar{W}_m$ of W_m does not contain equilibria other than $u \equiv 0$. Together with the Lyapunov functional $V(u)$, this implies that $\bar{W}_m = W_m \subseteq X$ is an embedding of $\mathbb{R}^m$ without boundary; see [Hen81].

Blow-up under the p-growth assumption (1.5) follows from the dichotomy of proposition 2.1 (i), (ii). Indeed absence of nontrivial equilibria in $\bar{W}_m = W_m$

prevents convergence option (ii) and hence enforces finite time blow-up (i). This proves the proposition. $\bowtie$

3 Critical values

In this section we prove theorem 1.1. We give an outline first. We construct an initial condition u^0 in the m-dimensional fast unstable manifold W_m of $u \equiv 0$ with m prescribed critical values

$$0 < u_1 > u_2 < u_3 >< \ldots \tag{3.1}$$

of the associated solution $u(t, \cdot)$ at blow-up time $t = T(u^0)$ and at non-prescribed locations

$$0 < x_1 < x_2 < x_3 < \ldots < \pi. \tag{3.2}$$

This corresponds to the choice $\iota' = +1$. The case $\iota' = -1$ is analogous, replacing u by $-u$ and $f(u)$ by $-f(-u)$. The general case of nonstrict inequalities in (3.1), (3.2) will be resolved in lemma 3.2 below, by approximation.

We construct u^0 by Brouwer degree. We set up a continuous map $\tilde{\mathbf{v}}$ defined on a hemisphere domain

$$u^0 \in S^{m-1}_{\rho,+} \subseteq W^{\mathrm{loc}}_m \setminus W^{\mathrm{loc}}_{m-1}. \tag{3.3}$$

which is constructed as follows. Recall that $W_{m-1} \subseteq W_m$ are the fast unstable manifolds investigated in proposition 2.3, locally parametrized over their tangent spaces $E_{m-1} \subseteq E_m$. Thus W^{loc}_{m-1} intersects a small sphere S^{m-1}_{ρ} in

$W_m^{\rm loc}$ around $u \equiv 0$ in the equator S_ρ^{m-2}. By proposition 2.3 (iii), we have

$$\lim_{t \to -\infty} u(t,\cdot)/||u(t,\cdot)|| \in \mathrm{span}\{\sin mx\}, \tag{3.4}$$

for $u^0 \in W_m \setminus W_{m-1}$, and in particular

$$\lim_{t \to -\infty} \mu(u(t,\cdot)) = m. \tag{3.5}$$

For $S_{\rho,+}^{m-1}$ we choose the hemisphere of u^0 such that the limit (3.4) provides the positive multiple of $\sin mx$. The values of the map $\tilde{\mathbf{v}}(u^0)$ encode the critical values $u_1, ..., u_m$ of the blow-up profile $u(T(u^0),\cdot)$. We compactify, vectorize, and normalize these values, denoting

$$\begin{aligned} v_j &:= \text{arc tan } u_j \\ \mathbf{v} &:= (v_1, ..., v_m) \\ \tilde{\mathbf{v}} &:= \mathbf{v}/|\mathbf{v}|_2 \end{aligned} \tag{3.6}$$

Note that inequalities (3.1) for the critical values $u_1, ..., u_m$ impose the restriction

$$\mu(u(T(u^0),\cdot)) = m. \tag{3.7}$$

Under this restriction $\mathbf{v} \neq 0$ is well-defined. In (3.13) below, we lift the restriction (3.7) and extend the definition of $\tilde{\mathbf{v}}$ to a map

$$\tilde{\mathbf{v}} : S_{\rho,+}^{m-1} \to \bar{D}^{m-1}. \tag{3.8}$$

Here $D^{m-1} \subset S^{m-1} \subset \mathbb{R}^m$ denotes the unit vectors satisfying the inequalities (3.1), and $\bar{D}^{m-1}$ denotes their closure. Note that D^{m-1}, $\bar{D}^{m-1}$ are homeomorphic to $(m-1)$-dimensional disks. Together with approximation lemma 3.2 below, the proof of theorem 1.1 now reduces to showing

$$D^{m-1} \subset \text{range } \tilde{\mathbf{v}}, \tag{3.9}$$

by Brouwer degree.

We now outline how to properly extend the definition of $\tilde{\mathbf{v}}$ and which homotopies to use for our computation of Brouwer degree. As long as $\mu(u(t,\cdot)) = m$ is maximal, we can associate critical values

$$\mathbf{u} = \mathbf{u}(t, u^0) = (u_1(t), \ldots, u_m(t)) \tag{3.10}$$

and critical locations

$$\mathbf{x} = \mathbf{x}(t, u^0) = (x_1(t), \ldots, x_m(t)) \tag{3.11}$$

to the solution profile $u(t,\cdot)$, as was done in (1.19) for the special case $t = T(u^0)$. Since $u^0 \in S^{m-1}_{\rho,+} \subseteq W_m \setminus W_{m-1}$, we have $\lim\limits_{t\to-\infty} \mu(u(t,\cdot)) = m$, by (3.5). Therefore, definitions (3.10), (3.11) are valid for $-\infty < t \le t_m(u^0)$, that is, up to the first dropping time

$$t_m(u^0) := \inf\{t \in (-\infty, T(u^0)\} \mid \mu(u(t,\cdot)) < m\} \tag{3.12}$$

of μ. (In case $\mu(u(T(u^0),\cdot)) = m$, we leave t_m undefined.) Note how neighboring locations of $x_j(t)$ and the associated values $u_j(t)$ coalesce at $t = t_m(u^0)$. Compactifying, vectorizing, and normalizing as in (3.6) we thus obtain a map

$$\begin{aligned} \tilde{\mathbf{v}}: \quad & \mathcal{D} \to \bar{D}^{m-1} \\ & \mathcal{D} := \{(t, u^0);\; u^0 \in S^{m-1}_{\rho,+}, -\infty < t \le T(u^0)\} \end{aligned} \tag{3.13}$$

using the extended definition

$$\tilde{\mathbf{v}}(t, u^0) := \tilde{\mathbf{v}}(t_m(u^0), u^0), \qquad \text{for } t_m(u^0) \le t \le T(u^0) \tag{3.14}$$

in case $\mu(u(t,\cdot))$ ever drops below m.

In proposition 2.2 we have shown continuity of the blow-up time $T(u^0)$ and of the associated blow-up profile. In continuity lemma 3.1 below we show that the first dropping time $t_m(u^0)$ and the map $\tilde{\mathbf{v}}$ are also continuous on their respective domains of definition. To prove surjectivity (3.9) at blow-up time $t = T(u^0)$, we compute the Brouwer degree

$$\deg\ (\tilde{\mathbf{v}}(T(\cdot),\cdot), S^{m-1}_{\rho,\epsilon}, \tilde{\mathbf{v}}^0) \neq 0 \tag{3.15}$$

for any arbitrarily chosen $\tilde{\mathbf{v}}^0 \in D^{m-1}$. Here $\epsilon > 0$ is chosen small enough and $S^{m-1}_{\rho,\epsilon}$ is the closed $(m-1)$–disk of points in the (topological) upper hemisphere $S^{m-1}_{\rho,+}$ staying a distance at least ϵ from the equator $\partial S^{m-1}_{\rho,+} = S^{m-2}_{\rho}$ of S^{m-1}_{ρ}; see (3.3). Since the equator lies in W_{m-1}, where $\mu \leq m-1$ for all past history, we can in fact choose ϵ small enough such that

$$\mu \leq m-1 \text{ on } \partial S^{m-1}_{\rho,\epsilon} \tag{3.16}$$

In particular, $t_m < 0$ is defined on $\partial S^{m-1}_{\rho,\epsilon}$. Coalescence of neighboring values u_j at $t = t_m < 0$ implies

$$\tilde{\mathbf{v}}(t,u^0) = \tilde{\mathbf{v}}(t_m(u^0),u^0) \in \partial \bar{D}^{m-1}, \tag{3.17}$$

for all $u^0 \in \partial S^{m-1}_{\rho,\epsilon}$ and all $0 \leq t \leq T(u^0)$. Invoking a standard homotopy

$$h(\tau,u^0) := \tilde{\mathbf{v}}(\tau T(u^0),u^0), \tag{3.18}$$

for $0 \leq \tau \leq 1$, we obtain

$$\deg\ (\tilde{\mathbf{v}}(T(\cdot),\cdot), S^{m-1}_{\rho,\epsilon}, \tilde{\mathbf{v}}^0) = \deg\ (\tilde{\mathbf{v}}(0,\cdot), S^{m-1}_{\rho,\epsilon}, \tilde{\mathbf{v}}^0) \tag{3.19}$$

by (3.17) and homotopy invariance of Brouwer degree. Similarly, the degree is independent of the particular choice of $\tilde{\mathbf{v}}^0 \in D^{m-1}$. Moreover, by standard deformation of the domain $S^{m-1}_{\rho,\epsilon}$ the degree does not depend on the choice of small enough $\rho, \epsilon > 0$.

This first homotopy only uses continuity of T with respect to u_0.

Our next homotopy will involve the nonlinearity f, but will only be performed locally, near the trivial equilibrium $u \equiv 0$, where blow-up is not an issue. We deform our superlinear $f = f^0$ to its linearization $f^1(u) = \lambda u,\ \lambda := f'(0) > m^2$ at $u \equiv 0$:

$$f^\tau(u) := \tau f'(0)u + (1-\tau)f(u), \tag{3.20}$$

for $0 \leq \tau \leq 1$. The local fast unstable manifolds $W_m = W_m^{f^\tau}$, the hemispherical disks $S^{m-1}_{\rho,\epsilon} = S^{m-1,f^\tau}_{\rho,\epsilon}$, and the dropping times $t_m(u^0) = t_m^{f^\tau}(u^0)$ all depend continuously on f^τ. Here we consider the disks $S^{m-1,f^\tau}_{\rho,\epsilon}$ as parametrized over $E_m = \operatorname{span}\{\sin x, ..., \sin mx\}$ as was described in proposition 2.3. Note that (3.17) holds throughout the homotopy, because $\mu < m$ holds true on a neighborhood in H^2 of $S^{m-2,f^1}_{\rho,\epsilon}$, and therefore holds on $\partial S^{m-1,f^\tau}_{\rho,\epsilon}$ throughout the homotopy. Indeed the tangent spaces to W^u_m and W^u_{m-1} at $u \equiv 0$ remain untouched by the homotopy (3.20).

By superlinear growth of the "hard spring" f^τ, the trivial solution $u \equiv 0$ remains the only equilibrium in the closure $\bar{W}_m^{f^\tau}$ of $W_m^{f^\tau}$, for all $0 \leq \tau \leq 1$. By continuous dependence of domains and mappings, and keeping (3.17) in mind, homotopy invariance of Brouwer degree implies

$$\deg\ (\tilde{\mathbf{v}}(0,\cdot), S^{m-1,f}_{\rho,\epsilon}, \tilde{\mathbf{v}}^0) = \deg\ (\tilde{\mathbf{v}}(0,\cdot), S^{m-1,\lambda u}_{\rho,\epsilon}, \tilde{\mathbf{v}}^0). \tag{3.21}$$

For readability we have suppressed the dependence of $\tilde{\mathbf{v}}(0,\cdot)$ on f^τ here.

The target $\tilde{\mathbf{v}}^0$ remains fixed. With this final, linearizing homotopy, we have achieved that

$$W_m^{\lambda u} = E_m = \text{span}\{\sin x, ..., \sin mx\} \tag{3.22}$$

coincides with the trigonometric eigenspace E_m. An explicit computation for the specific choice

$$\tilde{\mathbf{v}}^0 = (1, -1, ..., (-1)^{m-1})/\sqrt{m} \in D^{m-1} \tag{3.23}$$

in trigonometric lemma 3.3 below shows that

$$\deg\ (\tilde{\mathbf{v}}(0, \cdot), S_{\rho,\epsilon}^{m-1,\lambda u}, \tilde{\mathbf{v}}^0) \neq 0. \tag{3.24}$$

Because the disk D^{m-1} is connected, (3.24) remains true for any $\tilde{\mathbf{v}}^0 \in D^{m-1}$. Combining the homotopy equalities (3.21) and (3.19), this proves

$$\deg\ (\tilde{\mathbf{v}}(T(\cdot), \cdot), S_{\rho,\epsilon}^{m-1,f}, \tilde{\mathbf{v}}^0) \neq 0, \tag{3.25}$$

for arbitrary choices of $\tilde{\mathbf{v}}^0 \in D^{m-1}$, as was claimed in (3.15). This proves that $D^{m-1} \subseteq \text{range}\ \tilde{\mathbf{v}}$, as was claimed in (3.9). Since the extended definition of $\tilde{\mathbf{v}}(T(\cdot), \cdot)$ via the dropping time t_m only produces values in ∂D^{m-1}, we thus have constructed initial conditions $u^0 \in W_m$ with prescribed critical values (3.1) at blow-up time $t = T(u^0)$.

Approximation lemma 3.2 below then completes the proof of theorem 1.1.

It remains to prove continuity lemma 3.1, approximation lemma 3.2, and the degree trigonometric lemma 3.3. Rather than repeating the technical setting for each technical lemma, we refer to the above proof of theorem 1.1 for notational set-up and assumptions.

The extended, compactified and normalized map $\tilde{\mathbf{v}}$ evaluates critical values along solution profiles $u(t,\cdot)$ and was defined in (3.3)–(3.8), (3.10)–(3.14) above.

Lemma 3.1 *The map*

$$\tilde{\mathbf{v}} : \mathcal{D} \to \bar{D}^{m-1} \tag{3.26}$$

is continuous. Moreover, the map $\tilde{\mathbf{v}}(0,\cdot)$ depends continuously on f in the C^2_{loc}-topology and on the appropriate domains of definition $S^{m-1}_{\rho,\epsilon} = S^{m-1,f}_{\rho,\epsilon}$.

Proof: From proposition 2.2 we recall continuity of the blow-up time $T = T(u^0, f)$ for (u^0, f) in the blow-up space $\mathcal{B}$, where finite time blow-up occurs; see (2.2), (2.3). To prove continuity of $\tilde{\mathbf{v}}$ in the relevant domain $-\infty < t \leq T(u^0, f)$, $u^0 \in S^{m-1,f}_{\rho,\epsilon}$, we address the cases $-\infty < t < T(u^0, f)$ and $t = T(u^0, f)$ separately. We also distinguish the case of maximal minmax number $\mu = m$, at t, and the case $\mu \leq m - 1$.

Consider the case $-\infty < t < T(u^0, f)$ and $\mu = m$ first. By continuous dependence of solutions $u(t,\cdot) \in H^1_0 \subseteq C^0$ on (u^0, f), for $0 \leq t < T(u^0, f)$, and by backwards continuity on the finite-dimensional unstable manifold W^u, the critical values $u_j(t) = u(t, x_j(t))$ and the critical points $x_j(t)$ depend continuously on t, u^0, and f, as long as $\mu = m$ does not drop. By definition (3.6) – (3.8) the map $\tilde{\mathbf{v}}$ is then also continuous in t, u^0, f.

Next consider the case $-\infty < t < T(u^0, f)$ with $\mu \leq m - 1$. Then the first dropping time $t_m = t_m(u^0, f)$ has occurred at or before t, that is, $t_m \leq t$. Because all zeros of $u_x(t,\cdot)$ are simple, immediately before and after dropping, continuous dependence of solutions $u(t,\cdot) \in H^2 \cap H^1_0 \subseteq C^1$ then implies continuity of the dropping time $t_m = t_m(u^0, f)$. In particular, the domain of definition of t_m is open. By definition (3.12) – (3.14) the map $\tilde{\mathbf{v}}$ is then also continuous in t, u^0, f.

In case $t = T(u^0, f)$ and $\mu = m$, we argue as for $-\infty < t < T(u^0, f)$ and $\mu = m$, using continuity of the blow-up time $T(u^0, f)$ and of the compactified blow-up profile $\arctan u(t, \cdot)$ with respect to u^0, f.

In case $t = T(u^0, f)$ and $\mu \leq m - 1$, finally, we first suppose the dropping time $t_m = t_m(u^0, f) < T(u^0, f)$ is defined. Then we may argue by continuity of $t_m(u^0, f)$, as in the case $-\infty < t < T(u^0, f)$ and $\mu \leq m - 1$ above. Suppose next $t_m(u^0, f) < T(u^0, f)$ is not defined. The dropping of μ and the blow-up of $u(t, \cdot)$ must then occur simultaneously, at $t = T(u^0, f)$. In particular, $\mu(u(t, \cdot)) = m$ for all $-\infty < t < T(u^0, f)$. Choose a sequence $u_n^0 \to u^0$, $f_n \to f$, $T(u_n^0, f_n) > t^n \to t = T(u^0, f)$ such that $t^n < t_m(u_n^0, f_n)$ remain below the dropping times t_m, eventually. Then $\tilde{\mathbf{v}}(u_n^0, t_n, f_n) \to \tilde{\mathbf{v}}(u^0, t, f)$, by continuous dependence of solutions and definition (3.6) – (3.8) of the map $\tilde{\mathbf{v}}$.

This proves the lemma. $\bowtie$

Lemma 3.2 *Let* $\mathbf{u}'$ *satisfy nonstrict inequalities (1.20),* $\iota = +1$, *such that* $u'_j = \pm\infty$ *for some* j. *Then there exists* $u^0 \in W_m$ *such that*

$$\mathbf{u}(T(u^0), u^0) = \mathbf{u}'^0 \quad \textit{and} \tag{3.27}$$

$$\mu(u(t, \cdot)) = m \tag{3.28}$$

for all $-\infty < t < T(u^0)$.

Proof: For $\mathbf{u}' = (u'_1, ..., u'_m)$ with pairwise disjoint neighboring values u'_j, u'_{j+1} the lemma is proved (modulo trigonometric lemma 3.3 below); see (3.25). We can therefore approximate $\mathbf{u}'$ by

$$\mathbf{u}' = \lim_{n\to\infty} \mathbf{u}^n \tag{3.29}$$

such that the lemma holds true for each $\mathbf{u^n}$. This provides initial conditions $u^{0,n} \in S^{m-1}_{\rho,\varepsilon}$ and solutions $u^n(t,\cdot)$ such that

$$\begin{aligned} \mathbf{u}(T(u^{0,n}), u^{0,n}) &= \mathbf{u^n}, \text{ and} \\ \mu(u^n(t,\cdot)) &= m, \end{aligned} \tag{3.30}$$

for all $-\infty < t \le T(u^{0,n})$. Without loss of generality $u^{0,n} \to u^0$ converges. Lemma 3.1 implies convergence

$$\begin{aligned} \lim_{n\to\infty} T(u^{0,n}) &= T(u^0) \\ \mathbf{u}(T(u^0), u^0) = \lim_{n\to\infty} \mathbf{u}(T(u^{0,n}), u^{0,n}) &= \mathbf{u}', \end{aligned} \tag{3.31}$$

where $u(t,\cdot)$ is the solution associated to $u^0 \in S^{m-1}_{\rho,\epsilon}$. By continuity of finite dropping times $t_m < T$, we conclude

$$\mu(u(t,\cdot)) = m$$

for all $-\infty < t < T(u^0)$. This proves the lemma. ⋈

Lemma 3.3 *For linear $f(u) = \lambda u$ with $\lambda > m^2$ and for*

$$\tilde{\mathbf{v}}^0 = (1, -1, ..., (-1)^{m-1})/\sqrt{m},$$

as in (3.23), we have

$$\deg\ (\tilde{\mathbf{v}}(0,\cdot), S^{m-1}_{\rho,\epsilon}, \tilde{\mathbf{v}}^0) \neq 0. \tag{3.32}$$

Proof: Since $\tilde{\mathbf{v}}^0 \in D^{m-1}$, rather than ∂D^{m-1}, the extended definition (3.15) via dropping time does not interfere: the value $\tilde{\mathbf{v}}^0$ can only be attained by u^0 such that $\mu(u^0) = m$. In particular,

$$u^0 \in E_m = \text{span}\{\sin\ x, ..., \sin\ mx\} \tag{3.33}$$

itself possesses m nondegenerate extrema of alternating sign and identical absolute value. Since $z(u^1 - u^2) \leq m - 1$ for any $u^1, u^2 \in E_m$, this implies

$$u^0 = \alpha \sin\ (mx) \tag{3.34}$$

for some $\alpha > 0$.

To compute the degree (3.32), it is therefore sufficient to determine the local degree of $\tilde{\mathbf{v}}(0, \cdot)$ at $u^0 = \sin\ (mx)$ with respect to the normal plane $< u^0 >^{\perp} = E_{m-1}$ to u^0 at u^0. This degree is given by the sign of the determinant of the linearization of $\tilde{\mathbf{v}}$, in the nondegenerate case. Differentiating $u^0(x_j) = u_j$ at a nondegenerate extremum

$$(u^0 + \epsilon\eta)_x(x_j + \epsilon\xi_j) = 0 \tag{3.35}$$

with respect to ϵ at $\epsilon = 0$, we obtain

$$Du_j \cdot \eta = \eta(x_j) \tag{3.36}$$

for the value derivative at u^0. Note that $\tilde{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|_{\mathbf{2}}$ is constant along radial rays, and $v_j = \arctan\ u_j$ is just a diffeomorphic transformation. To show our claim on nonzero degree it is therefore sufficient to show that the matrix

$$M := \begin{pmatrix} \sin\ x_1 & \sin 2x_1 & ... & \sin(m-1)x_1 & \sin(mx_1) \\ \vdots & \vdots & & \vdots & \vdots \\ \sin\ x_m & \sin 2x_m & ... & \sin(m-1)x_m & \sin(mx_m) \end{pmatrix} \tag{3.37}$$

possesses nonzero determinant when evaluated at the extremal locations

$$x_j = (j - \frac{1}{2})\pi/m, \quad j = 1, ..., m \tag{3.38}$$

of sin mx, where cos mx vanishes. The k-th column accounts for variations $\eta = \sin kx,\ k < m$. The last augmenting column accounts for radial collapse, when passing from $\mathbf{v}$ to $\tilde{\mathbf{v}}$ near $\tilde{\mathbf{v}}^0$.

It remains to show that ker $M = \{0\}$ is trivial. Suppose a linear combination $e \in E_m$ of the first m eigenfunction of u_{xx} on $x \in [0, \pi]$ with Dirichlet boundary conditions vanishes at m points $x_1, ..., x_m \in (0, \pi)$. But $z(e) \leq m - 1$, by nonincrease of the zero number; see [Stu36]. Therefore $e = 0$. This proves ker $M = \{0\}$, completing the proof of the lemma and of theorem 1.1. $\bowtie$

4 Critical points

In this section we prove theorem 1.2. With slight adaptations our proof follows the lines of section 3. We first construct an initial condition $u^0 \in W_m$ with m critical values $u_1, ..., u_m$ at essentially – up to codimension one – prescribed locations

$$0 < x_1 < x_2 < ... < x_m < \pi. \tag{4.1}$$

at blow-up time $t = T(u^0)$. Again we consider only the case $\iota' = +1$, since $\iota' = -1$ is analogous. The more degenerate case of nonstrict inequalities in (4.1) will be resolved in lemma 4.1 below, by approximation.

Again, we construct u^0 by Brouwer degree. This time we set up a continuous map $\boldsymbol{\sigma}$ with the same hemisphere domain

$$u^0 \in S^{m-1}_{\rho,+} \subseteq W_m \setminus W_{m-1} \tag{4.2}$$

as was used in section 3; see (3.3)-(3.5). For critical locations

$$\mathbf{x} = \mathbf{x}(t, u^0) = (x_1, ..., x_m) \tag{4.3}$$

of the solution profile $u(t, \cdot)$, the values

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(t, u^0) = (\sigma_0, ..., \sigma_m) \tag{4.4}$$

are defined by (1.23), (1.25);

$$x_{j+1} - x_j = \begin{cases} (\pi - \vartheta)\sigma_j & j \text{ odd} \\ & \text{for} \\ \vartheta\sigma_j & j \text{ even} \end{cases} \tag{4.5}$$

Here $j = 0, ..., m,\ x_0 = 0,\ x_{m+1} = \pi$, the value $\vartheta = \vartheta(t, u^0)$ is chosen in $(0, \pi)$, and we assume

$$\mu(u(t, \cdot)) = m \tag{4.6}$$

with strict inequalities $x_{j+1} - x_j > 0$, as in (4.1). Note that $\sigma_j > 0$ sum up to 1 for even/odd j, respectively.

Let

$$\Sigma^k := \{s \in \mathbb{R}^{k+1};\ s_j > 0, \textstyle\sum s_j = 1\} \tag{4.7}$$

denote the k-simplex with closure $\overline{\Sigma}^k$. Then

$$\boldsymbol{\sigma}(t, u^0): \qquad S^{m-1}_{\rho,+} \to \overline{\Sigma}^{[(m-1)/2]} \times \overline{\Sigma}^{[m/2]} \tag{4.8}$$

if (4.6) holds. The range space of (4.8) is homeomorphic to an $(m-1)$–disk. As in section 3, we extend the map (4.8) to $-\infty < t \leq T(u^0)$ by freezing it at the value $\boldsymbol{\sigma}(t_m(u^0), u^0)$ for t above the value $t_m(u^0)$ where μ might drop below m; see (3.12)–(3.14). Note that $\boldsymbol{\sigma}$ is continuous in its domain of definition $\mathcal{D}$, by the proof of continuity lemma 3.1.

To prove theorem 1.2, we show surjectivity

$$\text{range } \boldsymbol{\sigma}(T(\cdot), \cdot) \supseteq \Sigma^{[(m-1)/2]} \times \Sigma^{[m/2]}, \tag{4.9}$$

by Brouwer degree. Imitating (3.15) we claim

$$\deg\ (\boldsymbol{\sigma}(T(\cdot), \cdot), S^{m-1}_{\rho,\epsilon}, \boldsymbol{\sigma}^0) \neq 0 \tag{4.10}$$

for small ϵ, ρ, and any $\boldsymbol{\sigma}^0$ in the right hand side of (4.9). Again, $\boldsymbol{\sigma}(t, \cdot)$ maps $\partial S^{m-1}_{\rho,\epsilon}$ to the boundary of $\Sigma^{[(m-1)/2]} \times \Sigma^{[m/2]}$, for all $0 \leq t \leq T(u^0)$.

We compute the degree (4.10) using the exact same homotopies as in (3.18)–(3.21). In lemma 4.2 below we show

$$\deg\ (\boldsymbol{\sigma}(0, \cdot), S^{m-1}_{\rho,\epsilon}, \boldsymbol{\sigma}^0) \neq 0 \tag{4.11}$$

for the reduced linear case $f(u) = \lambda u$ and any arbitrarily chosen $\boldsymbol{\sigma}^0 \in \Sigma^{[(m-1)/2]} \times \Sigma^{[m/2]}$. Homotopy invariance of degree then implies (4.10) for the original nonlinearity $f = f(u)$ and, by Brouwer degree, the surjectivity (4.9). Invoking approximation lemma 4.1 then completes the proof of theorem 1.2.

It remains to prove approximation lemma 4.1 and the second trigonometric degree lemma 4.2. For technical setting, assumptions, and notational set-up see the above proof of theorem 1.2 and section 3.

Lemma 4.1 *Let* $\boldsymbol{\sigma}' \in \overline{\Sigma}^{[(m-1)/2]} \times \overline{\Sigma}^{[m/2]}$ *be given. Then there exists* $u^0 \in W_m$ *such that*

$$\boldsymbol{\sigma}(T(u^0), u^0) = \boldsymbol{\sigma}' \quad \text{and} \tag{4.12}$$

$$\mu(u(t, \cdot)) = m \tag{4.13}$$

for all $-\infty < t < T(u^0)$.

Proof: Replacing **u** by $\boldsymbol{\sigma}$ and approximating $\boldsymbol{\sigma}'$ by vectors $\boldsymbol{\sigma}^n$ from the interior $\Sigma^{[(m-1)/2]} \times \Sigma^{[m/2]}$, the proof of approximation lemma 3.2 applies verbatim. $\bowtie$

Lemma 4.2 *Consider linear* $f(u) = \lambda u$ *with* $\lambda > m^2$. *Choose* $\boldsymbol{\sigma}^0 = (\boldsymbol{\sigma}_0^0, ..., \boldsymbol{\sigma}_m^0) \in \Sigma^{[(m-1)/2]} \times \Sigma^{[m/2]}$ *such that*

$$\sigma_j = 2/m \tag{4.14}$$

for $1 \leq j < m$, *and* $\sigma_0^0 = \sigma_m^0 = 1/m$. *Then*

$$\deg\ (\boldsymbol{\sigma}(0, \cdot), S^{m-1}_{\rho,\epsilon}, \boldsymbol{\sigma}^0) \neq 0. \tag{4.15}$$

Proof: We first have to determine all initial conditions $u^0 \in S^{m-1}_{\rho,\epsilon} \subseteq E_m = \text{span}\ \{\sin\ x, ..., \sin\ mx\}$ such that

$$\boldsymbol{\sigma}(0, u^0) = \boldsymbol{\sigma}^0 \tag{4.16}$$

We show that the only solution is of the form

$$u^0 = \alpha \sin\ mx\ , \tag{4.17}$$

$\alpha > 0$. We then show that the local degree of $\boldsymbol{\sigma}(0, \cdot)$ at u^0 is nonzero. Note that our choice of $\boldsymbol{\sigma}^0$ implies $\mu = m$ so that we are working with the original, non-extended definition (4.4)–(4.5) of $\boldsymbol{\sigma}$.

To determine $u^0 \in E_m$, we first observe that (4.17) is an obvious solution of (4.16). Indeed this follows from (4.5), putting $\vartheta = \pi/2$. We claim that other solutions do not exist. Suppose therefore that $u^0 \in E_m \setminus \{0\}$ solves (4.16). Even/odd-numbered intervals in the partition $0 = x_0, x_1, ..., x_m, x_{m+1} = \pi$ of $x \in [0, \pi]$ may still have different total lengths $\vartheta, \pi - \vartheta \in (0, \pi)$ respectively, although the lengths $\vartheta, \pi - \vartheta$ are equally split among intervals of the same parity. Nevertheless, we can choose coefficients α_0, β_0 such that the zeros $x \in (0, \pi)$ of

$$\beta_0 + \alpha_0 \cos\ mx \tag{4.18}$$

coincide with $x_1, ..., x_m$. Since $z(u^1 - u^2) \leq m$ for u^1, u^2 in

$$E'_m := \text{span}\ \{1, \cos\ x, ..., cos\ mx\}, \tag{4.19}$$

in fact (4.18) provides the only function in E'_m with those prescribed zeros, up to scalar multiples of the coefficient vector (α_0, β_0). Since $u^0_x \in E'_m$ with coefficient $\beta_0 = 0$, we conclude $u^0 = \alpha_0 \sin\ mx$ and $\vartheta = \pi - \vartheta = \pi/2$.

It remains to compute the local degree of $\boldsymbol{\sigma}(0, \cdot)$ at $u^0 = \sin\ mx$. This degree is given by the sign of the linearization of $\boldsymbol{\sigma}(0, \cdot)$, in the nondegenerate case. Differentiating

$$(u^0 + \epsilon\eta)_x(x_j + \epsilon\xi_j) = 0 \tag{4.20}$$

implicitly with respect to ϵ at $\epsilon = 0$, as in (3.35), we obtain

$$Dx_j \cdot \eta = -\eta_x(x_j)/u^0_{xx}(x_j) = (-1)^j \eta_x(x_j). \tag{4.21}$$

Since $(\boldsymbol{\sigma}, \vartheta) \mapsto \mathbf{x}$ is a diffeomorphic parametrization of the critical locations $\mathbf{x} = (x_1, \ldots, x_m)$, by (4.5), the lemma is proved once we show

$$(4.22)\ \mathbf{x}_\vartheta \notin \text{range} \begin{pmatrix} -\cos\ x_1 & \ldots & -(m-1)\cos(m-1)x_1 \\ \cos\ x_2 & \ldots & (m-1)\cos(m-1)x_2 \\ \vdots & & \vdots \\ (-1)^m \cos\ x_m & \ldots & (-1)^m(m-1)\cos(m-1)x_m \end{pmatrix}$$

Here $\mathbf{x}_\vartheta$, the partial derivative of $\mathbf{x} = \mathbf{x}(\boldsymbol{\sigma}, \vartheta)$ with respect to ϑ at $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0, \vartheta = \pi/2$ is given explicitly from its definition (4.5) as

$$(\mathbf{x}_\vartheta)_j = (-1)^{j+1}/m, \tag{4.23}$$

$j = 1, ..., m$. Therefore (4.22) is equivalent to showing that the augmented square matrix

$$M = \begin{pmatrix} 1 & \cos\ x_1 & \ldots & \cos(m-1)x_1 \\ \vdots & \vdots & & \vdots \\ 1 & \cos\ x_m & \ldots & \cos(m-1)x_m \end{pmatrix} \tag{4.24}$$

possesses trivial kernel. Similarly to our proof of lemma 3.3, this is immediate from interpolation, because $z(u^1 - u^2) \leq m-1$ for any choice of $u^1, u^2 \in E'_{m-1}$; see [Stu36]. This proves the lemma and completes the proof of theorem 1.2. $\bowtie$

5 Conclusion and discussion

In this final section, we first review and interpret our results in the case of degenerate blow-up, where neighboring critical values u_j or critical locations x_j coalesce. We then explore possibilities of describing blow-up shapes $u(T(u^0), \cdot)$ by other quantities rather than just extremal values $\mathbf{u} = (u_1, ..., u_m)$ alone or locations $\mathbf{x} = (x_1, ..., x_m)$ alone. We raise an open question concerning the global parametrization of the fast unstable manifold W_m by its tangent eigenspace E_m, in this context. We conclude with a discussion of our rather restrictive assumptions (1.3)–(1.5) on the nonlinearity $f = f(u)$.

Concerning degenerate blow-up, we illustrate the implications of our proof by approximation lemmas 3.2 and 4.1. Suppose for example that $m \geq 5$ and

$$\infty > u_1 = u_2 = u_3 \ > \ u_4 > -\infty \tag{5.1}$$

at blow-up time $T(u^0)$. Our approximation is based on a sequence $u^{0,n} \to u^0$, such that

$$0 < u_1^n > u_2^n < u_3^n > u_4^n > -\infty \tag{5.2}$$

and $\mu(u(T(u^{0,n}), \cdot) = m$ at blow-up times $T(u^{0,n}) \to T(u^0)$. Note that $u_j^n \to u_j$. For the solution profiles $u(t, \cdot)$ associated to u^0, this implies strict inequalities

$$0 < u_1 > u_2 < u_3 > u_4 \tag{5.3}$$

and $\mu(u(t, \cdot)) = m$, for all $-\infty < t < T(u^0)$. We see how the degenerate profile $u_1 = u_2 = u_3$ arises at blow-up time. Since $u(T(u^0), \cdot)$ cannot remain constant on intervals, we can also infer $x_1 = x_2 = x_3$ at $t = T(u^0)$. The approximation encoded by

$$\infty > u_1 > u_2 = u_3 = u_4 > -\infty, \tag{5.4}$$

on the other hand, would produce a solution $u(t,\cdot)$ where the degeneracy arises at the second location $x_2 = x_3 = x_4$, rather than the first.

Similar remarks apply to prescribed multi-point blow-up like

$$+\infty = u_1 = u_2 = u_3 > u_4 = -\infty. \tag{5.5}$$

In that case, we can still conclude that (5.2), (5.3) hold for the approximating profiles $u^n(t,\cdot)$ and for $u(t,\cdot)$ itself. Again, we see the critical locations x_1, x_2, x_3 merge for $t \to T(u^0)$, with two local maxima disappearing to $+\infty$ and dragging the in-between local minimum behind. Simultaneously, the next minimum value escapes to $-\infty$. Scenarios like

$$\begin{aligned} &+\infty = u_1 > u_2 = u_3 = u_4 = -\infty, \\ &+\infty > u_1 > u_2 = u_3 = u_4 = -\infty, \\ &0 < u_1 = u_2 < u_3 = \infty > u_4 > -\infty, \end{aligned} \tag{5.6}$$

etcetera, can easily be construed. Prescribing

$$\begin{aligned} &u_1 = +\infty > u_2 < u_3 = +\infty, \\ &u_1 = +\infty > u_2 = -\infty \end{aligned} \tag{5.7}$$

keeps blow up distances $x_3 - x_1$, $x_2 - x_1$ positive, respectively. We do not plan to exhaust all combinatorial possibilities, here.

We have already deplored the schism between a description of blow-up shape by critical values u_j versus critical locations x_j. Although both the value vector $\mathbf{u}$ and the location vector $\mathbf{x}$ can be prescribed, up to codimension one, they seem unrelated. A similar statement holds for $\mathbf{x}$.

On the other hand, the map

$$W_m \ni u^0 \mapsto (x_{2_{j-1}}(t, u^0), u_{2_{j-1}}(t, u^0))_{j=1,...,m/2} \tag{5.8}$$

is injective for even $m = \mu(u(t, \cdot))$ and fixed t. Indeed, $z(u^1 - u^2) < m$ on W_m, by proposition 2.3 (iv). On the other hand, $z(u^1 - u^2)$ would have to drop by at least 2 at each of the interior locations x_j, $j = 1, ..., m/2$, if ever the right hand sides of (5.8) agree for $u^1 \not\equiv u^2$ replacing u^0. Since this is impossible, we conclude $u^1 \equiv u^2$ and injectivity of (5.8).

Hence the map (5.8) of, say, maxima locations and values determines locations and values of minima as well. Prescribing these values at blow-up may be a viable project. It should be noted, however, that x_1, x_m cannot get arbitrarily close to the boundary $x \in \{0, \pi\}$; see [FM86] and, more specifically for the class of equations considered here, [MF07] again. Simple-minded surjectivity therefore must fail. Other geometric characteristics, for example based on normalized relative L^q-masses of local peaks, also come to mind in this context.

We have proved surjectivity of the compactified and normalized $\mathbf{u}$-map $\tilde{\mathbf{v}}$ of local extrema, at blow-up time T. However, surjectivity of the compactified and normalized $\mathbf{u}$-map $\tilde{\mathbf{v}}$ still allows for several initial conditions u^0 to produce the same vector $\mathbf{u}$. How about injectivity? How about uniqueness, rather than mere existence, of those initial conditions u^0 which blow up with prescribed shape $\mathbf{u}$? Indeed the results of section 3 can be adapted to show uniqueness of points in W_m with prescribed finite extremal vector $\mathbf{u}$ such that $\mu = m$ is maximal. The arguments are similar to the trigonometric interpolation result of section 3. At present we are not able to extend such a parametrization by shape to include the blow-up horizon $t = T$. Even in the simplest linear case, however, bijectivity as established in section 3 proves an unusual trigonometric interpolation result by m prescribed extreme values.

Alternatively, we may consider the globally injective parametrization

$$P : W_m \to E_m \tag{5.9}$$

of the fast unstable manifold W_m by orthogonal projection onto its tangent space at $u \equiv 0$; see proposition 2.3 (v). Under monotonicity and p-growth assumptions (1.3)–(1.5), any solution on $W_m \setminus \{0\}$ blows up in finite time. Still, W_m may be a graph over E_m, and may even remain globally Lipschitz with its infinite-dimensional $E_m^\perp$-component measured in a suitable norm. This would imply that blow-up is primarily driven by the E_m-components. In contrast, it could also happen that blow-up on W_m occurs in $E_m^\perp$ while at least parts of the finite-dimensional E_m-component remain bounded. While E_m represents fastest growth near the origin $n \equiv 0$, it is not clear whether the same E_m-dominated growth remains dominant at infinity.

Technically, this question is related to dropping some of our overly restrictive assumptions on the nonlinearity $f(u)$. While the p-growth assumptions seem natural to provide the blow-up dichotomy and single-point blow-up of proposition 2.1, the assumptions (1.3), (1.4) play a different role.

Consider the set $\mathcal{B}_m^f$ of $u^0 \in X$ for which a solution $u(t, \cdot)$ exists for $-\infty <$

$t < T(u^0)$, which remains bounded and satisfies $\mu \leq m$ for all negative times. Clearly $\mathcal{B}_m = W_m$ under our assumptions. Dropping (1.3), (1.4), let $\mathcal{A}_m^f \subseteq \mathcal{B}_m^f$ denote the subset of those $u^0 \in \mathcal{B}_m$ for which $T(u^0) = +\infty$. Proposition 2.1 provides a global bound for these solutions in $\{\mu \leq m\}$. Solutions in $\mathcal{B}_m^f \setminus \mathcal{A}_m^f$ blow up in finite time. Geometrically, we expect $\mathcal{A}_m^f$ to be homeomorphic to an m-ball and $\mathcal{B}_m^f$ to be an embedded $\mathbb{R}^m$, as before. Using zero numbers, it can be shown that

$$P : \mathcal{B}_m^f \to E_m \tag{5.10}$$

is still injective. The graph $\mathcal{B}_m^f$ should depend continuously on f. All this geometry would allow us to set up the same maps $\tilde{\mathbf{v}}$ and $\boldsymbol{\sigma}$, as before, on hemispheres slightly outside $\mathcal{A}_m^f$. Performing a homotopy to our prototype $f(u) = \lambda u + |u|^{p-1}u$ then extends theorems 1.1, 1.2 to blow-up initial conditions $u^0 \in \mathcal{B}_m \setminus \mathcal{A}_m$, circumventing assumptions (1.3), (1.4). We do not carry out the necessary details here.

As a final reminder we admit that corresponding questions for nonlinearities $f(x, u, u_x)$ as well as nonlinear diffusion $(d(u_x))_x$ have been left unattended, so far.

References

[Ang88] S. Angenent. The zero set of a solution of a parabolic equation. *Crelle J. reine angew. Math.*, **390**:79–96, (1988).

[BF86] P. Brunovský and B. Fiedler. Numbers of zeros on invariant manifolds in reaction-diffusion equations. *Nonlin. Analysis, TMA*, **10**:179–194, (1986).

[BR78] G. Birkhoff and G.-C. Rota. *Ordinary differential equations. 3rd ed.* New York etc.: John Wiley & Sons, 1978.

[CL84] T. Cazenave and P.-L. Lions. Solutions globales d'équations de la chaleur semi linéaires. (Global solutions of semi-linear heat equations). *Commun. Partial Differ. Equations*, 9:955–978, 1984.

[CM89] X.-Y. Chen and H. Matano. Convergence, asymptotic periodicity and finite point blow-up in one-dimensional heat equations. *J. Diff. Eq.*, **78**:160–190, (1989).

[FM86] Avner Friedman and Bryce McLeod. Blow-up of solutions of nonlinear degenerate parabolic equations. *Arch. Ration. Mech. Analysis*, 96:55–80, 1986.

[FR00] B. Fiedler and C. Rocha. Orbit equivalence of global attractors of semilinear parabolic differential equations. *Trans. Am. Math. Soc.*, 352(1):257–284, 2000.

[Fri64] A. Friedman. *Partial Differential Equations of Parabolic Type.* Prentice Hall, 1964.

[Gal04] Victor A. Galaktionov. *Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications.* Chapman & Hall/CRC, Boca Raton , 2004.

[Hen81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Lect. Notes Math. **840**. Springer-Verlag, New York, 1981.

[Mat82] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semi-linear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sec. IA*, **29**:401–441, (1982).

[Mer92] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. *Commun. Pure Appl. Math.*, 45(3):263–300, 1992.

[MF07] H. Matano and B. Fiedler. Continuity of blow-up profiles in one-dimensional nonlinear heat equations. *in preparation*, (2007).

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.

[Qui03] P. Quittner. Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems. *Houston J. Math.*, 29(3):757–799, 2003.

[Sch90] R. Schaaf. *Global Solution Branches of Two Point Boundary Value Problems*. Lect. Notes Math. **1458**. Springer-Verlag, New York, 1990.

[SGKM95] A. A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov. *Blow-up in Quasilinear Parabolic Equations*, volume 19 of *Expositions in Mathematics*. de Gruyter, Berlin, New York, 1995.

[Stu36] C. Sturm. Sur une classe d'équations à différences partielles. *J. Math. Pure Appl.*, **1**:373–444, (1836).

[Ura67] M. Urabe. *Nonlinear autonomous oscillations. Analytical theory.* Mathematics in Science and Engineering. 34. New York-London: Academic Press XI, 330 p. , 1967.