Nonlinear Sturm global attractors: unstable manifold decompositions as regular CW-complexes

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Abstract

We study global attractors $\mathcal{A}_f$ of scalar partial differential equations $u_t = u_{xx} + f(x, u, u_x)$ on the unit interval with, say, Neumann boundary. Due to nodal properties of differences of solutions, which amount to a nonlinear Sturm property, we call $\mathcal{A}_f$ a Sturm global attractor. We assume all equilibria v to be hyperbolic. Due to a gradient-like structure we can then write

$$(*) \qquad \mathcal{A}_f = \bigcup_v W^u(v)$$

as a dynamic decomposition into finitely many disjoint invariant sets: the unstable manifolds $W^u(v)$ of the equilibria v. Based on our previous Schoenflies result [FiRo13], we prove that the dynamic decomposition (*) is in fact a regular finite CW-complex with cells $W^u(v)$, in the Sturm case. We call this complex the *regular dynamic complex* or *Sturm complex* of the Sturm attractor $\mathcal{A}_f$.

We characterize the planar Sturm complexes by bipolar orientations of their 1-skeletons. We also show that any regular finite CW-complex which is the closure of a single 3-cell arises as a Sturm complex. We include a preliminary discussion of the tetrahedron and the octahedron as Sturm complexes.

1 Introduction

Sturm global attractors $\mathcal{A}_f$ are the global attractors of scalar parabolic equations

$$u_t = u_{xx} + f(x, u, u_x) \tag{1.1}$$

on the unit interval $0 < x < 1$. Just to be specific we consider Neumann boundary conditions $u_x = 0$ at $x = 0, 1$. Standard semigroup theory provides local solutions $u(t, x)$ for $t \geq 0$ and given initial data at $t = 0$ in suitable solution spaces $u(t, \cdot) \in X \subseteq C^1([0, 1], \mathbb{R})$. Under suitable dissipativeness assumptions on $f \in C^2$, any solution eventually enters a fixed large ball in X. In fact that large ball of initial conditions itself limits onto the maximal compact and invariant subset $\mathcal{A}_f$ which is called the global attractor. In particular $\mathcal{A}_f$ is contractible. In general, the global attractor consists of all *eternal solutions*, i.e. of all solutions $u(t, \cdot)$ which exist globally and remain uniformly bounded for all real times $t \in \mathbb{R}$, both in the positive and in the negative (backwards) time direction. Since (1.1) also possesses a Lyapunov function, alias a variational structure, the eternal solutions $u(t, \cdot)$ turn out to possess forward and backward limits, i.e.

$$\lim_{t \to \pm\infty} u(t, \cdot) = v_\pm \,. \tag{1.2}$$

In other words, the α- and ω-limit sets of $u(t, \cdot)$ are two distinct equilibria. We call $u(t, \cdot)$ a *heteroclinic* or *connecting* orbit and write $v_- \leadsto v_+$ for such heteroclinically connected equilibria. Equilibria $v = v(x)$ are time-independent solutions, of course, and hence satisfy the ODE

$$0 = v_{xx} + f(x, v, v_x)\,, \tag{1.3}$$

for $0 \leq x \leq 1$, again with Neumann boundary. See [He81, Pa83, Ta79] for a general background, [Ma78, MaNa97, Ze68, Fietal14] for the gradient-like Lyapunov structure of (1.1) under separated boundary conditions, and [BaVi92, ChVi02, Edetal94, Ha88, Haetal02, La91, Ra02, SeYo02, Te88] for global attractors in general.

We attach the name of Sturm to the PDE (1.1), and to its global attractor $\mathcal{A}_f$ because of a crucial nodal property of its solutions which we express by the *zero number* z. Let $0 \leq z(\varphi) \leq \infty$ count the number of strict sign changes of $\varphi : [0, 1] \to \mathbb{R}$, $\varphi \not\equiv 0$. Then

$$t \longmapsto z(u^1(t, \cdot) - u^2(t, \cdot))\,, \tag{1.4}$$

is finite and nonincreasing with time t, for $t > 0$ and any two distinct solutions u^1, u^2 of (1.1). Moreover z drops strictly with increasing t, at any multiple zero of $x \longmapsto u^1(t_0, x) - u^2(t_0, x)$. See Sturm [St1836] for a linear autonomous version. The case $z = 0$ is known as strong monotonicity or parabolic comparison principle for scalar parabolic equations, and holds in any space dimension. The full Sturm structure (1.4), however, restricts applicability to one space dimension, and a few types of delay equations and of tridiagonal Jacobi type ODE systems. The dynamic consequences of the Sturm structure, however, are enormous. For a first introduction see [Ma82, BrFi88, BrFi89, Ro91, FiSc03, Ga04] and the many references there. The present paper can also be read as an introduction to the labyrinth of nonlinear Sturm theory, with dynamic decompositions of Sturm global attractors as a guiding Ariadne thread.

In a series of papers, we have given a combinatorial description of Sturm global attractors $\mathcal{A}_f$; see [FiRo96, FiRo99, FiRo00]. Here and below we assume that all equilibria v of (1.1), (1.3) are *hyperbolic*, i.e. without eigenvalues (of) zero (real part) of their linearization. This generic nondegeneracy assumption provides an odd number N of equilibria, labelled such that

$$v_1 < v_2 < \cdots < v_N \qquad \text{at} \quad x = 0\,, \tag{1.5}$$

all in $\mathcal{A}_f$. The combinatorial description is based on the *Sturm permutation* $\sigma = \sigma_f \in S_N$ which was introduced by Fusco and Rocha in [FuRo91] and is defined by

$$v_{\sigma(1)} < v_{\sigma(2)} < \cdots < v_{\sigma(N)} \qquad \text{at} \quad x = 1\,. \tag{1.6}$$

Using a shooting approach to the ODE boundary value problem (1.3), the Sturm permutations $\sigma = \sigma_f$ have been characterized as *dissipative Morse meanders* in [FiRo99]; see also sections 3, 4 below. In [FiRo96] we have shown how to determine which equilibria $v_\pm$ possess a heteroclinic orbit connection (1.2), explicitly and purely combinatorially from σ. More geometrically, global Sturm attractors $\mathcal{A}_f$ and $\mathcal{A}_g$ with the same Sturm permutation $\sigma_f = \sigma_g$ are C^0 orbit-equivalent [FiRo00]. (As an aside we hasten to admit that some different and even nonconjugate Sturm permutations give rise to C^0 orbit-equivalent Sturm attractors.)

In all our previous papers heteroclinic orbits were described by the *connection graph* $\mathcal{H}_f$ with vertices given by the set $\mathcal{E}_f$ of equilibria, all hyperbolic. Let $i(v) = \dim W^u(v)$ denote the *Morse index* of v, i.e. the dimension of the unstable manifold W^u of v. Then the edges of the directed connection graph $\mathcal{H}_f$ are given by the unique heteroclinic orbits

$u : v_- \leadsto v_+$ between equilibria of adjacent Morse index $i(v_+) = i(v_-)-1$. In other words, an edge between such vertices $v_\pm$ exists if, and only if, $v_\pm$ possess a heteroclinic orbit connecting them. The "connects to" relation $\leadsto$ is transitive and satisfies a cascading principle, [BrFi89]. Therefore it is sufficient to know the connection graph $\mathcal{H}_f$ in order to conclude for any pair $v_\pm$ of equilibria whether or not they possess a heteroclinic connecting orbit: $v_- \leadsto v_+$ if and only if there exists a path from v_- to v_+ in $\mathcal{H}_f$.

For planar Sturm attractors $\mathcal{A}_f$, i.e. for equilibrium sets $\mathcal{E}_f$ with a maximal Morse index two [Br90, Jo89, Ro91], a slightly more geometric approach has been initiated in [FiRo08, FiRo09, FiRo10]. It was clarified which planar graphs $\mathcal{H}$ do arise as connection graphs $\mathcal{H} = \mathcal{H}_f$ of planar Sturm attractors $\mathcal{A}_f$, and which ones don't. Meanwhile a *Schoenflies theorem* has also been proved to hold for the closure $\overline{W}^u(v) \subseteq X$ of the unstable manifold W^u of any hyperbolic equilibrium v; see [FiRo13]. In particular $\overline{W}^u(v)$ is the homeomorphic embedding of a closed unit ball $B^{i(v)}$ of dimension $i(v)$ of v. This allows us to reformulate the combinatorial results of [FiRo08, FiRo09, FiRo10] in a more geometric and topological language, as follows.

Consider a *regular finite* CW-*complex*

$$\mathcal{C} = \bigcup_{j=1}^{N} c_j\,, \tag{1.7}$$

i.e. a finite disjoint union of *cell interiors* c_j. Here the closures $\bar{c}_j = h_j(\bar{B}_j)$ are the homeomorphic images under h_j of closed unit balls $\bar{B}_j$. For positive dimensions of $\bar{B}_j$ we let $c_j = h_j(B_j)$ denote the image of the interior B_j of $\bar{B}_j$ and call $\dim B_j$ the dimension of the (open) cell c_j. For dimension zero we write $B_j := \bar{B}_j$ so that the 0-cell $c_j = h_j(B_j)$ is just a point. The *m-skeleton* $\mathcal{C}^m$ of $\mathcal{C}$ consists of all cells of dimension at most m. The main requirement for a (finite) CW-complex now is that the boundary $(m-1)$-sphere $S_j = \partial B_j = \bar{B}_j \setminus B_j$ of any m-ball B_j maps into the $(m-1)$-skeleton, i.e.

$$\partial \bar{c}_j = h_j(S_j) \subseteq \mathcal{C}^{m-1} \tag{1.8}$$

under the continuous map $h_j : \bar{B}_j \to \bar{c}_j$, for any m-cell c_j of positive dimension m. The restricted map $h_j|S_j$ is only required to be continuous, for CW-complexes. For *regular* CW-complexes, in contrast, $h_j|S_j$ is required to be a homeomorphism onto the image $h_j(S_j)$. Then h_j is called

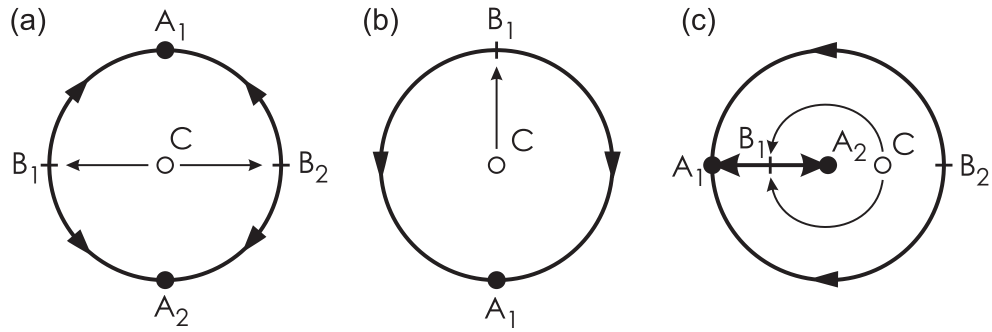

Figure 1.1: *Three planar unstable manifolds of an equilibrium C with Morse index 2. Stable equilibria are A_1, A_2 and Morse index 1 saddles are B_1, B_2.*
The Chafee-Infante type Sturm global attractor of case (a), left, is a planar regular finite CW-complex. Cases (b) and (c), center and right, are finite CW-complexes but are not regular.

the *attaching* (or *gluing*) *homeomorphism.* See [FrPi90] for a background on this terminology.

In variational or gradient-like settings with hyperbolicity of equilibria it is very tempting to view the disjoint dynamic decomposition

$$\mathcal{A} = \bigcup_{v \in \mathcal{E}} W^u(v) \tag{1.9}$$

of the global attractor $\mathcal{A}$ into unstable manifolds W^u of equilibria v as some kind of a finite regular CW-complex.

See figure 1.1 for two caveat examples. The Chafee-Infante attractor (a), left, arises from PDE (1.1) with nonlinearity $f(u) = \lambda u(1-u^2)$ for parameters $\pi^2 < \lambda < 4\pi^2$ and is the only finite regular CW-complex in fig. 1.1. The Sturm permutation is $\sigma_f = (1\ 4\ 3\ 2\ 5)$. The other two examples, (b) and (c), are finite CW-complexes but are not regular. In example (b), in fact, the one-point boundary $\partial W^u(B_1) = \{A_1\}$ of the saddle unstable manifold is not the two-point 0-sphere. In case (c), the spiked boundary $\partial W^u(C) = \{A_1, A_2\} \cup W^u(B_1) \cup W^u(B_2)$ is a continuous, but not homeomorphic, image of the circle S^1. In fact, the spike $\overline{W}^u(B_1)$ is traversed twice. For a more advanced example on $\mathcal{A} = S^1 \times S^2$ where the finite dynamic decomposition (1.9) is not even a CW-complex see [BaHu04]. That example is not Morse-Smale, i.e. nontransverse stable and unstable manifolds of hyperbolic equilibria occur. For comments on the Morse-Smale case see [BaHu04], remark 6.36.

In view of such warning examples we still stubbornly focus on the seemingly exceptional situation when the attractor decomposition (1.9) into unstable manifolds, as cells, does provide a *regular* finite CW-complex. Just in case this actually happens, ever, we will then call the dynamic decomposition (1.9) of the global attractor $\mathcal{A}$ a *regular dynamic complex*. Well, it does happen!

Theorem 1.1. *Let* $\mathcal{A} = \mathcal{A}_f$ *be a Sturm global attractor of* (1.1) *and assume all equilibria* $v_1, \ldots, v_N$ *are hyperbolic. Then*

$$\mathcal{A}_f = \bigcup_{j=1}^{N} W^u(v_j) \tag{1.10}$$

is a regular *dynamic complex, i.e. the dynamic decomposition* (1.10) *of* $\mathcal{A}_f$ *is a finite* regular *CW-complex with (open) cells given by the unstable manifolds* $W^u(v_j)$ *of the equilibria* v_j*.*

The proof is closely related to the Schoenflies result [FiRo13] and will be given in section 2. By theorem 1.1 we can define the *Sturm complex* $\mathcal{C}_f$ to be the regular dynamic complex $\mathcal{C}_f = \bigcup_{v\in\mathcal{E}_f} W^u(v)$ of the Sturm global attractor $\mathcal{A}_f$, provided all equilibria $v \in \mathcal{E}_f$ are hyperbolic. A planar Sturm complex $\mathcal{C}_f$, for example, is the regular dynamic complex of a planar $\mathcal{A}_f$, i.e. of a Sturm global attractor for which all equilibria $v \in \mathcal{E}_f$ have Morse indices $i(v) \leq 2$.

In view of theorem 1.1 our results on the planar connection graphs $\mathcal{H}_f$ of planar Sturm attractors $\mathcal{A}_f$ obtained in [FiRo08, FiRo09, FiRo10] can now be reformulated as follows.

Theorem 1.2. *A regular finite CW-complex* $\mathcal{C}$ *is a planar Sturm complex* $\mathcal{C} = \mathcal{C}_f$*, if and only if* $\mathcal{C} \subseteq \mathbb{R}^2$ *is planar, contractible, and the 1-skeleton* $\mathcal{C}^1$ *of* $\mathcal{C}$ *possesses a bipolar orientation.*

Here the 1-skeleton $\mathcal{C}^1$ is considered as a graph with the sink 0-cells as vertices, and with the 1-cell unstable manifolds $W^u(v)$ of the saddles v with Morse index $i(v) = 1$ as edges. An orientation of the edges is called bipolar if it does not admit directed cycles, i.e. defines a partial order $<$, and possesses exactly one maximal vertex $\bar{v}$ and one minimal vertex $\underline{v}$ with respect to that order. Both extrema are required to lie on the boundary of $\mathcal{C}$. We then say that the bipolar orientation is from $\underline{v}$ to $\bar{v}$. See fig. 1.2 for simple examples. More intricate bipolar orientations are

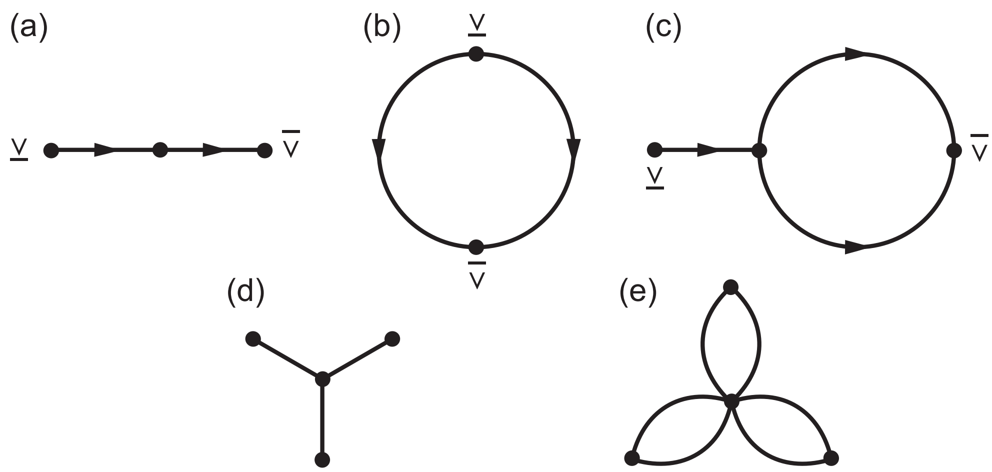

Figure 1.2: *Examples of bipolar orientations of planar 1-skeletons, (a)–(c). Examples (d) and (e) do not possess any bipolar orientation.*

discussed in figs. 3.1 and 6.1–6.3. See the survey [Fretal95] for a characterization of bipolarity via 2-connectivity of graphs. Section 3 develops further details, and section 4 contains an application of theorem 1.2.

As an illustration of our approach to Sturm attractors $\mathcal{A}_f$, and as a first step beyond the planar case, consider the special case of $\mathcal{A}_f$ homeomorphic to a 3-ball and with a unique equilibrium $v_* \in \mathcal{E}_f$ of Morse index $i(v_*) = 3$. Of course we keep the requirement that all equilibria are hyperbolic. Again by the Schoenflies result [FiRo13], and by theorem 1.1, we then see how $\mathcal{A}_f = \overline{W}^u(v_*)$ is the closure of the unstable manifold of v_*. In other words, $\mathcal{A}_f$ consists of a single closed 3-cell with 2-sphere boundary; see (1.7)–(1.9). Equivalently, v_* connects heteroclinically to all other equilibria, i.e. to $\mathcal{E}_f \setminus \{v_*\}$; see section 2.

Theorem 1.3. *A regular finite CW-complex $\mathcal{C}$ is the Sturm complex $\mathcal{C} = \mathcal{C}_f$ of a 3-dimensional Sturm global attractor $\mathcal{A}_f = \overline{W}^u(v_*)$ with only hyperbolic equilibria and with Morse index $i(v_*) = 3$, if and only if $\mathcal{C}$ is the closure of a single 3-cell.*

In contrast to the planar case of theorem 1.2, there are no additional conditions imposed, e.g., on any kind of bipolar orientation of the 1-skeleton $\mathcal{C}^1$ inside the 2-skeleton $\mathcal{C}^2$ which constitutes the 2-sphere boundary of the single 3-cell $W^u(v_*)$. The only-if-part of theorem 1.3 therefore is just a corollary to theorem 1.1. The proof of the if-part, however, reduces to theorem 1.2 via a judicious choice of such a bipolar orientation as is omitted in the statement of theorem 1.3. The main point of theorem 1.3, in

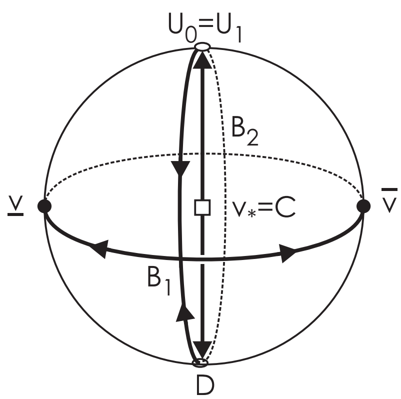

Figure 1.3: *The Chafee-Infante 3-ball Sturm global attractor* $\mathcal{A}_f = \overline{W}^u(v_*)$, *with* $i = 0$ *sinks* $\underline{v}, \bar{v}$; *with* $i = 1$ *saddles* B_0, B_1; *and with* $i = 2$ *sources* $U_0 = U_1, D$. *See also sections 4, 5 for this notation.*

conclusion, is that any 2-sphere finite regular CW-complex does arise as the boundary of the unstable manifold of the unique equilibrium v_* with Morse index 3 in a suitable 3-ball Sturm attractor $\mathcal{A}_f = \operatorname{clos} W^u(v_*)$.

As a first example we consider the well-studied 3-dimensional Chafee-Infante attractor $\mathcal{A}_f$ of (1.1), which arises for cubic $f(u) = \lambda u(1-u^2)$ and parameters $4\pi^2 < \lambda < 9\pi^2$. It consists of the unstable manifolds of the 3-dimensionally unstable equilibrium $v_* \equiv 0$ and of 2 equilibria each, for Morse indices $i = 0, 1, 2$. The Sturm permutation is $\sigma_f = (1\ 6\ 3\ 4\ 5\ 2\ 7)$. See fig. 1.3 for a sketch of the Sturm complex $\mathcal{C}_f$. Note how the 2-sphere boundary consists of two closed 2-cells which are 2-gons.

The remaining paper is organized as follows. In section 2 we sketch the proof of theorem 1.1 on the regular dynamic complex $\mathcal{C}_f$ of Sturm global attractors $\mathcal{A}_f$. In section 3 we adapt the planar results of [FiRo08, FiRo09, FiRo10] to prove theorem 1.2 on planar Sturm complexes $\mathcal{C}_f$. Section 4 outlines our strategy of proof for the if-part of Sturm ball theorem 1.3. Section 5 establishes the heteroclinic effects on the two additional saddle-node hyperbolic equilibria $C = v_* \equiv 0$ and D introduced in outline section 4, respectively. This will complete the proof of theorem 1.3. We conclude with a brief discussion of some examples, refinements, and desiderata, in section 6.

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2 The Sturm complex

In this section we prove theorem 1.1, i.e. we show that every Sturm global attractor

$$\mathcal{A}_f = \bigcup_{v \in \mathcal{E}_f} W^u(v) \tag{2.1}$$

with only hyperbolic equilibria $\mathcal{E}_f = \{v_1, \ldots, v_N\}$ is a finite regular CW-complex with closed cells

$$\bar{c}_j = \operatorname{clos} W^u(v_j)\,, \tag{2.2}$$

$j = 1, \ldots, N$.

Finiteness is obvious. By the Schoenflies result [FiRo13] we already know that all $\bar{c}_j = h_j(\bar{B}^{i(v_j)})$ are indeed homeomorphic images of closed balls $\bar{B}^{i(v_j)}$ of the Morse index dimension $i(v_j) = \dim W^u(v_j)$. The cell interiors $c_j = h_j(B^{i(v_j)}) = W^u(v_j)$ are disjoint, by definition of the unstable manifolds $W^u(v_j)$. The cell boundaries

$$\partial\bar{c}_j = h_j(S^{i(v_j)-1}) \tag{2.3}$$

are homeomorphic images of the sphere boundaries $S^{i-1} = \partial B^i$. In lemma 2.2 below we show

$$\partial W^u(v_-) = \bigcup_{v_+ :\, v_- \leadsto v_+} W^u(v_+)\,, \tag{2.4}$$

i.e. the boundary of the unstable manifold of any equilibrium v_- consists of precisely the unstable manifolds $W^u(v_+)$ of all those other equilibria v_+ which v_- connects to, heteroclinically.

Already Henry and Angenent independently showed that stable and unstable manifolds of hyperbolic equilibria in Sturm systems are necessarily transverse; see [An86, He85]. This Morse-Smale property is a surprising consequence of the zero number Sturm property. By a simple dimension count, $v_- \leadsto v_+$ then implies

$$i(v_-) > i(v_+)\,. \tag{2.5}$$

In other words, the boundary of any cell is contained in the lower-dimensional skeleton of $\mathcal{C} = \bigcup_{j=1}^{N} c_j$:

$$\partial \bar{c}_j \subseteq \mathcal{C}^{i(v_j)-1} \,. \tag{2.6}$$

This shows that (2.1) is a finite regular CW-complex and hence proves theorem 1.1.

To prove (2.4) and lemma 2.2 we follow a very elegant solution of the connecting orbit problem due to Wolfrum; see [Wo02]. We call two equilibria $v_\pm$ *k-adjacent*, if $z(v_+ - v_-) = k$ and there does not exist any equilibrium w with $w(0)$ strictly between $v_\pm(0)$ such that

$$z(v_+ - w) = k = z(v_- - w) \,. \tag{2.7}$$

Should such an equilibrium w exist, we call it a *blocking* equilibrium. In symbols we write $v_- \prec_k v_+$ for k-adjacent equilibria with $v_-(0) < v_+(0)$, and $v_- \succ_k v_+$ if $v_-(0) > v_+(0)$.

Lemma 2.1. [Wo02] *Let $v_\pm$ be distinct equilibria with $i(v_-) \geq i(v_+)$ and $z(v_+ - v_-) = k$. Then*

$$v_- \leadsto v_+ \tag{2.8}$$

possess a heteroclinic orbit $u(t, \cdot)$ from v_- to v_+ if, and only if, $v_\pm$ are k-adjacent and their Morse indices satisfy

$$i(v_-) > k \geq i(v_+) \,. \tag{2.9}$$

Moreover there then exists a unique heteroclinic orbit $u(t, \cdot)$ from v_- to v_+ such that

$$z(u(t, \cdot) - v_\pm) = k \tag{2.10}$$

holds for all positive and negative real t.

We observe that the existence of a blocking equilibrium w indeed blocks the existence of a heteroclinic orbit u: $v_- \leadsto v_+$, by dropping of the Sturmian zero number $z(u(t, \cdot) - w)$. For large $t > 0$, existence of u would indeed lead to a contradiction as follows:

$$\begin{aligned} k = z(v_- - w) = &\ z(u(-t, \cdot) - w) \geq \\ \geq &\ z(u(+t, \cdot) - w) = \ z(v_+ - w) = k \end{aligned} \tag{2.11}$$

and hence $z(u(t,\cdot)-w)\equiv k$ for all real t. But choosing $t=t_0$ such that $u(t_0,0)=w(0)$ at $x=0$, the Neumann boundary condition enforces a multiple zero at $x=0$, and hence $t\mapsto z(u(t,\cdot)-w)$ must drop strictly at $t=t_0$; in contradiction to (2.11). An early variant of this blocking argument can be found in [BrFi88, BrFi89] and later in [FiRo96]. The Wolfrum argument is significantly more advanced, taking into account the transversality properties of fast stable and unstable manifolds as well.

Lemma 2.2. *Let $\mathcal{A}_f$ be a Sturm global attractor, let all equilibria be hyperbolic and consider any equilibrium v_-. Then the boundary of the unstable manifold $W^u(v_-)$ is given by* (2.4) *above.*

Proof.
The right hand side $\bigcup_{v_-\rightsquigarrow v_+} W^u(v_+)$ of (2.4) is contained in the left hand side $\partial W^u(v_-)$ by the λ-Lemma, which is a general property for Morse-Smale systems. See [PaMe82] in finite dimensions and [Ol83, Haetal02] in infinite dimensions. It remains to show, conversely, that any $u_0\in\partial W^u(v_-)$ is contained in $\bigcup_{v_-\rightsquigarrow v_+} W^u(v_+)$, as required for CW-complexes. Since $\partial W^u(v_-)\subseteq\mathcal{A}_f$ are both compact and invariant, forward and backward in time, we can pass to the equilibrium $v_+ := \alpha(u_0)\in\partial W^u(v_-)$ of the α-limit set of u_0. By definition $u_0\in W^u(v_+)$, and it only remains to show

$$v_-\rightsquigarrow v_+\,. \tag{2.12}$$

We proceed indirectly. By the Wolfrum lemma 2.1, failure of (2.12) implies the existence of a blocking equilibrium w, between $v_\pm$ at $x=0$, such that

$$z(w-v_\pm)=k:=z(v_+-v_-)\,. \tag{2.13}$$

On the other hand $v_+\in\partial W^u(v_-)$ implies the existence of trajectories $u(t,\cdot)\in W^u(v_-)$ which pass as close to v_+ as we may wish, say at time $t_0=0$. For large negative $-t<0$ nonincrease of the zero number $t\mapsto z(u(t,\cdot)-w)$ then implies

$$\begin{aligned} k=z(v_--w)= &\ \ z(u(-t,\cdot)-w)\geq \\ \geq &\ \ z(u(0,\cdot)-w)= \ \ z(v_+-w)=k\,, \end{aligned} \tag{2.14}$$

quite similarly to (2.11). However, dropping of z ensues again when $u(t_0,0)=w(0)$ at the Neumann boundary $x=0$. This contradiction proves the last remaining claim (2.12), the lemma, and theorem 1.1. $\bowtie$

3 Planar Sturm complexes

In this section we prove theorem 1.2. To prove the if-part we invoke the results of [FiRo08, FiRo09]. Suppose the 1-skeleton $\mathcal{C}^1$ of the given finite planar regular CW-complex $\mathcal{C}$ is bipolar. Then [FiRo08] implies that there exists a planar Sturm global attractor $\mathcal{A}_f$ such that the 1-skeleton $\mathcal{C}_f^1$ of the associated Sturm complex $\mathcal{C}_f$ coincides with $\mathcal{C}^1$. By planarity of $\mathcal{A}_f$ and $\mathcal{C}$, however, the (open) 2-cells of the contractible complex $\mathcal{C}$ and the 2-dimensional remaining unstable manifolds $W^u(v)$ of the contractible global attractor $\mathcal{A}_f$, projected into the plane as in [Br90, Jo89, Ro91], coincide. Indeed they are both given by the bounded components, alias faces, of $\mathbb{R}^2 \setminus \mathcal{C}^1$. Therefore $\mathcal{C}_f = \mathcal{C}$, proving the if-part of theorem 1.2. The only-if-part is also easy and can be settled as follows. Let $\mathcal{C} = \mathcal{C}_f$ be the Sturm complex of the planar Sturm global attractor $\mathcal{A}_f$, as given by theorem 1.1. Since the cells of $\mathcal{C}_f$ are the unstable manifolds of $\mathcal{A}_f$, and since $\mathcal{A}_f$ is planar by [Br90, Jo89, Ro91], the Sturm complex $\mathcal{C}_f$ is planar. Moreover $\mathcal{C}_f$, as $\mathcal{A}_f$, is contractible. In lemma 3.1 we show that the 1-skeleton $\mathcal{C}_f^1$ of any, not necessarily planar, Sturm complex $\mathcal{C}_f$ is always bipolar. This settles the only-if-part of theorem 1.2.

Lemma 3.1. *Let $\mathcal{C}_f^1$ be the 1-skeleton of the Sturm complex $\mathcal{C}_f$ of a Sturm global attractor $\mathcal{A}_f$ with only hyperbolic equilibria. Then the graph $\mathcal{C}_f^1$ possesses a bipolar orientation.*

Proof.
By definition the edges of the 1-skeleton $\mathcal{C}_f^1$ of $\mathcal{C}_f$ are the one-dimensional unstable manifolds $W^u(v)$ of the saddles $i(v) = 1$. By [BrFi86], $z(u^1 - u^2) = 0$ for any two distinct u^1, u^2 on the closure of the same edge. In other words either $u^1(x) > u^2(x)$ for all $0 \leq x \leq 1$, or $u^1 < u^2$ everywhere. We can therefore define an orientation of any edge, e.g. towards larger u^j. Trivially this order extends to the two end points of any edge, i.e. to the two sinks which the saddle v connects to heteroclinically. By definition, the orientation associated to this upwards order is cycle free on $\mathcal{C}_f^1$. Therefore it only remains to identify the maximal and minimal vertices $\bar{v}$ and $\underline{v}$. We show there exists only one (locally) maximal vertex $\bar{v} \in \mathcal{C}_f^1$ with respect to our order; the argument for minima $\underline{v}$ is analogous.

Starting the PDE (1.1) with any large enough x-independent initial condition $u|_{t=0} = u_0 > 0$ and letting $t \to +\infty$, we see how $u(t, \cdot) \to \bar{v}$ has to converge to an equilibrium $\bar{v}$, which satisfies $\bar{v} > w$ for any other equilibrium $w \in \mathcal{E}_f \setminus \{\bar{v}\}$. Indeed $u_0 > w$ and $z(u(t, \cdot) - w) = 0$ cannot drop any further. In particular, $\bar{v} = v_N$ is the maximal vertex; see (1.5). Moreover $\bar{v}$ is monotonically accessible from above, and hence lies on the boundary of $\mathcal{C}_f$. General results on monotone dynamical systems, e.g. [Ma79], show stability of $\bar{v}$ and hence $\bar{v} = v_N \in \mathcal{C}^0 \subseteq \mathcal{C}^1$.

Next suppose, indirectly, that $v \neq \bar{v}$ is any other locally maximal vertex in $\mathcal{C}_f^1$. Then $v \in \mathcal{C}_f^0$ is likewise stable, by definition. Moreover $v < \bar{v}$ implies that there exists an unstable equilibrium $v < w < \bar{v}$ such that w connects to v downwards, by a monotonically decreasing heteroclinic orbit. This contradicts local maximality of v. Therefore $\bar{v} = v_N$ is the unique locally maximal vertex $\bar{v} \in \mathcal{C}_f^1$. This proves the lemma, along with theorem 1.2. $\bowtie$

In [FiRo08, FiRo09] we actually associate a permutation σ to the 1-skeleton $\mathcal{C}^1$ of any planar, finite, regular CW-complex $\mathcal{C}$ with a bipolar orientation of $\mathcal{C}^1$. The permutation σ then is the Sturm permutation $\sigma = \sigma_f$ of a suitable planar Sturm global attractor $\mathcal{A}_f$. Moreover the prescribed bipolar orientation coincides with the monotonicity orientation defined in the proof of the above lemma. It is this planar construction which will prove our three-dimensional theorem 1.3, in the following sections. For many examples of this construction see [FiRo10].

The precise recipe constructs the Sturm permutation

$$\sigma := h_0^{-1} \circ h_1 \tag{3.1}$$

via a pair (h_0, h_1) of ZS-*Hamiltonian paths* with the following vertex set $\mathcal{V}$. In the planar complex $\mathcal{C}$, we call vertices of the 0-skeleton $\mathcal{C}^0$ sinks, edges of the 1-skeleton saddles, and the faces of the 2-skeleton sources. Together we represent these objects by the vertices of $\mathcal{V}$: as midpoints, for each edge, and as some interior point for each face. We construct the quadrangulation G_2 of $\mathcal{C}^1$ with these vertices: we connect each saddle with two edges to its neighboring sinks, and each source with edges to all saddles on its face boundary. See figure 3.1 (a) for an example of the quadrangulation G_2 where the 1-skeleton is a planar tetrahedron.

Following [FiRo10], we call a Hamiltonian path h_0 in the filled graph G_2 a *boundary Z-Hamiltonian path* if properties (a)-(c) below all hold. Properties (b), (c) restrict the path h_0 as it traverses the source w of any face F. Let $\ldots v_{-2}\, v_{-1}\, w\, v_1\, v_2 \ldots$ denote the vertex sequence, oriented along h_0. Then $v_{\pm 1}$ are saddles on the face boundary ∂F. The vertices $v_{\pm 2}$ are sinks or other sources outside F. If v_{-2} or v_{+2} is a sink then it belongs to ∂F. Since ∂F contains at least four vertices, and $v_1 v_2$

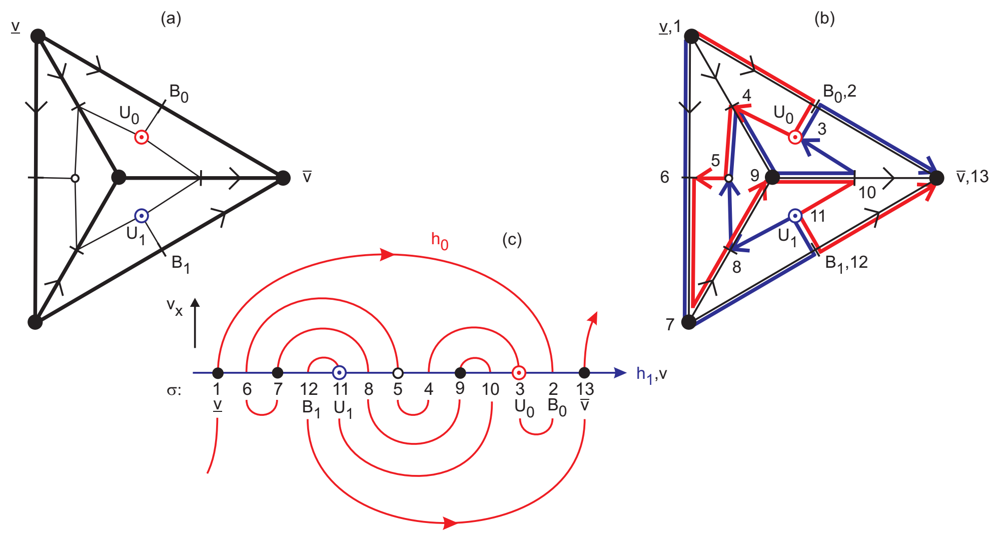

Figure 3.1: *The planar tetrahedron. Top left, (a), a bipolar orientation with minimum* $\underline{v} = v_1$ *and maximum* $\bar{v} = v_{13}$*. Thin: the quadrangulation* G_2*. Top right, (b), the associated ZS-Hamiltonian pair* (h_0, h_1) *with* h_0 *in red,* h_1 *in blue. Bottom, (c), the meander of the associated Sturm permutation* $\sigma = h_0^{-1} \circ h_1 = (1\ \ 6\ \ 7\ \ 12\ \ 11\ \ 8\ \ 5\ \ 4\ \ 9\ \ 10\ \ 3\ \ 2\ \ 13)$*. For* B_i, U_i *see section 6.*

are immediate successors, we can then speak of a clockwise or counter-clockwise direction of the arc v_1v_2 from v_1 to v_2, uniquely and similarly for $v_{-2}\,v_{-1}$. Specifically we require

(a) **"1-skeleton":**
On the 1-skeleton, h_0 follows the bipolar orientation. In particular h_0 starts at the minimal vertex and terminates at the maximal vertex.

(b) **"No right turn exit":**
Whenever $h_0 = \ldots wv_1v_2 \ldots$ exits any Morse source w of a face F, then v_1v_2 are not both on ∂F in clockwise direction.

(c) **"No left turn entry":**
Whenever $h_0 = \ldots v_{-2}\,v_{-1}\,w \ldots$ enters any Morse source w of a face F, then $v_{-2}\,v_{-1}$ are not both on ∂F in counter clockwise direction.

The letter Z graphically indicates the admissible behaviour, in case both the exit arc v_1v_2, on top, and the entry arc $v_{-2}\,v_{-1}$, on bottom, are on ∂F: right turn entry and left turn exit. Note, however, that h_0 is also permitted to connect Morse sources of adjacent faces through the bisecting Morse saddle of a shared edge, without creating an arc on ∂F.

By plain reflection κ we can also define *(boundary) S-Hamiltonian paths* h_1. We simply call h_1 an S-Hamiltonian path for G_2 if the reflected path $h_0 := \kappa\, h_1$ is Z-Hamiltonian for the reflected graph $\kappa\, G_2$. In other words the S-Hamiltonian path h_1 is neither permitted right turns, upon face entry, nor left turns upon exit. By a *(boundary) ZS-Hamiltonian pair* (h_0, h_1) we mean a Z-Hamiltonian path h_0 and an S-Hamiltonian path h_1 in G_2, both of which start at the same boundary vertex $\underline{v}$ and terminate at the same, distinct, boundary vertex $\bar{v}$ in G. See figure 3.1 (b) for the unique ZS-Hamiltonian pair associated to the bipolar orientation of the tetrahedron of figure 3.1 (a). See [FiRo10] for many more examples, including all plane Platonic polyhedra.

In fig. 3.1 (c) we illustrate the resulting Sturm permutation $\sigma = h_0^{-1} \circ h_1$ to be a *meander permutation*: it results from labelling the intersection points of a Jordan curve h_0 with the horizontal axis h_1, first sequentially along h_0, and then reading those labels off along h_1. In [FiRo99] it was proved that a general permutation $\sigma \in S_n$ is a Sturm permutation $\sigma = \sigma_f$ arising from (1.1), if and only if σ is a dissipative Morse meander. Here dissipativeness requires $\sigma(1) = 1$ and $\sigma(N) = N$. The Morse condition requires certain net windings to remain positive along the meander; see (4.7) below. From the Sturm point of view, the meander h_0 arises, in the (v, v_x)-plane, as the shooting image of the "initial" Neumann boundary, at $x = 0$, when evaluated at $x = 1$; see (1.3). Intersections with the horizontal x-axis h_1, at $x = 1$, are equilibria v_j. Labelling intersections sequentially along h_0, i.e. at $x = 0$, provides the ordering (1.5). Reading the labels along h_1, at $x = 1$, shows (1.6). The net-positive winding of h_0 amounts to nonnegative Morse indices of the equilibria. A right turn of h_0 in fact increases the Morse index by 1, from one equilibrium to the next, whereas a left turn decreases the Morse index by 1. See again fig. 3.1 (c) for an example.

4 Theorem 1.3: outline of proof

The only-if-part of theorem 1.3 is an immediate consequence of theorem 1.1, as we have already noticed. Indeed suppose $\mathcal{A}_f = \overline{W}^u(v_*)$ with $i(v_*) = 3$. Then the Sturm complex $\mathcal{C} = \mathcal{C}_f = \mathcal{A}_f = \bigcup_{v \in \mathcal{E}_f} W^u(v)$ is the closure of the single 3-cell given by $W^u(v_*)$.

The if-part of theorem 1.3 is considerably more involved. Given any regular finite CW-complex $\mathcal{C} = \bigcup c_j$ which is the closure $\mathcal{C} = \bar{c}_*$ of a single 3-cell we have to realize that complex as the Sturm complex $\mathcal{C} = \mathcal{C}_f = \bigcup W^u(v_j)$ of the Sturm global attractor $\mathcal{A}_f$, for some suitable nonlinearity f of the PDE (1.1). We will prove this by reduction to the 2-dimensional case of theorem 1.2 and the previous section. It may seem surprising, however, that we do not impose any bipolarity condition anywhere.

Our construction is based on the S^2-boundary $\partial c_* = S^2 = \mathcal{C}^2$ of the single open 3-cell c_*, i.e. on the 2-skeleton of the finite regular CW-complex $\mathcal{C}$. We now define *unbranched n-gons*, and then proceed with our construction depending on whether or not $\mathcal{C}^2$ contains any face which is an unbranched n-gon. An unbranched n-gon is a 2-cell d of $\mathcal{C}^2$ for which at most two of the $n \geq 2$ distinct 0-vertices in ∂d possess degree 3 or larger, in the 1-skeleton $\mathcal{C}^1$. Here the (edge) degree of a vertex v counts the number of edges attached to v. Vertices of degree 0 are excluded by connectedness of $\mathcal{C}^2$. Vertices of degree 1 are excluded because $\mathcal{C}^2$ is regular; see fig. 1.1 (c). Hence 2 is the minimal degree of any vertex.

In the easy case when $\partial c_* = \mathcal{C}^2$ possesses an unbranched n-gon d, we fix any pair $\underline{v}, \bar{v}$ of 0-cells in ∂d such that all $n-2$ remaining 0-cell vertices in $\partial d \setminus \{\underline{v}, \bar{v}\}$ possess degree 2. We then turn to the more demanding case when unbranched n-gons are absent, omitting further analogous details on the easier case of unbranched n-gons. The case of unbranched 2-gons arises, for example, in the 3-dimensional Chafee-Infante attractor (1.1) with cubic $f(u) = \lambda u(1-u^2)$ for $4\pi^2 < \lambda < 9\pi^2$.

Henceforth we assume absence of unbranched n-gons. We fix any open branched 2-cell n-gon $d \in \mathcal{C}^2$. We also fix any edge adjacent pair $\underline{v}, \bar{v}$ of vertices, alias 0-cells, in the 1-skeleton n-gon boundary $\partial d \subseteq \mathcal{C}^1$ of d. Then the complex

$$\mathcal{C}_* := \mathcal{C} \setminus \{c_*, d\} \subset S^2 \tag{4.1}$$

with deleted 3-cell c_* and 2-cell d is a planar finite regular contractible CW-complex, e.g. by stereographic projection from anywhere inside d.

In lemma 4.1 below we show that the 1-skeleton $\mathcal{C}_*^1$ of the planar CW-complex $\mathcal{C}_*$ can be endowed with a bipolar orientation which points towards the interior of $\mathcal{C}_*$ from all 0-cells in $\partial\mathcal{C}_* = \partial d$, except possibly at $\bar{v}$. Indeed that lemma applies because we have assumed the absence of unbranched n-gons, and because the n-gon $\partial\mathcal{C}_* = \partial d$ is homeomorphic to a circle, by regularity of the 2-sphere CW-complex $\mathcal{C}^2$. We then follow the proof of theorem 1.2, in section 3 above, to realize $\mathcal{C}_* = \mathcal{C}_{f_*}$ as the Sturm complex of a planar Sturm global attractor $\mathcal{A}_{f_*}$. In particular we derive the general form of the Sturm meander of the associated Sturm permutation $\sigma_* = \sigma_{f_*} \in S_{N-2}$; see fig. 4.1, right. The central step, then, is the insertion of two new vertices C and D to modify the permutation $\sigma_* \in S_{N-2}$ and obtain a dissipative Morse meander $\sigma \in S_N$, see lemma 4.2 below. Sturm realization [FiRo99] then implies that $\sigma = \sigma_f$ is the Sturm permutation for a suitable nonlinearity f of (1.1). In section 5 we show how the cells $c' = W^u(C)$ and $d' = W^u(D)$, respectively, do not interfere with the Sturm complex $\mathcal{C}_* = C_{f_*} \subset \mathcal{C}_f$. Rather, c' and d' attach to $\mathcal{C}_*$ precisely as the previously removed cells c_* and d did. This will then prove

$$\mathcal{C} = \mathcal{C}_* \cup \{c_*, d\} = \mathcal{C}_* \cup \{c', d'\} = \mathcal{C}_f\,, \tag{4.2}$$

and will thus establish the if-part of theorem 1.3, as well.

Lemma 4.1. *Let $\mathcal{C}_* \subseteq \mathbb{R}^2$ be a planar finite regular CW-complex such that the boundary $\partial\mathcal{C}_*$ is homeomorphic to a circle S^1. Assume $\mathcal{C}_*$ does not contain any unbranched n-gon. Let $\underline{v}, \bar{v} \in \partial\mathcal{C}_*$ be any two distinct edge adjacent 0-cells. Then there exists a bipolar orientation of the 1-skeleton $\mathcal{C}_*^1$ from $\underline{v}$ to $\bar{v}$ such that all edges between 0-cells in $\partial\mathcal{C}_* \setminus \{\bar{v}\}$ and 0-cells in $\mathcal{C}_* \setminus \partial\mathcal{C}_*$ are oriented away from $\partial\mathcal{C}_*$.*

Proof.
In the 1-skeleton $\mathcal{C}_*^1$, we first collapse all 0-cells in the S^1-boundary $\partial\mathcal{C}^*$, except $\bar{v}$, and all boundary edges among them into a single new minimum 0-cell $\underset{\sim}{v}$. The resulting planar CW-complex $\mathcal{C}_0$ is still regular with S^1-boundary. Indeed edge loops to $\underset{\sim}{v}$ do not arise, because $\underline{v}, \bar{v} \in \partial\mathcal{C}_* = S^1$ are edge adjacent on one side of S^1 and, most importantly, because $\mathcal{C}_*$ does not possess any unbranched n-gon face – which might create a loop to $\underset{\sim}{v}$.

The 1-skeleton $\mathcal{C}_0^1$ of the topological disk $\mathcal{C}_0$ is 2-connected. By [Fretal95], $\mathcal{C}_0^1$ therefore possesses a bipolar orientation from $\underset{\sim}{v}$ to $\bar{v}$. Here we use adjacency of $\underset{\sim}{v}, \bar{v}$ along a boundary edge and the precise definitions of [Fretal95]. We now expand $\underset{\sim}{v}$ again, to the boundary 0-cells $(\mathcal{C}_*^0 \setminus \{\bar{v}\}) \cap \partial\mathcal{C}_*$ and the edges of $\mathcal{C}_*^1$ among them, with the obvious edge orientation of the two boundary paths in $S^1 = \partial\mathcal{C}_*$ from $\underline{v}$ to $\bar{v}$. By definition of the bipolar orientation of $\mathcal{C}_0^1$ from $\underset{\sim}{v}$ to $\bar{v}$, all edges between 0-cells in

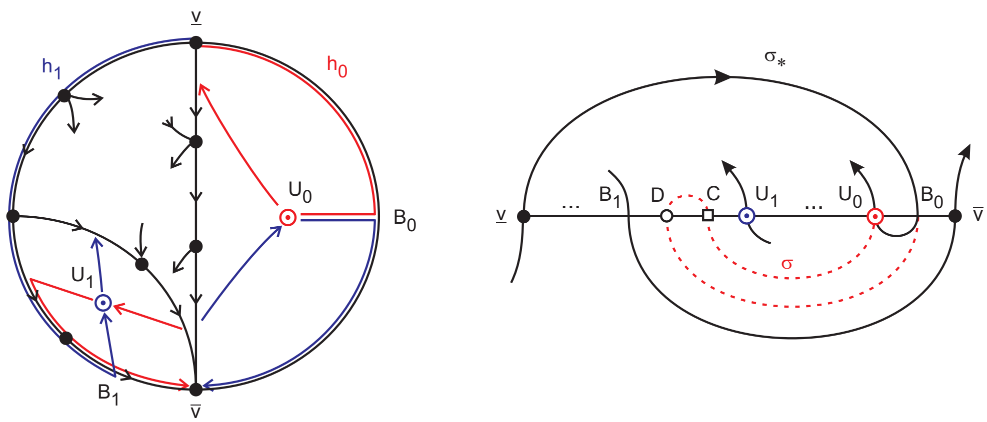

Figure 4.1: *Left: The planar oriented 2-complex $\mathcal{C}_*$ with its ZS-Hamiltonian pair h_0 (red), h_1 (blue). Right: The corresponding meander σ_*. Corresponding equilibria are labelled identically. The 3d meander σ adds the intersections C, D and is marked in dashed red.*

$\partial\mathcal{C}_* \setminus \{\bar{v}\}$, formerly just $\underline{v}$, and 0-cells in $\mathcal{C}_* \setminus \partial\mathcal{C}_*$ are now oriented away from $\partial\mathcal{C}_*$. This proves the lemma. $\bowtie$

In fig. 4.1, left, we sketch the complex $\mathcal{C}_*$ of the above construction, endowed with a bipolar orientation of the 1-skeleton as provided in lemma 4.1, and up to a reflection which does not impede the argument in any substantial way. The distinguished poles $\underline{v}, \bar{v}$ are joined by the edge of the saddle B_0. The face adjacent to this edge is U_0. We denote the face adjacent to $\bar{v}$ and the other edge B_1 in the $\mathcal{C}_*$-boundary S^1 by U_1. We recall that $\mathcal{C}_*$ is a planar Sturm complex $\mathcal{C}_* = \mathcal{C}_{f_*}$, by [FiRo08, FiRo09].

As in the examples of [FiRo10], we obtain the corresponding meander of fig. 4.1, right. Indeed the Z-Hamiltonian path h_0, alias the meander curve, must traverse the face U_0 as $\underline{v}B_0U_0\ldots$. Similarly the S-Hamiltonian path h_1 is bound to follow $\ldots U_0B_0\bar{v}$. This shows that the first two arches of h_0, from $\underline{v}$ to B_0 and from B_0 to U_0, successively hit the second to last intersection B_0, and the third to last intersection U_0, along the horizontal h_1-axis. Similar arguments in the face U_1 show that the last h_0 arch $\ldots B_1\bar{v}$ emanates from the saddle B_1 which immediately precedes the source U_1 on the path $h_1 = \ldots B_1U_1\ldots$. Indeed U_1 is the first source on h_1 because the preceding left S^1-boundary of $\mathcal{C}_*$, from $\underline{v}$ to B_1, only offers oriented edges which leave that boundary. While the Z-path h_0 could accept, the S-path h_1 has to decline any such edge seduction from the boundary. This establishes the sketch of fig. 4.1, right, for the meander σ_* of the planar Sturm complex $\mathcal{C}_* = \mathcal{C} \setminus \{c_*, d\}$.

Lemma 4.2. *Assume σ_* is a planar Sturm permutation as indicated in fig. 4.1, right. Let the paths h_0, h_1 be the ZS-Hamiltonian pair of σ_* with the following modifications for the inclusion of two new vertices C, D:*

$$h_0 = \underline{v} \dots B_0 DCU_0 \dots \bar{v}\,; \tag{4.3}$$

$$h_1 = \underline{v} \dots B_1 DCU_1 \dots \bar{v}\,. \tag{4.4}$$

In other words, the new vertices D, C are inserted between B_0, U_0 and B_1, D_1, in that order, respectively. Then $\sigma := h_0^{-1} \circ h_1$ is a dissipative Morse meander, and hence a Sturm permutation. The Morse indices of C, D are

$$i(C) = 3\,; \qquad i(D) = 2\,. \tag{4.5}$$

All other Morse indices remain unchanged.

Proof.
We have to show that h_0, h_1 define a dissipative Morse meander σ; see [FiRo99].

Since both h_0 and h_1 run from $\underline{v}$ to $\bar{v}$, the permutation $\sigma = h_0^{-1} h_1$ fixes $\underline{v}, \bar{v}$. Hence σ is dissipative.

The paths h_0, h_1 replace the arch B_0U_0, adjacent to the unmodified arch $B_1\bar{v}$, by the extended loop B_0DCU_0. Note that the new arches DB_0 and CU_0 successively follow the arch $B_1\bar{v}$ on the same side. Thus the extension of B_0U_0 to B_0DCU_0 along B_1B_0 extends the given Jordan curve meander of σ_* to a Jordan curve meander of σ. This proves σ is a meander.

It remains to show that the resulting Morse numbers i_C, i_D of the vertices C, D of σ are nonnegative. We already know

$$i_{B_0} = 1\,, \qquad i_{U_0} = 2\,, \tag{4.6}$$

because the Morse numbers coincide with the Morse indices, for the Sturm permutation σ_*, and because B_0 corresponds to an edge in the Sturm complex $\mathcal{C}_*$ while U_0 corresponds to a 2-cell face. Since right turns of the meander curve increase the Morse numbers by one, (4.6) implies (4.5). All other Morse numbers remain unchanged. Indeed this follows from the explicit recursion

$$i_{m+1} = i_m + (-1)^{m+1}\,\mathrm{sign}\,(\sigma^{-1}(m+1) - \sigma^{-1}(m)) \tag{4.7}$$

of [FiRo96], equation (2.3), where $i_m = i(v_m)$ is the Morse index of the m-th equilibrium v_m along the path h_0 at $x = 0$, and $\sigma^{-1}(m)$ describe the relative position of v_m along the path h_1, i.e. at the other boundary $x = 1$. This proves the lemma. $\bowtie$

5 The saddle-node equilibria C and D

In this section we complete the proof of theorem 1.3. Let $\mathcal{C}_f$ denote the Sturm complex of the Sturm permutation $\sigma = \sigma_f$ as constructed in the previous section from the planar reduced complex $\mathcal{C}_* = \mathcal{C}_{f_*}$. Recall how we have added two equilibria C, D to the Sturm permutation $\sigma_* = \sigma_{f_*}$ of $\mathcal{C}_* = \mathcal{C} \setminus \{c_*, d\}$, to construct σ. See fig. 4.1. In the main lemma 5.5 below we indeed show how the heteroclinic orbits, and hence the cells $c' = W^u(C)$ and $d' = W^u(D)$, respectively, attach to $\mathcal{C}_*$ as to replace the missing cells c_* and d. This shows $\mathcal{C} = \mathcal{C}_f$, as we summarize at the end of this section. We prepare the proof of lemma 5.5 in the string of lemmas 5.1–5.4. In lemma 5.1 we collect some properties of the zero number z_* on the equilibria $\mathcal{E}_* = \mathcal{E}_{f_*}$ of the reduced planar Sturm complex $\mathcal{C}_*$. Lemma 5.2 shows how zero numbers z on $\mathcal{E}_*$ coincide with those on $\mathcal{E} = \mathcal{E}_f$. Zero numbers of equilibria in $\mathcal{E}_f$ with respect to C and D, respectively, also coincide with each other. Lemma 5.3 explores some preliminary heteroclinic orbits of C in $\mathcal{C}_f$. As a final preparation, lemma 5.4 determines all zero numbers with respect to C, or D in $\mathcal{C}_f$.

We fix some notation. Whenever we write expressions like $z(v_1 - v_2)$, we tacitly assume the argument of z not to be identically zero. With z we refer to $f, \mathcal{A}_f, \mathcal{E}_f, \sigma$ while z_* refers to $f_*, \mathcal{A}_*, \mathcal{E}_*, \sigma_*$. We occasionally use the signed versions

$$z(\varphi) = n_\pm \tag{5.1}$$

for the zero number $z(\varphi) = n$: the index $+$ of n indicates $\varphi > 0$, at $x = 0$, while $-$ indicates $\varphi < 0$. We decompose the set $\mathcal{M}$ of boundary equilibria in $S^1 = \partial\mathcal{C}_*$ disjointly into

$$\mathcal{M} = \{\underline{v}, \bar{v}\} \cup \mathcal{M}_- \cup \mathcal{M}_+. \tag{5.2}$$

Here $\mathcal{M}_- := \{B_0\}$, in the notation of section 4, is the single equilibrium on the right boundary of the disk $\mathcal{C}_*$ in fig. 4.1. The other set $\mathcal{M}_+$ collects the remaining equilibria on the left boundary. In terms of fig. 4.1 we therefore arrive at the boundary orders of equilibria in $\mathcal{E}_f$ as sketched in fig. 5.1.

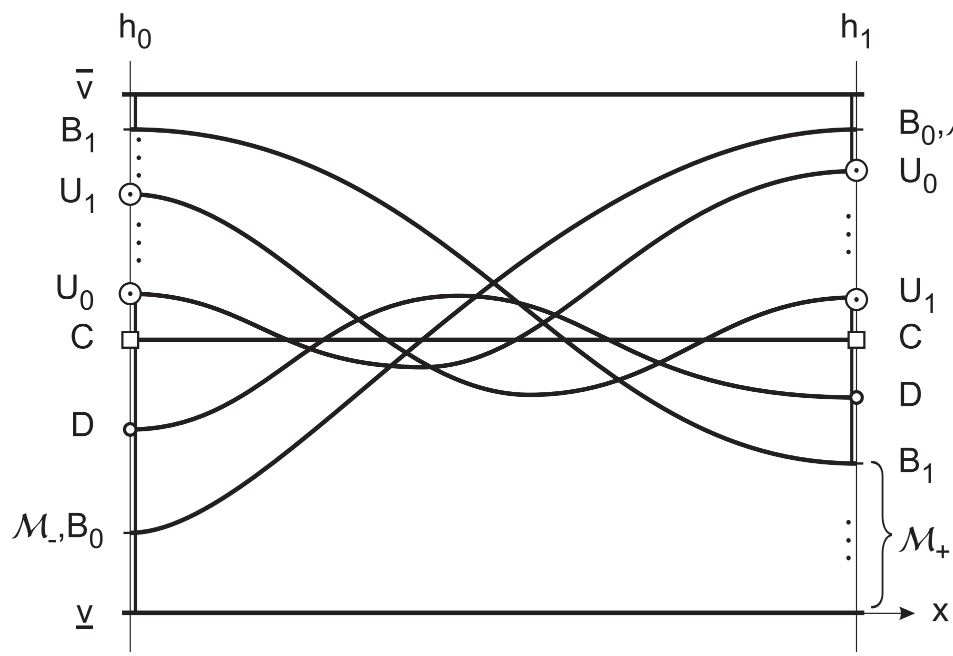

Figure 5.1: *Boundary ordering of equilibria $v = v(x) \in \mathcal{E}$ at $x = 0$, by h_0, and at $x = 1$, by h_1, according to the (h_0, h_1) meander of fig. 4.1. Vertical double lines at $x = 0, 1$ indicate the absence of intermediate equilibria.*

Lemma 5.1. *Let $v_1, v_2 \in \mathcal{E}_*$. Then*

$$z_*(v_1 - v_2) \leq 1 \tag{5.3}$$

$$z_*(v_1 - \underline{v}) = 0_+ \, ; \qquad z_*(v_1 - \bar{v}) = 0_- \, . \tag{5.4}$$

If $v_1, v_2 \in \mathcal{M}_+$, then

$$z_*(v_1 - v_2) = 0 \, . \tag{5.5}$$

Proof.
Claim (5.3) follows because the Sturm global attractor $\mathcal{A}_*$ is planar; see [Br90, Jo89, Ro91]. Claim (5.4) follows because $\underline{v}, \bar{v}$ are the lowest and highest equilibrium of $\mathcal{A}_*$, respectively, in the monotone order of $z = 0$; see also the proof of lemma 3.1. Claim (5.5) follows because the boundary 1-skeleton $S^1 = \partial\mathcal{C}_*$ possesses a strict monotone order, on either side of the poles $\underline{v}, \bar{v}$, given by the edges, alias the one-dimensional unstable manifolds, of the saddle equilibria there; see lemma 3.1 again. This proves the lemma. ⋈

Lemma 5.2. *Let $v^1, v^2 \in \mathcal{E}_*$. Then*

$$z(v^1 - v^2) = z_*(v^1 - v^2)\,, \tag{5.6}$$

$$z(v^1 - C) = z(v^1 - D)\,, \tag{5.7}$$

where all zero numbers are understood to be signed, as in (5.1)*.*

Proof.
Comparing the orders at $x = 0$ in fig. 5.1, the claims for the signed versions of z follow from the unsigned versions, which we prove next.

To prove claim (5.6) we invoke equation (2.2) of [FiRo96] for $v^1 = v_n$, $v^2 = v_m$,

$$\begin{aligned} z(v_n - v_m) = i(v_m) + &\quad \tfrac{1}{2}[(-1)^n \operatorname{sign}(\pi^{-1}(n) - \pi^{-1}(m)) - 1] + \\ + &\quad \sum_{j=m+1}^{n-1} (-1)^j \operatorname{sign}(\pi^{-1}(j) - \pi^{-1}(m))\,. \end{aligned} \tag{5.8}$$

Let us compare the results for $\pi := \sigma$ with those of $\pi := \sigma_*$. If the equilibria C, D do not appear between v_m and v_n, at $x = 0$, then (5.8) yields identical results in either case. Here we use that the Morse numbers of σ and of σ_* coincide on $\mathcal{E}_*$; see lemma 4.2.

Since C, D are adjacent at $x = 0$, the only other case to consider is that C, D both appear between $v_m, v_n \in \mathcal{E}_*$ at $x = 0$. Their contributions to the sum in (5.8) then cancel for $\pi = \sigma$, because $D = v_j$ and $C = v_{j+1}$ for some j. Again this proves claim (5.6).

To prove claim (5.7) we may assume $v^1 = v_m,\ D = v_n,\ C = v_{n+1}$, without loss of generality; see also [FiRo96], (2.6). Then (5.8) implies

$$\begin{aligned} z(C - v^1) - z(D - v^1) &= \\ &= \tfrac{1}{2}[(-1)^{n+1} \operatorname{sign}(\sigma^{-1}(n+1) - \sigma^{-1}(m))] + \\ &\quad + \tfrac{1}{2}[(-1)^n \operatorname{sign}(\sigma^{-1}(n) - \sigma^{-1}(m))] = \\ &= 0. \end{aligned} \tag{5.9}$$

This last cancellation occurs because the position $\sigma^{-1}(m)$ of v_m along h_1, i.e. at $x = 1$, cannot be between $\sigma^{-1}(n+1), \sigma^{-1}(n)$ alias adjacent C, D. This proves (5.7), and the lemma. $\bowtie$

As a corollary to lemma 5.2 we observe that equilibria $v_1, v_2 \in \mathcal{E}_* \subset \mathcal{E}_f$ with a heteroclinic orbit $v_1 \leadsto v_2$ in the extended Sturm attractor $\mathcal{A}_f$ are already connected by a heteroclinic orbit in $\mathcal{A}_*$. Indeed blocking in $\mathcal{E}_*$ implies blocking in $\mathcal{E}_f$ because the zero numbers do not change.

Lemma 5.3. *In the notation of section 4 we have*

$$z(D - C) = 2_- \,; \tag{5.10}$$

$$C \rightsquigarrow D\,; \tag{5.11}$$

$$C \rightsquigarrow U_0 \rightsquigarrow B_0 \rightsquigarrow \underline{v} \quad \text{and} \quad B_0 \rightsquigarrow \bar{v} \tag{5.12}$$

in $\mathcal{A}_f$. In particular C possesses heteroclinic orbits to, both, $\underline{v}$ and $\bar{v}$.

Proof.
By the liberalism of [FiRo96, Wo02], equilibria which are adjacent at $x = 0$, or at $x = 1$, with no other equilibria between them, possess a heteroclinic orbit running from the higher to the lower Morse index. See also lemma 2.1. In view of fig. 4.1 this proves claims (5.11) and (5.12). The heteroclinic orbit $u(t, \cdot)$ from C to D satisfies $z(u(t, \cdot) - C) < i(C) = 3$ and $z(u(t, \cdot) - D) \geq i(D) = 2$. Letting $t \to \pm\infty$ and using monotonicity of the zero number, this implies

$$z(u(t, \cdot) - C) = z(u(t, \cdot) - D) = z(D - C) = 2_- \,, \tag{5.13}$$

for all t. This proves (5.10), and the lemma. ⋈

Lemma 5.4. *Let $v \in \mathcal{E}_*$. Then*

$$z(v - D) = z(v - C) = \begin{cases} 0 & \text{if} \quad v \in \{\underline{v}, \bar{v}\}\,, \\ 1_\pm & \text{if} \quad v \in \mathcal{M}_\pm\,, \\ 2_+ & \qquad \text{otherwise.} \end{cases} \tag{5.14}$$

Proof.
In view of (5.7) we only consider $z(v - C)$. Dissipativeness of σ implies $z(v - C) = 0$ for $v \in \{\underline{v}, \bar{v}\}$. See also the proof of lemma 3.1. We show, conversely, that $z(v - C) = 0$ implies $v \in \{\underline{v}, \bar{v}\}$. We then show that $z(v - C) = 1_\pm$ if and only if $v \in \mathcal{M}_\pm$. Because [Br90, Jo89, Ro91] imply $z(v_1 - v_2) \leq 2$ on the 3-dimensional Sturm global attractor, this will prove the lemma.

We first show that $z(v - C) = 0_+$ implies $v = \bar{v}$; the arguments for 0_- and $\underline{v}$ are analogous. Indeed suppose, indirectly, that there exists $v \neq \bar{v}$ with $z(v - C) = 0_+$. Since $z(\bar{v} - v) = 0_+$, also, this implies that v blocks any heteroclinic orbit $C \rightsquigarrow \bar{v}$. This contradicts lemma 5.3, and hence proves $v = \bar{v}$.

Next we address $v \in \mathcal{M}_\pm$ to show $z(v - C) = 1_\pm$. If $v \in \mathcal{M}_- = \{B_0\}$, then $v < C$ at $x = 0$ and $v > C$ at $x = 1$. Since $z(v - C) \leq 2$ on $\mathcal{A}$ this proves $z(v - C) = 1_-$. The argument for any $v \in \mathcal{M}_+$ is analogous.

Conversely, suppose $z(v - C) = 1$. In case $v < C$ at $x = 0$, fig. 5.1 implies $v = B_0 \in \mathcal{M}_-$. Indeed any other option fails because $z(\underline{v} - C) = 0$ and $z(D - C) = 2$; see (5.10). In case $z(v - C) = 1$ and $v > C$ at $x = 0$ we must have $v < C$ at $x = 1$ and hence $v \in \mathcal{M}_+$, again by fig. 5.1. Indeed any other option $v = \underline{v}, D$ fails as above. This proves the lemma. $\bowtie$

Lemma 5.5. *The following three statements hold true.*

(i) All heteroclinic connectivity in $\mathcal{C}_*$ persists in $\mathcal{C}_f$.

(ii) $C \rightsquigarrow v$ for all $v \in \mathcal{E}_f \setminus \{C\}$.

(iii) $D \rightsquigarrow v$ if and only if $v \in \partial\mathcal{C}_* \cap \mathcal{E}_f$.

Proof.
To show claim (i) we first recall that neither the Morse indices $i(v_1)$ nor the signed zero numbers $z(v_1 - v_2)$ of $v_1, v_2 \in \mathcal{E}_*$ change when passing from $\mathcal{C}_*, \sigma_*$ to $\mathcal{C}_f, \sigma$; see lemmas 4.2 and 5.2. Since the Morse indices and zero number blocking determine all heteroclinic orbits, by [FiRo96, Wo02], we only have to show that C, D do not block any heteroclinic orbits $v_1 \rightsquigarrow v_2$ existing in $\mathcal{E}_*$; see also lemma 2.1. By lemma 5.2, (5.7) it is sufficient to show this claim for C.

First suppose $v_1 \in \mathcal{C}_* \setminus \partial\mathcal{C}_*$. Then $k := z(v_1 - v_2) = z_*(v_1 - v_2) \leq 1$, by lemma 5.1. On the other hand $z(v_1 - C) = 2$, by lemma 5.4. Therefore C does not interfere with k-adjacency of v_1, v_2, and lemma 2.1 implies $v_1 \rightsquigarrow v_2$ in $\mathcal{C}_f$ as well.

Since $i(\underline{v}) = i(\bar{v}) = 0$ are sinks, the only remaining possibility is $v_1 \in \mathcal{M}_\pm$ with $i(v_1) = 1$. Then $k := z(v_1 - v_2) = z_*(v_1 - v_2) = 0$, whereas lemma 5.4 implies $z(v_1 - C) = 1$. Again we conclude $v_1 \rightsquigarrow v_2$ in $\mathcal{C}_f$. This proves claim (i).

To show claim (ii) we first observe that any equilibrium v in the 1-skeleton $\mathcal{C}_*^1$ is on the boundary of some 2-cell face $W^u(w)$ of $\mathcal{C}_*$, because $\mathcal{C}_*$ is contractible and $\partial\mathcal{C}_* = S^1$. In particular $w \rightsquigarrow v$; see (2.4) and lemma 2.2. By transitivity of $\leadsto$ it is therefore sufficient to show $C \leadsto w$ for any $w \in \mathcal{E}_*$ with $i(w) = 2$. But then lemma 5.4 implies $z(w - C) = 2_+$. Hence $z(w - v) \leq 1$ for any other $v \in \mathcal{E}_*$ implies that v cannot block 2-adjacency of C, w. Likewise $v := D$ cannot block 2-adjacency, because $z(D - C) = 2_-$ by (5.10). Therefore lemma 2.1 implies $C \leadsto w$ and proves claim (ii).

It remains to show claim (iii), i.e. $D \leadsto v$ for any $v \in \partial\mathcal{C}_* \cap \mathcal{E}_f = \mathcal{M}_\pm \cup \{\underline{v}, \bar{v}\}$, and no others. The second claim follows from the first because $\partial W^u(D) = \partial\mathcal{C}_*$ if, and only if, $\partial W^u(D) = S^1 \supseteq \partial\mathcal{C}_*$ contains the circle $\partial\mathcal{C}_*$.

By lemma 5.4, $z(v - D) = 0$ if and only if $v \in \{\underline{v}, \bar{v}\}$. Therefore lemma 2.1 implies $D \leadsto \underline{v}, \bar{v}$.

Next consider any $v \in \mathcal{M}_\pm$. To show $D \leadsto v$, in view of lemma 2.1, we only have to show that D, v are 1-adjacent. Lemma 5.4 implies $z(v - D) = 1_\pm$ because $v \in \mathcal{M}_\pm$. We show 1-adjacency indirectly and suppose there exists a blocking equilibrium $w \in \mathcal{E}_f$ such that $z(w - D) = z(v - D) = z(v - w) = 1_\pm$. Then lemma 5.4 implies $w \in \mathcal{M}_\pm$ is in the same set as v. Since $\mathcal{M}_- = \{B_0\}$ does not accommodate both v and w, this implies $v, w \in \mathcal{M}_+$. But then $z(v - w) = 0$, as we have shown above. This contradiction proves claim (iii), and the lemma. $\bowtie$

We now summarize the proof of theorem 1.3, based on lemma 5.5 above. We have to show that the Sturm complex $\mathcal{C}_f$ constructed above coincides with the prescribed finite regular CW-complex $\mathcal{C} = \bar{c}_*$ of the single 3-cell $\bar{c}_*$.

Up to a choice of orientation, we first identify the 3-cell $c' := W^u(C)$ with the open 3-cell c_*, by an orientation preserving homeomorphism. The orientation of the 3-dimensional unstable manifold $W^u(C)$ can be reversed for that, if necessary. Indeed we may replace x by $1 - x$, i.e. $f(x, u, p)$ by $f(1 - x, u, -p)$, if necessary, to effect an orientation reversal over the span of $\{1, \cos \pi x, \cos 2\pi x\}$ which parametrizes $W^u(C)$.

Identifying $\bar{c}_* = S^2 \cup c_* = \bar{c}' = \operatorname{clos} W^u(C)$, by the Schoenflies theorem [FiRo13] we then have to show that the boundary 2-sphere complex $S^2 = \partial c_*$ coincides with the Sturm complex of $\partial W^u(C)$. After careful selection and deletion of a 2-cell d from $\partial\mathcal{C} = \partial c_*$ we have shown that

$$\mathcal{C}_* := \mathcal{C} \setminus \{c_*, d\} = \mathcal{C}_{f_*} \tag{5.15}$$

is a planar Sturm complex, in case $\partial\mathcal{C}$ does not contain any unbranched n-gon; see sections 3, 4. We have left the easier case where d can be chosen to be an unbranched n-gon to our reader. Note how the orientations of $c_* = c'$ fix the orientations of all cells of the Sturm complex $\mathcal{C}_{f_*}$ to coincide with those of the reduced complex $\mathcal{C}_*$.

The final missing 2-cell d of $\mathcal{C}$, previously deleted, closes $\partial\mathcal{C}$ to be the 2-sphere boundary $\partial c_* = \partial c'$ of the 3-cell $c_* = c'$. Likewise, the 2-cell $d' = W^u(D) \in \partial c'$ of the full Sturm complex $\mathcal{C}_f$ achieves that goal. Hence we may identify the cell closures $\bar{d} = \bar{d}'$, again by an orientation preserving homeomorphism. This completes the proof of theorem 1.3.

6 Discussion

We comment on our attempts to construct Sturm complexes $\mathcal{C}_f$ which coincide with a given CW-complex $\mathcal{C}$. By theorem 1.1, the prescribed CW-complex $\mathcal{C}$ must be finite regular. Planar Sturm complexes $\mathcal{C}$ have been characterized by bipolar orientations, in theorem 1.2. In the present section we apply the construction of sections 4, 5 to produce some 3-dimensional Sturm complexes $\mathcal{C}_f = \mathcal{C}$ which coincide with the 3-dimensional tetrahedron and octahedron, respectively. We briefly discuss other Sturm realizations of the tetrahedron, the octahedron, and the cube which do not arise from the above construction. The section concludes with some cases where $\mathcal{C}_f = \bar{c}_*$ is the closure of a single m-cell of dimension $m \geq 4$, and with the *Snoopy sandwich*: a 3-dimensional regular CW-complex which involves two 3-cells, but which is *not* of Sturm type.

As a first example we construct the 3-dimensional tetrahedron $\mathcal{C} = \mathcal{C}_f$ from the planar tetrahedron $\mathcal{C}_* = \mathcal{C}_{f_*}$ of section 3, fig. 3.1. We consider the exterior of fig. 3.1 (b) as the omitted 2-cell $d = W^u(D)$ in the 2-sphere one-point compactification of the plane. We have already indicated the relevant equilibria $\underline{v}, \bar{v}, B_0, B_1, U_0, U_1$ of the template fig. 4.1 in fig. 3.1. In fig. 6.1 (a) we construct the meander $\sigma = \sigma_f$ of the 3-tetrahedron $\mathcal{C}_f = \mathcal{C}_* \cup W^u(C) \cup W^u(D)$ by inserting the CD-loop as indicated in fig. 4.1. In particular

$$\sigma_f = (1\ 8\ 9\ 14\ 3\ 4\ 13\ 10\ 7\ 6\ 11\ 12\ 5\ 2\ 15)\,. \tag{6.1}$$

Fig. 6.1 (b) shows the resulting Hamiltonian paths h_0, h_1 in the 3-tetrahedron. Note how h_0, h_1 both traverse the (omitted) exterior 2-cell $d = W^u(D)$ before traversing the reduced planar complex $\mathcal{C}_*$. By lemma 5.4 the S^1-boundary $\partial d = \partial\mathcal{C}_* = \mathcal{M}_+ \cup \mathcal{M}_- \cup \{\underline{v}, \bar{v}\}$, indicated in green in fig. 6.1 (b) and in subsequent figures, in fact coincides with the boundary

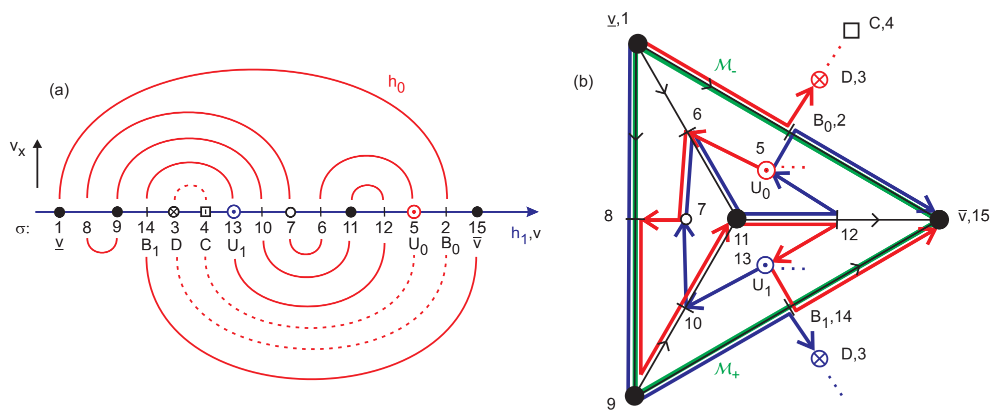

Figure 6.1: *A first 3-dimensional tetrahedron $\mathcal{C}_f$. Left, (a), the Sturm meander and permutation σ_f. Right, (b), the Hamiltonian paths h_0, h_1 in $\mathcal{C}_*$. The surface 2-sphere is the compactified plane. The exterior face $d = W^u(D)$ is represented by multiple copies of D, to be identified. The interior 3-cell $c_* = W^u(C)$ is omitted; the paths h_0, h_1 proceed from the downwell source D through C to the upwell sources U_0, U_1, respectively.*

of the 2-dimensional fast unstable manifold of C; compare [FiRo13].

There is another Sturm realization $\mathcal{C}_f = \mathcal{C}$ of the 3-tetrahedron complex $\mathcal{C}$, with the same bipolar orientation of the 1-skeleton; see fig. 6.2. The main difference is the (green) S^1-boundary of the fast unstable manifold of C. There are now two distinct downwell sources D_0, D_1 preceding C, in addition to the two distinct upwell sources U_0, U_1 following C on the Hamiltonian paths h_0, h_1, respectively. The resulting Sturm permutation reads

$$(6.2) \qquad \sigma_f = (1\ 14\ 9\ 6\ 5\ 10\ 13\ 2\ 3\ 12\ 11\ 4\ 7\ 8\ 15)\,.$$

Up to O(3)-symmetry, any other choices of (necessarily adjacent) poles $\underline{v}, \bar{v}$ and compatible bipolar orientation of the 1-skeleton $\mathcal{C}^1$ coincide with the two cases discussed above and do not lead to new Sturm realizations $\mathcal{C}_f$ of the 3-tetrahedron complex $\mathcal{C}$.

Let us consider the octahedral complex next, again in its planar variant $\mathcal{C}_*$ and in its 3-dimensional version $\mathcal{C} = \mathcal{C}_* \cup d \cup c_*$. See fig. 6.3 for the complexes (a), (c) and their meanders (b), (d), respectively. We obtain the Sturm permutation

$$(6.3) \qquad \sigma_{f_*} = (1 \quad \begin{array}{cccccccccccc} 8 & 9 & 24 & & & 23 & 20 & 19 & 10 & 7 & 6 & 11 & 18 \\ 17 & 12 & 5 & 4 & 13 & 16 & 21 & 22 & 15 & 14 & 3 & 2 & 25) \end{array}$$

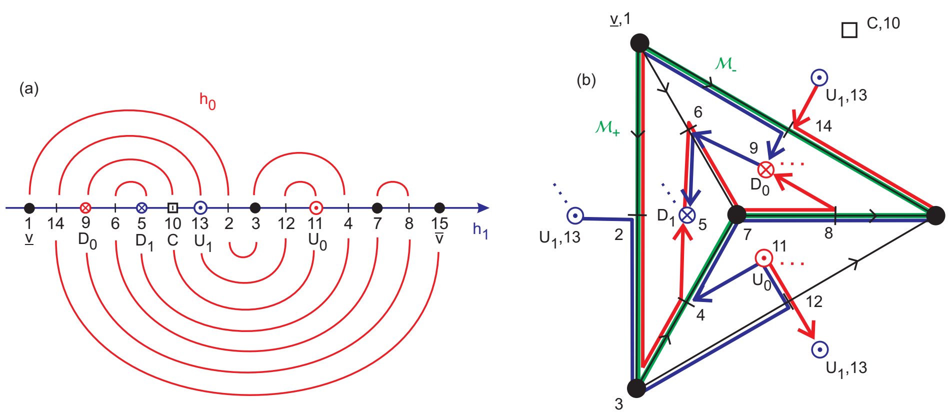

Figure 6.2: *The second Sturm realization $\mathcal{C}_f$ of the 3-tetrahedron $\mathcal{C}$. See fig. 6.1 for notation. Note the new meridian S^1-boundary $\mathcal{M}_\pm$ (green) of the fast unstable manifold of C. The three copies of U_1 are identical.*

for the planar Sturm complex $\mathcal{C}_*$ with the indicated bipolar orientation from $\underline{v}$ to the edge adjacent vertex $\bar{v}$; see fig. 6.3 (a). The construction of section 4 then yields the Sturm permutation

$$\sigma_f = (1 \;\; \begin{array}{ccccccccccccc} 10 & 11 & 26 & 3 & 4 & 25 & 22 & 21 & 12 & 9 & 8 & 13 & 20 \\ 19 & 14 & 7 & 6 & 15 & 18 & 23 & 24 & 17 & 16 & 5 & 2 & 27) \end{array} \tag{6.4}$$

for the full 3-octahedron Sturm complex $\mathcal{C} = \mathcal{C}_f$. Note how the inserted equilibria D, C with labels $3, 4$ fill the gap in the planar Sturm permutation of (6.3) and increase all original labels ≥ 3 by two.

Even for the given *meridian decomposition* $S^1 = \partial d = \mathcal{M}_+ \cup \mathcal{M}_- \cup \{\underline{v}, \bar{v}\}$ we may choose other compatible bipolar orientations of the inner triangle of the planar 1-skeleton $\mathcal{C}_*^1$. Our construction then results in variants of the Sturm permutations σ_{f_*} and σ_f. As in the case of the tetrahedron, we may also explore other meridian decompositions, where the above S^1-boundary of the 2-dimensional fast unstable manifold of $v_* = C$ is not the boundary of a single face. Again this leads to new variants of Sturm permutations σ_{f_*} and σ_f. Similarly, we may construct all 3-dimensional Platonic solids from their plane counterparts as discussed in [FiRo10].

We were very surprised, however, that we could not find a single 3-octahedral Sturm complex $\mathcal{C}_f = \mathcal{C}$ such that the polar vertices $\underline{v}$ and $\bar{v}$ were antipodes of the 1-skeleton $\mathcal{C}^1$, rather than edge adjacent. We ran into this observation by mindlessly checking pairs (h_0, h_1), which arise from 70944 Hamiltonian paths in $\mathcal{C}$ from $\underline{v}$ to its antipode $\bar{v}$, for the Sturm properties of $\sigma := h_0^{-1} \circ h_1$. Alas, to no avail. This is in marked

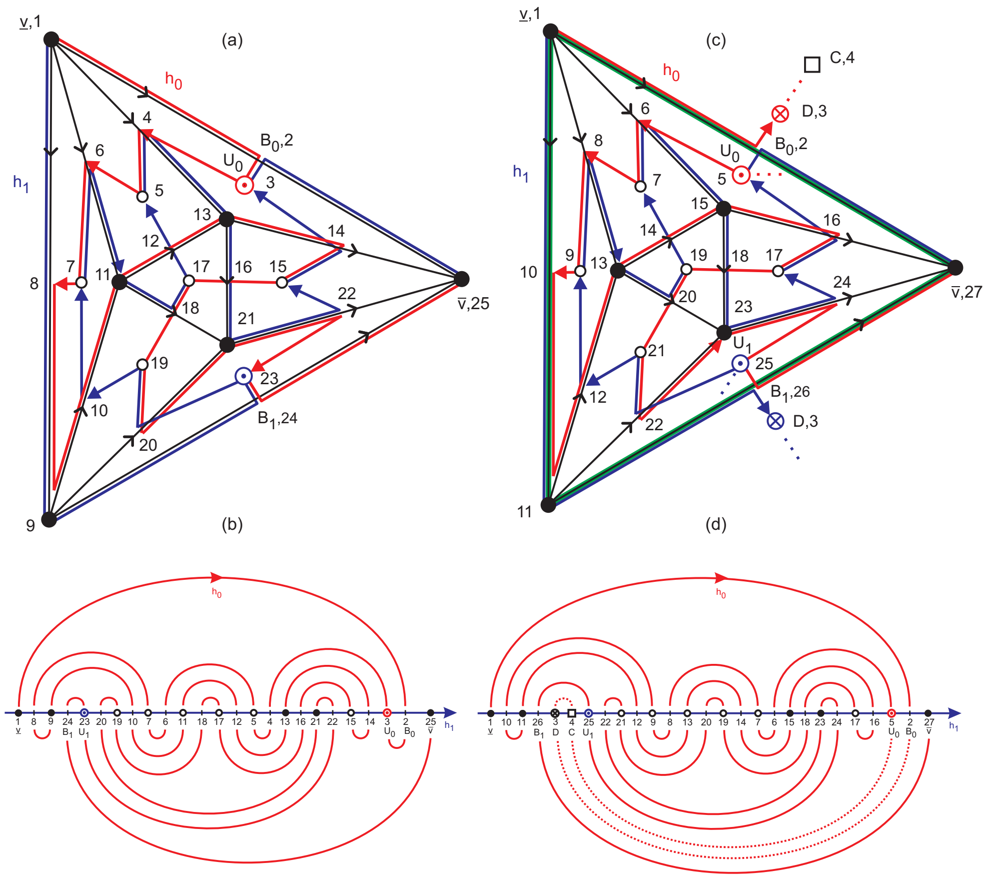

Figure 6.3: *Sturm realizations of the planar octahedron $\mathcal{C}_*$ and the 3-octahedron $\mathcal{C} = \mathcal{C}_* \cup d \cup c_*$. Top left, (a), the ZS-Hamiltonian pair (h_0, h_1) of $\mathcal{C}_*$ generated by a bipolar orientation of the 1-skeleton $\mathcal{C}_*^1$ from $\underline{v}$ to $\bar{v}$. Bottom left, (b), the associated Sturm meander and Sturm permutation σ. Bottom right, (d), the same meander, augmented by the (dashed) CD-loop. Top right, (c), the associated 3-octahedron Sturm complex $\mathcal{C} = \bar{c}_*$ with downwell face $d = W^u(D)$, $D = v_3$, with 3-cell $c_* = W^u(C)$, $C = v_4$, and upwell faces of U_0, U_1 for h_0, h_1, respectively.*

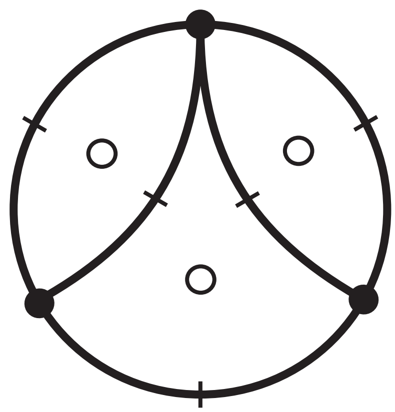

Figure 6.4: *The planar "Snoopy" Sturm global attractor.*

contrast with the 65552 Hamiltonian paths from $\underline{v}$ to adjacent $\bar{v}$, which provided a few distinct Sturm complexes which are octahedra. Also, these results are in striking contrast with the 3-cube complex, dual to the 3-octahedron which admits realizations as a Sturm complex for any relative position of the bipolarity vertices $\underline{v}$ and $\bar{v}$.

We plan to give a full explanation of these remarkable phenomena elsewhere. This requires a detailed study of the relation between

- the meridian decomposition of $S^2 = \partial c_* = \partial W^u(C)$ by the S^1-boundary of the fast unstable manifold of C,
- the bipolar orientation of the 1-skeleton $\mathcal{C}^1$, and
- the geometry of the downwell and upwell 2-cells on ∂c_* with respect to the S^1 meridian.

In particular this will allow us to fully describe all Sturm complexes $\mathcal{C} = \bar{c}_*$ which are the closure of a single 3-cell: not only as some finite regular cell complexes in the spirit of theorem 1.3, but with respect to their fine structure in terms of bipolarity and their meridian decomposition.

Impatiently we have found Sturm realizations of simplices and hypercubes in any dimension n, as well as some 4-dimensional Sturm octahedra. The n-simplex results, for example, from the meander with $2^{n-1}, 2^{n-2}, \ldots, 2, 1$ nested arches on top, next to each other, and $2^n - 1$ nested arches on bottom. See fig. 6.2 (a) for the case $n = 3$. Similarly, the n-hypercube results inductively from $3^{n-1}, 3^{n-2}, \ldots, 3, 1$ nested arches on top and $1, 3, \ldots, 3^{n-2}, 3^{n-1}$ on bottom, left to right. It may even be worth noting that these configurations are single connected curves. A more diligent and systematic approach will aim for a characterization of all 3-dimensional Sturm complexes.

As our final example, let us consider the planar "Snoopy" attractor $\mathcal{C}_*$ of fig. 6.4. It consists of three 2-cells glued to form a 2-disk as indicated. There are two realizations as a planar Sturm global attractor. Our construction above shows how to obtain the 3-cell Sturm global attractor $\mathcal{C} = \mathcal{C}_* \cup d \cup c_*$ which results when we attach the compactified Snoopy exterior d as a single 2-cell to obtain $S^2 = \partial c_*$. However, let us now glue two copies of such a Snoopy 3-cell with the 2-dimensional part $\mathcal{C}_*$ as their shared 2-cells. Then the resulting finite regular CW-complex $\mathcal{C}$, a *Snoopy sandwich,* is not a Sturm complex any more.

This simple final caveat may serve as an invitation to further explore the strange and fascinating lands of Sturm global attractors – a link between PDE global dynamics and the discrete geometry of some, but not all, finite regular CW-complexes.

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