A PERMUTATION CHARACTERIZATION OF STURM GLOBAL ATTRACTORS OF HAMILTONIAN TYPE

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Abstract. We consider Neumann boundary value problems of the form $u_t = u_{xx} + f$ on the interval $0 \leq x \leq \pi$ for dissipative nonlinearities $f = f(u)$. A permutation characterization for the global attractors of the semiflows generated by these equations is well known, even in the much more general case $f = f(x, u, u_x)$. We present a permutation characterization for the global attractors in the restrictive class of nonlinearities $f = f(u)$. In this class the stationary solutions of the parabolic equation satisfy the second order ODE $v'' + f(v) = 0$ and we obtain the permutation characterization from a characterization of the set of 2π-periodic orbits of this planar Hamiltonian system. Our results are based on a diligent discussion of this mere pendulum equation.

1. Introduction

Scalar semilinear parabolic equations defined on an interval, for several types of boundary conditions, have been extensively considered in the literature. For a standard reference see [Hen81]. Under suitable assumptions on the nonlinearity, these equations generate global dissipative semiflows on appropriate function spaces. The corresponding dynamical systems exhibit global attractors, which collect relevant asymptotic information on the behavior of the semiflow. We refer to [Hal88, BV92, HMO02] for general references on global attractors. For surveys more specific to parabolic equations see also [FS02, Rau02].

The simplest case, where the characterization of global attractors has been most succesful, are equations

$$u_t = u_{xx} + f(x, u, u_x) \tag{1.1}$$

defined on the interval $(0, \pi)$ with Neumann boundary conditions

$$u_x(t, 0) = u_x(t, \pi) = 0 \ . \tag{1.2}$$

When the nonlinearity $f = f(x, u, u_x)$ satisfies

$$f \in C^2 \text{ is dissipative }, \tag{1.3}$$

then equations (1.1), (1.2), generate a global dissipative semiflow $u(t, \cdot)$ on a state space X with a global attractor $\mathcal{A}_f \subset X$. Here *global semiflow* means that any solutions $u(t, \cdot)$ is defined for all time $t \geq 0$. The *dissipative* property of the semiflow requires the existence of a fixed large ball in X in which any solution $u(t, \cdot)$ stays eventually, for all time $t \geq t_0(u(0, \cdot))$.

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Specific sufficient conditions for dissipativeness of the nonlinearity f are the following sign and growth conditions

$$f(x,v,0)\cdot v < 0 \text{ for all large } |v| \ , \tag{1.4}$$

$$|f(x,v,p)| < C(|v|)(1+|p|^\gamma) \ , \tag{1.5}$$

for all (x,v,p), some suitable constant $0<\gamma<2$, and a continuous function $C(|v|)$. Then (1.1), (1.2), generates a global dissipative semiflow on the Sobolev space of functions with square integrable second derivative and satisfying Neumann boundary conditions; see [Ama85, Hen81, Paz83],

$$X = H^2([0,\pi],\mathbb{R}) \cap \{u_x = 0 \text{ at } x = 0,\pi\} \ . \tag{1.6}$$

The specific global attractors $\mathcal{A}_f \subset X$ arising in our setting (1.1), (1.2), are called *Sturm attractors.*

The semiflow generated by (1.1), (1.2), possesses a gradient-like structure due to the existence of a Lyapunov function $\mathcal{V}$ of the form

$$\mathcal{V}(u) = \int_0^\pi a(x,u,u_x)\ dx \ , \tag{1.7}$$

which is strictly decreasing along nonequilibrium solutions $u = u(t,\cdot)$; see for example [Mat78, Mat88, Zel68]. Consequently, the Sturm attractor $\mathcal{A}_f$ is in general composed of equilibrium solutions and heteroclinic orbit connections between equilibria.

The equilibrium solutions of the PDE (1.1), (1.2), correspond to the solutions of the ODE boundary value problem

$$\begin{aligned} v_{xx} + f(x,v,v_x) = 0 \quad &, \ 0<x<\pi \\ v_x = 0 \quad &, \ x = 0 \text{ or } \pi \ . \end{aligned} \tag{1.8}$$

Nondegeneracy of an equilibrium $v = v(x)$ requires v to be *hyperbolic*, that is, $\lambda = 0$ is not an eigenvalue of the linearized problem at v

$$\lambda u = u_{xx} + \partial_p f(x,v,v_x)u_x + \partial_v f(x,v,v_x)u \ , \ 0<x<\pi \ , \tag{1.9}$$

with Neumann boundary conditions. For generic f it turns out that the global attractor $\mathcal{A}_f$ has the *Morse-Smale property*: all PDE equilibria are hyperbolic and the intersections between their respective stable and unstable manifolds are transverse; see [Hen85, Ang86]. For the intricate case of periodic boundary conditions, which also features time periodic orbits of rotating wave type, see [CR08, JR08].

In the following, in addition to the dissipativeness condition (1.3) on f, we make the generic assumption that all PDE equilibria of (1.1), (1.2), are hyperbolic. Let

$$\text{Sturm}(x,u,u_x) \tag{1.10}$$

denote the *Sturm class* of $f(x,u,u_x)$, i.e., the set of all C^2-smooth nonlinearities f depending on x, u, and u_x, which satisfy these two conditions. Similarly, we define the Sturm class of $f(u)$,

$$\text{Sturm}(u) \ , \tag{1.11}$$

to consist of all nonlinearities $f \in \text{Sturm}(x,u,u_x)$ which are in fact independent of both x and u_x. Notations like $\text{Sturm}(x,u)$ or $\text{Sturm}(u,u_x)$ are quite self-explanatory. Note however, that these Sturm classes will depend on the underlying boundary conditions of the PDE (1.1). Indeed the required dissipativeness and hyperbolicity properties both depend on the specific boundary conditions. In the present paper we focus on Neumann boundary conditions (1.2). We aim at a purely combinatorial characterization of nonlinearities f, and of global attractors $\mathcal{A}_f$, in the Sturm class Sturm(u). See [FR99] where this modeling question was already mentioned. The case of periodic boundary conditions and Sturm class Sturm(u, u_x) will be considered elsewhere.

In our scalar and one-dimensional case $f \in$ Sturm(x, u, u_x) the single central and crucially most important object in the complete characterization of all Sturm attractors $\mathcal{A}_f$ are the Sturm permutations, which we define next.

As a consequence of hyperbolicity there is only a finite (odd) number of equilibria of (1.1), (1.2),

$$\mathcal{E} = \{v_k : 1 \leq k \leq n\} \ , \tag{1.12}$$

where $v_k = v_k(x)$ denote the solutions of (1.8). The Sturm permutation then arises from the ordering of the PDE hyperbolic equilibria (i.e. ODE solutions) v_k by their values on the boundaries at $x = 0$ and $x = \pi$, respectively. Let us label the equilibria such that their order at $x = 0$ is

$$v_1(0) < v_2(0) < \cdots < v_n(0) \ . \tag{1.13}$$

Then the Sturm permutation $\sigma = \sigma_f \in \mathcal{S}(n)$ is defined by the ordering at $x = \pi$

$$v_{\sigma(1)}(\pi) < v_{\sigma(2)}(\pi) < \cdots < v_{\sigma(n)}(\pi) \ . \tag{1.14}$$

Therefore, an arbitrary permutation $\sigma \in \mathcal{S}(n)$ is a *Sturm permutation* if, and only if, $\sigma = \sigma_f$ for some dissipative nonlinearity $f \in$ Sturm(x, u, u_x). In [FR99] the set of Sturm permutations $\sigma \in \mathcal{S}(n)$ was completely characterized in purely combinatorial terms. This characterization will be recalled in Section 2. Moreover, Sturm attractors $\mathcal{A}_f$, $\mathcal{A}_g$ with equal ODE Sturm permutations $\sigma_f = \sigma_g$ are C^0-orbit equivalent, viz dynamically indistinguishable, on the PDE level; see [FR00]. In fact, a simple explicit algorithm based on Sturm permutations σ_f alone determines which equilibria in $\mathcal{A}_f$ possess a PDE heteroclinic orbit connection, and which do not; see [FR96, Wol02b]. We emphasize that all input information involved here concerns only ODE arguments. *In consequence this ODE information alone, as encoded in the Sturm permutation σ_f, is sufficient to completely determine the PDE dynamics of the global attractors $\mathcal{A}_f$.* For further references see [FR91, Fie94, FR96, Fie96, FR99, FR00] in the case of semilinear parabolic equations, and [Roc94] in a class of finite dimensional analogues called Jacobi systems in [FO88]. See also [Roc91, Fi&al02, Wol02a, Wol02b, HW05, FR08, FR09a, FR09b] for applications of this characterization.

Our main goal in the present paper is therefore a complete and purely combinatorial characterization of those permutations $\sigma \in \mathcal{S}(n)$ which arise as Sturm permutations $\sigma = \sigma_f$ in the restricted Hamiltonian class $f \in$ Sturm(u). See Theorem 1 below. The heteroclinic orbits in this case were first identified by [BF88], [BF89], albeit without the proper context of Sturm permutations introduced in [FR91].

In the restricted Hamiltonian class $f \in$ Sturm(u), equation (1.1) takes the form

$$u_t = u_{xx} + f(u) \ . \tag{1.15}$$

The ODE corresponding to the stationary problem (1.8) becomes the Hamiltonian "pendulum" equation

$$v'' + f(v) = 0 \ . \tag{1.16}$$

Indeed the corresponding planar ODE system

$$\begin{aligned} v' &= p \\ p' &= -f(v) \end{aligned} \tag{1.17}$$

possesses a Hamiltonian $H = H(v,p)$ given by

$$H(v,p) = \frac{1}{2}p^2 + F(v) \ , \tag{1.18}$$

where $F(v) = \int_0^v f(s)ds$ denotes the potential associated to f. For this reason, we call $\mathcal{A}_f$ of *Hamiltonian type* when $f \in \mathrm{Sturm}(u)$ even though the (semi)flow on $\mathcal{A}_f$ is a gradient flow.

Consider any equilibrium solution $v_k \in \mathcal{E}$ of the Neumann boundary value problem (1.8) with $f = f(v)$. Since (1.16) is reversible with respect to $x \to -x$, we may extend v_k to $-\infty < x < +\infty$ by reflection through the boundaries to obtain a periodic solution of (1.16) with (possibly nonminimal) period 2π. Moreover, a characterization for the set of 2π-periodic orbits of (1.16) has already been established in [Roc07]. Therefore, this correspondence between equilibria of (1.1), (1.2), and 2π-periodic orbits of the planar Hamiltonian system (1.17) becomes instrumental for our characterization of Sturm global attractors $\mathcal{A}_f$ of Hamiltonian type. For completeness, we review the characterization results for the set of 2π-periodic orbits of (1.16) in Section 3.

From the zeros $e_1, \dots, e_m$ of f we obtain the set $\mathcal{Z}$ of equilibria of the ODE system (1.17),

$$\mathcal{Z} = \{(e_j, 0), j = 1, \dots, m\} \ . \tag{1.19}$$

Clearly, the zeros e_j correspond to the spatially homogeneous equilibrium solutions v_k of (1.15), (1.2). Due to dissipativeness of f and nondegeneracy of equilibria, we note $f'(e_j) < 0$ for j odd and $f'(e_j) > 0$ for j even. This implies that the equilibria $(e_j, 0)$ with odd j are ODE saddles, and the equilibria $(e_j, 0)$ with even j are ODE centers. We also note that the connected component of $(e_j, 0)$ in the critical energy level $\{(u,p) : H = F(e_j)\}$ consists of:

- only $(e_j, 0)$, for the local minima e_j of F. They occur exactly when j is even and correspond to the Hamiltonian ODE centers;
- $(e_j, 0)$, together with the attached separatrices of (1.17) for the local maxima e_j of F. They occur exactly when j is odd and correspond to the Hamiltonian ODE saddles.

The first and last ODE equilibria, $(e_1, 0)$ and $(e_n, 0)$, are ODE saddles, each with two unbounded separatrices attached. All other separatrices are unbounded, or are homoclinic loops to one ODE saddle, or are heteroclinic orbits between distinct ODE saddles. For later use it will be convenient to note that all heteroclinic orbits can easily be eliminated by arbitrarily small smooth perturbations $f + \varepsilon g \in \mathrm{Sturm}(u)$ of compact support in g. Indeed it is sufficient to choose g with primitive function G such that the m critical values of H and F, which correspond to the zeros e_j, for $\varepsilon = 0$, all become distinct, for $F + \varepsilon G$ and small $\varepsilon > 0$.

Based on the permutation $\sigma \in \mathcal{S}(n)$, only, we define the *Morse numbers* $i_k = i_k(\sigma)$, for $1 \leq k \leq n$, by

$$i_k(\sigma) := \sum_{j=1}^{k-1} (-1)^{j+1} \operatorname{sign}(\sigma^{-1}(j+1) - \sigma^{-1}(j)) \ , \tag{1.20}$$

where an empty sum denotes zero. For motivation, suppose $\sigma = \sigma_f$ is a Sturm permutation. Then the Morse numbers $i_k(\sigma)$ coincide with the Morse indices $i(v_k)$ of the PDE equilibria v_k, that is, with the number of strictly positive eigenvalues λ of the linearization (1.9) around $v = v_k$. By analogy, we call a point $k \in \{1, \ldots, n\}$ of any general permutation $\sigma \in \mathcal{S}(n)$ a σ-*stable* point, if

$$i_k(\sigma) = 0 \ . \tag{1.21}$$

We caution the reader that our notion of a σ-stable point in a Sturm permutation $\sigma = \sigma_f$ refers to PDE asymptotic stability, in the sense of (1.1), (1.2). In the Hamiltonian case $f \in \text{Sturm}(u)$ which we are actually aiming for, these σ-stable points k will — quite confusingly — be given by the ODE saddles $(e_j, 0)$ of the Hamiltonian system (1.17). Indeed, stable equilibrium solutions v_k of the PDE (1.1), (1.2) must necessarily be spatially homogeneous for nonlinearities $f = f(u)$ (see [Cha75]). Therefore σ-stable points k can only correspond to ODE equilibria $(e_j, 0)$, viz ODE saddles or ODE centers. Since ODE centers $(e_j, 0)$ are characterized by $f'(e_j) > 0$, they are all unstable with respect to spatially homogeneous perturbations. In conclusion $(e_j, 0)$ must be an ODE saddle. Conversely, all ODE saddles $(e_j, 0)$ give rise to PDE asymptotically stable, spatially homogeneous equilibria v_k, which appear as σ-stable points k in the Sturm permutation σ associated to the Hamiltonian nonlinearity $f = f(u)$.

Our main result, Theorem 1 below, provides a completely combinatorial characterization for a general permutation $\sigma \in \mathcal{S}(n)$ to belong to the Hamiltonian class $f \in \text{Sturm}(u)$, i.e. to be realized as the Sturm permutation $\sigma = \sigma_f$ for some such f. This characterization consists of two ingredients: the *involution property* and the *integrability property*. We first introduce the involution property and show that it necessarily holds true for any $\sigma = \sigma_f$ in the Hamiltonian class $f \in \text{Sturm}(u)$. Necessity of the integrability property, introduced next, will be deferred to Section 4. Sufficiency of both conditions is proved in Sections 5 and 6.

We address the *involution property* of $\sigma = \sigma_f$, $f \in \text{Sturm}(u)$ first, which requires

$$\sigma^2 := \sigma \circ \sigma = \text{id} \ . \tag{1.22}$$

Of course this is a purely combinatorial condition: any permutation $\sigma \in \mathcal{S}(n)$ satisfies the involution property (1.22) if, and only if, σ has a unique decomposition into a (possibly empty) commuting product of $q \geq 0$ disjoint 2-cycles

$$\sigma_f = (\underline{c}_1 \ \overline{c}_1) \ldots (\underline{c}_q \ \overline{c}_q) \ . \tag{1.23}$$

As a warm-up for the remaining paper, we digress here briefly to show why any Sturm permutation $\sigma = \sigma_f$ in the Hamiltonian class $f \in \text{Sturm}(u)$

is indeed an involution. By definition $\sigma = \sigma_f$ keeps track of the orders of the hyperbolic equilibria $v_1(x), \ldots, v_n(x)$ on the Neumann boundaries $x = 0$ and $x = \pi$. From the ODE point of view in (1.17), each of these Neumann PDE equilibria corresponds to exactly one of three distinct types of ODE orbits in the phase plane (v, p):

(i) to ODE equilibria $(v, p) = (e_j, 0)$;
(ii) to orbits $(v, p) = (v_j(x), v_j'(x))$ of period π (not necessarily minimal);
(iii) to orbits $(v, p) = (v_j(x), v_j'(x))$ which are not π-periodic, but possess (not necessarily minimal) period 2π after a reflection extension to all $x \in \mathbb{R}$.

Here and below periodic orbits (v, v') are nonstationary periodic ODE orbits. Unless specified otherwise, periods are not meant to be minimal periods.

Cases (ii) and (iii) are easily distinguished in terms of the minimal period $T > 0$ of the periodic orbit (v_j, v_j'). Indeed

$$T = 2\pi/\ell \tag{1.24}$$

where ℓ is even in case (ii), and odd in case (iii). The Sturm permutation keeps track of the intersection points of these ODE periodic orbits with the v-axis $p = v'(x) = 0$, at $x = 0, \pi$. In cases (i) and (ii) the resulting boundary values of $v(x)$ coincide, at $x = 0$ and $x = \pi$. In cases (ii) and (iii), one and the same 2π-periodic orbit (v_j, v_j') gives rise to exactly two distinct PDE equilibria $\underline{v}_j(x)$ and $\overline{v}_j(x) = \underline{v}_j(x + T/2)$, which differ by a phase shift in x over half the minimal period T. For definiteness let

$$\begin{aligned} \underline{v}_j(0) &= \min v_j \ , \\ \overline{v}_j(0) &= \max v_j \ . \end{aligned} \tag{1.25}$$

In case (ii) we obtain identical boundary values

$$\underline{v}_j(0) = \underline{v}_j(\pi) \quad , \quad \overline{v}_j(0) = \overline{v}_j(\pi) \tag{1.26}$$

because π itself is a period of v_j. Only in case (iii) the boundary values differ, because π is not a period of v_j, even though 2π admittedly is:

$$\begin{aligned} \underline{v}_j(0) &= \min v_j \ = \overline{v}_j(\pi) \ , \\ \overline{v}_j(0) &= \max v_j = \underline{v}_j(\pi) \ . \end{aligned} \tag{1.27}$$

Here we have used that $\underline{v}_j, \overline{v}_j$ are related by phase shift over $T/2$ and that $\pi = \ell T/2$ with ℓ odd in case (iii); see (1.24).

Clearly case (iii) therefore corresponds to a 2-cycle $(\underline{c}_j \ \overline{c}_j)$ of $\sigma = \sigma_f$ by (1.27). Case (i), and by (1.26) also case (ii), correspond to σ-fixed points. This proves that any Sturm permutation $\sigma = \sigma_f$ in the Hamiltonian class $f \in \mathrm{Sturm}(u)$ is indeed an involution.

The involution property of σ_f in the Hamiltonian class $f \in \mathrm{Sturm}(u)$ alone, although necessary, is not sufficient to characterize this class of Sturm permutations. In fact, the 2-cycles of the decomposition (1.23) must also obey some elementary topological restrictions due to the nesting configuration of the 2π-periodic orbits of (1.17) in the phase plane. In particular:

(a) different periodic orbits cannot intersect;
(b) orbits belonging to the same connected region of the set of all periodic orbits are nested with a total ordering; and

(c) the boundaries of the connected regions are composed of homoclinic loops to saddle points (generically excluding heteroclinic orbits). For an illustration see Figure 3.3 further below.

Motivated by the consideration of the phase portrait of (1.17) we introduce the following definitions for the integer boundary labels of 2-cycles $(\underline{c}\ \overline{c})$.

For $\alpha \neq \beta$ we say that integer 2-cycles $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ in $\{1, \ldots, n\}$ are *intersecting* if the corresponding open intervals in $\mathbb{R}$ have a nonempty intersection,

$$(\underline{c}_\alpha, \overline{c}_\alpha) \cap (\underline{c}_\beta, \overline{c}_\beta) \neq \emptyset\ . \tag{1.28}$$

Intersecting $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ are called *nested* if one of these intervals contains the other, or equivalently

$$(\underline{c}_\beta - \underline{c}_\alpha)(\overline{c}_\alpha - \overline{c}_\beta) > 0\ . \tag{1.29}$$

Nested $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ are called *centered* if the midpoints of these intervals coincide, or equivalently

$$\underline{c}_\beta - \underline{c}_\alpha = \overline{c}_\alpha - \overline{c}_\beta\ . \tag{1.30}$$

We define the *σ-stable core* C_α of a 2-cycle $(\underline{c}_\alpha\ \overline{c}_\alpha)$ as the set of its interior σ-stable points, alias ODE saddles:

$$C_\alpha = \{k :\ i_k(\sigma) = 0 \text{ and } \underline{c}_\alpha < k < \overline{c}_\alpha\}\ . \tag{1.31}$$

The notion of σ-stable core induces an equivalence relation on the set of nested 2-cycles. Two nested 2-cycles $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ are called *core-equivalent* if they share the same σ-stable core, $C_\alpha = C_\beta$.

Finally, we arrive at a definition which is central to our permutation characterization in the class $f \in \mathrm{Sturm}(u)$. An involution permutation $\sigma \in \mathcal{S}(n)$ is called *integrable* if:

(I.1) any two intersecting 2-cycles are nested,
(I.2) any two core-equivalent 2-cycles are centered, and
(I.3) any pair of non-nested 2-cycles is separated by a σ-stable point.

Then, our main result is the following purely combinatorial characterization of Sturm permutations in the Hamiltonian class $f \in \mathrm{Sturm}(u)$.

Theorem 1. *A Sturm permutation $\sigma \in \mathcal{S}(n)$ is in the Hamiltonian class of $f \in \mathrm{Sturm}(u)$, if and only if σ is an integrable involution.*

Throughout this paper we use arguments involving PDE and ODE dynamics, in addition to pure combinatorics. To help keeping track of the involved arguments we will use the following ground rules. Unless otherwise specified, all arguments related to dynamics will refer to ODE dynamics. For example, a center, a saddle, or a periodic orbit will correspond, respectively, to a center, a saddle, or a periodic solution of an ODE. Moreover, to avoid misunderstandings, the term cycle will be used only in the combinatorial sense. Closed ODE orbits will be called periodic, rather than cycles. In addition, all arguments related to combinatorics that could be misinterpreted as dynamics will explicitly mention a permutation. For example, a label $k \in \{1, \ldots, n\}$ for which $\sigma(k) = k$ will be called a *σ-fixed point.*

The proof of Theorem 1 will be presented in Sections 4, 5 and 6. In the final Section 7 we present some concluding remarks and list examples derived from this permutation characterization.

2. Sturm permutations in the general class $f \in \text{Sturm}(x, u, u_x)$

We recall the characterization of Sturm permutations, (1.13), (1.14), for the general Neumann problem (1.1), (1.2), where $f \in \text{Sturm}(x, u, u_x)$. In order to do this we also recall the purely combinatorial definitions of dissipative, Morse and meander permutations. The main result of this section asserts that Sturm permutations are precisely those permutations which are dissipative Morse meanders [FR99]; see Theorem 2 below.

A permutation $\sigma \in \mathcal{S}(n)$ is called *dissipative* whenever n is odd and σ satisfies

$$\sigma(1) = 1 \ , \ \sigma(n) = n \ . \tag{2.1}$$

By the parabolic comparison principle and the dissipative condition (1.3) Sturm permutations are dissipative, indeed.

We recall that, given any permutation $\sigma \in \mathcal{S}(n)$, the Morse numbers $i_k = i_k(\sigma)$, for $1 \leq k \leq n$, are defined by (1.20). Note $i_1 = 0$ by the empty sum, and $i_n = 0$ for dissipative σ. The permutation $\sigma \in \mathcal{S}(n)$ is called *Morse permutation* if the numbers $i_1, \ldots, i_n$ are all nonnegative.

For any permutation $\sigma \in \mathcal{S}(n)$ consider $j, k \in \{1, \ldots, n\}$ such that $\sigma^{-1}(j)$ is between $\sigma^{-1}(k)$ and $\sigma^{-1}(k+1)$ with j of the same even/odd parity as k, that is $(-1)^j = (-1)^k$. We call σ a *meander permutation* if $\sigma^{-1}(j+1)$ is also between $\sigma^{-1}(k)$ and $\sigma^{-1}(k+1)$, for all such j, k. A more geometric definition that puts the term meander in evidence is the following: consider a C^1 Jordan curve Γ intersecting the horizontal axis in n ordered points with strict crossings, i.e. a *meander* as introduced by Arnold in [Arn88]. Let us label the crossings by $1, \ldots, n$ in increasing order along the curve Γ. Along the horizontal axis, proceeding left to right, the same labels appear in the possibly different order $\sigma(1), \ldots, \sigma(n)$; see Figure 2.1 right. Any permutation σ arising by such a construction is called a meander permutation. Note that our two definitions are equivalent, by the Jordan curve theorem. Indeed the crossing label k appears at position $\sigma^{-1}(k)$ along the horizontal axis.

These definitions turn up in connection with the solution of the ODE Neumann boundary value problem (1.8) by the shooting method. Consider the flow of (1.8) on the phase plane (v, v_x) with initial conditions at $x = 0$ on the horizontal Neumann axis $\{(v, v_x) : v_x = 0\}$. The diffeomorphic image of this v-axis at $x = \pi$, under the ODE flow, is called the *shooting curve* Γ_f and corresponds to the meander Γ. The meander $\Gamma = \Gamma_f$ intersects the v-axis at the n points $(v_k(\pi), 0)$ corresponding to the n Neumann equilibrium solutions v_k of the PDE (1.1), i.e. the equilibria (1.12). The ordering of the equilibria along the meander determines the meander permutation $\sigma = \sigma_f \in \mathcal{S}(n)$ introduced in (1.13), (1.14). In particular, Sturm permutations are meander permutations. Less trivially, the numbers $i_k(\sigma)$, $k = 1, \ldots, n$, defined in (1.20) correspond to the nonnegative Morse indices of the equilibria v_k for (1.1),

$$i(v_k) = i_k(\sigma) \ . \tag{2.2}$$

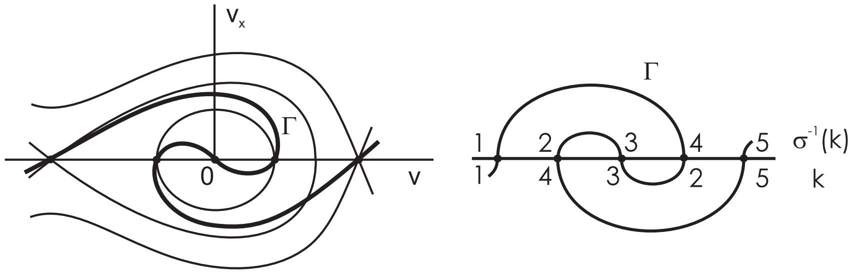

FIGURE 2.1. A meander Γ corresponding to the meander permutation $\sigma = (1\ 4\ 3\ 2\ 5) \in \mathcal{S}(5)$. Left: phase plane ODE flow and shooting meander $\Gamma = \Gamma_f$. Right: stylized meander and Sturm permutation $\sigma = \sigma_f \in \mathcal{S}(5)$.

Therefore Sturm permutations are dissipative Morse meanders.

Figure 2.1 illustrates a meander $\Gamma = \Gamma_f$ corresponding to a problem with $n = 5$ equilibria. The associated meander permutation $\sigma = \sigma_f \in \mathcal{S}(5)$, represented as a permutation of $\{1, 2, 3, 4, 5\}$ and also in cycle notation, is given by

$$\sigma = \{1, 4, 3, 2, 5\} = (2\ 4)\ . \tag{2.3}$$

Above we have indicated why Sturm permutations necessarily possess the dissipative, Morse and meander properties. But these conditions are also sufficient: a complete characterization of Sturm permutations for (1.1) in purely combinatorial terms is contained in the following

Theorem 2 ([FR99], Theorem 1.2). *A permutation $\sigma \in \mathcal{S}(n)$ is a Sturm permutation $\sigma = \sigma_f$ in the Sturm class $f \in \mathrm{Sturm}(x, u, u_x)$ if, and only if, σ is a dissipative Morse meander permutation.*

For completeness we recall that, in addition to the Morse indices $i(v_k)$, the Sturm permutation $\sigma \in \mathcal{S}(n)$ also determines the numbers of zeros $z(v_j - v_k)$ of the difference $v_j - v_k$ between any two different equilibria $v_j, v_k \in \mathcal{E}$,

$$z(v_j - v_k) = z_{jk}(\sigma)\ . \tag{2.4}$$

In particular, for adjacent equilibria we have

$$z(v_{k+1} - v_k) = \min\{i_{k+1}(\sigma), i_k(\sigma)\}\ . \tag{2.5}$$

We refer to [FR96] for details.

In the sequel we will quite often use the following immediate consequence of the explicit definition (1.20) of the Morse numbers $i_k(\sigma)$:

Lemma 1. *The numbers $k \in \{1, \dots, n\}$ and $i_k(\sigma)$ have opposite even/odd parity.*

Proof: Indeed, from (1.20) we obtain $i_k(\sigma) = k - 1 (\mathrm{mod}\, 2)$.

3. Period maps and period lap numbers

In this Section we review the characterization of 2π-periodic ODE solutions of autonomous planar Hamiltonian pendulum equations (1.16) by means of the period map $T = T_f$. See [Roc07] for complete details. For motivation we recall our proof, in Section 1, that all Sturm permutations $\sigma = \sigma_f$ in the Hamiltonian Sturm class $f \in \mathrm{Sturm}(u)$ are involutions. That proof critically relied on the fact that all PDE equilibria $v_j(x)$ on $0 \leq x \leq \pi$ are 2π-periodic ODE solutions $(v_j(x), v_j'(x))$, due to Neumann boundary conditions.

Generically in f the potential function F appearing in the Hamiltonian (1.18) is a Morse function, that is, all critical points e_j of F are nondegenerate: $f(e_j) = 0$ implies $f'(e_j) \neq 0$. The assumption $f = f(u) \in \mathrm{Sturm}(u)$ guarantees this nondegeneracy. In fact the Sturm class $\mathrm{Sturm}(u)$ ensures hyperbolicity of all the PDE equilibria, not just the homogeneous ones given by the ODE saddles and centers $(e_j, 0)$. Possibly after an additional small perturbation of f to $f + \varepsilon g$ in the Sturm class $\mathrm{Sturm}(u)$, as we have noted in the introduction, all critical values of F can be assumed distinct: $f(e_j) = f(e_k) = 0$ for $j \neq k$ implies $F(e_j) \neq F(e_k)$. In the present section, as in [Roc07], we impose this additional genericity condition to hold for f itself.

Since the critical values $F(e_j)$ are assumed to be distinct we can define the *Morse type* $\mu \in \mathcal{S}(n)$ of the potential F to be given by the ordering of the critical values

$$F(e_{\mu(1)}) < F(e_{\mu(2)}) < \cdots < F(e_{\mu(m)}) \ , \tag{3.1}$$

as contrasted with the ordering $e_1 < \cdots < e_m$ of the critical points themselves. See Figure 3.1 below for an example.

In general, the Hamiltonian ODE pendulum flow (1.16) possesses open bounded regions filled up with periodic solutions. The regions are the bounded connected components, in the (v, v_x)-plane, of the preimage, under the Hamiltonian H, of the regular values of H. The regions occur in two different types:

- punctured disks (like $\mathcal{C}_2$ and $\mathcal{C}_3$ in Figure 3.1), and
- annular regions (like $\mathcal{C}_1$ in Figure 3.1).

Each punctured disk consists of the periodic solutions encircling one, and only one, of the centers $(e_\alpha, 0)$, for even $\alpha = 2, 4, \ldots, (m-1)$. Therefore there are $(m-1)/2$ such disks. Their outer boundaries each consist of a single saddle with an attached homoclinic separatrix loop. The annular regions consist of periodic solutions enclosing more than one equilibrium. They are bounded by saddles and their homoclinic ODE separatrices. More precisely, as for the punctured disks, the outer boundary of each annulus is given by a single homoclinic separatrix loop attached to some ODE saddle $(e_j, 0)$. The inner boundary, in contrast, consists of a single ODE saddle $(e_k, 0)$ with a figure ∞ of two attached homoclinic loops. The number a of annular regions, as well as their precise geometric nesting with each other and with the punctured disks, depends only on the Morse type μ of F. A

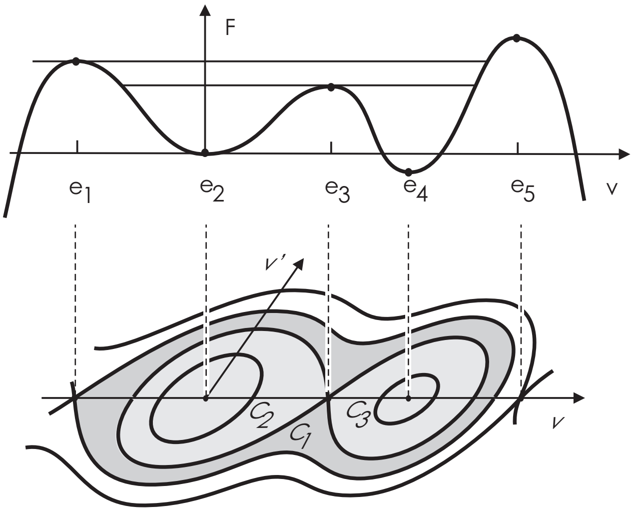

FIGURE 3.1. A phase portrait corresponding to a potential F with five stationary points. Note that e_1, e_3, e_5 are separatrix saddles with $F(e_3) < F(e_1) < F(e_5)$. The remaining critical points e_2, e_4 of even index are local minima, alias phase plane centers.

sharp upper estimate on a is given by

$$a \leq (m-3)/2 \ . \tag{3.2}$$

Indeed each annular region can be mapped injectively to its unique inner ODE saddle $(e_k, 0)$. These inner saddles $(e_k, 0)$ must differ from $(e_1, 0)$ and $(e_m, 0)$: by dissipativity of f, any figure ∞ pair of attached homoclinic loops entails additional ODE centers to exist both to the right and to the left of $(e_k, 0)$. With $(m+1)/2 - 2$ remaining saddles $(e_k, 0)$ to choose from, this proves (3.2). In Figure 3.1 we show a phase portrait with two punctured disks and one annular region corresponding to a nonlinearity f with five zeros e_j.

Let the open domain $D \subset \mathbb{R}$ denote the set of those real v_0 for which $(v_0, 0)$ lies on a nonstationary periodic ODE orbit of (1.17). The *period map*

$$T_f : D \to (0, \infty) \tag{3.3}$$

associates to each $v_0 \in D$ the minimal period of $(v_0, 0)$. The nonconstant 2π-periodic orbits of (1.17) are obtained from the solutions of

$$T_f(v_0) = 2\pi/\ell \ , \quad \text{with } \ell \in \mathbb{N} \ . \tag{3.4}$$

For definiteness, in the Neumann representation of the periodic orbits we only consider the corresponding solutions $v = v(x)$ with minima at $x = 0$, that is, $v(0) = v_0$, $v'(0) = 0$ and $v''(0) > 0$. In this way, to each 2π-periodic orbit v of minimal period $2\pi/\ell$ we associate the positive integer $\ell = \ell(v)$ which we call the *period lap number*; see [Mat82, Roc07] (this is half of the usual lap number).

Since all equilibria of (1.1), (1.2) for $f \in \text{Sturm}(u)$ are hyperbolic, (1.17) has a finite number of nondegenerate 2π-periodic orbits. It is useful to recall here that a nonconstant 2π-periodic orbit with Neumann initial condition

$(v_0, 0)$ is nondegenerate if, and only if

$$T_f'(v_0) \neq 0 \; ; \tag{3.5}$$

see [STW80].

We have already noticed that the Morse type μ of F determines the phase plane nesting configuration of the connected regions of periodic orbits. In fact, the combinatorial structure given by the nesting of the periodic orbits establishes a total ordering on the set of connected regions. To define this total ordering more formally we consider a representation of the connected regions by a *regular bracket structure*, [Lan03]. This is the structure of the parentheses which is ordinarily used in arithmetic expressions to indicate the hierarchical order of computations. For example, the configuration of connected regions for the phase portrait shown in Figure 3.1 is represented by

$$(()()) \; . \tag{3.6}$$

Note how innermost bracket pairs, only, correspond to the outer homoclinic loop boundaries of punctured disks. Each remaining bracket pair corresponds to the outer homoclinic loop boundary of an annular region. In particular, a regular bracket structure arising in connection with the Morse type μ of a potential F

- contains all the information regarding the types of the connected regions: punctured disks or annular regions;
- is uniquely determined by the Morse type μ of F.

Different Morse types μ, however, may still give rise to identical bracket structures. For example, the bracket structure remains unaffected by changes of the precise relative order of F at local minima. Similarly, the precise relative order of the local maxima of F does not always matter.

Let

$$\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_s\} \tag{3.7}$$

denote the set of nonstationary 2π-periodic orbits of the ODE (1.17). If v_{k_β} is any of the two PDE equilibrium solutions of (1.15), (1.2), corresponding to $\mathbf{p}_\beta \in \mathcal{P}$, then we also define its period lap number $\ell(v_{k_\beta})$ to coincide with the period lap number of $\mathbf{p}_\beta$ determined by the period map T_f in (3.4). This period lap number $\ell(v_{k_\beta})$ coincides with the usual lap number $\hat{\ell}(v_{k_\beta}) := z(v'_{k_\beta}) + 1$. It sounds complicated, but is actually simple. By (3.4), the period lap number $\ell(\mathbf{p}_\beta)$ of a 2π-periodic orbit $\mathbf{p}_\beta$ counts its number of complete cyclings for $0 \leq x \leq 2\pi$. The standard lap number $\hat{\ell}(v_{k_\beta})$ counts the number of half-cyclings of the equilibrium $(v_{k_\beta}(x), v'_{k_\beta}(x))$ on the half-interval $0 \leq x \leq \pi$, from Neumann to Neumann boundary condition. Of course $\hat{\ell}(v_{k_\beta}) = \ell(\mathbf{p}_\beta)$. In other words, the period lap number $\ell(v_{k_\beta}) = z(v'_{k_\beta}) + 1$ counts the number of intervals of strict monotonicity of $v_{k_\beta}(x)$, for $0 \leq x \leq \pi$. Therefore the period lap number for equilibria coincides with the standard lap number as introduced by Matano, [Mat82].

We briefly recall how period lap numbers can also be computed directly from the Sturm permutation σ_f via (2.4). In fact, if $v_j \in \mathcal{E}$ is any equilibrium of (1.15), (1.2), encircled by $\mathbf{p}_\beta$ in the phase plane, we have

$$\ell(v_{k_\beta}) = z(v_{k_\beta} - v_j) = z_{k_\beta j}(\sigma_f) \; ; \tag{3.8}$$

see [BF89] and Section 2.

Let $K = (m-1)/2 + a$ denote the total number of connected regions $\mathcal{C}_r, 1 \leq r \leq K$, of spatially nonhomogeneous periodic orbits: $(m-1)/2$ punctured disks, as generated and counted by their ODE centers, and a annuli. We obtain a total ordering for this set of connected regions using the one to one correspondence with the set of left brackets in the regular bracket structure. Inside each connected region, separately, the subset of 2π-periodic orbits is also totally ordered by the nesting relation.

Let s_r denote the number of nonstationary 2π-periodic orbits in region $\mathcal{C}_r$, and let

$$S_r = (\ell_1^r, \dots, \ell_{s_r}^r) \tag{3.9}$$

denote the sequence of their period lap numbers, ordered by decreasing size in the nesting relation. Empty sequences are allowed. For brevity we supress the superscript r. In this way, to each period map T_f we associate an ordered collection of ordered sequences

$$S = (S_r)_{1 \leq r \leq K} \ , \tag{3.10}$$

which we call the *lap signature* of T_f.

As a consequence of the continuity of the period map T_f the lap signature of T_f cannot be arbitrary. See [Roc07] for details. In fact, each sequence $S_r = (\ell_1, \dots, \ell_{s_r})$ must satisfy the following conditions.

(S.1) *Outer boundary condition*: We must have

$$\ell_1 = 1 \ . \tag{3.11}$$

Indeed ℓ_1 refers to the slowest, outermost periodic orbit and $\ell_1 = 1$ holds because $T_f(v) \to +\infty$ as v approaches the outer boundary of $\mathcal{C}_r$ determined by the outer homoclinic separatrix.

(S.2) *Neighbor jump condition*: Continuity of T_f implies the impossibility of jumps larger than 1 in successive period lap numbers; see (3.4). Therefore, we have

$$|\ell_{j+1} - \ell_j| \leq 1 \ , \ j = 1, \dots, s_r - 1 \ . \tag{3.12}$$

(S.3) *Alternate jump condition*: If $\ell_{j_0-1} \neq \ell_{j_0} = \cdots = \ell_{j_1} \neq \ell_{j_1+1}$ for $1 \leq j_0 \leq j_1 < s_r$, then

$$(\ell_{j_0} - \ell_{j_0-1})(\ell_{j_1+1} - \ell_{j_1}) = (-1)^{j_1 - j_0} \ . \tag{3.13}$$

Here we define $\ell_0 := 0$ for $j_0 = 1$. Indeed (3.5) implies that successive periodic orbits with the same value of T_f must have alternate signs of T_f'. Therefore, successive jumps in the period lap numbers must have the sign parity of the number of successive repeated values of T_f between the jumps. See Figure 3.2 for illustrations of both parities.

(S.4) *Inner boundary condition*: If, finally, S_r corresponds to an annular region then we must also have

$$\ell_{s_r} = 1 \text{ with } s_r \text{ even} \ . \tag{3.14}$$

Indeed, $\ell_{s_r} = 1$ refers to the innermost periodic orbit and $T_f(v) \to +\infty$ also holds as v approaches the figure ∞ pair of homoclinic loops which constitutes the *inner* boundary of any annular region. Due to (3.5) the number s_r of periodic orbits in the annular region must then be even.

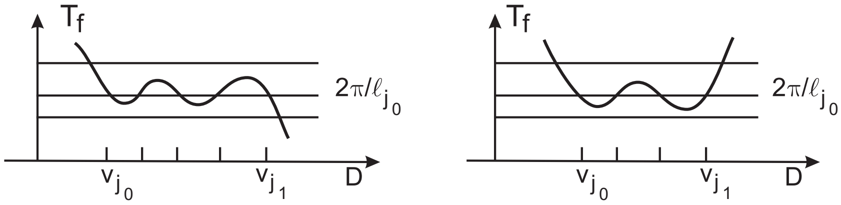

FIGURE 3.2. Period maps T_f. Left: $j_1 - j_0$ even. Right: $j_1 - j_0$ odd.

It turns out that these necessary conditions are also sufficient to describe the set of lap signatures of period maps. In fact, consider any regular bracket structure with K components and any ordered collection S of sequences (3.9). Then, if these sequences satisfy the above conditions (3.11-3.13), the collection S of sequences is the lap signature of a period map T_f. This characterization is contained in the following

Theorem 3 ([Roc07], Proposition 3). *Consider any regular bracket structure B with K pairs of brackets, corresponding to a Morse type μ. Consider any collection of sequences*

$$S = (S_r)_{1 \le r \le K} \ . \tag{3.15}$$

Then, S is the lap signature of a nondegenerate period map $T = T_f$ of a nonlinearity $f \in$ Sturm(u) for which the potential F has the Morse type μ corresponding to B if, and only if:

(i) *each S_r corresponding to an innermost bracket pair satisfies the outer boundary condition (S.1), the neighbor jump condition (S.2), and the alternate jump condition (S.3), as given by (3.11), (3.12) and (3.13);*
(ii) *each S_r corresponding to the remaining bracket pairs satisfies conditions (S.1-S.3) as before, and in addition the inner boundary condition (S.4) as given by (3.14).*

Let us reformulate Theorem 3 slightly. Again consider a bracket structure B with $K \ge 0$ pairs of brackets, and any collection of sequences $S = (S_r)_{1 \le r \le K}$, i.e. $S_r = (\ell_1^r, \dots, \ell_{s_r}^r)$ with arbitrary positive integers s_r and ℓ_j^r, which need not be lap numbers of anything, a priori. We call S an *abstract signature* associated to the bracket structure B, if properties (i) and (ii) of Theorem 3 both hold. With this definition we arrive at

Theorem 4. *Consider any regular bracket structure B with K pairs of brackets, corresponding to a Morse type μ, and any collection of sequences $S = (S_r)_{1 \le r \le K}$. Then, S is the lap signature of a nondegenerate period map $T = T_f$ of a nonlinearity $f \in$ Sturm(u) with Morse type μ if, and only if, S is an abstract signature associated to the bracket structure B.*

In Figure 3.3 we present a phase portrait with three connected regions like Figure 3.1, and with five 2π-periodic orbits. The phase portrait has the bracket representation (3.6). A lap signature for a period map corresponding to this phase portrait is the following collection of sequences:

$$S_1 = (1,1) \ , \ S_2 = (1) \ , \ S_3 = (1,2) \ . \tag{3.16}$$

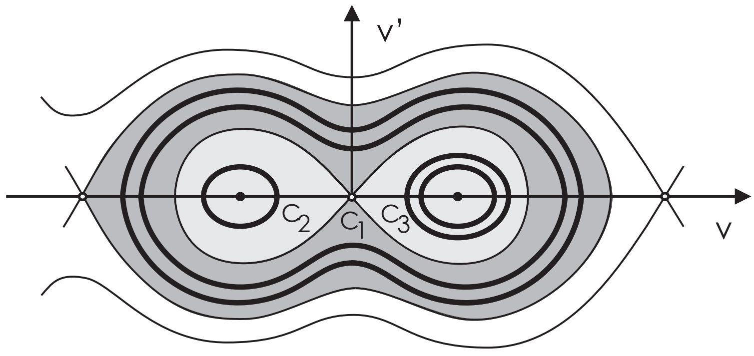

FIGURE 3.3. A phase portrait with five equilibria, five 2π-periodic orbits and three homoclinic region boundaries. Annular region $\mathcal{C}_1$: dark gray. Punctured disk regions $\mathcal{C}_2$, $\mathcal{C}_3$: light gray.

Here, according to the ordering of the left parentheses in the representation (3.6), S_2 and S_3 correspond to the 2π-periodic orbits in the disk regions $\mathcal{C}_2$ and $\mathcal{C}_3$ respectively, and the sequence S_1 corresponds to the 2π-periodic orbits in the annular region $\mathcal{C}_1$.

To complete the characterization of the equilibria of (1.15), (1.2) by means of the period map T_f we recall the relation between the Morse indices and the values of T_f next. See for example [BC84]. See also [FRW04] where the relation between the Morse index and the period lap number is discussed. Consider the pair of equilibria $\underline{v}_{k_\beta}, \overline{v}_{k_\beta} \in \mathcal{E}$ of (1.15), (1.2), corresponding to any nonstationary 2π-periodic orbit $\mathbf{p}_\beta \in \mathcal{P}$ of (1.17). Let $v_{k_\beta} = \underline{v}_{k_\beta}$ denote the equilibrium solution with minimum at $x = 0$, i.e. $v''_{k_\beta}(0) > 0$. Let $v_0 := v_{k_\beta}(0)$ denote that minimum. Then $i(v_{k_\beta}) = i(\underline{v}_{k_\beta}) = i(\overline{v}_{k_\beta})$ and

$$i(v_{k_\beta}) = \begin{cases} 2\pi/T_f(v_0) & \text{if} \quad T'_f(v_0) < 0 \ , \\ 2\pi/T_f(v_0) + 1 & \text{if} \quad T'_f(v_0) > 0 \ . \end{cases} \tag{3.17}$$

The spatially homogeneous, x-independent, equilibria $v_{k_\alpha} \in \mathcal{E}$ of (1.15), (1.2) correspond to the saddles and centers $(e_\alpha, 0) \in \mathcal{Z}$ of (1.17). Then, for the saddles $v_{k_\alpha} = e_\alpha$ (with odd α) we have

$$i(v_{k_\alpha}) = 0 \ . \tag{3.18}$$

For the centers $v_{k_\alpha} = e_\alpha$ (with even α) we have

$$i(v_{k_\alpha}) = [2\pi/\tilde{T}_f(e_\alpha)] + 1 \ , \tag{3.19}$$

where $[\cdot]$ denotes the integer part, and $\tilde{T}_f$ is the continuous extension of T_f to the center e_α given by

$$\tilde{T}_f(e_\alpha) := 2\pi/\sqrt{f'(e_\alpha)} \ . \tag{3.20}$$

All Morse indices satisfy $i(v_k) = i_k(\sigma)$ as obtained from (1.20).

We recall that PDE hyperbolicity in the sense of (1.1), (1.2), of the ODE centers $(e_\alpha, 0) \in \mathcal{Z}$, ($\alpha$ even), in the sense of (1.17), is also determined from the period map T_f. In fact, PDE hyperbolicity of the equilibrium $v_{k_\alpha} = e_\alpha$ corresponds to non-degeneracy of $(e_\alpha, 0)$ which occurs if, and only if, $f'(e_\alpha)$ is not a squared integer.

4. From Hamiltonian phase planes to integrable Sturm involutions

In this section we prove that, in addition to the involution property already established in (1.22), our integrability conditions (I.1-I.3) of Theorem 1 are necessary for a Sturm permutation $\sigma \in \mathcal{S}(n)$ to be generated by a nonlinearity f in the Hamiltonian Sturm class Sturm(u).

Proof of Theorem 1 (the "only if"–part): Let f be in the Hamiltonian Sturm class Sturm(u) and let $\sigma_f \in \mathcal{S}(n)$ denote the Sturm permutation obtained from (1.15), (1.2). In Section 1 we have already shown that σ_f is an involution and we consider its decomposition (1.22), $\sigma_f = (\underline{c}_1\ \overline{c}_1)\dots(\underline{c}_q\ \overline{c}_q)$, into a product of 2-cycles.

Let $\underline{c}_\alpha < \overline{c}_\alpha$ and $\underline{c}_\beta < \overline{c}_\beta$ denote the labels of two distinct 2-cycles $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ of σ_f. Without loss of generality, let these 2-cycles be labeled such that $\underline{c}_\alpha < \underline{c}_\beta$.

Recall from (1.27) and (3.7) that each nonstationary 2π-periodic orbit $\mathbf{p}_\beta \in \mathcal{P}$ of (1.17) corresponds to a pair of spatially nonhomogeneous Neumann equilibria $\underline{v}_{k_\beta}, \overline{v}_{k_\beta} \in \mathcal{E}$ of (1.15), (1.2), where $\underline{v}_{k_\beta}$ and $\overline{v}_{k_\beta}$ have, respectively, their minimum and maximum value at $x = 0$.

Each 2-cycle of σ_f corresponds to a periodic ODE orbit with (not necessarily minimal) period 2π, which does not possess π as one of its periods. Therefore the minimal period of $\underline{v}_{k_\beta}$ and $\overline{v}_{k_\beta}$ is $2\pi/\ell$ for some odd ℓ.

The σ_f-fixed points, in contrast, correspond to the ODE equilibria $(e_j, 0)$, both saddles and centers, plus those pairs $\underline{v}_{k_\beta}, \overline{v}_{k_\beta}$ which are generated by 2π-periodic orbits and do also possess π as one of the periods. Therefore the minimal period of $\underline{v}_{k_\beta}$ and $\overline{v}_{k_\beta}$ is $2\pi/\ell$ for some even ℓ, in this case.

Suppose the 2-cycles $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ of σ_f correspond to nested 2π-periodic orbits. Since distinct ODE orbits cannot intersect, the Jordan curve theorem then implies

$$(\underline{c}_\beta - \underline{c}_\alpha)(\overline{c}_\alpha - \overline{c}_\beta) > 0\ . \tag{4.1}$$

Therefore, any two intersecting 2-cycles are nested; see (1.29). This establishes the integrability condition (I.1).

For integrability conditions (I.2-I.3) we consider the potential F of f. To show that any two core-equivalent 2-cycles $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ are centered, as claimed in (I.2), consider the case

$$\underline{c}_\alpha < \underline{c}_\beta < \overline{c}_\beta < \overline{c}_\alpha\ , \tag{4.2}$$

without loss of generality. The equilibria $v_{\underline{c}_\alpha}$ and $v_{\underline{c}_\beta}$ have local minima at $x = 0$. Hence (1.16) implies $F'(v) = f(v) = -v'' < 0$ at $v_{\underline{c}_\alpha}(0)$ and $v_{\underline{c}_\beta}(0)$. Similarly $F' > 0$ at the maxima $v_{\overline{c}_\beta}(0)$ and $v_{\overline{c}_\alpha}(0)$. Since the 2-cycles are core-equivalent, F does not possess any local maxima, i.e. ODE saddles, in any of the intermediate ranges $(v_{\underline{c}_\alpha}(0), v_{\underline{c}_\beta}(0))$ and $(v_{\overline{c}_\beta}(0), v_{\overline{c}_\alpha}(0))$. Since the signs of F' coincide at the boundary of each of these ranges, this implies strict monotonicity of F on these ranges, and hence absence of ODE centers there. Therefore, all other entries of σ_f in the ranges $(\underline{c}_\alpha, \underline{c}_\beta)$ and $(\overline{c}_\beta, \overline{c}_\alpha)$ can only arise from periodic orbits, and hence must occur in pairs. But none of these pairs can occur in a single one of these ranges, only: otherwise the corresponding 2-cycle would force an ODE center to exist there – a contradiction. Hence, we must have

$$\underline{c}_\beta - \underline{c}_\alpha = \overline{c}_\alpha - \overline{c}_\beta \ . \tag{4.3}$$

and the core-equivalent 2-cycles are centered; see (1.30). This proves integrability property (I.2).

To show that any pair of non-nested 2-cycles $(\underline{c}_\alpha\ \overline{c}_\alpha)$ and $(\underline{c}_\beta\ \overline{c}_\beta)$ is separated by a σ-stable point, as claimed in integrability condition (I.3), consider

$$\underline{c}_\alpha < \overline{c}_\alpha < \underline{c}_\beta < \overline{c}_\beta \ . \tag{4.4}$$

As above we obtain that $F' > 0$ at the maximum $v_{\overline{c}_\alpha}(0)$ and $F' < 0$ at the minimum $v_{\underline{c}_\beta}(0)$. So F attains a local maximum in the interval $(v_{\overline{c}_\alpha}(0), v_{\underline{c}_\beta}(0))$. This corresponds to an ODE saddle, hence to a σ_f-fixed point k with $\overline{c}_\alpha < k < \underline{c}_\beta$ which is σ_f-stable. This establishes the integrability condition (I.3) and completes the proof of the "only if"–part of the theorem. □

5. Lap signatures and Morse indices

We now prepare the proof, deferred to Section 6, that our integrability conditions (I.1-I.3) of Theorem 1 are not only necessary but also sufficient for a Sturm involution σ to be generated by a nonlinearity $f = f(u)$ in the Hamiltonian Sturm class Sturm(u). With this goal in mind we present several lemmas which relate the lap signature sequences $S_r = (\ell_1, \ldots, \ell_{s_r})$, as discussed in Section 3, with the Morse indices $i(v_k)$ of the stationary solutions v_k of the PDE (1.1) under Neumann boundary conditions. We recall that the period lap number ℓ_j refers to the minimal period $T_f = 2\pi/\ell_j$ of the 2π-periodic ODE orbits.

Our first lemma in this section relates the period lap number ℓ of any 2π-periodic spatially nonhomogeneous ODE orbit to the Morse index $i \in \{\ell, \ell+1\}$ of any of the two associated PDE equilibria; see also [BF88]. Lemma 3 identifies fixed points c of $\sigma = \sigma_f$ for f from the Hamiltonian class $f = f(u)$ as homogeneous ODE equilibria, in case their Morse numbers i_c satisfy $i_c \in \{0, 1\}$. The next lemmas consider lap signature sequences $S_r = (\ell_1, \ldots, \ell_{s_r})$ corresponding to the regions $\mathcal{C}_r$ of ODE periodic orbits. Lemma 4 calculates the jump between the terminal value ℓ_{s_r} and the last entry ℓ_{j_0-1} in S_r which deviates from ℓ_{s_r}. That jump is related to the Morse index i of the innermost periodic orbit $\mathbf{p}^r_{s_r}$ in $\mathcal{C}_r$. This calculation prepares for checking the alternate jump condition (S.3), (3.13) for invented extended signature sequences, in Section 6. The special case of a punctured disk $\mathcal{C}_r$, where ℓ_{s_r} refers to the innermost 2π-periodic ODE orbit around a Hamiltonian ODE center, is summarized in Lemma 5. Lemmas 6 and 7, finally, address Morse indices of Sturm involutions.

We recall our notational convention by which $v_{k_\beta} = \underline{v}_{k_\beta}$, among the two PDE equilibria $\underline{v}_{k_\beta}$ and $\overline{v}_{k_\beta}$ associated to the same 2π-periodic solution $\mathbf{p}_\beta$, denotes the one which attains its minimum at $x = 0$.

Lemma 2. *Let the 2π-periodic ODE orbit $\mathbf{p}$ possess period lap number ℓ, and hence minimal period $T_f(v_0) = 2\pi/\ell$. Then both PDE equilibria $\underline{v}$, $\overline{v}$*

associated to $\mathbf{p}$ *possess period lap number* $\ell \geq 1$ *and Morse index*

$$i = i(\underline{v}) = i(\overline{v}) \in \{\ell, \ell+1\} \ . \tag{5.1}$$

More precisely

$$i = \begin{cases} \ell & \text{if} \quad T_f'(v_0) < 0 \ , \\ \ell+1 & \text{if} \quad T_f'(v_0) > 0 \ . \end{cases} \tag{5.2}$$

Proof: Already in Section 3 we have observed that $\underline{v}$, $\overline{v}$ possess period lap number ℓ. The reflection $u \mapsto -u$ interchanges the role of $\underline{v}$ and $\overline{v}$ without affecting T_f. Therefore it is sufficient to prove (5.2), which has already been observed in (3.17) and is based on an elementary Sturm-Liouville comparison argument which involves the derivative $\underline{v}_x$. This proves the lemma. Similar results can be found in the literature; in particular for (5.1) see [BF88], Lemma 5.1 and [Smo83], Lemma 24.16, and for (5.2) see [FRW04], Lemma 5.3. □

Lemma 3. *Let* $f = f(u)$ *be in the Hamiltonian Sturm class* Sturm(u). *Consider a Sturm permutation* $\sigma := \sigma_f \in \mathcal{S}(n)$ *and any* $c \in \{1, \ldots, n\}$. *Let* i_c *denote the Morse number of the associated PDE equilibrium* $v(x)$. *Assume either* $i_c = 0$, *or else* $i_c = 1$, $\sigma(c) = c$. *Then* $v(x) \equiv e$ *is a spatially homogeneous ODE equilibrium* $(e, 0)$ *of the phase plane of* f. *Moreover* $(e, 0)$ *is*

(a) *an ODE saddle, if* $i_c = 0$;
(b) *an ODE Hamiltonian center, if* $i_c = 1$ *and* $\sigma(c) = c$.

Proof: First consider case (a). By Lemma 2, $0 = i_c \in \{\ell, \ell+1\}$ cannot indicate a nonstationary ODE periodic orbit with $\ell \geq 1$. Therefore $v(x) \equiv e$ is spatially homogeneous. The linearization

$$\lambda u = u_{xx} + f'(e)u \tag{5.3}$$

at $v(x) \equiv e$ immediately shows $f'(e) < 0$ for $i_c = 0$. Therefore $(e, 0)$ is an ODE saddle.

In case (b) we claim that $\sigma(c) = c$ and $i_c = 1$ imply σ is a Hamiltonian center of f. Indeed, σ-fixed points correspond to π-periodic orbits of f in the ODE phase plane. Because all Hamiltonian saddles of f are σ-stable, with $i_c = 0$, we only have to exclude the possibility that c corresponds to a spatially nonhomogeneous π-periodic orbit. By (1.24) such orbits have $\ell \geq 2$ even. In fact, $i \in \{\ell, \ell+1\}$, $\ell \geq 1$ by Lemma 2, and hence $i_c(\sigma) \geq 2$. This contradiction to $i_c(\sigma) = 1$ proves that c corresponds to a Hamiltonian center of f. □

We now turn to our investigation of lap signature sequences

$$S_r = (\ell_1, \ldots, \ell_{s_r}) = (\ell_1, \ldots, \ell_{j_0-1}, \ell_{j_0}, \ldots, \ell_{s_r}) \tag{5.4}$$

with constant tails

$$\ell_{j_0-1} \neq \ell_{j_0} = \cdots = \ell_{s_r} \tag{5.5}$$

from some index $1 < j_0 \leq s_r$ on.

Lemma 4. *Let* $S_r = (\ell_1, \ldots, \ell_{s_r})$ *denote a sequence in the lap signature of* T_f *corresponding to any region* $\mathcal{C}_r$ *(either a punctured disk or an annulus)*

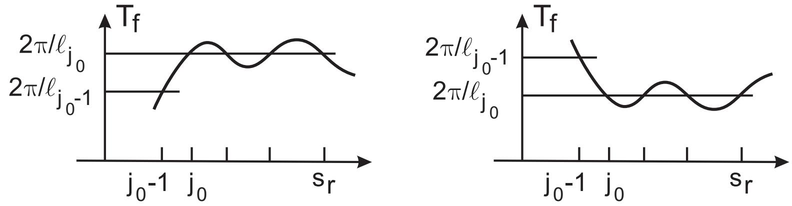

FIGURE 5.1. Period maps T_f with tails $j := s_r - (j_0 - 1)$ even. Left: $T_f'(v_0) < 0$. Right: $T_f'(v_0) > 0$.

of periodic orbits. Let $\mathbf{p}_\beta \in \mathcal{P}$, with associated equilibria $\underline{v}_{k_\beta}$, $\overline{v}_{k_\beta}$ denote the innermost 2π-periodic orbit with period lap number ℓ_{s_r}. Once again let v_{k_β} with Morse index $i(v_{k_\beta}) = i_{k_\beta} \in \{\ell_{s_r}, \ell_{s_r}+1\}$ denote the PDE equilibrium $\underline{v}_{k_\beta}$ associated to $\mathbf{p}_\beta$ with minimum v_0 at $x = 0$. Assume the lap signature sequence S_r satisfies (5.4), (5.5) above, for some $1 < j_0 \leq s_r$. Then

$$\ell_{j_0-1} = \ell_{s_r} + (-1)^{s_r-(j_0-1)} \quad \textit{if} \quad i_{k_\beta} = \ell_{s_r} \ , \tag{5.6}$$

$$\ell_{j_0-1} = \ell_{s_r} - (-1)^{s_r-(j_0-1)} \quad \textit{if} \quad i_{k_\beta} = \ell_{s_r} + 1 \ . \tag{5.7}$$

Furthermore s_r and i_{k_β} are of the same even/odd parity. In particular

$$\ell_{s_r} \equiv s_r \pmod 2, \qquad \textit{if} \quad i_{k_\beta} = \ell_{s_r} \ , \tag{5.8}$$

$$\ell_{s_r} \equiv s_r + 1 \pmod 2, \quad \textit{if} \quad i_{k_\beta} = \ell_{s_r} + 1 \ . \tag{5.9}$$

Proof: Let $j := s_r - (j_0 - 1)$ denote the maximal number of repeated lap numbers in the tail of S_r. Continuity of T_f and the hyperbolicity condition (3.5) imply

$$\ell_{s_r-j} - \ell_{s_r-j+1} = \begin{cases} (-1)^j & \text{if} \quad T_f'(v_0) < 0 \ , \\ (-1)^{j+1} & \text{if} \quad T_f'(v_0) > 0 \ . \end{cases} \tag{5.10}$$

See Figure 5.1. By (3.17) and (3.4), again, this shows (5.6) and (5.7).

Our proof of claims (5.8), (5.9) is based on an even/odd parity count, mod 2, of the total number s_r of 2π-periodic orbits $\mathbf{p} \in \mathcal{P} \cap \mathcal{C}_r$. Let ν_ℓ count the number of such orbits $\mathbf{p}$ with prescribed lap number ℓ,

$$s_r = \sum_{\ell \geq 1} \nu_\ell \ . \tag{5.11}$$

The graph of T_f provides the following parity counts, depending on ℓ:

(a) ν_ℓ is odd for $1 \leq \ell < \ell_{s_r}$;
(b) ν_ℓ is even for $\ell > \ell_{s_r} \geq 1$;
(c) ν_ℓ is odd for $\ell = \ell_{s_r}$, if $i(v_{k_\beta}) = \ell_{s_r}$;
(d) ν_ℓ is even for $\ell = \ell_{s_r}$, if $i(v_{k_\beta}) = \ell_{s_r} + 1$.

Properties (a)–(d) follow because $\ell_1 = 1$ with $T_f' < 0$. Claims (c)–(d) also use (5.2). For an illustration see Figure 5.2. Inserting (a)–(d) into (5.11), mod 2, implies

$$s_r = (\ell_{s_r} - 1) + (i(v_{k_\beta}) - \ell_{s_r} + 1) = i(v_{k_\beta}) \pmod 2 \ . \tag{5.12}$$

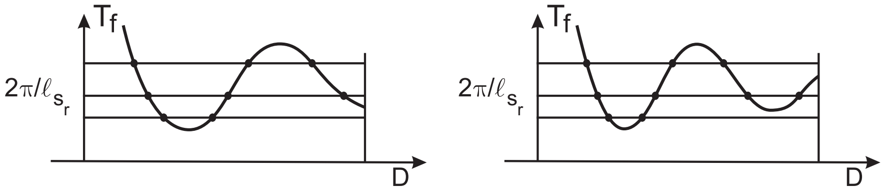

FIGURE 5.2. Period maps T_f to illustrate counting of periodic orbits. Left: $T_f'(v_0) < 0$. Right: $T_f'(v_0) > 0$.

Here the first parenthesis corresponds to case (a), and the second one to cases (c)–(d). Case (b) does not contribute, mod 2. This proves (5.8), (5.9) and the lemma. □

For example consider an annulus $\mathcal{C}_r$. Then $\ell_{s_r} = 1$ and $T_f'(v_0) > 0$ at the minimum v_0 of the innermost 2π-periodic orbit. In this case, (5.2) implies $i(v_{k_\beta}) = \ell_{s_r} + 1 = 2$. In particular s_r and $i(v_{k_\beta})$ are both even. With (5.7) our assumption $\ell_{s_r} = 1$ implies $j < s_r$ is odd, because $\ell_{s_r - j} > 0$.

The previous lemma treats annulus regions and punctured disks $\mathcal{C}_r$ alike. In the following lemma we refine our analysis of terminally constant lap signatures from (5.4), (5.5) for the case of a punctured disk $\mathcal{C}_r$ with an ODE Hamiltonian center $(e_\alpha, 0)$ at its center.

Lemma 5. *Let the assumptions of Lemma 4 hold for the punctured disk $\mathcal{C}_r$. In particular $\mathbf{p}_\beta$, with associated PDE equilibrium $v_{k_\beta} = \underline{v}_{k_\beta}$ of minimum v_0 at $x = 0$, denotes the innermost nonstationary periodic orbit around the Hamiltonian ODE center $(e_\alpha, 0)$. Let $v_{k_\alpha}(x) := e_\alpha$ be the spatially homogeneous PDE equilibrium corresponding to $(e_\alpha, 0)$. Then the Morse indices $i_{k_\alpha} := i(v_{k_\alpha})$, $i_{k_\beta} := i(v_{k_\beta})$ satisfy*

$$i_{k_\alpha}, i_{k_\beta} \in \{\ell_{s_r}, \ell_{s_r} + 1\} \ , \tag{5.13}$$

More specifically, $i_{k_\alpha} = \ell_{s_r}$ implies

$$i_{k_\beta} = \ell_{s_r} + 1 \ , \ \text{and} \tag{5.14}$$

$$\ell_{j_0 - 1} = \ell_{s_r} - (-1)^{s_r - (j_0 - 1)} \ . \tag{5.15}$$

The other alternative, $i_{k_\alpha} = \ell_{s_r} + 1$, implies

$$i_{k_\beta} = \ell_{s_r} \tag{5.16}$$

$$\ell_{j_0 - 1} = \ell_{s_r} + (-1)^{s_r - (j_0 - 1)} \ . \tag{5.17}$$

In particular (5.13) strengthens to $\{i_{k_\alpha}, i_{k_\beta}\} = \{\ell_{s_r}, \ell_{s_r} + 1\}$.

Proof: We first claim $k_\beta = k_\alpha - 1$. Indeed the PDE equilibrium v_{k_β} corresponds to the innermost 2π-periodic orbit $\mathbf{p}_\beta$ and attains its minimum $v_0 = v_{k_\beta}(0) < e_\alpha \equiv v_{k_\alpha}$ at $x = 0$. Since any further 2π-periodic orbits between $(v_0, 0)$ and $(e_\alpha, 0)$ are excluded, this proves the claim.

By definition of the lap signature, $\mathbf{p}_\beta$ has period lap number ℓ_{s_r} and

$$\ell_{s_r} = 2\pi / T_f(v_0) \ . \tag{5.18}$$

Lemma 2, (5.2) implies

$$i(v_{k_\beta}) = \begin{cases} \ell_{s_r} & \text{if} \quad T_f'(v_0) < 0 \ , \\ \ell_{s_r} + 1 & \text{if} \quad T_f'(v_0) > 0 \ . \end{cases} \tag{5.19}$$

For punctured disks, continuity of T_f implies

$$[2\pi/\tilde{T}_f(e_\alpha)] = \begin{cases} \ell_{s_r} & \text{if} \quad T_f'(v_0) < 0 \ , \\ \ell_{s_r} - 1 & \text{if} \quad T_f'(v_0) > 0 \ , \end{cases} \tag{5.20}$$

and (3.19) shows

$$i(v_{k_\alpha}) = \begin{cases} \ell_{s_r} + 1 & \text{if} \quad T_f'(v_0) < 0 \ , \\ \ell_{s_r} & \text{if} \quad T_f'(v_0) > 0 \ . \end{cases} \tag{5.21}$$

This establishes (5.13), (5.14) and (5.16). See again Figure 5.2 for an illustration.

To prove (5.15), (5.17) we just invoke Lemma 4 for the respective cases (5.14), (5.16). This proves the lemma. □

The next two lemmas deal with Sturm involutions $\sigma = \sigma_f$ with and without the integrability property.

Lemma 6. *Let $\sigma = \sigma_f \in \mathcal{S}(n)$ denote a Sturm involution with 2-cycle $(\underline{c}\ \overline{c})$. Then*

$$i_{\underline{c}}(\sigma) = i_{\overline{c}}(\sigma) \ . \tag{5.22}$$

Moreover, $\underline{c}$ and $\overline{c}$ possess the same even/odd parity.

Proof: Interchanging the role of the boundaries $x = 0$ and $x = \pi$ in (1.13), (1.14) by the transformation $x \mapsto \pi - x$, we obtain the inverse permutation σ^{-1}. Since the transformation does not alter Morse indices, all Sturm permutations satisfy

$$i_k(\sigma) = i_{\sigma(k)}(\sigma^{-1}) \ . \tag{5.23}$$

Since $\sigma = \sigma^{-1}$ is an involution and $\sigma(\underline{c}) = \overline{c}$ we obtain (5.22) for $k := \underline{c}$. Moreover, by Lemma 1 $i_k(\sigma)$ and k possess opposite parity. Therefore $\underline{c}$ and $\overline{c}$ are of equal parity. □

Lemma 7. *Let $\sigma = \sigma_f \in \mathcal{S}(n)$ denote an integrable Sturm involution. Then all σ-stable points k are σ-fixed points, $\sigma(k) = k$.*

Proof: Since σ is an involution, σ can be decomposed into disjoint 2-cycles as in (1.23). By integrability all intersecting 2-cycles are nested; see property (I.1). We prove the lemma by contradiction. Assume that $(\underline{c}\ \overline{c})$ is a 2-cycle of σ-stable points, i.e.

$$i_{\underline{c}}(\sigma) = i_{\overline{c}}(\sigma) = 0 \ . \tag{5.24}$$

Here we have used Lemma 6 to strengthen the negation of Lemma 7. Comparing (1.20) with $k = \underline{c}$ and $k = \underline{c} + 1$ for the involution σ we obtain

$$i_{\underline{c}+1}(\sigma) - i_{\underline{c}}(\sigma) = (-1)^{\underline{c}+1} \operatorname{sign}(\sigma(\underline{c}+1) - \sigma(\underline{c})) \ . \tag{5.25}$$

Since σ is a Morse permutation, $i_{\underline{c}}(\sigma) = 0$ implies $i_{\underline{c}+1}(\sigma) = 1$.

Suppose first that $\underline{c}+1 = \sigma(\underline{c}+1)$ is a σ-fixed point. With $\sigma(\underline{c}) = \overline{c}$ this yields

$$(-1)^{\underline{c}+1} = \mathrm{sign}(\underline{c}+1-\overline{c}) \ . \tag{5.26}$$

Moreover the σ-fixed point $\underline{c}+1$ satisfies $\underline{c} < \underline{c}+1 < \overline{c}$. Therefore $\underline{c}$ must be even, by (5.26). Hence $i_{\underline{c}}(\sigma)$ must be odd, by Lemma 1. This contradicts the assumption $i_{\underline{c}}(\sigma) = 0$.

If, on the other hand, $\underline{c}+1$ is in a 2-cycle $(\underline{c}+1\ \overline{\beta})$, then (5.25) implies

$$(-1)^{\underline{c}+1} = \mathrm{sign}(\overline{\beta}-\overline{c}) \ . \tag{5.27}$$

Since $\underline{c} < \underline{c}+1 < \overline{c}$, the 2-cycles $(\underline{c}+1\ \overline{\beta})$ and $(\underline{c}\ \overline{c})$ are intersecting. Because σ is integrable they must also be nested; see again property (I.1). In particular we have $\underline{c} < \overline{\beta} < \overline{c}$. But then (5.27) implies that $\underline{c}$ is even, again, and we reach a contradiction as before. □

6. From integrable Sturm involutions to a realization by Hamiltonian phase planes

In this section we complete the remaining "if"–part of our main Theorem 1. We start from a prescribed integrable Sturm involution σ. We have to construct a nonlinearity $f = f(u)$ in the Sturm class Sturm(u) such that the Sturm permutation σ_f of the nonlinearity $f(u)$ coincides with the prescribed integrable Sturm involution σ. A realization $\sigma = \sigma_h$ in the broader class $h = h(x, u, u_x) \in$ Sturm(x, u, u_x) is always guaranteed to exist.

Specifically, by Theorem 2 the Sturm permutation $\sigma = \sigma_h$ is a dissipative Morse meander. By our integrability assumption, the involution σ satisfies the integrability conditions (I.1-I.3) of Theorem 1. We recall that (I.1) demands nesting of overlapping 2-cycles. Condition (I.2) asserts symmetric nesting of 2-cycles sharing the same core of σ-stable points. Finally, (I.3) requires 2-cycles without overlap to be separated by σ-stable points.

Our proof proceeds by induction over the number of 2-cycles $(\underline{c}\ \overline{c})$ of the involution σ, according to the following outline.

In total absence of 2-cycles we conclude $\sigma = \mathrm{id}$. A simple ODE model $\dot{u} = g(u) \in \mathbb{R}$ is possible, e.g. of the form $g(u) = \sin u - \delta u^3$ with suitable coefficient $\delta > 0$ to accomodate any odd total number of n equilibria. For $f(u) := \varepsilon g(u)$ and sufficiently small $\varepsilon > 0$, the global attractor $\mathcal{A}_f$ of (1.1), (1.2), then coincides with this spatially homogeneous ODE model. Indeed the prescribed ODE model $\dot{u} = g(u)$ is then equivalent to the resulting PDE flow on a one-dimensional inertial manifold, for sufficiently small $\varepsilon > 0$; see [Hal88, Tem88].

If any 2-cycle $(\underline{c}\ \overline{c})$ is present in the integrable Sturm involution σ, the nesting property (I.1) of σ allows us to choose that 2-cycle to be of minimal range. In other words $\sigma(k) = k$, for all $\underline{c} < k < \overline{c}$. We will construct $f \in$ Sturm(u) such that $\sigma = \sigma_f$, via Theorem 3. We construct a regular bracket structure and lap signature $(S_r)_{1 \le r \le K}$ compatible with σ, such that $\sigma = \sigma_f$ for the nonlinearity $f = f(u)$ which Theorem 3 provides. To bring induction to bear, we first construct a *reduced permutation* $\hat{\sigma}$ on the elements

$$E_{\underline{c},\overline{c}} := \{1, \dots, \underline{c}, \overline{c}+1, \dots, n\} = \{1, \dots, n\} \setminus \{\underline{c}+1, \dots, \overline{c}\} \tag{6.1}$$

as follows. We simply let

$$\hat{\sigma}(k) := \begin{cases} \sigma(k) & \text{for} \quad k \in E_{\underline{c},\overline{c}} \setminus \{\underline{c}\} \\ \underline{c} & \phantom{\text{for}} \quad k = \underline{c} \end{cases} \tag{6.2}$$

In particular the 2-cycle $(\underline{c}\ \overline{c})$ of σ, together with its range $\{\underline{c}, \ldots, \overline{c}\}$, reduces to a single $\hat{\sigma}$-fixed point $\underline{c} = \hat{\sigma}(\underline{c})$.

In Lemma 8 below we show that $\hat{\sigma}$ remains an integrable Sturm involution, like σ itself was. Of course $\hat{\sigma}$ acts on an odd number $\hat{n} = n - (\overline{c} - \underline{c})$ of elements because Lemma 6 asserts that the constituents $\underline{c}, \overline{c}$ of the 2-cycle $(\underline{c}\ \overline{c})$ of σ are of equal even/odd parity. Moreover, $\hat{\sigma}$ features one 2-cycle less than σ itself. By induction hypothesis, we therefore have a realization of the reduced permutation $\hat{\sigma} = \hat{\sigma}_{\hat{f}}$ by a Hamiltonian nonlinearity $\hat{f} = \hat{f}(u)$ in the Sturm class Sturm(u). In particular $\hat{\sigma}, \hat{f}$ come fully equipped with a regular bracket structure $\hat{B}$, an associated Morse type $\hat{\mu}$ and a lap signature $\hat{S}_{\hat{f}}$. By Theorem 4 the lap signature $\hat{S}_{\hat{f}}$ can also be viewed as an abstract signature $\hat{S} := \hat{S}_{\hat{f}}$ with bracket structure $\hat{B}$. Theorems 3 and 4 provide the abstract signature $\hat{S}$ only in the restricted class of those Sturm Hamiltonian $\hat{f} \in$ Sturm(u), for which all critical $\hat{F}$-values at the zeros of $\hat{f}$ are distinct. This may not be the case for the $\hat{f} \in$ Sturm(u) provided by our induction hypothesis. Hyperbolicity of all PDE equilibria of $\hat{f}$, however, is already sufficient to ensure persistence of all PDE equilibria and, a forteriori, invariance of $\hat{\sigma} = \hat{\sigma}_{\hat{f}}$ itself, under small perturbations of $\hat{f}$ to $\hat{f} + \varepsilon\hat{g}$. Already in the introduction, we have noted that such perturbations are always able to make all critical values of $\hat{F} + \varepsilon\hat{G}$ at the zeros of $\hat{f} + \varepsilon\hat{g}$ distinct. Here and below we will therefore work with $\hat{f} + \varepsilon\hat{g}$ directly, and assume $\hat{f}$ itself provides the abstract signature $\hat{S}$, without any loss of generality.

To complete our proof of Theorem 1 it is therefore sufficient to provide a regular bracket structure B and an abstract signature S, from $\hat{B}$ and $\hat{S}$, such that B and S give rise to the prescribed integrable Sturm involution σ.

We achieve this goal by formally extending the bracket structure $\hat{B}$ and the signature $\hat{S}$ associated to $\hat{\sigma} = \hat{\sigma}_{\hat{f}}$. Our extensions are motivated geometrically, as we will briefly indicate in Section 7. For our proof, however, it is more convenient to decree an extended bracket structure B and an extended signature S, ex cathedra. In Lemmas 10, 11, 12 below we show how these formal extensions indeed provide abstract signatures which satisfy conditions (S.1–S.3) and, when applicable, (S.4); see (3.11)–(3.14). Therefore Theorem 4 provides a nonlinearity $f \in$ Sturm(u) with the abstract signature S and the bracket structure B as chosen, ex cathedra. In Lemma 13 we then conclude that B, S were in fact chosen such that $\sigma_f = \sigma$ is the prescribed original integrable Sturm involution σ. This will complete the induction step over the number of 2-cycles $(\underline{c}\ \overline{c})$ in σ, and will prove the "if"–part of our main Theorem 1.

Lemmas 10, 11, 12 below address the following complete list of three complementary cases:

(A) the pair $\underline{c}, \overline{c}$ is even;
(B) the pair $\underline{c}, \overline{c}$ is odd, and either $\overline{c} - \underline{c} = 2$, or else $\overline{c} - \underline{c} > 2$ and $i_{\underline{c}} > 2$;
(C) the pair $\underline{c}, \overline{c}$ is odd, $\overline{c} - \underline{c} > 2$ and $i_{\underline{c}} = 2$.

Here we excluded the remaining case, i.e. the pair $\underline{c}, \overline{c}$ is odd, $\overline{c} - \underline{c} > 2$ and $i_{\underline{c}} < 2$ (therefore $i_{\underline{c}} = 0$ by evenness due to Lemma 1). This is because Lemma 3 (a) in this case implies that $\underline{c}$ is a saddle, hence a σ-fixed point in contradiction with $(\underline{c}\ \overline{c})$ being a 2-cycle.

Lemma 8. *The reduced permutation $\hat{\sigma}$ introduced in (6.2) is an integrable Sturm involution.*

Proof: By Lemma 6 we have that $\underline{c}$ and $\overline{c}$ are of equal parity. Therefore $\hat{\sigma}$ is a meander like σ itself: neither the remaining orderings nor any of the parities did change. Moreover $\hat{\sigma}$ is dissipative, as was σ. In addition (1.20), (6.2), and (5.22) show that

$$i_k(\hat{\sigma}) = i_k(\sigma) \quad \text{for} \quad k \in E_{\underline{c},\overline{c}} \ . \tag{6.3}$$

Indeed the recursion (1.20) works identically for the remaining k. Therefore, $\hat{\sigma}$ is a Sturm permutation. Finally, deleting $(\underline{c}\ \overline{c})$ from the 2-cycle representation of σ we obtain the 2-cycle representation of $\hat{\sigma}$. By definitions (I.1)–(I.3) of integrability it follows that $\hat{\sigma}$ is also an integrable involution. This proves the lemma. □

We remark that (6.3) allows us to omit σ, $\hat{\sigma}$, henceforth, when we evaluate Morse numbers i_k for k appearing in both permutations. This allows us to interpret the cases (A)–(C) of our outline in terms of $i_{\underline{c}}(\hat{\sigma})$, rather than $i_{\underline{c}}(\sigma) = i_{\underline{c}}(\hat{\sigma})$.

By our induction hypothesis there is a regular bracket structure $\hat{B}$ and a Hamiltonian nonlinearity $\hat{f} \in \text{Sturm}(u)$ such that $\hat{\sigma} = \hat{\sigma}_{\hat{f}}$, i.e. a phase plane realization of $\hat{\sigma}$.

Lemma 9. *In the phase plane realization $\hat{f}$ of $\hat{\sigma}$, the equilibrium corresponding to the $\hat{\sigma}$-fixed point $\underline{c}$ is an ODE Hamiltonian center of $\hat{f}$.*

Proof: Obviously $\underline{c}$ cannot correspond to an ODE saddle. Indeed we have $i_{\underline{c}}(\hat{\sigma}) = i_{\underline{c}}(\sigma)$, by the remark above, and $i_{\underline{c}}(\sigma) > 0$ by Lemma 7.

To expose $\underline{c}$ as a Hamiltonian ODE center of $\hat{\sigma}_{\hat{f}}$ we consider the action of the permutation $\hat{\sigma}$ on the neighboring elements of $\underline{c}$. In particular we investigate whether or not $\underline{c} - 1$ or $\overline{c} + 1$ belong to some 2-cycle of $\hat{\sigma}$.

There are only two possibilities:

(i) Either there exists a 2-cycle of the form $(\underline{c} - 1\ \overline{\beta})$ or $(\underline{\beta}\ \overline{c} + 1)$; or

(ii) both $\underline{c} - 1$ and $\overline{c} + 1$ are $\hat{\sigma}$-fixed points.

Indeed, neighboring 2-cycles like $(\underline{\beta}\ \underline{c} - 1)$ are excluded by integrability property (I.3) and the absence of any separating point between $\underline{c} - 1$ and $\underline{c}$.

Alternative (i): We assume, without loss of generality, that the 2-cycle in the neighborhood of $\underline{c}$ has the form $(\underline{c} - 1\ \overline{\beta})$. This implies that $\overline{\beta} \geq \overline{c} + 1$. Indeed, by the integrability condition (I.1), $(\underline{c} - 1\ \overline{\beta})$ and $(\underline{c}\ \overline{c})$ are necessarily nested 2-cycles of σ. We distinguish the cases $\overline{\beta} = \overline{c} + 1$ and $\overline{\beta} > \overline{c} + 1$.

If $\overline{\beta} = \overline{c} + 1$, then $(\underline{c} - 1\ \overline{c} + 1)$ is a 2-cycle of $\hat{\sigma}_{\hat{f}}$ with mid point $\underline{c}$; recall that $\hat{\sigma}$ acts on $E_{\underline{c},\overline{c}} = \{1, \ldots, \underline{c} - 1, \underline{c}, \overline{c} + 1, \ldots, n\}$. We conclude that $\underline{c}$ corresponds to the unique ODE equilibrium point of $\hat{f}$ encircled by the

2π-periodic orbit corresponding to the 2-cycle $(\underline{c}-1\ \overline{c}+1)$ of $\hat{\sigma}_{\hat{f}}$. Therefore, $\underline{c}$ corresponds to a Hamiltonian center of $\hat{f}$.

Consider the case $\overline{\beta} > \overline{c}+1$ of alternative (i) next. We claim that there exists a $\hat{\sigma}$-stable (and hence, by Lemma 7, $\hat{\sigma}$-fixed) point $s \in \{\overline{c}+1, \ldots, \overline{\beta}-1\}$ such that the points in $\{\overline{c}+1, \ldots, s\}$ are all $\hat{\sigma}$-fixed, as is the neighboring fixed point $\underline{c}$ of $\hat{\sigma}$. In particular the Morse indices of $\{\underline{c}, \overline{c}+1, \ldots, s\}$ alternate in $\{0, 1\}$, by σ-stability of s and by recursion formula (1.20). Since $i_{\underline{c}} > 0$, this implies $i_{\underline{c}} = 1$, for the $\hat{\sigma}$-fixed point $\underline{c}$. By Lemma 3 this proves $\underline{c}$ corresponds to a Hamiltonian center of f.

It remains to prove the above claim, by construction of the $\hat{\sigma}$-stable point s. If the 2-cycle $(\underline{c}-1\ \overline{\beta})$ of σ contains further nested interior 2-cycles γ to the right of the 2-cycle $(\underline{c}\ \overline{c})$ of σ, then we invoke integrability property (I.3) of σ with respect to $(\underline{c}\ \overline{c})$ and the leftmost such 2-cycle γ. This provides a separating σ-stable point s, which is also $\hat{\sigma}$-stable and possesses the required attributes.

In absence of such further 2-cycles γ, we invoke integrability property (I.2) of σ: since the 2-cycles $(\underline{c}\ \overline{c})$ and $(\underline{c}-1\ \overline{\beta})$ are not centered, due to $\overline{\beta} > \overline{c}+1$, they are not core equivalent. Hence there exists a σ-stable point $s \in \{\overline{c}+1, \ldots, \overline{\beta}-1\}$ with the required attributes. In either case, the construction of s proves the lemma.

Alternative (ii): When both $\underline{c}-1$ and $\overline{c}+1$ are $\hat{\sigma}$-fixed points, we consider the points $\underline{c}-j$ and $\underline{c}+j$ for $j = 2, 3, \ldots$, and repeat the above pingpong between (i) and (ii) until we obtain either a 2-cycle as in alternative (i) or, since $\hat{\sigma}$ is dissipative, we reach a $\hat{\sigma}$-fixed point with index zero. Then, we obtain a sequence of $\hat{\sigma}$-fixed points with alternating indices 0 and 1, and argue again as in (i). This completes the proof that the $\hat{\sigma}$-fixed point $\underline{c}$ corresponds to an ODE Hamiltonian center. □

Let $\hat{S} = (\hat{S}_r)_{1\leq r\leq \hat{K}}$ denote the lap signature obtained from the realization $\hat{f}$ of the regular bracket structure $\hat{B}$ and the reduced integrable Sturm involution $\hat{\sigma}$. We extend this phase plane realization constructing a regular bracket structure B and an abstract signature $S = (S_r)_{1\leq r\leq K}$ corresponding to a realization of $\sigma = \sigma_f$.

As we have indicated in the outline to this section, we address the three complementary cases (A)–(C) in Lemmas 10–12, respectively. All these cases are completed in Lemma 13, which shows $\sigma = \sigma_f$ provided that $f \in \mathrm{Sturm}(u)$ is chosen as a Hamiltonian realization of the extended bracket structure B of $\hat{B}$ and the extended abstract signature S of $\hat{S}$.

Throughout let

$$\hat{S}_q = (\ell_1, \ldots, \ell_{\hat{s}_q}) \tag{6.4}$$

be the lap signature sequence corresponding to the innermost bracket pair q, which describes the punctured disk $\hat{\mathfrak{C}}_q$ of periodic orbits around the Hamiltonian ODE center $(e_\alpha, 0)$ of $\hat{f}$ associated to $\underline{c}$.

Lemma 10. *Let $\underline{c}$, $\overline{c}$ be even. Define the extended regular bracket structure $B := \hat{B}$ and the extended collection of sequences $S = (S_r)_{1\leq r\leq K}$ with*

$K := \hat{K}$ and $S_r := \hat{S}_r$, except for replacing the sequence $\hat{S}_q$ by

$$S_q = (\ell_1, \dots, \ell_{\hat{s}_q}, i_{\underline{c}}, i_{\underline{c}} + 1, \dots) \ , \tag{6.5}$$

with a total number of $(\overline{c} - \underline{c})/2 - 1 \geq 0$ identical trailing entries $i_{\underline{c}} + 1$. Then B is a regular bracket structure corresponding to a Morse type μ and S is an abstract signature associated to B.

Proof: Only $\hat{S}_q$ got replaced by S_q, in the lap signature $\hat{S}$. Moreover $\hat{S}_q$ is the lap signature sequence corresponding to the punctured disk $\hat{\mathcal{C}}_q$ and hence $\hat{S}_q$ corresponds to an innermost bracket of $\hat{B}$. Therefore S_q corresponds to an innermost bracket of B. Hence it is sufficient to show that S_q satisfies the signature conditions (S.1)–(S.3) specified in (3.11)–(3.13).

Signature condition (S.1), (3.11) on outer boundaries requires the first entry of S_q to equal 1. If $\hat{s}_q \geq 1$, i.e. if $\hat{S}_q \neq (\ )$ is not the empty sequence, this is obvious because $\ell_1 = 1$ for the signature sequence $\hat{S}_q$. Let us therefore suppose $\hat{s}_q = 0$, $\hat{S}_q = (\ )$, which implies $S_q = (i_{\underline{c}}, i_{\underline{c}} + 1, \dots)$ by (6.5). This sequence satisfies the outer boundary condition (S.1) if, and only if, $i_{\underline{c}} = 1$ which therefore is what we claim. In fact, if $\hat{S}_q = (\ )$ the region $\hat{\mathcal{C}}_q$ is a punctured disk without any 2π-periodic orbits encircling the Hamiltonian ODE center $v_{\underline{c}}$. Therefore, one of the neighbors of $\underline{c}$, namely $\underline{c} - 1$ or $\overline{c} + 1$, must be an ODE saddle, by the definition of the regular bracket structure $\hat{B}$ of $\hat{S}$, $\hat{f}$, $\hat{\sigma} = \hat{\sigma}_{\hat{f}}$. In particular that neighbor is $\hat{\sigma}$-stable, see (3.18). Hence (1.20) for $\hat{\sigma}$ and $i_{\underline{c}} \geq 0$ imply $i_{\underline{c}} = 1$, as claimed. This proves signature condition (S.1) for the extended collection of sequences S.

Signature condition (S.2), (3.12) on neighbor jumps has to be checked in the form

$$|\ell_{\hat{s}_q} - i_{\underline{c}}| \leq 1 \ , \tag{6.6}$$

only. By Lemma 9, $\underline{c} = k_\alpha$ is a Hamiltonian ODE center for $\hat{\sigma}_{\hat{f}}$. Therefore claim (6.6) is immediate from Lemma 5 applied to $\hat{\sigma}_{\hat{f}}$ with $k_\alpha = \underline{c}$, $s_r = \hat{s}_q$; see (5.13). The condition becomes void, of course, when $\hat{S}_q = (\ )$ is empty.

Signature condition (S.3), (3.13) on alternate jumps has to be checked in cases $j_1 = \hat{s}_q, \hat{s}_q + 1$, only – again for nonempty $\hat{S}_q$. Lemma 5, applied to $\hat{\sigma}$ with $k_\alpha = \underline{c}$, $s_r = \hat{s}_q$ and the signature sequence $\hat{S}_q$, takes care of these cases as follows; see Table 1. In the first two rows we specify and distinguish the three subcases $j_1 = \hat{s}_q$; $j_1 = \hat{s}_q + 1$ and $j_0 = j_1$; $j_1 = \hat{s}_q + 1$ and $j_0 < j_1$. The entry for $i_{\underline{c}}$, alias i_{k_α}, follows from conclusion (5.13), $i_{\underline{c}} \in \{\ell_{\hat{s}_q}, \ell_{\hat{s}_q} + 1\}$ of Lemma 5 applied to $\hat{\sigma}$, together with the assumption

$$\ell_{j_0 - 1} \neq \ell_{j_0} = \dots = \ell_{j_1} \neq \ell_{j_1 + 1} \ , \tag{6.7}$$

under which the alternate jump condition (3.13) in (S.3) is to be verified. For example consider the second column which corresponds to the case $j_1 = \hat{s}_q$, $1 \leq j_0 \leq \hat{s}_q$ as specified in the first and second rows of Table 1. Then the extended sequence (6.5) and (6.7) imply $i_{\underline{c}} = \ell_{j_1 + 1} \neq \ell_{j_1} = \ell_{\hat{s}_q}$. Lemma 5 applied to $\hat{\sigma}$ with $k_\alpha = \underline{c} = j_1 + 1$, $s_r = \hat{s}_q$ implies $i_{\underline{c}} = i_{k_\alpha} = \ell_{\hat{s}_q} + 1$, as we claim in the third row of column two of Table 1. Moreover, from $j_1 = \hat{s}_q$ it follows that $\ell_{j_1} = \ell_{\hat{s}_q}$ as claimed in the fourth row. To deal with row five, we have already noted $\ell_{j_1 + 1} = i_{\underline{c}} = i_{k_\alpha} = \ell_{\hat{s}_q} + 1$ in row three. With $\ell_{s_r} = \ell_{\hat{s}_q}$

j_1	$\hat{s}_q$	$\hat{s}_q + 1$	$\hat{s}_q + 1$
j_0	$1 \leq j_0 \leq \hat{s}_q$	$\hat{s}_q + 1$	$1 \leq j_0 < \hat{s}_q + 1$
$i_{\underline{c}} = i_{k_\alpha}$	$\ell_{\hat{s}_q} + 1$ (at $j_1 + 1$)	$\ell_{\hat{s}_q} + 1$ (at $j_0 = j_1$)	$\ell_{\hat{s}_q}$ (at $j_1 > j_0$)
ℓ_{j_1}	$\ell_{\hat{s}_q}$	$i_{k_\alpha} = \ell_{\hat{s}_q} + 1$	$i_{k_\alpha} = \ell_{\hat{s}_q}$
ℓ_{j_1+1}	$i_{k_\alpha} = \ell_{\hat{s}_q} + 1$	$i_{k_\alpha} + 1 = \ell_{\hat{s}_q} + 2$	$i_{k_\alpha} + 1 = \ell_{\hat{s}_q} + 1$
ℓ_{j_0-1}	$\ell_{\hat{s}_q} + (-1)^{\hat{s}_q-(j_0-1)}$, by (5.17)	$\ell_{\hat{s}_q}$, by $j_0 = j_1$	$\ell_{\hat{s}_q} - (-1)^{\hat{s}_q-(j_0-1)}$, by (5.15)
$(\ell_{j_1} - \ell_{j_0-1})(\ell_{j_1+1} - \ell_{j_1})$	$-(-1)^{\hat{s}_q-(j_0-1)}$	1	$(-1)^{\hat{s}_q-(j_0-1)}$
$(-1)^{j_1-j_0}$	$(-1)^{\hat{s}_q-j_0}$	1	$(-1)^{\hat{s}_q+1-j_0}$

TABLE 1. Quantities involved in the alternate jump condition $(\ell_{j_1} - \ell_{j_0-1})(\ell_{j_1+1} - \ell_{j_1}) = (-1)^{j_1-j_0}$ in the proof of Lemma 10; see (S.3) and (3.13). The three right columns cover all arising cases. The two bottom rows successfully compare those quantities which are claimed to coincide, in Lemma 10.

the entry in row six follows from Lemma 5, (5.17). The remaining columns are similarly and straightforwardly true.

Comparing the resulting bottom lines proves the alternate jump condition (S.3) for the extended sequence S_q, and completes the lemma. □

We now address case (B) of our previous outline. We again use the same extension $B = \hat{B}$ employed in Lemma 10.

Lemma 11. *Let $\underline{c}, \overline{c}$ be odd, and either $\overline{c} - \underline{c} = 2$, or else $\overline{c} - \underline{c} > 2$ and $i_{\underline{c}}(\sigma) > 2$. Define the extended regular bracket structure $B := \hat{B}$ and the extended collection of sequences $S = (S_r)_{1 \le r \le K}$ with $K := \hat{K}$ and $S_r := \hat{S}_r$, except for replacing $\hat{S}_q$ by*

$$S_q = (\ell_1, \dots, \ell_{\hat{s}_q}, i_{\underline{c}} - 1, i_{\underline{c}} - 2, \dots) \ , \tag{6.8}$$

with a total number of $(\overline{c} - \underline{c})/2 - 1 \ge 0$ identical trailing entries $i_{\underline{c}} - 2$. Then B is a regular bracket structure corresponding to a Morse type μ and S is an abstract signature associated to B.

Proof: Again by Theorem 4, we only need to show that S_q satisfies the abstract signature conditions (S.1)–(S.3) specified in (3.11)–(3.13).

As in the previous lemma, signature condition (S.1), (3.11) on outer boundaries is immediately verified for the extended signature S. In fact the relation $\hat{s}_q \ge 1$, corresponding to $\hat{S}_q \neq (\ )$, leads to $\ell_1 = 1$ for the signature sequence $\hat{S}_q$, as before. The case $\hat{s}_q = 0$, corresponding to $\hat{S}_q = (\ )$, is now void. To see this just recall that, by Lemma 1, $\underline{c} = k$ and $i_{\underline{c}}$ must possess opposite parity. On the other hand, $\hat{s}_q = 0$ implies $i_{\underline{c}} = 1$, by the arguments already given in the proof of the previous lemma. This contradicts oddness of $\underline{c}$.

Signature condition (S.2), (3.12) on neighbor jumps now needs only to be checked in the form

$$|\ell_{\hat{s}_q} - i_{\underline{c}} + 1| \le 1 \ , \tag{6.9}$$

i.e. for $j = \hat{s}_q$ and $\ell_{j+1} = i_{\underline{c}} - 1$. Let us apply Lemma 5 to $\hat{\sigma}$ again, with $k_\alpha = \underline{c}$ and $s_r = \hat{s}_q$. By Lemma 5, (5.13) we have $i_{\underline{c}} = i_{k_\alpha} \in \{\ell_{\hat{s}_q}, \ell_{\hat{s}_q} + 1\} = \{\ell_j, \ell_j + 1\}$. This proves (6.9).

Signature condition (S.3), (3.13) on alternate jumps has to be checked in cases $j_1 = \hat{s}_q, \hat{s}_q + 1$, only. As before we take care of all cases in the form of a table; see Table 2. In the first two rows we consider the three subcases $j_1 = \hat{s}_q$; $j_1 = \hat{s}_q + 1$ and $j_0 = j_1$; $j_1 = \hat{s}_q + 1$ and $j_0 < j_1$. Then the entry for $i_{\underline{c}}$ follows again from Lemma 5 conclusion (5.13), $i_{\underline{c}} \in \{\ell_{\hat{s}_q}, \ell_{\hat{s}_q} + 1\}$, and the assumption

$$\ell_{j_0 - 1} \neq \ell_{j_0} = \dots = \ell_{j_1} \neq \ell_{j_1 + 1} \ . \tag{6.10}$$

The entries for ℓ_{j_1}, ℓ_{j_1+1} now follow from the extended signature sequence S_q as specified in (6.8). More diligently, let us consider the third column, this time, which corresponds to the case $j_1 = \hat{s}_q + 1 = j_0$. The new ex cathedra choice (6.8) of the extended sequence S_q and (6.10) imply $i_{\underline{c}} = \ell_{j_1} + 1 \neq \ell_{j_0 - 1} + 1 = \ell_{\hat{s}_q} + 1$. Lemma 5, (5.13) applied to $\hat{\sigma}$ with $k_\alpha = \underline{c}$, $s_r = \hat{s}_q$ therefore implies $i_{\underline{c}} = i_{k_\alpha} = \ell_{\hat{s}_q}$, rather than $\ell_{\hat{s}_q} + 1$. This proves the entry in row three. Rows four and five follow from the ex cathedra definition of the extended sequence S_q via $\ell_{j_1} = \ell_{\hat{s}_q + 1} = i_{\underline{c}} - 1 = i_{k_\alpha} - 1 = \ell_{\hat{s}_q} - 1$ and

$\ell_{j_1+1} = i_{\underline{c}} - 2 = \ell_{j_1} - 1$. Row six follows because $\ell_{j_0-1} = \ell_{j_1-1} = \ell_{\hat{s}_q}$. Row eight is obvious, by $j_0 = j_1$, and row seven follows from the entries above. The other two columns are left at our reader's discretion. The comparison between the resulting bottom lines proves the alternate jump condition (S.3) for the extended sequence S_q, and completes the lemma. □

We now address the remaining case (C), which we recall is the odd parity $\underline{c}$ case with $\overline{c} - \underline{c} > 2$ and $i_{\underline{c}}(\sigma) = 2$. We first remark that, under these conditions, our involution σ possesses σ-stable points in $\{\underline{c}, \underline{c}+1, \ldots, \overline{c}\}$. In fact, from (1.20) applied to $k = \underline{c}+1$ and $k = \underline{c}$ we obtain

$$i_{\underline{c}+1}(\sigma) = i_{\underline{c}}(\sigma) + (-1)^{\underline{c}+1} \operatorname{sign}(\sigma^{-1}(\underline{c}+1) - \sigma^{-1}(\underline{c})) \ . \tag{6.11}$$

Recall $\sigma(\underline{c}) = \overline{c}$ and

$$\sigma(\underline{c}+j) = \underline{c}+j \quad , \quad j = 1, \ldots, (\overline{c}-\underline{c}) - 1 \ . \tag{6.12}$$

With $j = 1$ this implies

$$i_{\underline{c}+1}(\sigma) = i_{\underline{c}}(\sigma) + (-1)^{\underline{c}+1} \operatorname{sign}(\underline{c}+1-\overline{c}) = 2 + \operatorname{sign}(\underline{c}+1-\overline{c}) = 1 \tag{6.13}$$

because $\underline{c}$ is odd and $i_{\underline{c}}(\sigma) = 2$. Likewise, (1.20) applied to $k = \underline{c}+j$ and $k = \underline{c}+1$ implies

$$i_{\underline{c}+j}(\sigma) = i_{\underline{c}+1}(\sigma) + \sum_{k=1}^{j-1} (-1)^{\underline{c}+k+1} \operatorname{sign}(\sigma(\underline{c}+k+1) - \sigma(\underline{c}+k)) \ . \tag{6.14}$$

Inserting (6.12), (6.13) and using that $\underline{c}$ is odd, we conclude that the Morse indices of intermediate points between $\underline{c}$ and $\overline{c}$ alternate,

$$i_{\underline{c}+j}(\sigma) = 1 + \sum_{k=1}^{j-1} (-1)^k = \begin{cases} 1 & \text{if } j \text{ is odd} \ , \\ 0 & \text{if } j \text{ is even} \ . \end{cases} \tag{6.15}$$

Therefore, the σ-fixed intermediate points $\underline{c}+j$ with even $1 < j < (\overline{c}-\underline{c})-1$ are all σ-stable.

We use an extension B of the bracket structure $\hat{B}$ that reflects the existence of these σ-stable points. Consider the innermost bracket pair q corresponding to the punctured disk $\hat{\mathcal{C}}_q$ of $\hat{f}$ associated to $\underline{c}$. The regular bracket structure B is obtained from $\hat{B}$ by insertion of a regular bracket structure of the form

$$(\ldots(()())()\ldots)() \ , \tag{6.16}$$

with an even number $r_0 := (\overline{c}-\underline{c}) - 2$ of bracket pairs, just inside the braquet pair q. More precisely, the bracket structure B_{r_0} of (6.16) is defined inductively as follows. For $r_0 = 2$ start with two bracket pairs $B_2 = ()()$. For given B_{r_0-2}, define $B_{r_0} = (B_{r_0-2})()$.

An extended Morse type μ associated to the extended bracket structure B is obtained from $\hat{\mu}$ by replacing the local minimum at q by r_0+1 nearby local extrema: $r_0/2$ corresponding to local maxima and $r_0/2+1$ corresponding to local minima. The ordering μ of these points specified by (6.16) corresponds to a monotonically increasing order of the local maxima; see illustration in Figure 6.1.

For the extended ex cathedra collection of sequences S we choose to preserve all the sequences $\hat{S}_r, r \neq q$, arising from the lap signature $\hat{S}$, and modify

j_1	$\hat{s}_q$	$\hat{s}_q+1$	$\hat{s}_q+1$
j_0	$1 \leq j_0 \leq \hat{s}_q$	$\hat{s}_q+1$	$1 \leq j_0 < \hat{s}_q+1$
$i_{\underline{c}} = i_{k_\alpha}$	$\ell_{\hat{s}_q}$ (at j_1+1)	$\ell_{\hat{s}_q}$ (at $j_0=j_1$)	$\ell_{\hat{s}_q}+1$ (at $j_1>j_0$)
ℓ_{j_1}	$\ell_{\hat{s}_q}$	$i_{k_\alpha}-1=\ell_{\hat{s}_q}-1$	$i_{k_\alpha}-1=\ell_{\hat{s}_q}$
ℓ_{j_1+1}	$i_{k_\alpha}-1=\ell_{\hat{s}_q}-1$	$i_{k_\alpha}-2=\ell_{\hat{s}_q}-2$	$i_{k_\alpha}-2=\ell_{\hat{s}_q}-1$
ℓ_{j_0-1}	$\ell_{\hat{s}_q}-(-1)^{\hat{s}_q-(j_0-1)}$, by (5.15)	$\ell_{\hat{s}_q}$, by $j_0=j_1$	$\ell_{\hat{s}_q}+(-1)^{\hat{s}_q-(j_0-1)}$, by (5.17)
$(\ell_{j_1}-\ell_{j_0-1})(\ell_{j_1+1}-\ell_{j_1})$	$-(-1)^{\hat{s}_q-(j_0-1)}$	1	$(-1)^{\hat{s}_q-(j_0-1)}$
$(-1)^{j_1-j_0}$	$(-1)^{\hat{s}_q-j_0}$	1	$(-1)^{\hat{s}_q+1-j_0}$

TABLE 2. Quantities involved in the alternate jump condition $(\ell_{j_1}-\ell_{j_0-1})(\ell_{j_1+1}-\ell_{j_1}) = (-1)^{j_1-j_0}$ in the proof of Lemma 11; see (S.3) and (3.13). Again the three right columns cover all occurring cases. Once more the coincidence of the two bottom rows proves the claim in Lemma 11.

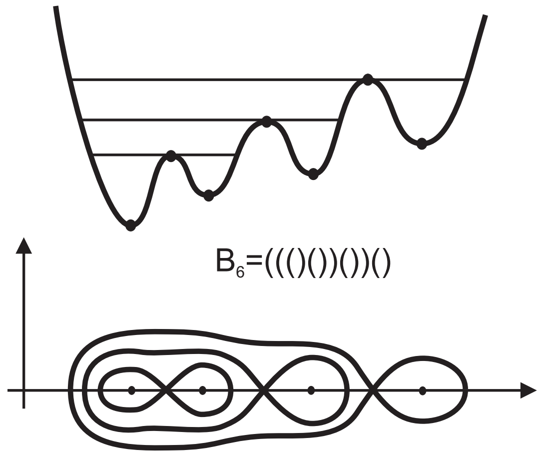

FIGURE 6.1. Illustration of the Morse type μ for B_{r_0} with $r_0 = 6$.

$\hat{S}_q$ appropriately. In addition, we insert r_0 empty sequences corresponding to the new pairs of brackets (6.16) in B.

Lemma 12. *Let $\underline{c}, \overline{c}$ be odd, $\overline{c} - \underline{c} > 2$ and $i_{\underline{c}}(\sigma) = 2$. Define the extended regular bracket structure B by inserting r_0 bracket pairs of the form (6.16) into the innermost bracket pair q of $\hat{B}$. Define the extended collection of sequences $S = (S_r)_{1 \le r \le K}$, with $K := \hat{K} + r_0$, by*

$$S_r = \hat{S}_r \ , \ r < q \ , \tag{6.17}$$

$$S_q = (\ell_1, \ldots, \ell_{\hat{s}_q}, 1) \ , \tag{6.18}$$

$$S_r = (\ ) \ , \ q < r \le q + r_0 \ , \tag{6.19}$$

$$S_r = \hat{S}_{r - r_0} \ , \ r > q + r_0 \ . \tag{6.20}$$

Then B is a regular bracket structure corresponding to a Morse type μ and the extended collection S is an abstract signature associated to B.

Proof: Since $\hat{B}$ and (6.16) are regular bracket structures, so is B. The extended Morse type μ associated to B, with $r_0/2$ new local maximum values of F and $r_0/2 + 1$ new local minimum values, was already described.

We remark that our choice of B implies that the first $r_0/2 - 1$ pairs of brackets in (6.16) correspond to annular regions and the remaining $r_0/2+1$ pairs correspond to punctured disks. An illustration is provided by Figure 6.1.

To show that S is an abstract signature it is now sufficient to show that the annular region S_q, given by (6.18), satisfies the signature conditions (S.1)–(S.4) specified by (3.11)–(3.14). In fact, all the new sequences $S_r, r = q+1, \ldots, q+r_0$, are empty, and all the remaining sequences S_r are obtained from $\hat{S}_r, r \neq q$, unchanged. Moreover, the bracket pair q in B is not an innermost bracket pair since it now contains the bracket structure (6.16). Therefore, inner boundary condition (3.14) in (S.4) also has to be verified for the now annular region S_q.

As in the proof of Lemma 11, oddness of $\underline{c}, \overline{c}$ implies that the case $\hat{s}_q = 0$ corresponding to $\hat{S}_q = (\ )$ is void. Therefore $\hat{s}_q \geq 1$ and the outer boundary condition (S.1) for S_q is obvious because $\ell_1 = 1$ for the nonempty lap signature sequence $\hat{S}_q$.

Signature condition (S.2), (3.12) on neighbor jumps only has to be checked in the form

$$|\ell_{\hat{s}_q} - 1| \leq 1 \ . \tag{6.21}$$

This follows again from Lemma 5, (5.13) applied to $k_\alpha = \underline{c}$, $s_r = \hat{s}_q$ and from $i_{\underline{c}}(\sigma) = 2$.

Signature condition (S.3), (3.13) on alternate jumps under the assumption

$$\ell_{j_0-1} \neq \ell_{j_0} = \cdots = \ell_{j_1} \neq \ell_{j_1+1} \ , \tag{6.22}$$

has to be checked only for $j_1 = \hat{s}_q$ and $1 \leq j_0 \leq \hat{s}_q$. Again, Lemma 5 applied to $\hat{\sigma}$ with $k_\alpha = \underline{c}, s_r = \hat{s}_q$ and the signature sequence $\hat{S}_q$, takes care of this case as shown on Table 3. In the first two rows we specify the unique case $j_1 = \hat{s}_q$ and $1 \leq j_0 \leq \hat{s}_q$. The third row specifies the assumption $i_{\underline{c}} = 2$. Rows four and five result from the last inequality in (6.22). Lemma 5, (5.15) now implies the entry in row six. Equality of the two bottom lines proves the alternate jump condition (S.3) as claimed.

Finally, the inner boundary condition (S.4), (3.14) follows from (6.18) and Lemma 4. In fact, we have $\ell_{s_q} = 1$ with $s_q := \hat{s}_q + 1$. Moreover, since $i_{\underline{c}} = 2$ is even, (1.20) implies that $i_{\underline{c}-1}$ is odd. Therefore, Lemma 4 applied to $\hat{\sigma}$ and $\hat{S}_q$ with $s_r = \hat{s}_q$, $k_\beta = \underline{c} - 1$ implies that $\hat{s}_q$ is odd. In fact, oddness of $\hat{s}_q$ follows from (5.8) if $\ell_{\hat{s}_q}$ is odd, and from (5.9) if $\ell_{\hat{s}_q}$ is even. Oddness of $\hat{s}_q$ in both cases establishes even parity of $s_q := \hat{s}_q + 1$ and completes the proof of the inner boundary condition (S.4) and the lemma. □

As discussed in the outline of this section, Lemmas 10–12 provide a nonlinearity $f \in \mathrm{Sturm}(u)$ with the regular bracket structure B and lap signature S, as chosen ex cathedra, in all three cases (A)–(C). To complete the proof of Theorem 1 we now complement the previous lemmas by showing that the Sturm permutation σ_f obtained from the nonlinearity f in fact coincides with the prescribed integrable Sturm involution: $\sigma_f = \sigma$.

Lemma 13. *Let* $f \in \mathrm{Sturm}(u)$ *denote a nonlinearity with regular bracket structure* B *and lap signature* S*, as obtained from Lemmas 10, 11, 12. Then the Sturm permutation obtained from* f *satisfies*

$$\sigma_f = \sigma \ . \tag{6.23}$$

Proof: We show that our Sturm permutation σ_f coincides with the given integrable Sturm involution σ in all the three cases (A)–(C) considered in Lemmas 10–12.

We first remark that σ_f is easily computed from the regular bracket structure B and lap signature S. In fact, the total number K of bracket pairs decomposes into

$$K = d + a \ . \tag{6.24}$$

j_1	$\hat{s}_q$
j_0	$1 \leq j_0 \leq \hat{s}_q$
$i_{\underline{c}} = i_{k_\alpha}$	2
ℓ_{j_1}	$\ell_{\hat{s}_q} = 2$
ℓ_{j_1+1}	1
ℓ_{j_0-1}	$\ell_{\hat{s}_q} - (-1)^{\hat{s}_q-(j_0-1)}$, by (5.15)
$(\ell_{j_1} - \ell_{j_0-1})(\ell_{j_1+1} - \ell_{j_1})$	$-(-1)^{\hat{s}_q-(j_0-1)}$
$(-1)^{j_1-j_0}$	$(-1)^{\hat{s}_q-j_0}$

TABLE 3. Quantities involved in the alternate jump condition $(\ell_{j_1} - \ell_{j_0-1})(\ell_{j_1+1} - \ell_{j_1}) = (-1)^{j_1-j_0}$ in the proof of Lemma 12; see (S.3) and (3.13).The right column covers the unique arising case. Equality of the two bottom rows proves the assertion in Lemma 12.

Here d denotes the number of innermost bracket pairs, i.e. the number of punctured disk regions $\mathcal{C}_r$ of periodic orbits of (1.16), and a denotes the total number of annular regions. Recall $0 \leq a \leq d-1$.

Hence, the phase portrait (1.17) of f possesses $2d+1$ ODE equilibrium solutions, which correspond to d ODE Hamiltonian centers and $d+1$ ODE saddles. All these ODE equilibria must correspond to fixed points of σ_f. The sequences $S_r = (\ell_1^r, \ldots, \ell_{s_r}^r), 1 \leq r \leq K$, then contain information on the relative positions and period lap numbers of the spatially nonhomogeneous 2π-periodic orbits $\mathbf{p}_\beta$. We recall that for each 2π-periodic orbit $\mathbf{p}_\beta$ there are two Neumann equilibrium solutions $\underline{v}_{k_\beta}$, $\overline{v}_{k_\beta}$ of (1.15), (1.2). Hence, the total number of PDE equilibria for this Neumann problem is

$$n = 2d + 1 + 2\sum_{r=1}^{K} s_r \ . \tag{6.25}$$

We recall that odd period lap numbers correspond to 2-cycles of σ_f, whereas period lap numbers of even parity correspond to σ_f-fixed points. Therefore, to compute $\sigma_f \in \mathcal{S}(n)$ from B and S we just need to check the even/odd parity of the period lap numbers $\ell^r_j, 1 \leq j \leq s_r$, which appear in the sequences S_r.

Since B and S are extensions of the regular bracket structure $\hat{B}$ and lap signature $\hat{S}$, respectively, the permutation σ_f restricted to $E_{\underline{c},\overline{c}} \setminus \{\underline{c}\}$ is identical to the reduced permutation $\hat{\sigma}$. In view of (6.2) this implies

$$\sigma_f(k) = \sigma(k)\ ,\ k \in E_{\underline{c},\overline{c}} \setminus \{\underline{c}\}\ . \tag{6.26}$$

To compute $\sigma_f(k)$ for the remaining labels $k \in \{\underline{c}, \ldots, \overline{c}\}$ we consider the extensions B, S, separately for each of the complementary cases (A)–(C) introduced in the outline.

In case (A), treated in Lemma 10, we only have to consider the lap signature sequence (6.5). Since $\underline{c}$ was assumed to be even, Lemma 1 implies that $i_{\underline{c}}(\sigma)$ is odd. By our choice (6.5) of the lap signature S, the Morse index $i_{\underline{c}} = \ell_{\hat{s}_q+1}$ is also the lap number, at label $\underline{c}$. Since $\ell_{\hat{s}_q+1} = \ell_{\underline{c}}$ is odd, $\underline{c}$ belongs to a 2-cycle of σ_f. The remaining identical entries $i_{\underline{c}}+1$ of the lap signature S exist only if $\overline{c} - \underline{c} > 2$, and are all even. This implies that $\underline{c}$ is followed by $(\overline{c}-\underline{c})/2-1 \geq 0$ points which are σ_f-fixed. They correspond to the (possibly empty set of) equilibrium solutions $v_{\underline{c}+l} = \underline{v}_{\underline{c}+l}, 1 \leq l \leq (\overline{c} - \underline{c})/2 - 1$. This exhausts the entries of S_r. Hence, the next point $\underline{c} + (\overline{c} - \underline{c})/2 = (\overline{c} + \underline{c})/2$, alias the midpoint of the punctured disk region $\mathcal{C}_q$, corresponds to an ODE Hamiltonian center and is also σ_f-fixed. This midpoint is then followed by $(\overline{c} - \underline{c})/2 - 1$ additional σ_f-fixed points corresponding to the equilibrium solutions $\overline{v}_{\underline{c}+l}, 1 \leq l \leq (\overline{c} - \underline{c})/2 - 1$, appearing in reverse order. In this way the total number of σ_f-fixed points in S_q is $(\overline{c} - \underline{c}) - 1$. The last remaining point is $\overline{c}$, which therefore completes the 2-cycle $(\underline{c}\ \overline{c})$ of σ_f. This shows that $\sigma_f = \sigma$ in case (A).

Case (B), considered in Lemma 11, is entirely similar to the previous case. Again we only have to consider the lap signature sequence (6.8). Since $\underline{c}$ is now assumed odd, Lemma 1 implies that also $i_{\underline{c}}(\sigma) - 1$ is odd, again. The exact same arguments as above then show that $\sigma_f = \sigma$ also prevails in case (B).

We conclude with case (C), treated in Lemma 12. Here we have to address the more involved lap signature sequences $S_r,\ q \leq r \leq q + r_0$, introduced in (6.18)–(6.19). The sequence S_q assigns to the point $\underline{c} = \hat{s}_q+1$ an odd period lap number, $\ell = 1$. Hence, $\underline{c}$ again belongs to a 2-cycle of σ_f. Furthermore, the equally involved regular bracket structure B asserts that the sequence S_q corresponds to an annular region $\mathcal{C}_r$ of periodic orbits which encircles all the regions corresponding to the remaining sequences $S_r, q+1 \leq r \leq q+r_0$, (6.19). Since these remaining sequences are all empty, the $2r_0 = (\overline{c} - \underline{c}) - 2$ points following $\underline{c}$ cannot correspond to spatially nonhomogeneous 2π-periodic orbits. By construction (6.16) of B and by the associated Morse type μ, these points alternate between minima and maxima of the potential F. This implies that $\underline{c} + 1, \ldots, \overline{c} - 1$ are σ_f-fixed points which alternate between ODE Hamiltonian centers (the even labels) and ODE saddles (the odd labels). The remaining point $\overline{c}$ again completes the 2-cycle $(\underline{c}\ \overline{c})$ of σ_f. Recall that $\underline{c}$ is odd here. This shows that $\sigma_f = \sigma$ in case (C) and completes the proof of the lemma, and of Theorem 1. □

7. Concluding remarks

We begin this last section with a brief geometric motivation for our ex cathedra extensions B, S introduced in Lemmas 11–13. Since the arguments in cases (A)–(C) are similar, we only address the case (A) treated in Lemma 10. Therefore, we assume $\underline{c}, \overline{c}$ are even and indicate why the period lap numbers of the new 2π-periodic orbits, arising in the σ-configuration of the disk region $\mathcal{C}_q$, should satisfy

$$\ell_{\hat{s}_q+1} = i_{\underline{c}}(\sigma)\ , \tag{7.1}$$

$$\ell_{\hat{s}_q+1+l} = i_{\underline{c}}(\sigma) + 1 \quad , \quad 1 \le l \le (\overline{c} - \underline{c})/2 - 1\ , \tag{7.2}$$

as decreed in Lemma 10. Of course our rationale behind (7.1), (7.2) is that these properties must necessarily hold if ever $\sigma = \sigma_f$ is to be a Sturm permutation in the Sturm class $f \in \mathrm{Sturm}(u)$ of Hamiltonian vector fields with $f = f(u)$. Therefore we assume $\sigma = \sigma_f$ is already realized.

First observe that even parity of $\underline{c}$ and (1.20) implies

$$i_{\underline{c}+1}(\sigma) = i_{\underline{c}}(\sigma) + 1\ . \tag{7.3}$$

In fact, comparing (1.20) with $k = \underline{c}$ and $k = \underline{c} + 1$ for the permutation σ as in (5.25), we obtain

$$\begin{aligned} i_{\underline{c}+1}(\sigma) - i_{\underline{c}}(\sigma) &= (-1)^{\underline{c}+1} \operatorname{sign}((\sigma^{-1}(\underline{c}+1) - \sigma^{-1}(\underline{c})) \\ &= -\operatorname{sign}(\underline{c} + 1 - \overline{c})\ . \end{aligned} \tag{7.4}$$

Since $\overline{c} > \underline{c} + 1$, this yields (7.3).

Then, the extended phase plane organization arising from $\sigma = \sigma_f$ has to be compatible with the replacement of the Hamiltonian center $\underline{c}$ of $\hat{f}$ by the 2-cycle $(\underline{c}\ \overline{c})$ and an odd number of σ-fixed points,

$$\underline{c} + 1, \ldots, \overline{c} - 1\ . \tag{7.5}$$

This configuration should contain one Hamiltonian ODE center, corresponding to the midpoint $(\underline{c} + \overline{c})/2$, and a 2π-periodic orbit corresponding to the 2-cycle $(\underline{c}\ \overline{c})$. By (3.8) with $k_\beta = \underline{c}$ and $j = \underline{c} + 1$ the period lap number of this periodic orbit should satisfy

$$\ell(v_{\underline{c}}) = z(v_{\underline{c}} - v_{\underline{c}+1})\ . \tag{7.6}$$

Therefore, (2.5) with $k = \underline{c}$ implies

$$\ell(v_{\underline{c}}) = \min\{i_{\underline{c}}(\sigma), i_{\underline{c}+1}(\sigma)\}\ . \tag{7.7}$$

Since $\ell_{\hat{s}_q+1} = \ell(v_{\underline{c}})$, from (7.7) and (7.3) we obtain (7.1).

The remaining σ-fixed points (7.5) have alternating indices

$$i_{\underline{c}+l}(\sigma) = \begin{cases} i_{\underline{c}}(\sigma) + 1 & \text{if } l \text{ is odd}\ , \\ i_{\underline{c}}(\sigma) + 2 & \text{if } l \text{ is even}\ . \end{cases} \tag{7.8}$$

This is due to the adjacency of the labels. In fact, from (1.20) with $k = \underline{c} + l$ and $k = \underline{c} + l - 1$, and recalling the even parity of $\underline{c}$, we obtain

$$\begin{aligned} i_{\underline{c}+l}(\sigma) - i_{\underline{c}+l-1}(\sigma) &= (-1)^{\underline{c}+l} \operatorname{sign}((\sigma^{-1}(\underline{c}+l) - \sigma^{-1}(\underline{c}+l-1)) \\ &= (-1)^l\ . \end{aligned} \tag{7.9}$$

$n = 1$:	
$\sigma_1 = \{1\} = \mathrm{id}$;	$\tilde{\sigma}_1$
$n = 3$:	
$\sigma_1 = \{1, 2, 3\} = \mathrm{id}$;	$\tilde{\sigma}_1$
$n = 5$:	
$\sigma_1 = \{1, 2, 3, 4, 5\} = \mathrm{id}$;	$\tilde{\sigma}_1$
$\sigma_2 = \{1, 4, 3, 2, 5\} = (2\ 4)$;	$\tilde{\sigma}_2$
$n = 7$:	
$\sigma_1 = \{1, 2, 3, 4, 5, 6, 7\} = \mathrm{id}$;	$\tilde{\sigma}_1$
$\sigma_2 = \{1, 2, 3, 6, 5, 4, 7\} = (4\ 6)$;	$\tilde{\sigma}_2$
$\sigma_3 = \{1, 4, 5, 6, 3, 2, 7\} = (2\ 4\ 6)(3\ 5)$; $*$	
$\sigma_4 = \{1, 6, 3, 4, 5, 2, 7\} = (2\ 6)$;	$\tilde{\sigma}_3$
$\sigma_5 = \{1, 6, 5, 4, 3, 2, 7\} = (2\ 6)(3\ 5)$;	$\tilde{\sigma}_4$

TABLE 4. List of all Sturm permutations $\sigma_k \in \mathcal{S}(n)$ with $n \leq 7$. On the right are indicated the integrable involutions $\tilde{\sigma}_k \in \mathcal{S}(n)$. An asterix indicates that σ_k is not an involution.

Starting with (7.3) and using (7.9) recursively for $l = 2, \dots, \overline{c} - \underline{c} - 1$, yields (7.8).

Then, our σ-configuration must also contain $r_1 := (\overline{c} - \underline{c})/2 - 1 \geq 0$ additional 2π-periodic orbits corresponding to the pairs of equilibria

$$(v_{\underline{c}+1}, v_{\overline{c}-1}), \dots, (v_{(\underline{c}+\overline{c})/2-1}, v_{(\underline{c}+\overline{c})/2+1}) \ , \tag{7.10}$$

alias

$$(\underline{v}_{\underline{c}+1}, \overline{v}_{\underline{c}+1}), \dots, (\underline{v}_{\underline{c}+r_1}, \overline{v}_{\underline{c}+r_1}) \ . \tag{7.11}$$

To determine the period lap numbers of these periodic orbits we use again (3.8), now with $k_\beta = \underline{c} + l$ and $j = \underline{c} + l + 1$, obtaining

$$\ell(v_{\underline{c}+l}) = z(v_{\underline{c}+l} - v_{\underline{c}+l+1}) \ . \tag{7.12}$$

Therefore, (2.5) with $k = \underline{c} + l$ implies

$$\ell(v_{\underline{c}+l}) = \min\{i_{\underline{c}+l}(\sigma), i_{\underline{c}+l+1}(\sigma)\} \ . \tag{7.13}$$

In view of (7.8) this yields

$$\ell(v_{\underline{c}+l}) = i_{\underline{c}}(\sigma) + 1 \quad , \quad 1 \leq l \leq r_1 \ . \tag{7.14}$$

Since $\ell_{\hat{s}_q+1+l} = \ell(v_{\underline{c}+l})$, (7.14) implies (7.2) as expected.

In Table 4 we list all Sturm permutations $\sigma_k \in \mathcal{S}(n)$ with $n \leq 7$; see [Fie94]. By direct inspection, we can check which ones are also integrable involutions, and relabel them as $\tilde{\sigma}_k \in \mathcal{S}(n)$. All permutations $\sigma_k \in \mathcal{S}(n), n \leq 7$ turn out to be integrable involutions, except for $\sigma_3 = (2\ 4\ 6)(3\ 5)$, $n = 7$, which is not an involution. In view of Theorem 1, all integrable involutions $\tilde{\sigma}_k \in \mathcal{S}(n), n \leq 7$, listed in Table 4 are Sturm permutations in the Hamiltonian class $f \in \mathrm{Sturm}(u)$.

$\sigma_1 = \{1,2,3,4,5,6,7,8,9\} = \mathrm{id}$;		$\tilde{\sigma}_1$
$\sigma_2 = \{1,2,3,4,5,8,7,6,9\} = (6\ 8)$;		$\tilde{\sigma}_2$
$\sigma_3 = \{1,2,3,6,5,4,7,8,9\} = (4\ 6)$;		$\tilde{\sigma}_3$
$\sigma_4 = \{1,2,3,6,7,8,5,4,9\} = (4\ 6\ 8)(5\ 7)$;	$*$	
$\sigma_5 = \{1,2,3,8,5,6,7,4,9\} = (4\ 8)$;		$\tilde{\sigma}_4$
$\sigma_6 = \{1,2,3,8,7,6,5,4,9\} = (4\ 8)(5\ 7)$;		$\tilde{\sigma}_5$
$\sigma_7 = \{1,4,3,2,5,8,7,6,9\} = (2\ 4)(6\ 8)$;		$\tilde{\sigma}_6$
$\sigma_8 = \{1,4,5,6,7,8,3,2,9\} = (2\ 4\ 6\ 8)(3\ 5\ 7)$;	$*$	
$\sigma_9 = \{1,4,5,8,7,6,3,2,9\} = (2\ 4\ 8)(3\ 5\ 7)$;	$*$	
$\sigma_{10} = \{1,6,7,8,3,4,5,2,9\} = (2\ 6\ 4\ 8)(3\ 7\ 5)$;	$*$	
$\sigma_{11} = \{1,6,7,8,5,2,3,4,9\} = (2\ 6)(3\ 7)(4\ 8)$;	$(*)$	
$\sigma_{12} = \{1,6,7,8,5,4,3,2,9\} = (2\ 6\ 4\ 8)(3\ 7)$;	$*$	
$\sigma_{13} = \{1,8,3,4,5,6,7,2,9\} = (2\ 8)$;		$\tilde{\sigma}_7$
$\sigma_{14} = \{1,8,3,4,7,6,5,2,9\} = (2\ 8)(5\ 7)$;	$(*)$	
$\sigma_{15} = \{1,8,3,6,5,4,7,2,9\} = (2\ 8)(4\ 6)$;		$\tilde{\sigma}_8$
$\sigma_{16} = \{1,8,5,6,7,4,3,2,9\} = (2\ 8)(3\ 5\ 7)(4\ 6)$;	$*$	
$\sigma_{17} = \{1,8,7,4,5,6,3,2,9\} = (2\ 8)(3\ 7)$;		$\tilde{\sigma}_9$
$\sigma_{18} = \{1,8,7,6,5,4,3,2,9\} = (2\ 8)(3\ 7)(4\ 6)$;		$\tilde{\sigma}_{10}$

Table 5. List of all Sturm permutations in $\sigma_k \in \mathcal{S}(9)$ and the corresponding integrable involutions $\tilde{\sigma}_k \in \mathcal{S}(9)$. An asterix indicates that σ_k is not an involution. An asterix in parentheses indicates that σ_k is an involution but is not integrable.

The number of meander permutations grows exponentially with n; see [LZ92]. For $n = 9$ there are 18 Sturm permutations $\sigma \in \mathcal{S}(9)$ up to trivial equivalence due to symmetries; see again [Fie94]. Table 5 lists them in lexicographical order, and Figure 7.1 shows the corresponding canonical meanders, see [Fie94] for details.

Only 10 of these permutations are integrable involutions. Obviously, σ_4, σ_8, σ_9, σ_{10}, σ_{12} and σ_{16} are not involutions. The Sturm permutations σ_{11} and σ_{14} are not integrable. In fact, any two of the three 2-cycles of $\sigma_{11} = (2\ 6)(3\ 7)(4\ 8)$ are intersecting, but none are nested. This blatantly violates integrability condition (I.1). In contrast the 2-cycles of $\sigma_{14} = (2\ 8)(5\ 7)$ are nested and sink-equivalent. But they are not centered and therefore violate integrability condition (I.2).

This shows the independence of integrability conditions (I.1) and (I.2) in our definition of an integrable permutation. To show that condition (I.3) is independent of (I.1) and (I.2), one has to consider much larger values of n. It turns out that the Sturm permutation

$$\sigma = (2\ 20)(4\ 8)(9\ 13)(14\ 18) \in \mathcal{S}(21) \tag{7.15}$$

is an involution for which all intersecting 2-cycles are nested and all sink-equivalent 2-cycles are centered. However, none of the pairs of nonintersecting 2-cycles are separated by σ-stable points. This Sturm permutation has only 3 σ-stable points (see Figure 7.2).

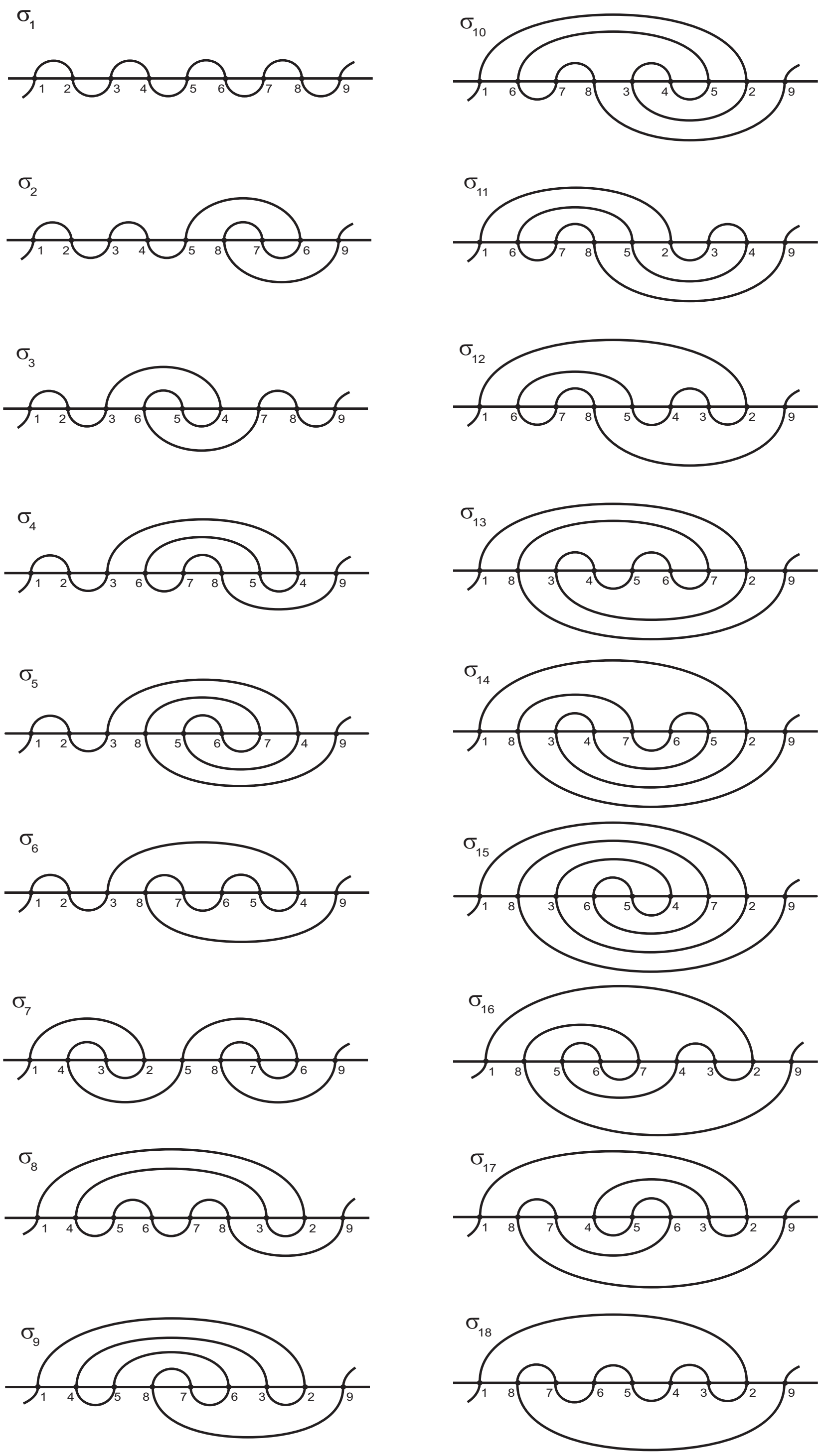

FIGURE 7.1. Sturm permutations in $\mathcal{S}(9)$, and their meanders.

As an additional application of our characterization of Sturm permutations in the Hamiltonian class $f \in \text{Sturm}(u)$ we examine the list of Sturm permutations in $\mathcal{S}(11)$. Up to trivial equivalence there are 75 Sturm global

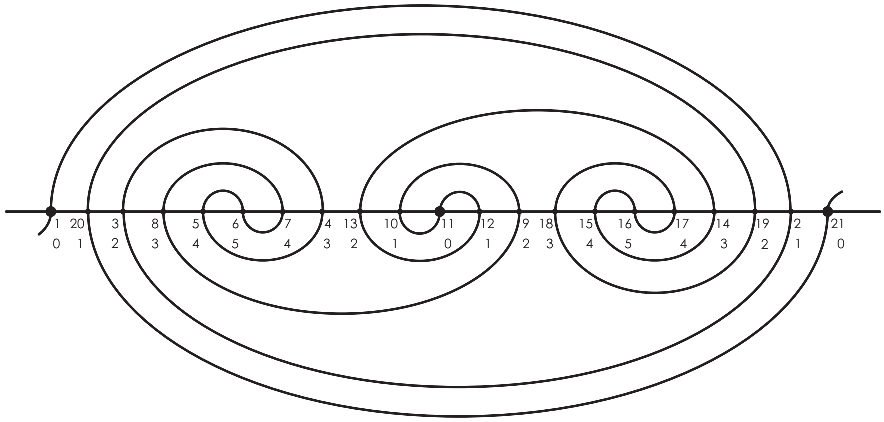

FIGURE 7.2. A Sturm permutation which is an involution, satisfies integrability conditions (I.1) and (I.2), but violates (I.3): the non-nested cycles (4 8), (9 13) and (14 18) are not separated by σ-stable points, here indicated by larger dots. The numbers on the bottom row denote the Morse indices of the corresponding equilibria.

$\tilde{\sigma}_1 = \text{id}$;	$\tilde{\sigma}_2 = (8\ 10)$;
$\tilde{\sigma}_3 = (6\ 8)$;	$\tilde{\sigma}_4 = (6\ 10)$;
$\tilde{\sigma}_5 = (6\ 10)(7\ 9)$;	$\tilde{\sigma}_6 = (4\ 6)(8\ 10)$;
$\tilde{\sigma}_7 = (4\ 8)$;	$\tilde{\sigma}_8 = (4\ 8)(5\ 7)$;
$\tilde{\sigma}_9 = (4\ 10)$;	$\tilde{\sigma}_{10} = (4\ 10)(6\ 8)$;
$\tilde{\sigma}_{11} = (4\ 10)(5\ 9)$;	$\tilde{\sigma}_{12} = (4\ 10)(5\ 9)(6\ 8)$;
$\tilde{\sigma}_{13} = (2\ 4)(8\ 10)$;	$\tilde{\sigma}_{14} = (2\ 4)(6\ 10)$;
$\tilde{\sigma}_{15} = (2\ 4)(6\ 10)(7\ 9)$;	$\tilde{\sigma}_{16} = (2\ 10)$;
$\tilde{\sigma}_{17} = (2\ 10)(5\ 7)$;	$\tilde{\sigma}_{18} = (2\ 10)(4\ 8)$;
$\tilde{\sigma}_{19} = (2\ 10)(4\ 8)(5\ 7)$;	$\tilde{\sigma}_{20} = (2\ 10)(3\ 9)$;
$\tilde{\sigma}_{21} = (2\ 10)(3\ 9)(6\ 8)$;	$\tilde{\sigma}_{22} = (2\ 10)(3\ 9)(4\ 8)$;
$\tilde{\sigma}_{23} = (2\ 10)(3\ 9)(4\ 8)(5\ 7)$.	

TABLE 6. List of all integrable Sturm involutions $\tilde{\sigma}_k \in \mathcal{S}(11)$.

attractors with 11 equilibria; see also [FR09b]. Our Theorem 1 then determines which ones are of Hamiltonian type; see Table 6. Among the 75 Sturm permutations $\sigma_k \in \mathcal{S}(11)$ there are 34 involutions, and only 23 are integrable. Table 6 lists these 23 integrable Sturm involutions $\tilde{\sigma}_k \in \mathcal{S}(11)$.

To complete this presentation we observe that Theorem 1 also holds in the Sturm class of reversible $f \in \text{Sturm}(u, u_x)$, that is, when f in the Sturm class $f = f(u, u_x)$ satisfies

$$f(v, -p) = f(v, p)\ . \tag{7.16}$$

In general, the ODE

$$v'' + f(v, v') = 0 \tag{7.17}$$

corresponding to the stationary problem (1.8) is not Hamiltonian. However, like the Hamiltonian equation (1.16), (7.17) under condition (7.16) is reversible with respect to the involutive coordinate reflection $x \to -x$. It is remarkable that (7.17) in this reversible case is also integrable. In addition, Ragazzo has shown in [Rag10] that an appropriate smooth change of variables transforms the reversible equation (7.17) into a Hamiltonian pendulum equation of the form (1.16).

The discussion of the phase portrait of the ODE (7.17) is therefore entirely similar to the discussion of (1.16). In fact, the set $\mathcal{C}$ of periodic orbits of (7.17) is open and bounded in the phase plane (v, v'). By reversibility, $\mathcal{C}$ is symmetric with respect to the v-axis. Furthermore, there also exists a period map $T = T_f : D^{\mathcal{C}} \to \mathbb{R}_+$ of class C^2, with $D^{\mathcal{C}} := \{a \in \mathbb{R} : (a, 0) \in \mathcal{C}\}$, which extends the period map introduced in Section 3 to (7.17); see Lemmas 4.2 and 4.4 of [FRW04]. The period map T_f associates to each $v_0 \in D^{\mathcal{C}}$ the minimal period of $(v_0, 0)$ for (7.17), and characterizes the spatially nonhomogeneous 2π-periodic solutions of the reversible ODE (7.17) as in the Hamiltonian case $f = f(u)$. In particular, the period map T_f for (7.17) has the following properties:

- T_f is continuous;
- nonconstant 2π-periodic orbits of (7.17) are obtained from the solutions of
$$T_f(v_0) = 2\pi/\ell \text{ with } \ell \in \mathbb{N} \; ; \tag{7.18}$$
- each such orbit is nondegenerate, or equivalently, the corresponding solution of the PDE (1.1), (1.2) is hyperbolic if, and only if
$$T'(v_0) \neq 0 \; . \tag{7.19}$$

Therefore, the lap signature S associated to the period map T_f and introduced in the Hamiltonian case $f \in \text{Sturm}(u)$ extends to the reversible case $f \in \text{Sturm}(u, u_x)$. In addition, due to the properties of T_f, the lap signature S has to satisfy the same conditions (S.1)–(S.4), (3.11)–(3.14) of Section 3, as before. This implies that conditions (i) and (ii) of Theorem 3 are again necessary conditions for a collection of sequences of positive integers S to be the signature of a period map T_f for (7.17). Moreover, since the Hamiltonian case $f = f(u)$ is a particular instance of a reversible nonlinearity in $\text{Sturm}(u, u_x)$, these conditions are also sufficient. This shows that Theorem 1 also holds true in the larger Sturm class of reversible $f \in \text{Sturm}(u, u_x)$.

For an outlook, consider nonlinearities f in the Hamiltonian reversible subclass of $f \in \text{Sturm}(u, u_x)$ – but now consider periodic boundary conditions. The heteroclinic orbit connections were determined in [FRW04], in this case. The permutation characterization introduced here then provides a geometric description of Sturm global attractors for the Morse-Smale semiflows generated by dissipative scalar parabolic equations (1.1), under periodic boundary conditions. This study will be conducted in forthcoming work.

References

[Ama85] H. Amann. Global existence for semilinear parabolic systems. *J. Reine Angew. Math.*, **360**: 47–83, (1985).

[Ang86] S. Angenent. The Morse-Smale property for a semi-linear parabolic equation. *J. Differential Equations*, **62**: 427–442, (1986).

[Arn88] V. I. Arnold. A branched covering $C\mathrm{P}^2 \to S^4$, hyperbolicity and projective topology. *Siberian Math. J.*, **29** (1988): 717–726, (1989).

[BC84] P. Brunovský and S.-N. Chow. Generic properties of stationary state solutions of reaction-diffusion equations. *J. Differential Equations*, **53**: 1–23, (1984).

[BF88] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations. *Dynamics Reported*, **1**: 57–89, (1988).

[BF89] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: The complete solution. *J. Differential Equations*, **81**: 106–135, (1989).

[BV92] A. V. Babin and M. I. Vishik. *Attractors of Evolution Equations*. North Holland, Amsterdam, 1992.

[Cha75] N. Chafee. Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions. *J. Differential Equations*, **18**: 111–134, (1975).

[CR08] R. Czaja and C. Rocha. Transversality in scalar reaction-diffusion equations on a circle. *J. Differential Equations*, **245**: 692–721, (2008).

[Fie94] B. Fiedler. Global attractors of one-dimensional parabolic equations: sixteen examples. *Tatra Mt. Math. Publ.*, **4**: 67–92, (1994).

[Fie96] B. Fiedler. Do global attractors depend on boundary conditions? *Doc. Math. J. DMV*, **1**: 215–228, (1996).

[FR96] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations. *J. Differential Equations*, **125**: 239–281, (1996).

[FR99] B. Fiedler and C. Rocha. Realization of meander permutations by boundary value problems. *J. Differential Equations*, **156**: 282–308, (1999).

[FR00] B. Fiedler and C. Rocha. Orbit equivalence of global attractors of semilinear parabolic differential equations. *Trans. Amer. Math. Soc.*, **352**: 257–284, (2000).

[FR08] B. Fiedler and C. Rocha. Connectivity and Design of Planar Global Attractors of Sturm Type. II: Connection Graphs. *J. Differential Equations*, **245**: 692–721, (2008).

[FR09a] B. Fiedler and C. Rocha. Connectivity and Design of Planar Global Attractors of Sturm Type. I: Bipolar Orientations and Hamiltonian Paths. *J. Reine Angew. Math.*, **635**: 71–96, (2009).

[FR09b] B. Fiedler and C. Rocha. Connectivity and Design of Planar Global Attractors of Sturm Type. III: Small and Platonic Examples. *J. Dynam. Differential Equations*, **22**: 509–532, (2010).

[Fi&al02] B. Fiedler, C. Rocha, D. Salazar and J. Solà-Morales. Dynamics of peacewise-autonomous bistable parabolic equations. In *Differential equations and dynamical systems (Lisbon, 2000), 151–163*, A. Galves, J. K. Hale, C. Rocha (eds.), Fields Inst. Commun., **31**, Amer. Math. Soc., Providence, RI, 2002.

[FRW04] B. Fiedler, C. Rocha and M. Wolfrum. Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle. *J. Differential Equations*, **201**: 99–138, (2004).

[FS02] B. Fiedler and A. Scheel. Dynamics of reaction-diffusion patterns. In *Trends in Nonlinear Analysis, Festschrift dedicated to Willi Jäger for his 60th birthday, 23–152*, M. Kirkilionis, R. Rannacher, F. Tomi (eds.). Springer-Verlag, Heidelberg, 2002.

[FR91] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *J. Differential Equations*, **91**: 75–94, (1991).

[FO88] G. Fusco and W. M. Oliva. Jacobi matrices and transversality. *Proc. of the Roy. Soc. Edinburgh*, **109A**: 231–243, (1988).

[Hal88] J. K. Hale. *Asymptotic Behavior of Dissipative Systems*. Math. Surv. **25**. AMS Publications, Providence, 1988.

[HMO02] J. K. Hale, L. T. Magalhães and W. M. Oliva. *Dynamics in Infinite Dimensions. Second edition.* Applied Mathematical Sciences, 47. Springer-Verlag, New York, 2002.

[HW05] J. Härterich and M. Wolfrum. Convergence in gradient-like systems with applications to PDE. *Discrete and Contin. Dyn. Syst.*, **12**: 531–554, (2005).

[Hen81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lect. Notes in Math, Vol. 840. Springer-Verlag, New York, 1981.

[Hen85] D. Henry. Some infinite dimensional Morse-Smale systems defined by parabolic differential equations. *J. Differential Equations*, **59**: 165–205, (1985).

[JR08] R. Joly and G. Raugel. Generic hyperbolicity of equilibria and periodic orbits of the parabolic equation on the circle. To appear in *Trans. Amer. Math. Soc.*, (2008).

[Lan03] S. K. Lando. Lectures on Generating Functions. Stud. Math. Lib. vol 23, American Mathematical Society, 2003.

[LZ92] S. K. Lando and A. K. Zvonkin. Meanders. *Selecta Math. Soviet.*, **11**: 117–144, (1992).

[Mat78] H. Matano. Convergence of solutions of one-dimensional semilinear parabolic equations. *J. Math. Kyoto Univ.*, **18**: 221–227, (1978).

[Mat82] H. Matano. Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29**: 401–441, (1982).

[Mat88] H. Matano. Asymptotic behavior of solutions of semilinear heat equations on S^1. In *Nonlinear Diffusion Equations and their Equilibrium States II, 139–162.* W.-M. Ni, L.A. Peletier, J. Serrin (eds.). Springer-Verlag, New York, 1988.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[Rag10] C. Ragazzo. Scalar autonomous second order ordinary differential equations. Preprint, (2010).

[Rau02] G. Raugel. Global attractors in partial differential equations. In *Handbook of dynamical systems, Vol. 2, 885–982*, B. Fiedler (ed.). North-Holland, Amsterdam, 2002.

[Roc91] C. Rocha. Properties of the attractor of a scalar parabolic PDE. *J. Dynam. Differential Equations*, **3**: 575–591, (1991).

[Roc94] C. Rocha. Bifurcations in discretized reaction-diffusion equations. *Resenhas IME-USP*, **1**: 403–419, (1994).

[Roc07] C. Rocha. Realization of period maps of planar Hamiltonian systems. *J. Dynam. Differential Equations*, **19**: 571–591, (2007).

[STW80] J. Smoller, A. Tromba and A. Wasserman. Non-degenerate solutions of boundary-value problems. *Nonlinear Anal.*, **4**: 207–216, (1980).

[Smo83] J. Smoller. Shock Waves and Reaction-Diffusion Equations. Springer-Verlag, New York, 1983.

[Tem88] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1988.

[Wol02a] M. Wolfrum. Geometry of heteroclinic cascades in scalar parabolic differential equations. *J. Dynam. Differential Equations*, **14**: 207–241, (2002).

[Wol02b] M. Wolfrum. A sequence of order relations, encoding heteroclinic connections in scalar parabolic PDE. *J. Differential Equations*, **183**: 56–78, (2002).

[Zel68] T. I. Zelenyak. Stabilization of solutions of boundary value problemsfor a second order parabolic equation with one space variable. *Differ. Equ.*, **4**: 17–22, (1968).

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