Connectivity and Design of Planar Global Attractors of Sturm Type.

I: Bipolar Orientations and Hamiltonian Paths

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Abstract

Based on a Morse-Smale structure we study planar global attractors $\mathcal{A}_f$ of the scalar reaction-advection-diffusion equation $u_t = u_{xx} + f(x, u, u_x)$ in one space dimension. We assume Neumann boundary conditions on the unit interval, dissipativeness of f, and hyperbolicity of equilibria. We call $\mathcal{A}_f$ Sturm attractor because our results strongly rely on nonlinear nodal properties of Sturm type.

The planar Sturm attractor consists of equilibria of Morse index 0, 1, or 2, and their heteroclinic connecting orbits. The unique heteroclinic orbits between adjacent Morse levels define a plane graph $\mathcal{C}_f$ which we call the connection graph. Its 1-skeleton $\mathcal{C}_f^1$ consists of the unstable manifolds (separatrices) of the index-1 Morse saddles.

We present two results which completely characterize the connection graphs $\mathcal{C}_f$ and their 1-skeletons $\mathcal{C}_f^1$, in purely graph theoretical terms. Connection graphs are characterized by the existence of pairs of Hamiltonian paths with certain chiral restrictions on face passages. Their 1-skeletons are characterized by the existence of cycle-free orientations with only one maximum and only one minimum. Such orientations are called *bipolar* in [FMR95].

In the present paper we show the equivalence of the two characterizations. Moreover we show that connection graphs of Sturm attractors indeed satisfy the required properties. In [FiRo07a] we show, conversely, how to design a planar Sturm attractor with prescribed plane connection graph or 1-skeleton of the required properties. In [FiRo07b] we describe all planar Sturm attractors with up to 11 equilibria. We also design planar Sturm attractors with prescribed Platonic 1-skeletons.

1 Introduction

Based on a Morse-Smale structure we look at the simplest class of parabolic partial differential equations from a global dynamical systems point of view. More specifically we study the global spatio-temporal dynamics of the following scalar reaction-advection-diffusion equation in one space dimension

$$u_t = u_{xx} + f(x, u, u_x). \tag{1.1}$$

Here $t \geq 0$ denotes time, $0 < x < 1$ denotes space, and we seek solutions $u = u(t,x) \in \mathbb{R}$. To be completely specific we also fix Neumann boundary conditions

$$u_x = 0 \;\text{ at }\; x = 0, \;\text{and}\; x = 1. \tag{1.2}$$

1-gon Our results will hold analogously, though, for other separated boundary conditions.

For nonlinearities $f = f(x, u, p)$ of class C^2, standard theory provides a local solution semigroup $u(t, \cdot) = \mathcal{T}(t)u_0, t \geq 0$, on initial conditions $u_0 \in X$. For the underlying Banach space X we choose the Sobolev space H^2, intersected with the Neumann condition (1.2). See for example [Ta79, He81, Pa83] for a general background.

Starting with Ladyzhenskaya, the *global attractor* $\mathcal{A} = \mathcal{A}_f$ of the semigroup $\mathcal{T} = \mathcal{T}_f$ has developed into a central object of study. Assume

$$f \in C^2 \text{ is dissipative.} \tag{1.3}$$

Here *dissipativeness* requires that there exists a fixed large ball in X, in which any solution $u(t, \cdot) = \mathcal{T}(t)u_0$ stays eventually, for all $t \geq t(u_0)$. In particular solutions exist globally for all $t \geq 0$. Since our dissipative semigroup $\mathcal{T}(t)$ is also compact the global attractor $\mathcal{A}$ possesses the following three equivalent characterizations

$$\begin{aligned} \mathcal{A} \;&=\; \text{the smallest set attracting all bounded sets} \\ &=\; \text{the largest compact invariant set} \\ &=\; \text{the set of all bounded solutions } u(t,\cdot), t \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Attractivity requires that for every $\delta > 0$ and any bounded subset $\mathcal{B}$ of X there exists a time $t_0 = t_0(\delta, \mathcal{B})$ such that $\mathcal{T}(t)\mathcal{B}$ remains in a δ-neighborhood of $\mathcal{A}$ for all $t \geq t_0$. *Invariance* is understood in both positive and negative time direction. Negative time invariance of $\mathcal{A}$ requires the existence of a past history $u(-t) \in \mathcal{A}$, $t \geq 0$, for any $u_0 = u(0) \in \mathcal{A}$, such that $\mathcal{T}(t)u(-t) = u_0$, for all $t \geq 0$. Similarly, the set of all bounded solutions is understood to consist of all $u_0 \in X$ with uniformly bounded forward orbit $u(t)$ and some uniformly bounded past history $u(-t)$, for all $t \geq 0$. For broad surveys on the theory of global attractors we refer to [BaVi92, ChVi02, Ed&al94, Ha88, Ha&al02, La91, Ra02, SeYo02, Te88] and the many references there. The specific attractors arising from our setting (1.1), (1.2) we call *Sturm attractors*. An explicit sufficient, but not necessary, condition for $f = f(x,u,p)$ to be dissipative is a sign condition $f(x,u)0 \cdot u < 0$, for large $|u|$, together with subquadratic growth of $f(x,u,p)$ in the gradient variable p.

Zelenyak [Ze68] first noted the *gradient-like structure* of the semigroup $\mathcal{T}(t)$; see also [Ma78, Ma88]. In fact there exists a Lyapunov function $\mathcal{V}$ of the form

$$\mathcal{V}(u) = \int_0^1 a(x,u,u_x)dx \tag{1.5}$$

which is strictly decreasing with time t along all solutions $u(t,\cdot) = \mathcal{T}(t)u_0$, except at equilibria. For nonlinearities $f = f(x,u)$ which do not contain advection terms u_x a well-known explicit form of a is $a(x,u,p) = \frac{1}{2}p^2 - F(x,u)$ with primitive $F_u := f$.

To exclude degenerate cases we assume *hyperbolicity* of all equilibria

$$0 = v_{xx} + f(x,v,v_x) \tag{1.6}$$

of (1.1) with Neumann boundary conditions $v_x = 0$ given by (1.2). As usual hyperbolicity of v means that the linearized Sturm-Liouville eigenvalue problem

$$\lambda u = u_{xx} + f_p(x,v(x),v_x(x))u_x + f_u(x,v(x),v_x(x))u, \tag{1.7}$$

again with Neumann boundary (1.2), possesses only the trivial solution $u \equiv 0$ for $\lambda = 0$. We call the number of positive eigenvalues λ the *unstable dimension* or *Morse index* $i = i(v)$ of the equilibrium v. We number eigenvalues $\lambda = \lambda_k$ such that

$$\lambda_0 > \ldots > \lambda_{i-1} > 0 > \lambda_i > \lambda_{i+1} > \ldots \tag{1.8}$$

Hyperbolic equilibria v come equipped with local *unstable* and *stable manifolds* $W^u(v)$ and $W^s(v)$ of dimension and codimension $i(v)$, respectively. As a consequence of the Lyapunov function (1.5), the Sturm attractors $\mathcal{A}$ of (1.1), (1.2) consist of just all unstable manifolds,

$$\mathcal{A} = \bigcup_{v\in\mathcal{E}} W^u(v), \tag{1.9}$$

where $\mathcal{E} = \{v_1, \ldots, v_N\}$ denotes the set of all equilibria. Note that $\mathcal{E}$ is finite by dissipativeness of f and hyperbolicity of equilibria. Morse inequalities, Leray-Schauder degree, or a shooting argument in fact show that N is odd. To prove (1.9) just observe that the α-limit set of any bounded past history in $\mathcal{A}$ must consist of a single equilibrium, due to the gradient-like structure and hyperbolicity. For the same reason, but now going forward in time, the ω-limit set of any forward bounded solution is a single equilibrium. Therefore the global attractor $\mathcal{A}$ consists entirely of equilibria and *heteroclinic orbits* $u(t,\cdot)$ which converge to different equilibria for $t \to \pm\infty$.

The *Morse-Smale property* requires transverse intersections of all stable and unstable manifolds of equilibria in addition to hyperbolicity and the gradient-like structure. It was a celebrated result of Angenent and Henry, independently, that this Morse-Smale transversality is, not an additional requirement but, a consequence of hyperbolicity of equilibria; see [He85, An86]. Surprisingly this fact is based on a generalization of the *Sturm nodal property*, first observed by [St1836] and very successfully revived by [Ma82]. Let $z(u) \le \infty$ denote the number of strict sign changes of $u \in X \setminus \{0\}$. Let $u^1(t,\cdot)$, $u^2(t,\cdot)$ denote any two nonidentical solutions of (1.1), (1.2). Then

$$t \mapsto z(u^1(t,\cdot) - u^2(t,\cdot)) \tag{1.10}$$

is finite, for any $t > 0$, nonincreasing with t, and drops strictly whenever multiple zeros $u^1 = u^2$, $u^1_x = u^2_x$ occur at any t_0, x_0. See [An88]. See [Fi94, FiRo96, FiRo99, FiRo00, FiSche03, Ga04, Ra02] for aspects of nonlinear Sturm theory. It is for this property, central to the entire analysis in the present paper, that we use the term Sturm attractor for the global attractors of (1.1), (1.2).

The main goal of the present paper is a description of all two-dimensional Sturm attractors $\mathcal{A}$, i.e., of all global attractors $\mathcal{A}_f$ of (1.1), (1.2), for dissipative nonlinearities f such that all equilibria are hyperbolic of Morse index at most two.

Our description will be based on the *connection graph* $\mathcal{C}_f$ of the global attractor $\mathcal{A}_f$. Vertices of $\mathcal{C}_f$ are the N equilibria $v_1, \ldots, v_N \in \mathcal{E}_f$ of $\mathcal{A}_f$. An edge of $\mathcal{C}_f$ between v_j, v_k indicates the existence of a heteroclinic orbit between equilibria v_j, v_k of adjacent Morse index $i(v_j) = i(v_k) \pm 1$. By Morse-Smale transversality of stable and unstable manifolds, heteroclinic orbits can only run from higher to strictly lower Morse indices. Therefore the connection graph $\mathcal{C}_f$ comes with a natural flow-defined edge orientation: edges can be oriented from higher to lower Morse index. As an aside we already note here that heteroclinic orbits between adjacent Morse levels turn out to be unique, whenever they exist, in the Sturm setting (1.1), (1.2).

We restrict attention to adjacent Morse levels, for the following two reasons. First, Morse-Smale systems possess a *transitivity property* of heteroclinic connections. Let $v_1 \leadsto v_2$ indicate that there exists a heteroclinic orbit from v_1 to v_2. Then $v_1 \leadsto v_2$ and $v_2 \leadsto v_3$ implies $v_1 \leadsto v_3$. The proof is based on the λ-Lemma; see for example [PdM82]. Second and conversely, special to the Sturm setting (1.1), (1.2), suppose $v_k \leadsto v_0$ with $i(v_k) = i(v_0) + k$. Then there exist further equilibria $v_1, \ldots, v_{k-1}$ such that $i(v_j) = i(v_0) + j$ and $v_k \leadsto v_{k-1} \leadsto \ldots \leadsto v_1 \leadsto v_0$ connect through successively adjacent Morse levels. This *cascading principle* was first observed in [BrFi89]; see also [Wo02]. Together, transitivity and cascading imply that our graph $\mathcal{C}_f$ of Morse-adjacent heteroclinic connections settles the question, for any pair of equilibria, of whether or not there exists a heteroclinic connection.

As a simplified variant of the full connection graph $\mathcal{C}_f$ we also introduce its undirected 1-*skeleton* $\mathcal{C}_f^1$. Vertices of $\mathcal{C}_f^1$ are the sink equilibria only, i.e., the equilibria v with Morse index $i(v) = 0$. Edges of $\mathcal{C}_f^1$ are the unstable manifolds $W^u(v)$ of *saddle* equilibria, i.e., of equilibria v with $i(v) = 1$. More precisely distinct sink vertices v_j, v_k of $\mathcal{C}_f^1$ are connected by an (undirected) edge if, and only if, there exists a saddle equilibrium w such that $w \leadsto v_j$ and $w \leadsto v_k$. The 1-skeleton $\mathcal{C}_f^1$ thus ignores *source* equilibria v in $\mathcal{C}_f$, with $i(v) = 2$, together with their emanating heteroclinics to saddle targets.

Planarity of the connection graphs $\mathcal{C}_f$, $\mathcal{C}_f^1$ does not come as a surprise for two-dimensional Sturm attractors $\mathcal{A}_f$. To formulate our main results on the structure of these connection graphs we therefore collect some terminology concerning plane graphs G, next. See also [BeWi97], sections 1.6 and 11.2. We call a graph G *plane*, $G \subseteq \mathbb{R}^2$, if its vertices v_j and edges $e_{jk} = v_jv_k$ are embedded in the plane as points and continuous curves, respectively, such that edges neither intersect nor self-intersect, except possible at their vertex end points v_j, v_k. A *loop* is an edge v_kv_k with identical end points v_k; we only consider graphs without loops, below. A *multigraph* is allowed to possess several edges e_{jk}^ℓ connecting the same pair of vertices v_j and v_k. Rather than assigning an integer weight to a single edge we represent multiple edges by multiple nonintersecting curves sharing the same end point vertices. We call any multigraph G *finite*, if G consists of finitely many vertices and edges. Any finite plane multigraph G decomposes its complement $\mathbb{R}^2 \backslash G$ into finitely many connected components called the *regions* or *faces* of G. Exactly one of the regions is unbounded, and its boundary vertices and edges are called the *boundary* ∂G of G. Unless unboundedness is stated explicitly by faces we always mean bounded faces below.

Any sequence $e_{k_0k_1}^{\ell_1}, e_{k_1k_2}^{\ell_2}, \ldots, e_{k_{r-1}k_r}^{\ell_r}$ of edges is called a *walk* of length r. If all edges are distinct, the walk is called a *trail*. If, in addition, all vertices are distinct, a trail is called *path*. In the exceptional case of $k_0 = k_r$ where the first and last vertex only are allowed to coincide, a walk, trail, or path is called *closed*. A *cycle* is a closed path. A (not necessarily closed) path which visits each vertex exactly once is called a *Hamiltonian path*. A *Hamiltonian cycle*, similarly, is a cycle which is Hamiltonian.

Directed or *oriented* multigraphs are multigraphs together with an orientation for each edge. Multigraphs can be oriented, i.e., can be assigned an edge orientation. Conversely any directed multigraph can be made undirected simply by forgetting the orientation. Directed walks, trails, di-cycles (i.e. directed cycles), and Hamiltonian paths are required to traverse edges in the given orientation. A vertex $\underline{v}$ of a directed multigraph is called a *directed source* (short: *di-source*) if all its edges point away from it. If all its edges point toward $\bar{v}$, then we call $\bar{v}$ a *di-sink*. We also call $\underline{v}$ a (local) maximum and $\bar{v}$ a (local) minimum. We caution the

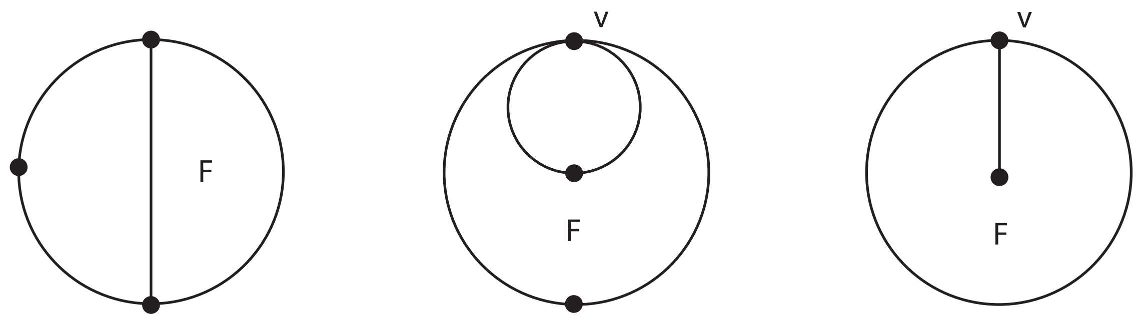

Figure 1.1: Cellular and non-cellular multigraphs. Left: cellular. Center: not cellular (doubly traversed vertex v on face boundary F). Right: not cellular (doubly traversed edge on face boundary).

reader that this notion for vertices in multigraphs differs from the Morse notion of source and sink equilibria in planar global attractors $\mathcal{A}_f$ based on their Morse index to be 0 or 2, respectively.

We call a graph G *connected*, if any two vertices v_{k_0}, v_{k_r} can be joined by a walk of suitable length r. For finite connected, plane multigraphs with N vertices, m edges and r bounded faces we recall the Euler characteristic

$$N - m + r = 1. \tag{1.11}$$

We call a plane multigraph *cellular* if each of its (bounded) faces F is bounded by an (undirected) cycle of distinct edges and vertices. See figure 1.1 for illustration. In other words, each bounded face F is the interior of a plane (topological) n-gon, for some $n \geq 2$. In particular the closure of each bounded face is homomorphic to a 2-disk.

Each boundary edge $e \subseteq \partial G$ is the boundary of at most one bounded face. Each other edge, called *interior*, is in the boundary of exactly two bounded faces. We note a slight asymmetry in the role of the unbounded face. Under compactification of $\mathbb{R}^2$ to the 2-sphere S^2, the previously unbounded open face will be homomorphic to an open 2-disk but will not necessarily become a cell of the resulting graph on S^2. The simplest connected example is the graph G of two vertices, v_1, v_2 with a single edge joining them.

We are now ready to state the first variant of our main result. We exclude the case of a trivial Sturm attractor $\mathcal{A}_f$ which consist of only one single globally attracting equilibrium.

Theorem 1.1 *A graph G is the 1-skeleton $\mathcal{C}_f^1$ of the connection graph $\mathcal{C}_f$ of some at most* *two-dimensional nontrivial Sturm attractor $\mathcal{A}_f$ with only hyperbolic equilibria if, and only if, G satisfies the following two properties:*

(i) G is a finite, connected, plane, cellular multigraph without loops, and

(ii) G possesses an orientation with exactly one di-sink $\bar{v}$ and one di-source $\underline{v}$, both on the boundary ∂G, and without di-cycles.

To say that one plane graph G "is" another plane graph $\tilde{G}$, here and below, indicates an *isomorphism*. The standard notion of graph isomorphism is a vertex bijection which preserves edges. For our plane graphs, we require a homeomorphism of the plane graph, including its bounded faces, which maps edge curves to edge curves and vertices to vertices. Combinatorially, it is sufficient to preserve face boundaries, in addition to the usual notion.

An orientation of G without di-cycles, as in part (ii) of theorem 1.1, equivalently defines a partial order on the vertices of G such that the orientation points downhill. In that sense we may call $\bar{v}, \underline{v}$ the unique minimum, maximum of this order, respectively. Such orientations are called *bipolar* with *poles* $\underline{v}, \bar{v}$ in the survey [FMR95], which also reviews several other applications.

We note that the full connection graph $\mathcal{C}_f$ and its flow-oriented variant can both be reconstructed uniquely from their 1-skeleton $\mathcal{C}_f^1$. In fact we detail this construction next for arbitrary finite, plane, cellular multigraphs G without loops. Motivated by $\mathcal{C}_f$ and its 1-skeleton we call the vertices of G *Morse sinks*. Starting from G bisect each edge by an additional vertex. Call the bisecting vertices *Morse saddles*. In each (bounded) face insert one additional vertex and call it a *Morse source*. Draw an edge from each Morse source to the n bisecting Morse saddles on the boundary of its face. We call the resulting undirected graph G_2 the *filled graph* of G. By construction G is the 1-skeleton of its filled graph G_2. Since this filling procedure decomposes each face of the 1-gon into quadrilaterals it is sometimes called a *quadrangulation*. Obviously there is a *"flow" directed* variant of this construction. We just orient bisected edges away from their bisecting Morse saddles, and edges in bounded faces of G away from their Morse sources.

To formulate our characterization of connection graphs $\mathcal{C}_f$, rather than their 1-skeletons $\mathcal{C}_f^1$, we need to introduce one last concept. Consider the filled graph G_2 of any finite, connected, plane, cellular multigraph G without loops. We call a Hamiltonian path h_0 in G_2 a *boundary Z-Hamiltonian path*, if the properties (a)–(c) below all hold. Properties (b), (c) restrict the path h_0 as it crosses through any Morse source w in a face F. Let $\ldots v_{-2}v_{-1}wv_1v_2\ldots$ denote the vertex sequence along h_0. Then $v_{\pm 1}$ are Morse saddles on the face boundary ∂F. The vertices $v_{\pm 2}$ are Morse sinks, or Morse sources other than w outside F. If v_{-2} or v_{+2} is a Morse sink then it belongs to ∂F. Since ∂F contains at least four vertices, and v_1v_2 are immediate successors, we can then speak of a clockwise or counter-clockwise direction of the arc v_1v_2 from v_1 to v_2, uniquely, and similarly for $v_{-2}v_{-1}$. Specifically we require

(a) "**Boundary**":

h_0 starts at some vertex $\underline{v}$ and terminates at another vertex $\bar{v}$, both in the boundary ∂G.

(b) "**No right turn exit**":

Whenever $h_0 = \ldots wv_1v_2 \ldots$ exits any Morse source w of a face F, then v_1v_2 are not both on ∂F in clockwise direction.

(c) "**No left turn entry**":

Whenever $h_0 = \ldots v_{-2}v_{-1}w \ldots$ enters any Morse source w of a face F, then $v_{-2}v_{-1}$ are not both on ∂F in clockwise direction.

The letter Z graphically indicates the admissible behavior in case both the exit arc v_1v_2, on top, and the entry arc $v_{-2}v_{-1}$, on bottom, are on ∂F: right turn entry and left turn exit. Note however that h_0 is also permitted to connect Morse sources of adjacent faces through the bisecting Morse saddle of a shared edge without creating arcs on ∂F at all. Also note that the reverse path $h_0^- = \ldots v_2v_1wv_{-1}v_{-2} \ldots$ of h_0 is boundary Z-Hamiltonian whenever h_0 is, albeit with reversed roles of the start and termination points $\underline{v}$ and $\bar{v}$.

By plain reflection κ we can also define (*boundary*) *S-Hamiltonian paths* h_1. We simply call h_1 S-Hamiltonian for G_2 if the reflected path $h_0 := \kappa h_1$ is Z-Hamiltonian for the reflected graph κG_2. In other words the S-Hamiltonian path h_1 is neither permitted right turns, upon face entry, nor left turns upon exit. By a (*boundary*) *ZS-Hamiltonian pair* (h_0, h_1) we mean a Z-Hamiltonian path h_0 and an S-Hamiltonian path h_1 in G_2, both of which start at the same vertex $\underline{v}$ and terminate at the same, distinct, vertex $\bar{v}$ in G. See figure 1.2 for examples

Our concept of ZS-Hamiltonian pairs (h_0, h_1) is motivated, as we shall see in sections 3, 4, by the fact that the ordering of equilibria v_k of the Sturm PDE (1.1), (1.2), alias vertices of the connection graph $\mathcal{C}_f$, by their boundary values $v_k(x)$ at $x = 0, 1$, respectively, defines a pair of Z- and S-Hamiltonian paths with properties (a)–(c).

Theorem 1.2 *A graph G_2 is the connection graph $\mathcal{C}_f$ of some nontrivial at most two-dimensional Sturm attractor $\mathcal{A}_f$ with only hyperbolic equilibria if, and only if, G_2 satisfies the following two properties:*

(i) G_2 is the filled graph of a finite, connected, plane, cellular multigraph G without loops, and

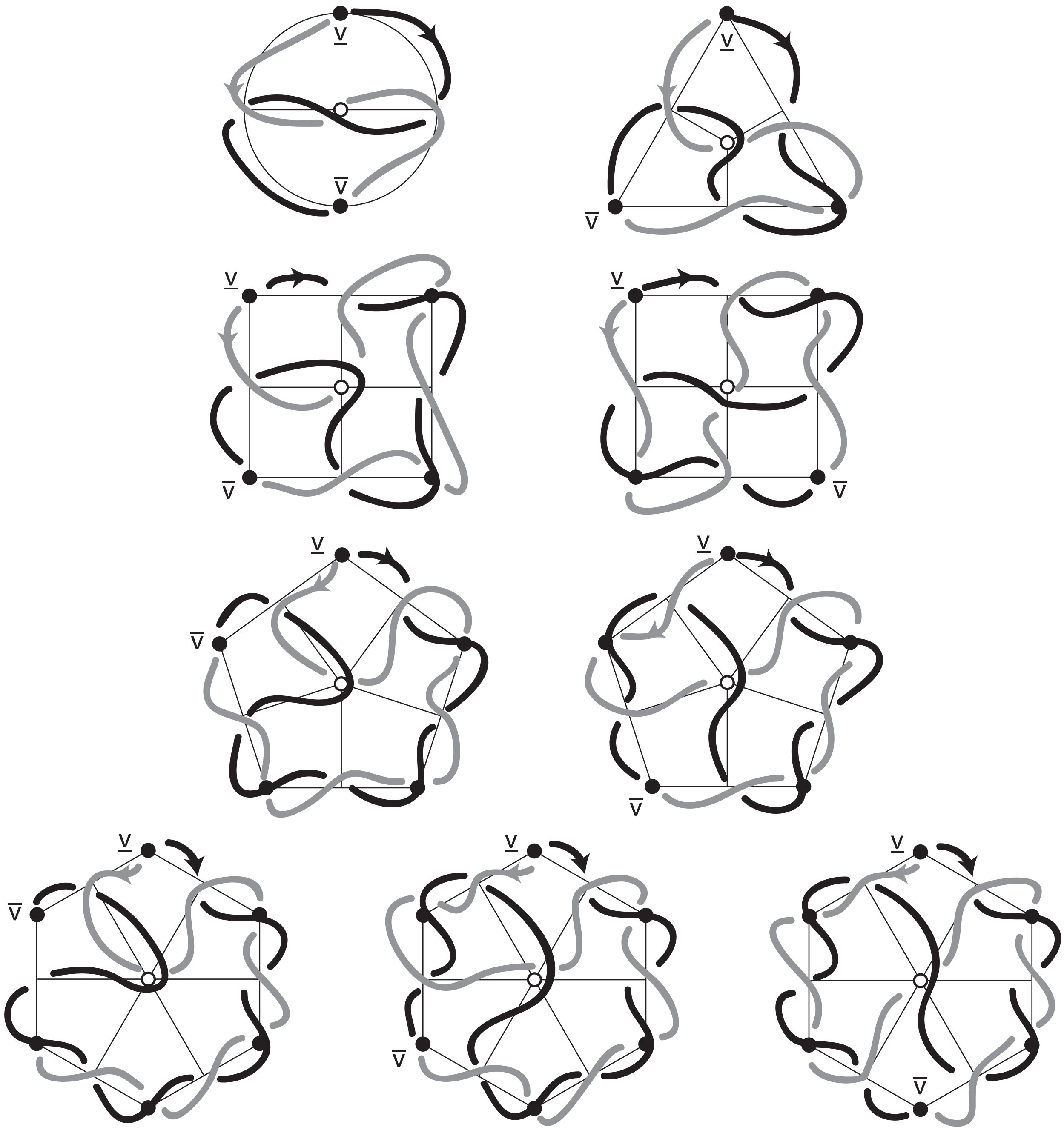

Figure 1.2: Boundary Hamiltonian pairs (h_0, h_1) for n-gons, $n = 2, \dots 6$. Path h_0 black, path h_1 gray.

(ii) G_2 *possesses a boundary* ZS*-Hamiltonian pair* (h_0, h_1) *which starts and ends at two distinct vertices* $\underline{v}$, $\bar{v}$ *in the boundary* ∂G.

The "flow" directed filled graph G_2 *then coincides with the flow directed connection graph* $\mathcal{C}_f$.

As a trivial corollary to theorem 1.2 we consider one-dimensional Sturm attractors $\mathcal{A}_f$, i.e., Sturm attractors without Morse source equilibria $i(v) = 2$. Then the connection graph $\mathcal{C}_f$, and in fact $\mathcal{A}_f$ itself, is homeomorphic to an interval with Morse sinks $\underline{v}, \bar{v}$, as end points and with Morse saddles $i(v) = 1$ alternating with Morse sinks, in between. Indeed the connection graph $\mathcal{C}_f$, in absence of any bounded faces, is a tree. By the Hamilton property of even a single Hamiltonian path h_0 the tree must be an interval as described above. The Z- and S-Hamiltonian paths h_0 and h_1 consist of identical paths $h_0 \equiv h_1$, in fact, and the attractor $\mathcal{A}_f$ is realized by spatially homogeneous ODE solutions $u(t, x)$.

Our strategy of proof for theorems 1.1 and 1.2 is the following. In Sections 2 and 3 we show the equivalence of theorems 1.1 and 1.2, based on purely graph theoretic considerations. We simply show the equivalence of properties (ii), as formulated in these theorems, for finite, plane, connected and loop-free cellular multigraphs G and their filled counterparts G_2. The point of our two formulations, later on, will be that orientations of G will define unique boundary Z- and S-Hamiltonian paths h_0 and h_1 in G_2, which can then be interpreted as the ordering of equilibria v_k in the connection graph $\mathcal{C}_f$ by their boundary values at $x = 0, 1$, respectively. In part II we survey previous results on the relation between these boundary orders h_0 h_1 and the connection graph $\mathcal{C}_f$; see [FiRo07a]. Section 4, below, provides an easy proof of the graph properties of G in theorem 1.1, for 1-skeletons $G = \mathcal{C}_f^1$ of connection graphs $G_2 := \mathcal{C}_f$ of planar Sturm attractors. The discussion in section 5 includes further graph theoretic properties of the bipolar 1-skeletons $\mathcal{C}_f^1$ following the beautiful survey [FMR95]. It then remains to prove the converse claim, namely that any pair G, G_2 with the properties of theorems 1.1 and 1.2 indeed arises as a 1-skeleton $\mathcal{C}_f^1$ and its connection graph $\mathcal{C}_f$, for some dissipative nonlinearity f with two-dimensional Sturm attractor $\mathcal{A}_f$ and hyperbolic equilibria. To achieve this goal we address the special case of an n-gon connection graph and attractor, in the sequel [FiRo07a]. We will first study the precise form of any general n-gon face within a planar Sturm attractor. We then conclude the proof of the remaining part by showing how our general construction of boundary ZS-Hamiltonian pairs (h_0, h_1) indeed gives rise to nonlinearities f with the prescribed connection graphs $\mathcal{C}_f^1$, $\mathcal{C}_f$. See section 5 for an outline of this remaining part of the proof. For illustration purposes we discuss and classify all plane Sturm attractors with up to 11 equilibria in part III [FiRo07b]. We also realize all classical plane Platonic graph there: tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

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2 From orientations to Hamiltonian pairs

In the terminology and notation of section 1 we consider plane graphs G together with their filled counterparts G_2. The current section proves the implication (i) $\Rightarrow$ (ii) of the following equivalence.

Theorem 2.1 *Let G be a finite, connected, plane, cellular multigraph without loops. Let G_2 denote the quadrangulation of G. Then the following two statements are equivalent.*

(i) G possesses an orientation G^d with exactly one di-source $\underline{v}$ and one di-sink $\bar{v}$, both in the boundary ∂G, and without directed cycles.

(ii) G_2 possesses a boundary ZS-Hamiltonian pair (h_0, h_1) between $\underline{v}$ and $\bar{v}$, in the sense of properties (a)–(c) preceding theorem 1.2

To show how (i) implies (ii) we start from an acyclic orientation G^d, alias a partial order, of the 1-skeleton G with boundary di-source $\underline{v}$ and boundary di-sink $\bar{v}$. Such orientations are called e *bipolar* with poles $\underline{v}, \bar{v}$ in [FMR95] Proposition 3.2(5) if, in addition, there is an edge e from $\underline{v}$ to $\bar{v}$. Planarity is not required there, but Hamiltonian paths are not discussed. Rather than quoting and bending we proceed directly - but with care. Of course $\underline{v}$ and $\bar{v}$, respectively, will serve as the start and termination vertex of the paths h_0, h_1 of the boundary ZS-Hamiltonian pair in the filled graph G_2. We only construct the Z-Hamiltonian path h_0, indicating the minor modifications for the S-path h_1 along our way.

The absence of di-critical vertices from the directed 1-skeleton G^d, other than the maximal start vertex $\underline{v}$ and the minimal termination vertex $\bar{v}$, will play a crucial role in our proof. We call a vertex v of a plane directed multigraph G *di-critical*, unless the edges pointing toward v and the edges pointing away from v each form nonempty and non-interspersed sets when

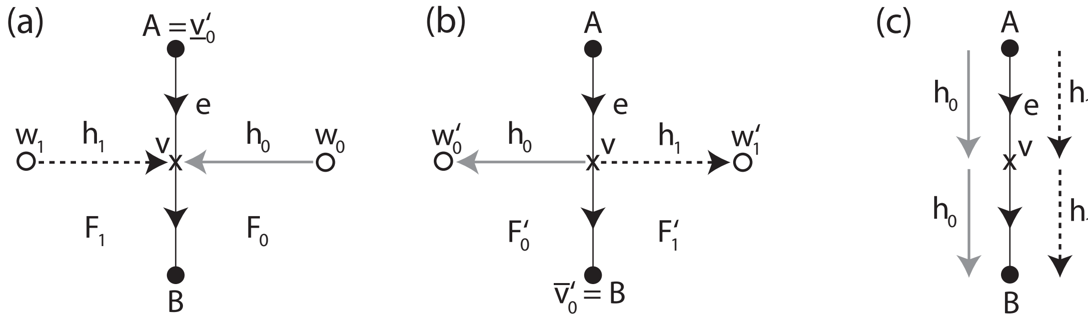

Figure 2.1: Hamiltonian paths, h_0, h_1 through Morse saddles v. For cases (a)–(c) see text.

going around v clockwise. In other words we can traverse a circle around v, say clockwise, such that we first meet all edges oriented toward v and then all edges oriented away from v. All equilibria in a plane, flow-oriented connection graph $\mathcal{C}_f$, for example, turn out to be di-critical vertices: sources, sinks, and also, by the geometry of their stable and unstable manifolds in $\mathcal{A}_f$, the saddles. The orientation G^d of the 1-skeleton G above will therefore differ fundamentally from any flow-defined orientation where any Morse-sink is a di-sink, for example.

Our poof proceeds along the following outline. Lemma 2.1 first observes the absence of di-critical vertices in the 1-skeleton G^d, other than $\underline{v}$, $\bar{v}$. Focusing on a single face F in G^d, lemma 2.2 then shows that each face boundary ∂F is oriented with exactly one maximum $\underline{v}'$ and one minimum $\bar{v}'$ within ∂F. A similar statement holds for $\underline{v}$, $\bar{v}$ on ∂G. The proof uses a duality construction to derive lemma 2.2 from lemma 2.1. See also [FMR95], section 5.

To construct the Z-Hamiltonian path h_0 from $\underline{v}$ to $\bar{v}$ in the filled graph G_2 we first specify how h_0 traverses each Morse saddle v. Let e denote the oriented edge AB of the 1-skeleton which v bisects, directed from Morse sink A to Morse sink B in G^d. See figure 2.1. The path h_0 proceeds from A to v along the given orientation of e as in figure 2.1(c), unless e is part of the boundary ∂F_0 of a bounded face F_0 *to the left of* the oriented edge e and $A = \underline{v}'_0$ in F_0. In that latter case, the path h_0 proceeds from the Morse source w_0 in F_0 to v. See figure 2.1(a). Similarly the path h_0 continues from v to B along e, unless e is part of the boundary $\partial F'_0$ of a bounded face F'_0 *to the right of* e and $B = \bar{v}'_0$ in F'_0. In that latter case, the path h_0 continues from v to the Morse source w'_0 in F'_0. See figure 2.1(b).

The modification of this construction for h_1 with faces F_1, F'_1 and Morse sources w_1, w'_1 in F_1, F'_1 is easy: we simply swap indices 0,1, switch the side requirements for F_1, F'_1 and keep all other defining properties in effect. See figure 2.1 again.

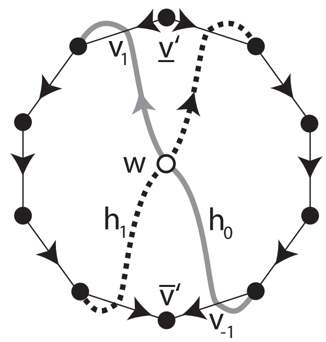

Figure 2.2: Face entry and exit of a Z-path h_0 (solid gray), and an S-path h_1 (dashed black). Continuation may not remain on face boundary.

By construction of the filled graph G_2 any edge e_2 in G_2 connects vertices of opposite Morse parity: one end point of e_2 is a Morse saddle and the other end point is either a Morse source or a Morse sink. Any Hamiltonian path h in G_2 must therefore traverse the vertices by alternating parity. In particular we can define the ZS-Hamiltonian pair (h_0, h_1) by the respective pieces of h_0, h_1 which traverse saddles, as above. In lemma 2.3 we show that the saddle pieces indeed fall into place to form a well-defined pair of Hamiltonian paths.

It is then obvious that properties (a)–(c) of boundary ZS-Hamiltonian pairs hold for (h_0, h_1). We only consider the Z-path h_0, with obvious modifications for the S-path h_1. By construction h_0 starts at the di-source $\underline{v}$ and terminates at the di-sink $\bar{v}$ in ∂G, which shows property (a). Now suppose $h_0 = \ldots wv_1v_2\ldots$ violates the "no right turn exit" property (b) at the Morse saddle $v_1 \in \partial F$, coming from the Morse source $w \in F$. Then the geometric situation of figure 2.1(a) holds, with $v = v_1$, $w_0 = w$, $F_0 = F$. Therefore $A = \underline{v}'_0$, to the right of v_1 in the clockwise direction on ∂F_0, is the maximum vertex in the oriented boundary ∂F_0. By figure 2.1(b), (c) this only leaves the possibilities $v_2 = w'_0$ or $v_2 = B$ for the continuation v_2 of the path $h_0 = \ldots wv_1v_2\ldots$ beyond the saddle $v = v_1$. This prevents h_0 from turning right on any face exit and thus establishes exit property (b) of a Z-Hamiltonian path. The arguments to establish entry property (c) are completely analogous.

From figure 2.1(a), (b) we also observe that the Z-Hamiltonian path h_0 enters any face F from the Morse saddle to the right of the minimum $\bar{v}'$ on ∂F and exits to the Morse saddle to the left of the maximum $\underline{v}'$ on ∂F. For an S-Hamiltonian path h_1, these sides are reversed. See figure 2.2.

To complete the proof of theorem 2.1 it only remains to prove lemmas 2.1–2.3 below.

Lemma 2.1 *Consider a graph G^d as in theorem 2.1 with boundary bipolar orientation as in* *theorem 2.1(i). Then G^d does not contain di-critical vertices other than $\underline{v}$ and $\bar{v}$.*

Proof: Our proof is indirect. Suppose A is a di-critical vertex in G^d, other than the maximum $\underline{v}$ and the minimum $\bar{v}$. Since $\underline{v}, \bar{v}$ are the only extrema in G^d, A must possess at least four edges $e^{\pm}_{1,2}$ adjacent to A such that e^-_j, e^+_j possess incoming and outgoing orientation, respectively. Moreover, these four edges must be interspersed: outgoing edges must alternate with incoming ones, when inspected cyclically around A. See figure 2.3. Following any oriented paths downward, along the outgoing directions e^+_j, until these paths first meet each other, we obtain an (undirected) closed Jordan curve Γ^+ with maximum at A. Indeed G^d contains only one minimum $\bar{v}$, and no directed cycles. Climbing e^-_j paths upward, against the orientation of G^d, we similarly construct a Jordan curve Γ^- with minimum at A. By construction, $\Gamma^+ \cap \Gamma^- = \{A\}$, in contradiction to the Jordan curve theorem. This proves the lemma. $\bowtie$

To shorten the remaining proofs, we introduce a slight variant G^* of the standard *dual graph* of G. Vertices of G^* inside ∂G are the Morse sources of the filled graph G_2 in the bounded faces of G. We replace the single vertex of the standard dual, representing the exterior of ∂G, by two vertices $\underline{v}^*$, $\bar{v}^*$ as follows. Edges e^* of G^* connect Morse sources of adjacent faces of G. Here distinct (bounded or unbounded) faces are called adjacent if their boundaries share at least one edge. We orient edges e^*, based on the oriented edge e which the adjacent faces share, such that the ordered pair (e^*, e) is oriented positively at the bisecting Morse saddle $\{v\} = e \cap e^*$. In other words e^* crosses e left to right. Then $\bar{v}^*$ terminates all edges e^* which point away from ∂G, to the outside, whereas $\underline{v}^*$ provides a start vertex for all edges e^* pointing toward ∂G from the outside. The ∂G part of lemma 2.2 below will ensure that this construction is possible in the plane without producing intersecting edges of G^*. See also figure 3.3 below and [FiRo07b] for some realistic examples.

The following elementary observations hold for our duality construction. Let G^- denote the graph G with all orientations reversed. Then

$$(G^-)^* = (G^*)^-; \tag{2.1}$$

$$G^{**} = G^- . \tag{2.2}$$

Moreover G^d, G^* does not possess directed cycles if, and only if, G^*, G^d does not possess local extrema other than $\{\underline{v}^*, \bar{v}^*\}$, $\{\underline{v}, \bar{v}\}$, respectively. Finally G^d, G^*, respectively, does not possess di-critical vertices other than $\{\underline{v}, \bar{v}\}$, $\{\underline{v}^*, \bar{v}^*\}$ if, and only if, any bounded face F^*, F of G^*, G^d possesses only one maximum $\underline{v}^{*\prime}, \underline{v}'$ and one minimum $\bar{v}^{*\prime}, \bar{v}'$ when restricting the underlying order to the boundary ∂F^*, ∂F, respectively.

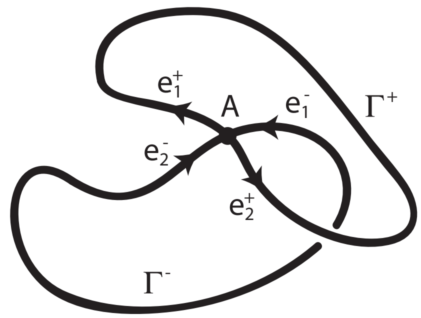

Figure 2.3: A di-critical vertex A with interspersed outgoing and incoming edges $e_{1,2}^{\pm}$.

Lemma 2.2 *Assume a boundary bipolar orientation G^d of a graph G is given, as in theorem 2.1, (i). Let F denote any bounded face of G. Then the orientation, when restricted only to the boundary ∂F, possesses exactly one di-source maximum $\underline{v}'$ and one di-sink minimum $\bar{v}'$.*

The same statement holds for the graph boundary ∂G, with $\underline{v}' = \underline{v}$ and $\bar{v}' = \bar{v}$.

Proof: We first address the statement on ∂G, indirectly and without using the above duality construction. Suppose $\underline{v}' \neq \underline{v}$ and $\bar{v}' \neq \bar{v}$ are additional local maxima and minima on ∂G (though not on G, of course). Without loss of generality we can choose $\underline{v}$, $\bar{v}'$, $\underline{v}'$ nearest possible to each other, so that $\underline{v}'$ connects to $\bar{v}'$, downhill, on ∂G, and $(\underline{v}, \underline{v}')$, $(\bar{v}, \bar{v}')$ mutually separate each other. See figure 2.4. As in the proof of lemma 2.2 we can then construct, in G^d, a descending Jordan path Γ_1 from $\bar{v}'$ to $\bar{v}$ and an ascending Jordan path Γ_2 from $\underline{v}'$ to $\underline{v}$. Here we use that the boundary minimum $\bar{v}' \neq \bar{v}$ must possess an outgoing edge because $\bar{v}$ is the unique local minimum in G^d. Analogously, $\underline{v}'$ possesses an incoming edge. By absence of di-cycles, and because $\underline{v}' > \bar{v}'$ in the orientation order of G^d, we obtain $\Gamma_1 \cap \Gamma_2 = \emptyset$, contradicting the Jordan curve theorem. This proves the claim about ∂G.

The remaining lemma, for G^d, follows from lemma 2.1 for the dual G^* introduced above.

$\bowtie$

Lemma 2.3 *The definitions of local paths h_0, h_1 through Morse saddles, as summarized in figure 2.1, define a boundary ZS-Hamiltonian pair (h_0, h_1) from $\underline{v}$ to $\bar{v}$ in the filled graph G_2. The direction of the paths h_0, h_1 respectively follows the orientation G_2^d of $G_2 \subseteq G \cup G^*$ inherited from G^d and the oriented duals G^*, G^{-*} introduced above.*

Proof: The orientation claims are immediate from figure 2.1 and the definition of the orientation of G^* and the reverse orientation G^{-*}.

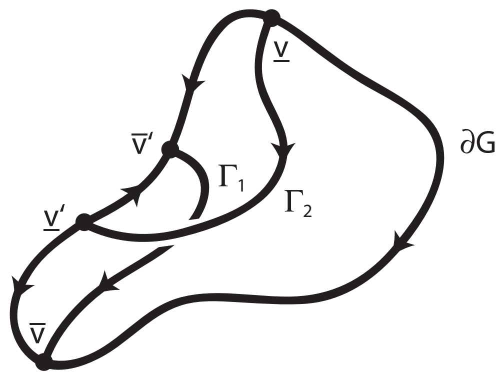

Figure 2.4: Absence of extrema on ∂G, besides $\underline{v}$ and $\bar{v}$.

By reflection, it suffices to consider h_0. It remains to show that the Morse saddle pieces of h_0 fall into place to define a Hamiltonian path from $\underline{v}$ to $\bar{v}$ in G_2. This reduces to the following two claims:

(a) the pieces glue to unique paths, locally through any Morse source and Morse sink, other than $\underline{v}, \bar{v}$;

(b) the glued pieces do not form any cycle.

We show claim (a) at Morse sources w, first. Let F denote the face of w with boundary extrema $\underline{v}', \bar{v}'$ as in lemma 2.2. Then h_0, when glued at w, traverses F from the saddle preceding $\bar{v}'$, clockwise on ∂F, to the saddle preceding $\underline{v}'$, by construction. See figure 2.2 again. This takes care of the Morse sources.

We next show claim (a) at Morse sinks v other than $\underline{v}, \bar{v}$. Duality $G \leftrightarrow G^*$ converts the claim to the previous case on G^*. We leave the instructive details to the reader.

At this stage our construction of h_0 might still consist of several connected components. All of them are paths. All will be cycles, except for one path running from $\underline{v}$ to $\bar{v}$. To prove h_0 is a Hamiltonian path we have to show absence of cyclic components. We prove claim (b) by contradiction: suppose there is a cycle h_0^c of h_0 as constructed in (a). If the interior of the cycle h_0^c contains any vertex, its component in h_0 will provide a strictly interior cycle. We can therefore descend to a case where the interior of h_0^c does not contain any further vertices of G_2. Thus $h_0^c = \partial F_2$ for some bounded face F_2 of G_2. Since $F_2 \subset F$, for some bounded face F of G, we see that F_2 must contain the Morse source w of F, as in the above analysis of case (a). The cyclic component h_0^c therefore must contain either the boundary extremum $\underline{v}'$ or $\bar{v}'$ on ∂F, together with both its adjacent vertices on ∂F. Since the orientation of h_0^c follows

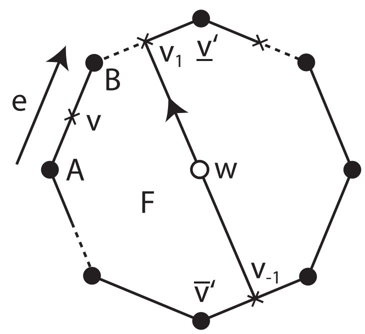

Figure 3.1: Hypothetical orientation conflict for an edge $e = AB$ on the boundary of a single face F.

the orientation of G^d, by construction, the appearance of an extremum in h_0^c contradicts it cyclicity. This proves the lemma. $\bowtie$

3 From Hamiltonian pairs to orientations

In this section we show the reverse implication, (ii) $\Rightarrow$ (i), under the assumptions of theorem 2.1. In the notation and terminology of sections 1 and 2, let h_0 be a boundary Z-Hamiltonian path in G_2 between Morse sink vertices $\underline{v}$ and $\bar{v}$ in ∂G, which satisfies properties (a)–(c) of section 1. We will then construct an orientation G^d on the 1-skeleton G of G_2, alias a partial order, with maximum $\underline{v}$ and minimum $\bar{v}$ as the only di-source and di-sink, respectively, and without di-cycles. In lemma 3.1 we will show that this orientation G^d is well-defined. In lemmas 3.2, 3.3 we show that G^d possesses the required properties: uniqueness of extrema $\underline{v}, \bar{v}$, and acyclicity. This completes the proof of theorem 2.1 and of the equivalence of theorems 1.1 and 1.2. In lemma 3.4 we observe that h_0, for the orientation G^d, coincides with the unique Z-Hamiltonian path constructed from such an order in section 2. Corollary 3.5 then provides a variant of theorem 1.2 by showing that the existence of a single boundary Z-Hamiltonian path h_0 in G already entails the existence of a boundary ZS-Hamiltonian pair (h_0, h_1).

The orientation of G^d induced by any boundary Z-Hamiltonian path h_0 in G_2 from $\underline{v}$ to $\bar{v}$ in ∂G is constructed as follows.

($\mathcal{O}1$) If h_0 traverses an edge e of G completely, then orient the edge in the direction of the path h_0.

($\mathcal{O}2$) If $h_0 = \ldots v_{-1}wv_1 \ldots$ traverses the Morse source w of any face F of G, then orient the face boundary ∂F in G with unique maximum $\underline{v}'$ and unique minimum $\bar{v}'$ given by the Morse sinks immediately following the exit vertex v_1 and the entry vertex v_{-1}, respectively, in clockwise direction.

Recall figures 2.1(c), 2.2 for an illustration of cases ($\mathcal{O}1$), ($\mathcal{O}2$), respectively.

Lemma 3.1 *Properties ($\mathcal{O}1$), ($\mathcal{O}2$) above define a unique orientation G^d on the 1-skeleton G of the filled graph G_2.*

Proof: Let $e = AB$ be an edge in G between Morse sinks A, B. We proceed, case by case, depending on the number of faces adjacent to e, and on the position of e on their face boundaries.

Case 0: *The edge e is not a face boundary.*

Then property ($\mathcal{O}1$) defines the orientation of e uniquely, and without conflict to ($\mathcal{O}2$).

Case 1: *The edge e is on the face boundary of a single face F.*
Then we rule out a conflict between ($\mathcal{O}1$) and ($\mathcal{O}2$), indirectly. Without loss of generality, we have to consider the geometric configuration of figure 3.1, possibly with $A = \bar{v}'$. By the Jordan curve theorem, however, the neighboring arcs AvB and $v_{-1}wv_1$ of h_0 must have opposite, rather than parallel, orientations. This contradiction rules out an orientation conflict on e.

Case 2: *The edge $e = AB$ is on the common face boundary of two adjacent faces F_1 and F_2.*
Let $\underline{v}'_j$ and $\bar{v}'_j$ denote the boundary maximum and minimum, respectively, as defined for face F_j, $j = 1, 2$, in property ($\mathcal{O}2$). Without loss of generality we have to exclude three subcases of potential orientation conflicts, in the geometric configurations of figure 3.2

Case 2.1: $\underline{v}'_1 \neq B$ and $\underline{v}'_2 \neq A$. Then the edge $e = AB$ must be contained entirely in the path h_0 and conflict is excluded by the argument of case 1.

Case 2.2: $\underline{v}'_1 = B$ and $\underline{v}'_2 = A$. Then the path h_0 joins the Morse sources w_1 and w_2 of the faces F_1 and F_2 via the bisecting Morse saddle v of AB. However, an orientation conflict of the Hamiltonian path h_0 itself arises at v, and excludes the present case.

Case 2.3: $\underline{v}'_1 = B$ and $\underline{v}'_2 \neq A$. Then the Z- Hamiltonian path $h_0 = \ldots w_1 v_{11} v_{12} \ldots$ cannot exit F_1 toward $v_{12} = w_2$ and therefore must proceed to $v_{12} = A$. The resulting parallel orientations of the neighboring h_0-arcs $v_{11}A$ and $w_2 v_{21}$ then provide a contradiction to the Jordan curve theorem, as in case 1. The case $\underline{v}'_1 \neq B$ and $\underline{v}'_2 = A$ is analogous.

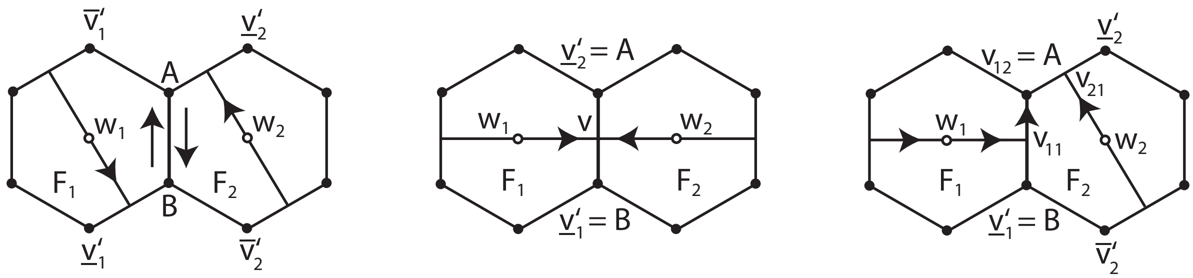

Figure 3.2: Hypothetical orientation conflicts for an edge $e = AB$ on the common boundary of two adjacent faces F_1 and F_2. Path h_0 in gray.

The above contradictions in all cases rule out all possible orientation conflicts, and complete the indirect proof of the lemma. $\bowtie$

By lemma 3.1, the Z-Hamiltonian path h_0 *generates* the orientation of G defined by ($\mathcal{O}1$), ($\mathcal{O}2$). We may then ask whether the direction of the path h_0 coincides with that orientation whenever h_0 progresses along part of an edge in G^d. The following lemma provides an affirmative answer.

Lemma 3.2 *The orientation G^d of G defined by properties ($\mathcal{O}1$) and ($\mathcal{O}2$) above is compatible with the direction of its generating Z-Hamiltonian path h_0. In particular this identifies $\underline{v}$ and $\bar{v}$ as the only possible di-source and di-sink in G^d.*

Proof: By property ($\mathcal{O}1$), the directions of h_0 and an edge e in G^d coincide, whenever that edge is traversed completely. By property ($\mathcal{O}2$), the directions also coincide, when e is traversed only partially; see figure 2.2. Since h_0 passes through any other vertex, only $\underline{v}$ and $\bar{v}$ are candidates for di-extrema. This proves the lemma. $\bowtie$

The above lemma does not yet prove that $\underline{v}$ is in fact a di-source, and $\bar{v}$ a di-sink. We address this question together with the absence of cycles.

Lemma 3.3 *The above orientation G^d of G does not possess directed cycles. In particular, $\underline{v}$ is the only di-source and $\bar{v}$ is the only di-sink in G^d.*

Proof: We first prove absence of cycles, indirectly. Suppose G^d contains a di-cycle Γ. Following forward or backward paths to the interior of Γ, if available, until they close up, we may descend to smaller cycles until we reach a minimal cycle Γ' without interior vertices of G. Here we use that such maximal directed paths may not terminate, except at a di-source or a di-sink. By lemma 3.2 these would have to coincide with $\underline{v}$, $\bar{v}$, which lie on the boundary of G, and hence outside the cycle Γ. Any minimal cycle Γ' then is a face boundary ∂F. By orientation property $(\mathcal{O}2)$, however, all face boundaries are acyclic. This proves absence of di-cycles.

Since G^d is an oriented graph without di-cycles, we may descend along any maximal directed path in G following the orientation. Any such path terminates at a di-sink, only. Since the possible di-sink $\bar{v}$ is the only candidate, by lemma 3.2, the path terminates at $\bar{v}$ and $\bar{v}$ is indeed a di-sink. Similarly, paths in G which ascend against the orientation terminate at the only di-source $\underline{v}$. This proves the lemma, and theorem 2.1. $\bowtie$

In section 2, we have constructed a unique boundary Z-Hamiltonian path h_0 in G_2 with properties (a)–(c) of section 1, from a given orientation G^d of the 1-skeleton G with di-source $\underline{v}$ and di-sink $\bar{v}$ on ∂G. In the present section, conversely, we have constructed a unique orientation G^d of G with properties $(\mathcal{O}1)$, $(\mathcal{O}2)$, from a given Z-Hamiltonian path h_0 in G_2 running between boundary Morse sinks $\underline{v}$ and $\bar{v} \in \partial G$. We now show that these constructions are inverses to each other.

Let $\mathcal{O}_{\underline{v},\bar{v}}(G)$ denote the class of orientations d of G with exactly one di-sink $\underline{v} \in \partial G$, one di-source $\bar{v} \in \partial G$, and without directed cycles. Let $Z\mathcal{H}_{\underline{v},\bar{v}}(G_2)$ denote the class of boundary Z-Hamiltonian paths of the quadrangulation G_2 from $\underline{v} \in \partial G$ to $\bar{v} \in \partial G$. Let

$$\begin{array}{rrcl} H_Z\colon & \mathcal{O}_{\underline{v},\bar{v}}(G) & \to & Z\mathcal{H}_{\underline{v},\bar{v}}(G_2) \\ & d & \mapsto & h_0 \end{array} \tag{3.1}$$

denote the construction of section 2, and

$$\begin{array}{rrcl} D_Z\colon & Z\mathcal{H}_{\underline{v},\bar{v}}(G_2) & \to & \mathcal{O}_{\underline{v},\bar{v}}(G) \\ & h_0 & \mapsto & d \end{array} \tag{3.2}$$

the construction of the present section.

Lemma 3.4 *Under the assumptions of theorem 2.1 and with the above notation, D and H are inverse maps to each other:*

$$H_Z = D_Z^{-1} \tag{3.3}$$

Proof: Suppressing Z, $\underline{v}$, $\bar{v}$ let $d \in \mathcal{O}$ and define $h_0 := H(d)$ by (a)–(c) of section 2. Then $(\mathcal{O}1)$ follows from figure 2.1(c) and $(\mathcal{O}2)$ follows from figure 2.2. Therefore $d = DH(d)$.

Conversely, let $d := D(h_0)$ be defined by $(\mathcal{O}1)$, $(\mathcal{O}2)$. Then figure 2.1(c) follows from $(\mathcal{O}1)$. Figure 2.2 follows from $(\mathcal{O}2)$ and, in turn, implies figures 2.1(a),(b). Therefore $h_0 = HD(h_0)$, and the lemma is proved. $\bowtie$

Let $S\mathcal{H}_{\underline{v},\bar{v}}(G_2)$ denote the boundary S-Hamiltonian paths h_1 from $\underline{v} \in \partial G$ to $\bar{v} \in \partial G$ in the filled graph G_2. Let κ denote the plane reflection with axis through $\underline{v}$, $\bar{v}$. Then

$$S\mathcal{H}_{\underline{v},\bar{v}}(G_2) = \kappa Z\mathcal{H}_{\underline{v},\bar{v}}(\kappa G_2), \tag{3.4}$$

by construction: reflection converts Z to S. If we consider the orientation class $\mathcal{O}_{\underline{v},\bar{v}}(G)$ as represented by oriented edge curves between vertices, then we also have

$$\mathcal{O}_{\underline{v},\bar{v}}(G) = \kappa\mathcal{O}_{\underline{v},\bar{v}}(\kappa G). \tag{3.5}$$

We can then define the maps H_S and D_S by conjugation,

$$D_S := \kappa D_Z \kappa; \qquad H_S := \kappa H_Z \kappa \tag{3.6}$$

and conclude that the commutator composition

$$H_S H_Z^{-1} = \kappa H_Z \kappa^{-1} H_Z^{-1}\colon\ Z\mathcal{H}_{\underline{v},\bar{v}}(G_2) \to S\mathcal{H}_{\underline{v},\bar{v}}(G_2) \tag{3.7}$$

is a bijection. Moreover H_S and D_S are inverses, just as H_Z and D_Z were. This proves the following corollary.

Corollary 3.5 *Under the assumptions of theorem 2.1, the graph G_2 possesses a boundary S-Hamiltonian path h_1 between $\underline{v}$ and $\bar{v}$, and thus a ZS-Hamiltonian pair (h_0, h_1), if and only if it possesses a boundary Z-Hamiltonian path h_0 between these vertices.*

We conclude this section with a short meditation on duality, as introduced before lemma 2.2. Given an orientation G^d of G, as in theorem 2.1(i), we have constructed an orientation of the slightly adapted dual graph G^* with a di-source $\underline{v}^*$ and a di-sink $\bar{v}^*$ in the exterior face of G. Let G_2^* denote the quadrangulation of G^*, as G_2 was the quadrangulation of G.

Corollary 3.6 *Under the assumptions of theorem 2.1 on G or, equivalently, on the undirected dual G^*, the quadrangulated dual graph G_2^* possesses a boundary ZS-Hamiltonian pair (h_0^*, h_1^*) between $\underline{v}^*$ and $\bar{v}^*$ if, and only if, G_2 possesses such a pair between $\underline{v}$ and $\bar{v}$.*

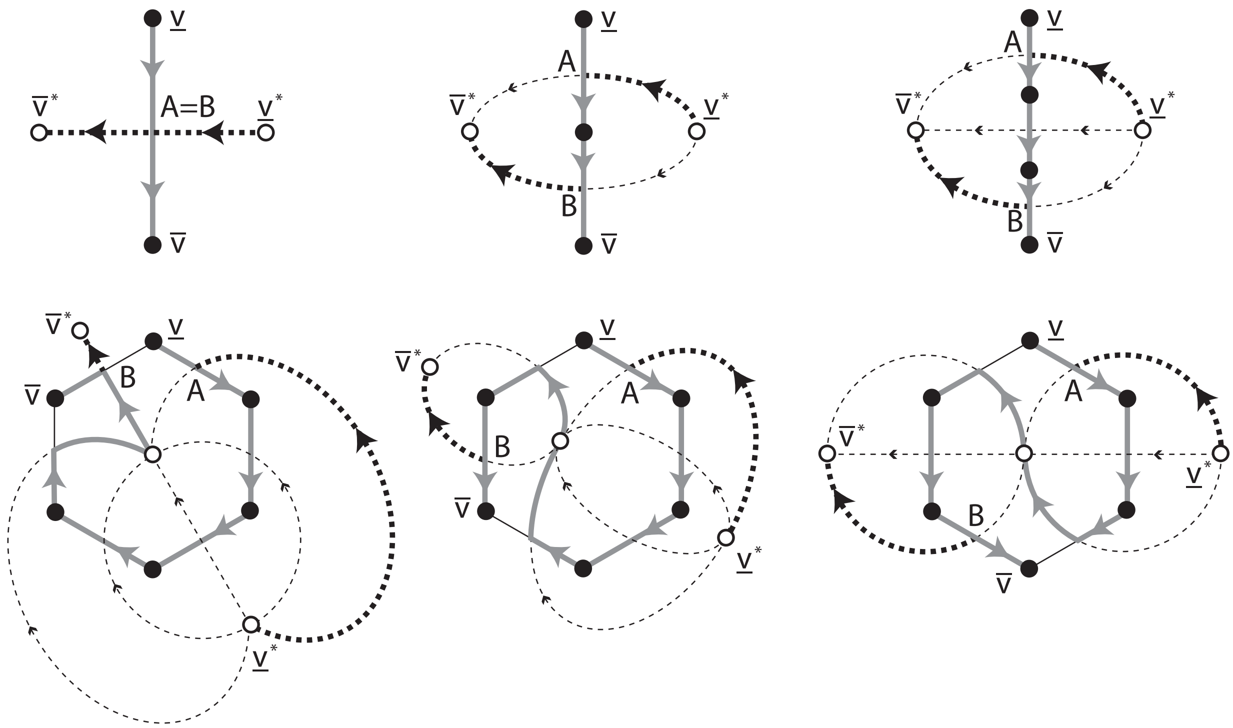

Figure 3.3: Some examples of graph duals G^* (dashed) of G^d (solid), and Z-Hamiltonian paths in G^d (gray solid) versus S-Hamiltonian paths in G^* (black dashed).

Proof: Theorem 2.1(i) holds for G^* with $\underline{v}^*, \bar{v}^*$ if, and only if, it holds for G^d, by construction of the dual orientation. $\bowtie$

We note that the above corollaries establish bijections

$$\begin{array}{rcl} Z\mathcal{H}_{\underline{v},\bar{v}}(G_2) & \rightarrow & Z\mathcal{H}_{\underline{v}^*,\bar{v}^*}(G_2^*) \\ S\mathcal{H}_{\underline{v},\bar{v}}(G_2) & \rightarrow & S\mathcal{H}_{\underline{v}^*,\bar{v}^*}(G_2^*) \end{array} \tag{3.8}$$

of the specific boundary ZS-Hamiltonian pairs in the filled graphs G_2 and G_2^*. The related bijection

$$\begin{array}{rcl} Z\mathcal{H}_{\underline{v},\bar{v}}(G_2) & \rightarrow & S\mathcal{H}_{\underline{v}^*,\bar{v}^*}(G_2^*) \\ h_0 & \mapsto & h_1^* \end{array} \tag{3.9}$$

is in fact most easily described. Simply replace the first edge $\underline{v}A$ and the last edge $B\bar{v}$, only, of a Z-Hamilton path

$$h_0 = \underline{v}A \dots B\bar{v} \;\in Z\mathcal{H}_{\underline{v},\bar{v}}(G_2) \tag{3.10}$$

by their counterparts $\underline{v}^*A$ and $B\bar{v^*}$, respectively:

$$h_1^* = \underline{v}^*A \dots B\bar{v}^*. \tag{3.11}$$

Similarly, we obtain the mirrored isomorphism

$$\begin{array}{rcl} S\mathcal{H}_{\underline{v},\bar{v}}(G_2) & \to & Z\mathcal{H}_{\underline{v},\bar{v}^*}(G_2^*) \\ h_1 & \mapsto & h_0^* \end{array} \tag{3.12}$$

replacing the first and last edges from/to $\underline{v}, \bar{v}$ by their $\underline{v}^*, \bar{v}^*$ counterparts. In particular, this construction shows how the directions of the Hamiltonian paths h_0, h_1, h_0^*, h_1^* all are compatible, both, with the orientations of G and G^*, and satisfy the orientation rules ($\mathcal{O}1$), ($\mathcal{O}2$) in both. In view of theorem 1.1 and 1.2 the bijections (3.8) will provide somewhat unusual "dualities" among planar global Sturm attractors; see [FiRo07b].

4 From Sturm attractors to skeleton orientations

In this section we prove the "only if" parts of theorems 1.1 and 1.2.We consider the 1-skeleton $\mathcal{C}_f^1$ of the connection graph $\mathcal{C}_f$ of any nontrivial, at most two-dimensional Sturm attractor $\mathcal{A}_f$ with only hyperbolic equilibria. The undirected graph $\mathcal{C}_f^1$ consists of the sink equilibria of $\mathcal{A}_f$, as vertices, and of the unstable manifolds of the saddle equilibria of $\mathcal{A}_f$, as edges. In lemmas 4.1–4.3, we show that the graph $G\!:= \mathcal{C}_f^1$ satisfies properties (i) and (ii) of theorem 1.1. Preparing for the proof of theorem 1.2, we also show in lemma 4.1 that the connection graph $G_2\!:= \mathcal{C}_f$ indeed possesses $G = \mathcal{C}_f^1$ as its 1-skeleton and is the filled graph of $\mathcal{C}_f^1$.

In the following lemma, we call a plane graph G *edge-cellular*, if the boundary ∂F of each bounded face is a cycle, except that vertices may be visited repeatedly. See figure 1.1 center, for an edge-cellular example, and figure 1.1 right, for an example which is not edge-cellular.

Lemma 4.1 *Under the above assumptions, the 1-skeleton $G\!:= \mathcal{C}_f^1$ of the connection graph $\mathcal{C}_f$ is a finite, connected, plane, edge-cellular multigraph without loops. The connection graph $\mathcal{C}_f$ itself is the quadrangulation $G_2 = \mathcal{C}_f$, of the 1-skeleton G.*

Proof: *Planarity* of any two-dimensional Sturm attractor $\mathcal{A}_f$, and hence of its 1-skeleton $\mathcal{C}_f^1 \subseteq \mathcal{C}_f \subseteq \mathcal{A}_f$, was shown by [Ro91] under separated boundary conditions. See also [Br90, Jo89] and, for the circle case $x \in S^1$, [MaNa97]. In fact let

$$(4.1) \qquad P\colon \quad \mathcal{A}_f \to \text{ span } \{\psi_0, \dots \psi_{n-1}\}$$

denote L^2 orthogonal projection onto the first n eigenfunctions of *any* Sturm-Liouville eigenvalue problem on $0 < x < 1$ with Neumann boundary conditions. Assume n is the maximal Morse index in the not necessarily planar Sturm attractor $\mathcal{A}_f$, with all equilibria assumed hyperbolic. Then the projection (4.1) is injective, and describes $\mathcal{A}_f$ as the C^1-graph of the function P^{-1} over the subset $P\mathcal{A}_f$ of span $\{\psi_0, \dots, \psi_{n-1}\}$. We identify $\mathcal{A}_f$ with its projection, for simplicity. Note the point of this construction: the local unstable manifolds $W^u(v)$ of equilibria v with maximal Morse index $i(v) = n$ in $\mathcal{A}_f$ are differentiable n-dimensional manifolds, parametrized locally over their respective tangent spaces, of course. The above results, however, provide a global parametrization.

Finiteness of $\mathcal{C}_f^1$, $\mathcal{C}_f$ follows because vertices are equilibria in the compact attractor $\mathcal{A}_f$ which are all assumed hyperbolic, hence isolated, and hence finite in number.

Connectedness of $\mathcal{C}_f^1$ and $\mathcal{C}_f$ follows from connectedness of $\mathcal{A}_f$ as the ω-limit set of a large ball in the underlying Sobolev space X. This also shows that $\mathcal{A}_f$ has trivial homology.

The 1-skeleton $\mathcal{C}_f^1$ is *loop-free*, because heteroclinic orbits between hyperbolic equilibria of adjacent Morse index are unique; see [BrFi88, BrFi89]. Indeed the two branches of the unstable manifolds $W^u(v)$ of saddles v, which form the edges in $\mathcal{C}_f^1$, cannot both connect to the same sink. The unstable manifolds $W^u(v_j)$ of several distinct saddles v_j, however, may well connect to the same pair of sinks, making $\mathcal{C}_f^1$ a multigraph.

To show $G = \mathcal{C}_f^1$ is *edge-cellular* with filled graph $G_2 = \mathcal{C}_f$, we first note that any bounded face F of G must be contained in (the projection of) $\mathcal{A}_f$. Indeed $\mathcal{A}_f$ caries only trivial homology. Since the closure clos F is bounded, connected, and invariant, the α-limit set $\alpha(F)$ in $\mathcal{A}_f$ is a single equilibrium w of Morse index $\alpha(w) = 2$. In other words, $F = W^u(w)$ is the unstable manifold of a single source w. By the λ-Lemma [PdM82] the edges of the face boundary ∂F in G are the unstable manifolds of the saddles v_j which w connects to by heteroclinic orbits u_j. By uniqueness of each such heteroclinic orbit u_j, between adjacent Morse levels 1 and 2, the face boundary ∂F is indeed a cycle of edges. The cycle may still run through the same sink, repeatedly, at this stage. See figure 1.1 center.

To see that $\mathcal{C}_f = G_2$ is the *quadrangulation* of $G = \mathcal{C}_f^1$ we observe that sinks, saddles, and sources in $\mathcal{C}_f$ indeed play the roles of Morse sinks, Morse saddles, and Morse sources in G_2. It only remains to show that the source w in any bounded face F does connect to each saddle v_j on the edge cycle ∂F. Starting with a small circle of initial conditions in $W^u(w)\setminus\{w\}$ around w, we see how any missing heteroclinic connection to any saddle $v_j \in \partial F$ would render the circle simply connected or even disconnected, after some finite positive time as it retracts to its ω-limit set in ∂F. This proves that the source $w \in F$ connects to each saddle $v_j \in \partial F$, and completes the lemma. $\bowtie$

To prove theorem 1.1(ii) for the 1-skeleton $G = \mathcal{C}_f^1$ of the connection graph $\mathcal{C}_f$ we have to define an orientation of the $\mathcal{C}_f^1$ edges $e = W^u(v)$ generated by their bisecting (Morse) saddles v. Let v_1, v_2 denote the (Morse) sink end points of e. We define

$$v_1 \prec v_2 \text{ if, and only if, } v_1(x) > v_2(x) \text{ for all } \ 0 < x < 1. \tag{4.2}$$

Let $u_j = u_j(t) \in X$ denote the heteroclinic connections from the saddle v to v_j, $j = 1, 2$. Then $0 \leq z(u_j(t) - v) < \dim W^u(v) = i(v) = 1$; see [FiRo96] and also [BrFi86]. Hence $u_j(t)$ is entirely above, or entirely below v, for all $t \in \mathbb{R}$, $0 < x < 1$. Passing to the limit $v_j = \lim\limits_{t\to+\infty} u_j(t)$ and recalling that $u_j(t)$ start to opposite sides of v, for $t \to -\infty$, we either conclude $v_1(x) < v(x) < v_2(x)$, for all $0 < x < 1$, or else the reverse inequalities hold. Therefore either $v_1 \succ v_2$, or else $v_1 \prec v_2$. This defines the orientation of e to point from v_1 to v_2, or from v_2 to v_1, respectively, as the "arrowheads" $\succ$ and $\prec$ may indicate.

Note how (4.2) defines a partial order $\preceq$ on not necessarily adjacent Morse sinks of $G = \mathcal{C}_f^1$. It is slightly unfortunate, but useful below, that a di-source $\underline{v}$ with respect to this orientation, alias a local maximum with respect to the order $\succ$ on G, is actually a minimum with respect to the pointwise partial order $<$ on X. This is caused by the boundary ZS-Hamiltonian paths h_0, h_1 following the orientation order $\succ$ downhill, but ordering the equilibria $v \in \mathcal{A}_f$ upward at the boundaries $x = 0, 1$.

By dissipativeness and the parabolic comparison principle, the nontrivial Sturm attractor $\mathcal{A}_f$ contains two distinguished equilibria $\underline{v}, \bar{v}$ such that

$$\underline{v} \succ v \succ \bar{v} \tag{4.3}$$

for any other equilibrium $v \in \mathcal{A}_f$. Indeed, the sinks $\underline{v}, \bar{v}$ can be obtained as ω-limit sets of constant initial conditions $u = \pm C$, at $t = 0$, for any sufficiently large $C > 0$.

Lemma 4.2 *Let the 1-skeleton $G = \mathcal{C}_f^1$ of the connection graph $G_2 = \mathcal{C}_f$ be endowed with the orientation d induced by the partial order $\prec$ of its Morse sinks.*

Then G possesses exactly one di-source $\underline{v}$, one di-sink $\bar{v}$, both on ∂G, and no di-cycles.

Proof: Absence of di-cycles is obvious because the orientation of G results from a partial order $\prec$. We show uniqueness of the di-sink $\bar{v}$, indirectly. Uniqueness of the di-source $\underline{v}$ is analogous.

By construction we have $\underline{v}, \bar{v} \in \partial G$. Suppose indirectly that $\tilde{v}$ is another di-sink. Then

$$\tilde{v}(x) < \bar{v}(x), \tag{4.4}$$

for all $0 < x < 1$, by definition of $\bar{v}$; see (4.3). Because (1.1), (1.2) is a strongly monotone dynamical system, in the sense of [Hi88, Ma87], there exists an equilibrium v above the

Morse sink $\tilde{v}$, and below $\bar{v}$ anyway, together with a monotone heteroclinic orbit $u = u(t, x)$ from v to $\tilde{v}$, such that

$$\tilde{v}(x) < u(t,x) < v(x) < \bar{v}(x) \tag{4.5}$$

for all $0 < x < 1$ and $t \in \mathbb{R}$. See [Po89, Po02] for detailed results which even carry over to several dimensions. By the cascading property of $\mathcal{A}_f$ mentioned in the introduction, we may assume v to be a saddle. This shows that an edge in $\mathcal{C}_f^1$ which contains the bisecting saddle $v \prec \tilde{v}$, is oriented away from $\tilde{v}$. Therefore $\tilde{v}$ cannot be a di-sink, and the lemma is proved. $\bowtie$

Lemma 4.3 *Under the assumptions of lemma 4.1, the 1-skeleton $G = \mathcal{C}_f^1$ of the connection graph $\mathcal{C}_f$ is not only edge-cellular but cellular.*

Proof: To exclude multiply traversed vertices v on a face boundary ∂F, as in figure 1.1 center, we proceed indirectly. Consider the orientation of G defined in lemma 4.2. If $v \in \partial F$ is multiply traversed, then the edge-cycle ∂F decomposes into at least two disjoint cycles through v, none of which can be a di-cycle. This produces an interior di-sink or di-source in a smallest cycle through v, and not on ∂G. This contradicts lemma 4.2. Therefore the edge cycle of ∂F passes through each vertex $v \in \partial F$ only once, and G is cellular. In particular, clos F is a topological disk and F is the interior of the (topological) n-gon ∂F. $\bowtie$

This completes the proofs of the “only if” parts of theorems 1.1 and 1.2.

5 Discussion

As an outlook on the sequels [FiRo07a, FiRo07b] of the present paper we briefly sketch the main line of argument in our proof of the “if” parts of theorems 1.1 and 1.2. The precise details are somewhat involved and of a quite different flavor from the present mostly graph oriented part and are therefore relegated to [FiRo07a].

We conclude with a discussion of further graph theoretic aspects of the bipolar 1-skeletons $\mathcal{C}_f^1$ of the connection graphs $\mathcal{C}_f$ of planar global Sturm attractors $\mathcal{A}_f$ following the beautiful survey [FMR95]. As an example we also include the simplest self-dual two-dimensional Sturm attractor. This attractor consists of 11 equilibria and their heteroclinic orbits. See [FiRo07b] for a complete classification of all Sturm attractors with up to 11 equilibria, and many more examples.

To prove the "if " part of theorems 1.1 and 1.2 we start from any pair G, G_2 of a plane graph G with the properties of theorem 1.1(i) and its quadrangulation G_2. We have to construct a dissipative nonlinearity $f = f(x, u, u_x)$ of (1.1) such that the associated connection graph $\mathcal{C}_f$ and its 1-skeleton $\mathcal{C}_f^1$ coincide with G_2 and G, respectively. In fact we have to address $G_2 = \mathcal{C}_f$ only, in view of the equivalence statement of theorem 2.1. In particular we are also given a boundary ZS-Hamiltonian pair (h_0, h_1) in G_2 from the di-source $\underline{v}$ to the di-sink $\bar{v}$ of the oriented 1-skeleton G^d.

The key role in our proof is played by the permutation

$$\pi = h_0^{-1} \circ h_1. \tag{5.1}$$

Here the Hamiltonian paths h_0, h_1 are viewed as permutations of the labeled vertices $v_1, \ldots, v_N$ of G_2. Note that π, unlike h_0 and h_1, is independent of the specific vertex labeling.

We then show that the permutation π is a dissipative Morse meander in the sense of [FiRo99]. Here *dissipative* means that $\pi(1) = 1$ and $\pi(N) = N$. The *Morse* property requires the integers

$$i_j := \sum_{\iota=1}^{j-1} (-1)^{\iota+1} \ \text{sign} \ \left(\pi^{-1}(\iota+1) - \pi^{-1}(\iota)\right) \tag{5.2}$$

to be nonnegative, for all $j = 2, \ldots, N$. The *meander* property, introduced in [ArVi89] in a completely different context, requires that there exist a C^1 Jordan curve $\mathcal{S}$ which intersects the horizontal axis tranversely at precisely N locations $k = 1, \ldots, N$. Numbering the intersections alternatively, by $j = 1, \ldots, N$ along the curve $\mathcal{S}$ is then required to yield $j = \pi(k)$.

It is fairly easy to see that any Sturm attractor $\mathcal{A}_f$ with equilibria $v_1, \ldots, v_N$ gives rise to a dissipative Morse meander permutation π_f. Following an idea implicit in (4.2) above, we may in fact define permutations h_0^f, h_1^f such that

$$\begin{aligned} v_{h_0^f(1)} < v_{h_0^f(2)} < \cdots < v_{h_0^f(N)} \quad &\text{at} \quad x = 0, \\ v_{h_1^f(1)} < v_{h_1^f(2)} y < \cdots < v_{h_1^f(N)} \quad &\text{at} \quad x = 1. \end{aligned} \tag{5.3}$$

Following Fusco and Rocha [FuRo91] we then define the *Sturm permutation*

$$\pi_f := (h_0^f)^{-1} \circ h_1^f. \tag{5.4}$$

By [FuRo91] the Sturm permutation is indeed a dissipative Morse meander. In fact the Morse numbers (5.2) turn out to be the Morse indices, $i_j = i(v_{h_0^f(j)}) \geq 0$. The meander property of π_f follows when we solve the ODE boundary value problem for the equilibria v_j by a shooting approach. Dissipativeness of π_f is an easy consequence of dissipativeness of f itself.

It is more demanding, but has also been achieved in [FiRo99], to show that, conversely, any dissipative Morse meander permutation π does arise as a Sturm permutation $\pi = \pi_f$ for a suitable dissipative nonlinearity $f = f(x, u, u_x)$ in (1.1). Once it has been established that π, defined via the ZS-Hamiltonian pair (h_0, h_1) in (5.1), indeed is a dissipative Morse meander, this therefore reveals $\pi = \pi_f$ as a true Sturm permutation. We can then invoke the results of [FiRo96] which specify the complete connection graph $\mathcal{C}_f$ once the Sturm permutation $\pi = \pi_f$ is known. The main remaining work in [FiRo07a] therefore will be to show that, yes indeed, the big excursion $G_2 \mapsto (h_0, h_1) \mapsto \pi = \pi_f \mapsto \mathcal{C}_f$ closes up by

$$\mathcal{C}_f = G_2. \tag{5.5}$$

Moreover we may label equilibria v_k such that $h_0 = h_0^f$ and $h_1 = h_1^f$ indeed specify the ordering (5.3) of $v_k(x)$ at the interval boundaries $x = 0$ and $x = 1$.

We conclude with a discussion of some graph theoretic aspects of the bipolar 1-skeletons $\mathcal{C}_f^1$ of connection graphs $\mathcal{C}_f$. Comparing with the survey [FMR95] we first note a slight discrepancy in the definition of bipolar graphs G and, in the plane case, their dual G^*. In [FMR95] an orientation d of a finite connected graph G with some distinguished oriented edge e is called *e-bipolar*, if the oriented graph G^d does not possess di-cycles and the endpoints $\underline{v}$ and $\bar{v}$ of e are the unique di-source vertex $\underline{v}$ and di-sink vertex $\bar{v}$ of the orientation, respectively. See [FMR95] Property 3.2(5). To match our definition in theorem 1.1(ii), which does not require adjaceny of the di-extrema $\underline{v}$ and $\bar{v}$, we therefore insert an artificial additional exterior edge e between the di-extrema $\underline{v}$ and $\bar{v}$ on the boundary of G.

For plane e-bipolar graphs G^d, with additional edge e, we may consider the standard oriented dual G^{d*} which represents the unbounded face of G^d by a single vertex v^* rather than by two such vertices $\underline{v}^*$ and $\bar{v}^*$. The additional exterior edge e, however, splits the unbounded exterior face in two and creates an additional edge e^* in G^{d*}. To reconcile G^{d*} with our dual $(G \backslash e)^*$ of $G \backslash e$ we only have to remove e^* from G^{d*}, keeping its end points $\underline{v}^*$ and $\bar{v}^* = v^*$ as vertices:

$$(G^d \backslash e)^* = G^{d*} \backslash e^*. \tag{5.6}$$

As a slight benefit we note that $(G^d \backslash e)^*$, as defined in the present paper, is bipolar with di-extrema $\underline{v}^*$ and $\bar{b}^*$. In contrast, G^{d*} is e^*-bipolar only after reversing the orientation of e^*; see [FMR95], Proposition 5.1.

Self-dual Sturm attractors are particular examples. Dimension one is trivial with a 1-skeleton of two sinks and a single edge; see the top left example in Figure 3.3. The simplest two-dimensional example features a 1-skeleton with 11 sinks; see Figure 5.1.

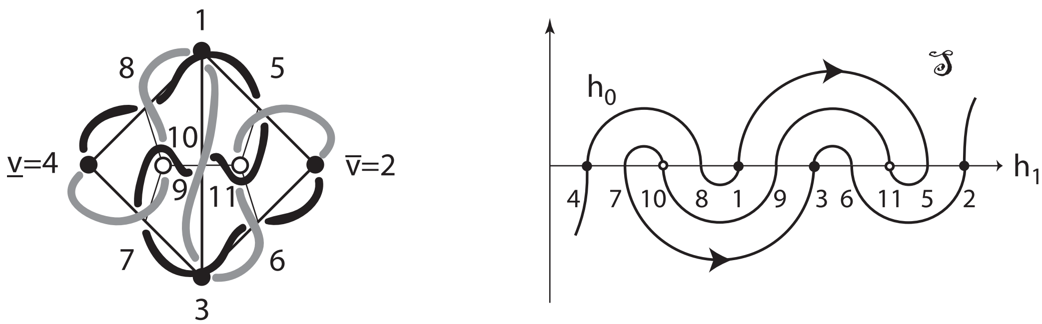

Figure 5.1: The simplest two-dimensional self dual Sturm attractor: connection graph (left) and shooting curve (right). ZS-Hamiltonian pair h_0 (black) and h_1 (gray).

This example was first encountered as the simplest non-pitchforkable Sturm attractor; see [Ro91] for a background. Combinatorially, pitchforkable attractors are characterized by the existence of a triplet of adjacent values $\pi(i-1), \pi(i), \pi(i+1)$ in the Sturm permutation π. In terms of ZS-Hamiltonian pairs they are characterized by paths h_0 and h_1 which coincide along at least three successive vertices in the quadrangulation G_2. The shooting curve $\mathcal{S}$, equivalently, is required to file sequentially through at least three successive intersections with the horizontal v-axis. The significance of non-pitchforkable Sturm attractors was that their heteroclinic orbits are not accessible to analysis via successive pitchfork bifurcations. Therefore the original approach by Conley, Smoller and Henry [CoSm83, He85] to the heteroclinic orbit problem failed for these Sturm attractors. We caution the combinatorially and dynamically well-versed reader that edge deletion and edge contraction on bi-polar 1-skeleton $\mathcal{C}_f^1$ as discussed in [FMR95], section 6 does not, in general, correspond to pitchfork bifurcations in the associated Sturm attractor $\mathcal{A}_f$. Only formally, such operations involve equilibria of Morse indices compatible with pitchfork bifurcations. The presence of deletable and contractable edges in the non-pitchforkable Rocha example, however, clearly demonstrates the discrepancy between these two notions. See [FiRo07a] for further discussion of the relation to bifurcation theory, and [FiRo07b] for a complete classification of planar Sturm attractors with up to 11 equilibria which indeed exhibits the Rocha example as the simplest planar self-dual case.

We conclude with some known results on existence and enumeration of e-bipolar orientations; see [FMR95] and the references there. Lempe et al. have shown in 1967 that e-bipolar orientations on (not necessarily plane) G *exist* if, and only if, the graph G is 2-*connected* (i.e., cannot be disconnected by removal of a single vertex). In view of the artificially added edge e, we can therefore view the original 1-skeleton $\mathcal{C}_f^1$ as a collecttion of 2-connected components which are arranged and connected sequentially along an interval. Even and Tarjan (1967,1986) have developed algorithms for finding some, but not necessarily all, e-bipolar orientations of any given, not necessarily plane, 2-connected graph G in only linear time. In the plane case, any e-bipolar orientation can be reached in linear time (Rosenstiehl, Tarjan 1986). This is a marked computational advantage of theorem 1.1 over any direct search for Hamiltonian paths in the spirit of theorem 1.2.

Enumeration results are surveyed in [FMR95], section 9. Let $2\sigma_e(G)$ denote the number of e-bipolar orientations of G. On the 1-skeleton $\mathcal{C}_f^1$ of the connection graph the artificial edge e only indicates the location of the di-extrema $\underline{v}, \bar{v}$, up to interchange, which are the end points of e. Remarkably the invariant $\sigma_e(G) = \sigma(G)$ is independent of e. In fact $\sigma(G)$ coincides with the beta invariant of Crapo and appears in the Tutte polynomial of G. For example consider adjacent $\underline{v}, \bar{v}$ on the boundary of a given1-skeleton G. We may than choose the "additional" edge e to already be present in G. In particular the number $\sigma_e(G)$ of bipolar orientations of G, alias the different Sturm permutations π_f which provide the 1-skeleton $\mathcal{C}_f^1 = G$ with prescribed extrema $\underline{v}, \bar{v}$, does not depend on the specific location of the adjacent extrema $\underline{v}, \bar{v}$ on the boundary of G.

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