Homoclinic period blow-up and the detection of multipulse solutions in lattice differential equations

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Abstract

In this article we study the dynamical behavior near solitary wave solutions of lattice differential equations (LDEs). We are interested in the case, where the solitary wave profile induces a homoclinic solution of the associated traveling-wave equation. Using exponential dichotomies we prove the existence of a C^0 function $\xi : \mathbb{R}^{m-1} \to \mathbb{R}^m$, such that the existence of a zero of ξ corresponds to the existence of a multi-pulse solution near the primary solitary wave solution. In particular, we can relate the existence of multi-round homoclinic solutions in the travelling wave equation (which hit a local Poincaré section m times before converging to a steady state) to the zero set of a specific C^0-function. This approach is known as Lin's method.

As an application we study the existence of periodic solutions in general and time-reversible lattice differential equations. Using Lin's method we can prove that the occurrence of symmetric solitary waves generically induces a family of large amplitude periodic solutions whose period becomes unbounded. This result is commonly known as the blue-sky catastrophe.

1 Introduction

Lattice differential equations appear in various areas such as mechanics, biology and physics. However, they are less amenable to analytic techniques than their continuous counterparts. As a consequence, the majority of works on traveling waves on lattices has been build upon continuum approximations, where the equation governing a traveling wave simply becomes an ordinary differential equation and dynamical systems methods become applicable. In sharp contrast to this scenario, the initial value problem associated to the traveling wave equation of the original one-dimensional lattice differential equation

$$\partial_t u^i(t) = F(u^{i-M}(t), \dots, u^i(t), \dots, u^{i+M}(t)) \quad (1)$$

is ill-posed. In fact, a traveling wave ansatz $u^i(t) = \psi(i + ct)$, $c \neq 0$, leads to a non-trivial advance delay equation

$$c\psi'(\xi) = F(\psi(\xi - M), \dots, \psi(\xi), \dots, \psi(\xi + M)), \quad (2)$$

where we have set $\xi = i - ct$; we refer to [20, 21, 22, 28, 5, 6, 7, 8] for background for these equations. Rather than working with (2) directly, a lot of authors preferred to use variational techniques instead [4, 25]. In particular, the existence of solitary waves

$$u^i(t) = \psi(i + c_* t) \tag{3}$$

has been provided in this way [4, 25], although the existence of small localized waves has also been proved by center manifold reductions [12, 13]. Here and in the following, a solitary wave denotes a traveling wave solution of (1) where the profile ψ satisfies $\psi(\xi) \to 0$ as $\xi \to \pm\infty$ (in particular, zero is a steady state). Instead of constructing such solutions, we want to study dynamical properties *near* solitary wave solution (3) in this work, which do not have to be of small amplitude. In order to motivate our interest note that in the case of ordinary differential equations one often encounters large periodic solutions near any homoclinic solution and even more complicated solutions when varying a parameter, see [15]. We are therefore interested in the question whether we can detect such solutions also in the traveling wave equation (2) despite its ill-posedness. Of course, flow-based concepts such as Poincaré-maps have to be neglected in this regard. To be more specific, we may be interested in the existence of periodic or m-homoclinic (respectively multi-round homoclinic) solutions, which hit a local Poincaré-section m times before approaching a steady state asymptotically. In this particular case variational methods are hard to apply. In fact, in a variational set up one looks for critical points in a suitable space that mainly incorporates the asymptotic behavior of the desired solution. But if we are also interested in the way *how* a homoclinic solution behaves on large but bounded time-intervals (such as intersecting a given Poincaré-section various times), then it is hard to incorporate this behavior in the definition of the space and still obtaining critical points.

We therefore proceed differently and set up a method (commonly referred to as Lin's method), that allows us to obtain a one-to-one correspondence between multipulse solutions (which induce m-homoclinic solutions in the ill-posed equation (2)) near the primary solitary wave and the zero set of a certain continuous function $\xi : \mathbb{R}^{m-1} \to \mathbb{R}^m$ (see the examples after theorem 5). Likewise, we can also relate the existence of large periodic solutions $u^j(t) = p(j+ct)$ to zeros of a specific C^0-function $\xi : \mathbb{R}^2 \to \mathbb{R}$, which depends differentiable on the parameter c. The strength of such a result is clear: In the context of the infinite-dimensional, ill-posed equation (2) it may be complicated to find a set up that allows one to look for m-homoclinic solutions or even periodic solutions directly, but in the realm of implicit function theorems and abstract index theory it may be possible on the other hand to prove the existence of zeros, which then immediately imply nontrivial solutions. We will construct the function ξ using Lyapunov-Schmidt reductions by constructing solutions of (2) having possible "jumps" near a given point of the homoclinic orbit within a prescribed finite-dimensional vector space. If all the jumps are zero, we infer a globally defined solution of the traveling wave equation.

The next theorem makes these statements precise and is the main result of this work (see also theorem 5 which lists all the relevant hypotheses explicitly). We will draw various interesting consequences from this result; in particular with respect to periodic solutions (see the theorems below). Moreover, we identify the variables of the jump function ξ as "'flight times"' of solutions segments, which stay close to the primary homoclinic solution ψ.

Theorem 1 (Lin's method for advance delay equations)
Fix $c \approx c_*$. Under generic assumptions there exist positive constants ω_* such that for all $\{\omega_j\}_{j\in\mathbb{Z}}$, with $\omega_j > \omega_*$ for all $j \in \mathbb{Z}$, there exist unique functions

$$\begin{aligned} x_+^j &: [-M, M+\omega_j) \to \mathbb{R}^N, \\ x_-^j &: [-M-\omega_{j-1}, M) \to \mathbb{R}^N, \qquad j \in \mathbb{Z}, \end{aligned}$$

such that

a) *x_+^j solves (5) on $(0, \omega_j)$ and x_-^j solves (5) on $(-\omega_{j-1}, 0)$ (see also section 3 for a precise definition of a solution).*

b) *$x_+^j(\omega_j + \bullet) = x_-^{j+1}(-\omega_j + \bullet)$ in $C^0([-M, M], \mathbb{R}^N)$.*

b) *The value $\xi_j(\omega_{j-1}, \omega_j) := x_+^j(\cdot) - x_-^j(\cdot) \in C^0([-M, M], \mathbb{R}^N)$ lies in a j-independent one-dimensional vector space. Moreover, ξ_j is a continuous function.*

c) *The orbit of $x_\pm^j$ is close to the orbit of ψ.*

Remark
We can also specify the form of the jump-functions ξ_j more explicitly (see the last equation (59) in theorem 5). However, this result forces us to define various quantities and we have therefore decided to postpone this result until theorem 5.

The theorem therefore provides the existence of solutions, which stay near the homoclinic solution ψ with possible jumps along a fixed one-dimensional vector space. Hence, if these jumps vanish for some sequence $\{\omega_i\}_i$ then we obtain a globally defined solution of (2). In order to get some geometrical feeling concerning the jump functions ξ_i, let us consider the particular case that (2) depends on an arbitrary parameter $\eta \approx 0$ (rather than the special case $\eta = c - c_*$) and possesses a homoclinic solution for $\eta = 0$. We may now be interested in the existence of homoclinic solutions near the primary when varying η slightly. It has already been shown in [9] that in this case the bifurcation function ξ^∞ under generic assumptions reads

$$\begin{aligned} \xi^\infty(\eta) &= \left(\int_{-\infty}^{\infty} \left\langle D_\eta F(\psi(s-M), \ldots, \psi(s+M), \eta)\big|_{\eta=0}, \tilde{\psi}(s) \right\rangle_{\mathbb{R}^N} ds \right) \cdot \eta \\ &+ \mathcal{O}(\eta^2), \end{aligned} \tag{4}$$

for some specific function $\tilde{\psi}$ (compare also with the last statement of theorem 5). In particular this means that any zero η of $\xi^\infty(\eta)$ induces a homoclinic solution. The bifurcation function ξ^∞ has been used in [9] in order to study a

homoclinic bifurcation scenario in a reversible advance delay equation which may appear as the travelling wave of a time-reversible LDE. As in the framework of ordinary differential equations, the leading order term in (4) (with respect to η) refers to the splitting of stable and unstable manifold of the steady state zero measured within a fixed one-dimensional vector space when varying the parameter η (again we refer to [9] for more details). Let us point out, that the notion of stable and unstable manifold, which has been introduced in [6, 7, 8], does not refer to equation (2) directly but rather to an related abstract setting (which will be introduced in section 3). We can now deduce from the bifurcation function ξ^∞ that as long as stable and unstable manifold split with non vanishing speed with respect to η (which is the generic case), then $\xi^\infty \neq 0$ and the homoclinic solution ψ does not persist for $\eta \neq 0$. Analogously to ordinary differential equations, this reflects the fact that homoclinic solutions are not a generic phenomenon but typically a codimension-one-phenomenon. Summarizing, this shows that the leading order term of the bifurcation function ξ^∞ has a very reasonable geometric interpretation. The jump-function ξ_i now provide a natural generalization of the limiting function ξ^∞ and we will in fact show that $\xi_i \to \xi^\infty$ as $\omega_{i-1}, \omega_i \to \infty$ (see theorem 5); hence, ξ^∞ simply coincides with the peculiar looking choice $\omega_i = \infty$ for all i in theorem 1. Compared to ξ^∞, the explicit representation of ξ_i (see theorem 5) additionally incorporates informations of the homoclinic solution ψ at its "tails" $\pm\infty$. As a consequence, the analysis of the zero-set of the jump functions will be influenced crucially in the way *how* the primary homoclinic solution ψ approaches the steady state zero as $\xi \to \pm\infty$. This behavior naturally relates to the leading eigenvalues of the linearization $D_1F(0, \eta)$ for $\eta = 0$, since by the results in [6, 7, 8] and [21] the homoclinic solution will generically approach the steady state along its leading eigenvectors. Again, this shows that we can make use of various geometrical information in order to study the zero-set of ξ_i which then determines the set of periodic, aperiodic or multiround homoclinic solutions near the primary homoclinic solution (we refer the reader to the discussion in section 8 for an outlook in this direction).
The contens of theorem 1 is commonly referred to as Lin's method and it is one of the achievements of this work to introduce this method in its general form in the framework of traveling wave equations of LDEs. This functional analytic approach was used by Palmer [24] to obtain solutions corresponding to a Smale horseshoe in the context of ordinary differential equations; this result was later extended by Sanstede et al [30] and Georgi [6] within the framework of elliptic equations on cylinders and LDEs, respectively. Hale and Lin [10] used Lin's method to obtain perturbations and continuations of heteroclinic orbits. It was subsequently developed more fully by Lin [16, 17, 18] and more recently by Sandstede [26, 27] and Sandstede and Scheel [28, 29, 30]. In the framework of LDEs Mallet-Paret [22] and Hoffman [11] used this functional analytic approach to address the problem of crystallographic pinning by a careful study of a singular perturbation problem.
As an application, we will use Lin's method (alias theorem 1) to discuss the existence of periodic solutions near the primary homoclinic solution ψ.

The next result makes this precise and infers the existence of large periodic solutions of (2) for wave speeds $c \approx c_*$ under generic conditions (which are listed more explicitly in the statement of theorem 7).

Theorem 2
Consider a LDE (1) and assume the existence of a solitary wave solution $u^i(t) = \psi(i + c_*t)$ *with* $\lim_{t\to\pm\infty} \psi(t) = 0$. *Moreover, we consider the case that zero is hyperbolic. Then, generically, for any* $\omega > \omega_*$ *there exists a traveling wave speed* $c = c(\omega)$ *such that the equation (1) possesses a periodic wave* $u^j(t) = p(j + ct)$. *Here,* p *is periodic with period* ω *and its orbit is close to the orbit of the primary homoclinic solution* ψ.

This theorem actually follows from the results of Mallet-Paret [22]. However, we think that its proof shows a nice application of Lin's method. In particular, we will have to study the zero-set of a suitable jump-function ξ by using a variant of the implicit function theorem (see the appendix).
The task of finding large amplitude periodic solutions near a homoclinic solution in fact simplifies greatly when the underlying lattice is additionally time-reversible (see section 4.3 for a precise definition). Examples of such lattices are the Klein-Gordon and the Fermi-Pasta Ulam lattice [9, 12, 13]. Roughly speaking this means that $\psi(\xi)$ is a solution of (2) whenever $R\psi(-\xi)$ is a solution. As we will see, the jump-functions ξ_j inherit a symmetry property in this case which makes it easier to study the zero-set, see [3] and section 7.1. We can prove the following result (see also theorem 6 which lists all hypotheses explicitly) which in the case of ordinary differential equations has been proved in [3] using the same methods.

Theorem 3 (Homoclinic period blow up in LDEs)
Consider a time-reversible LDE (1). Then, generically, the following happens: Near any symmetric solitary wave solution (i.e. ψ *is a symmetric solution of (2); see definition 5), which approaches a hyperbolic steady state asymptotically, there exists a family of periodic waves* $u_i^\kappa(t) = p^\kappa(i + c_*t)$ *of (1). Here,* p^κ *is a periodic solution of (2) for all* $\kappa > 0$. *Moreover, the orbit of* p^κ *is close to the orbit of* ψ *and for* $\kappa \searrow 0$ *the period of* p^κ *becomes unbounded.*

This result is known is the blue sky catastrophe in the case of ordinary differential equations, see [3, 2] and [15].
The set up of the article is as follows. We will introduce some notation in the next section. In section 3 we will then set up our problem in a suitable functional analytic framework and recall some known facts concerning advance-delay equations as well as the abstract formulation in section 4. In particular, we will state some results in the framework of reversible equations in section 4.3 which allow us to construct appropriate Poincaré-sections near the homoclinic solution. The statement of the main result as well as its proof is addressed in section 6. The theorems 3 and 2 are proved in the sections 7.1 and 7.2, respectively. We conclude the article with a brief discussion in section 8, where we comment on various generalizations of the obtained results.

2 Notation

In the following we will use the notation $x_t \in L^2([-M, M], \mathbb{R}^N)$ for an integrable function $x : \mathbb{R} \to \mathbb{R}^N$. This function is defined by $x_t(\theta) := x(t + \theta)$ for any $t \in \mathbb{R}$.
The following spaces will be frequently used throughout this paper:

$$\begin{aligned} Y &:= \mathbb{R}^N \times L^2([-M, M], \mathbb{R}^N), \\ Z^\infty &:= \mathbb{R}^N \times L^\infty([-M, M], \mathbb{R}^N) \\ X &:= \{(\xi, \varphi) \in Y \mid \varphi \in H^1([-M, M], \mathbb{R}^N) \text{ and } \varphi(0) = \xi\} \\ \tilde{X} &:= \{(\xi, \phi) \in \mathbb{R}^N \times C^0([-M, M], \mathbb{R}^N) : \phi(0) = \xi\} \end{aligned}$$

As a convention, if subspaces are furnished with an additional ˜, they are regarded as subspaces of $\tilde{X}$ with the induced norm. If they are furnished with an additional ˆ, they are viewed as subspaces of X.
Moreover, let $C^0 := C^0([-M, M], \mathbb{R}^N)$ and we define the spaces

$$X_{i-1,i} := C^0([-\omega_{i-1}, 0], \mathbb{R}^N \times C^0) \times C^0([0, \omega_i], \mathbb{R}^N \times C^0)$$

and

$$X^0_{i-1,i} := \{V = (V^-, V^+) \in X_{i-1,i} : V^\pm(0) \in \Gamma; V^+(0) - V^-(0) \in \tilde{Z}\}.$$

3 The abstract formulation

Let us consider the traveling wave equation

$$\dot{x}(t) = \frac{1}{c} F(x(t - M), \ldots, x(t), \ldots, x(t + M)) \tag{5}$$

for $c \neq 0$ where we assume from now on that $F \in C^k$ for some $k \geq 2$. We make the following definition.

Definition 1 (Solution)
We call a function $x \in L^2([-M, \tau), \mathbb{C}^N)$ a solution of (5) for some $M < \tau \leqslant \infty$ to the initial condition $\phi \in L^2([-M, M], \mathbb{C}^N)$, if $x \in H^1_{loc}([0, \tau), \mathbb{C})$, $x_0 = \phi$ and (5) is satisfied for almost every $t \in [0, \tau)$.

However, instead of working with (5) directly, we prefer to work with the abstract equation

$$\begin{aligned} \dot{U}(t) &= \mathcal{F}(\xi(t), \phi(t, \cdot)) \\ &= \begin{pmatrix} \frac{1}{c} F(\phi(t, -M), \ldots, \phi(t, 0), \ldots, \phi(t, M))) \\ \partial_\theta \phi(t, \theta) \end{pmatrix}, \end{aligned} \tag{6}$$

where $F \in BC^2(\mathbb{R}^{N(2M+1)} \times \mathbb{R}; \mathbb{R}^N)$ and $F(0) = 0$. This approach has first been used in [12, 13] although with a slightly different choice of state spaces X, Y. Let us now define what a strong and weak solution of (6) is.

Definition 2

- *We call a continuous function* $U(t) : [t_1, t_2) \to Y$ *a solution of (6) on* (t_1, t_2)*, where* $-\infty < t_1 < t_2 \leqslant \infty$*, if* $t \to U(t)$ *is continuous regarded as a map on* (t_1, t_2) *with values in* X*, if* $t \to U(t)$ *is differentiable regarded as a map on* (t_1, t_2) *with values in* Y *and (6) is satisfied on* (t_1, t_2)*.*

- *We call a differentiable function* $U(t) : (-\infty, t_2) \to Y$ *a solution of (6) on* $(-\infty, t_2)$ *and* $t_2 \in \mathbb{R}$*, if* $t \to U(t)$ *is continuous regarded as a map on* $(-\infty, t_2)$ *with values in* X *and (6) is satisfied on* $(-\infty, t_2)$*.*

- *We call a continuous function* $U : [t_1, t_2) \to \tilde{X}$ a weak *solution of (6), if*

$$U(t) = (x(t), x_t)$$

for some function $x \in C^0([t_1 - M, t_2 + M), \mathbb{R}^N) \cap C^1((t_1, t_2 + M), \mathbb{R}^N)$ *which solves the equation* $\dot{x}(t) = F(x_t, \lambda)$ *on* (t_1, t_2)*.*

The next lemma clarifies the connection between solutions of (6) and our original equation (5). The proof can be found in [5, 6, 7].

Lemma 1
Let

$$U(t) = \begin{pmatrix} \xi(t) \\ \varphi(t)(\cdot) \end{pmatrix}$$

be a solution of (6) on $(t_1 - M, t_2 + M)$*. Then* $\varphi(t)(\theta) = \xi(t + \theta)$ *for all* $t \in (t_1 - M, M + t_2)$ *and* $\theta \in [-M, M]$ *with* $t + \theta \in (t_1 - M, t_2 + M)$*. Furthermore* $\xi(t)$ *solves (5) on the interval* (t_1, t_2)*.*

4 Preliminary results

4.1 Linear equations

In this chapter we want to review some known facts about linear functional differential equations of mixed type which we will use in the sequel, see also [21, 28]. We investigate the linear equation

$$\dot{y}(t) = \frac{1}{c_*} D_1 F(h(t - M), \dots, h(t), \dots, h(t + M)) y_t =: L(t) y_t, \tag{7}$$

where we recall that $y_t(\theta) := y(t + \theta)$ for any $\theta \in [-M, M]$. Note that in any case $L(t)\phi$ for fixed t and $\phi \in C^0([-M, M], \mathbb{R}^N)$ has the form

$$L(t)\phi = \sum_{j=-M}^{M} L_j(t)\phi(j) \tag{8}$$

for some $L_j(\cdot) \in BC^0(\mathbb{R}, L(\mathbb{C}^N, \mathbb{C}^N))$. As in the nonlinear case we can relate equation (7) to the abstract equation

$$\partial_t V(t) = \mathcal{A}(t) V(t), \tag{9}$$

where the linear operator $\mathcal{A}(t) : X \subset Y \to Y$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} \xi \\ \varphi \end{pmatrix} = \begin{pmatrix} L(t)\varphi \\ \partial_\theta \varphi \end{pmatrix}$$

for $(\xi, \varphi) \in X$. Let us set $\mathcal{A}_+ := \lim_{t\to\infty} \mathcal{A}(t)$ (i.e. where $L(t)$ in the definition of $\mathcal{A}(t)$ is replaced by $L^+ := \lim_{t\to\infty} L(t)$). Then it is known that the spectrum of the densely defined operator $\mathcal{A}_+ : X \subset Y \to Y$ only consists of eigenvalues of finite multiplicity. Moreover, an element $\lambda_* \in \mathbb{C}$ is in spec$(\mathcal{A}_+)$, if the characteristic function vanishes at λ_*, that is, if

$$\det(\triangle(\lambda)) := \det\left[\lambda \cdot id - \sum_{j=-M}^{M} L_j^+(e^{+j\lambda} \cdot id)\right] = 0 \tag{10}$$

for $\lambda = \lambda_*$, where $L_j^+ := \lim_{t\to\infty} L_j(t)$. Furthermore, the algebraic multiplicity of λ_* as an eigenvalue of $\mathcal{A}_+$ (which is the dimension of its generalized eigenspace) coincides with the order of λ_* as a zero of $\det\triangle(\cdot)$; we refer to [5, 6, 28] for proofs of these statements.

Definition 3
We call a linear equation $\dot{x}(t) = Lx_t$ (respectively $\dot{V} = \mathcal{A}V$) **hyperbolic** *for some $L \in L(C^0, \mathbb{C}^N)$ (respectively $\mathcal{A} = (L, \partial_\theta) \in L(X, Y)$), if the characteristic equation*

$$det \triangle (\lambda) := det \left(\lambda \cdot id - \left(\sum_{k=1}^{m} L_k e^{\lambda k}\right)\right) = 0 \tag{11}$$

does not possess purely imaginary zeros $\lambda = is$ with $s \in \mathbb{R}$.

Let us now state a uniqueness-hypothesis which implies that two solutions $\tilde{y}, y \in H^1(\mathbb{R}, \mathbb{R}^N)$ of (7) are identical provided they coincide on some interval of length $2M$, see [5, 28]. We will need this hypothesis for the existence of exponential dichotomies.

Hypothesis 1
$det(A_{-M}(\cdot))$ and $det(A_M(\cdot))$ do not vanish identically on any nontrivial interval of $\mathbb{R}$.

The following result implies that on suitable subspaces the abstract equation (9) can be solved in forward- and backward time, respectively. The proof can again be found in [5, 6, 28].

Theorem 4 (Exponential dichotomy on $\mathbb{R}_+$)
Assume that the hypothesis 1 is satisfied and that the equation $\dot{V} = \mathcal{A}_+ V$ is hyperbolic. Then (9) possesses an exponential dichotomy on $\mathbb{R}_+$. That is, there exist constants $K, \alpha > 0$ and a family of strongly continuous projections $P(t) : Y \to Y$, $t \geqslant 0$, with the following properties. For $U \in Y$ and $t_0 \geqslant 0$

- *there exists a continuous function $\Phi_+^s(\cdot,\cdot)U : \{(t, t_0) : t \geqslant t_0; t, t_0 \geqslant 0\} \to Y$, such that $\Phi_+^{cs}(t_0, t_0)U = P(t_0)U$. Moreover, $\Phi^{cs}(t, t_0)U \in Rg(P(t))$ and $|\Phi_+^{cs}(t, t_0)U|_Y \leqslant Ke^{-\alpha|t-t_0|}|U|_Y$ for all $t \geqslant t_0 \geqslant 0$.*

- *There exists a continuous function $\Phi^u_+(\cdot,\cdot)U : \{(t,t_0) : t \leqslant t_0; t, t_0 \geqslant 0\} \to Y$, such that $\Phi^u_+(t_0,t_0)U = (id - P(t_0))U$. Moreover, $\Phi^u_+(t,t_0) \in ker(P(t))$ and $|\Phi^u_+(t,t_0)U|_Y \leqslant Ke^{-\alpha|t-t_0|}|U|_Y$ for all $t_0 \geqslant t \geqslant 0$.*

In the special case $U \in X$ the functions $t \mapsto \Phi^s_+(t,t_0)U$ and $t \mapsto \Phi^u_+(t,t_0)U$ define classical solutions of (9) on their domain of definition. In any case, if $U \in Rg(P(t_0))$ with $U = (\zeta, \phi(\cdot))$ the map $\Phi^s_+(t,t_0)U$ is of the form $(x(t), x_t)$ for $t > t_0$, $\Phi^s_+(t_0,t_0)U = U$ and $x(\cdot)$ defines a solution of (7) with $x_0 = \phi$. An analogous statement holds for $\Phi^u_+(t,t_0)U$.

Alternatively, there exist continuous solution operators $\Phi^s_-(t,t_0)$, $\Phi^u_-(t,t_0)$ for $t_0 \leqslant t_0 \leqslant 0$ and $t \leqslant t \leqslant 0$, respectively, which define strong solutions for initial values $U \in X$ and satisfy the estimates

$$\|\Phi^s_-(t,t_0)\|_{L(Y,Y)} \leqslant Ke^{-\alpha|t-t_0|}, \qquad \|\Phi^u_+(t,t_0)\|_{L(Y,Y)} \leqslant Ke^{-\delta|t-t_0|}.$$

We need the following result, which is proved in [5, 6, 7] and will be used in the sequel.

Lemma 2
Let $U \in \mathbb{R}^N \times C^0$. Then $\Phi^s_+(t,s)U \in Z^\infty$ and $\|\Phi^s_+(t,s)U\|_{Z^\infty} \leqslant Me^{-\alpha|t-s|}\|U\|_{Z^\infty}$. As a consequence, $\|\Phi^s_+(t,s)\|_{L(Z^\infty,Z^\infty)} \leqslant Me^{-\alpha|t-s|}$. The analogous statement is true for Φ^u_+ and $\Phi^{s/u}_-$.

4.2 Integral formulas, the weak* integral

When dealing with nonlinear equations and constructing stable and unstable manifolds we often have to deal with integral formulas of the kind

$$\int_0^t \Phi^s_+(t,s)G(s)ds, \tag{12}$$

where $G(s)$ is continuous as a map from $[0,\infty)$ to $\mathbb{R}^N \times C^0$. Typically, $G(s)$ has the form $G(s) = (g(s), 0)$ for some continuous function $g : \mathbb{R}_+ \to \mathbb{R}^N$. The integral expression (12) makes perfectly sense in Y, since the integrand is continuous with respect to s and values in Y. However, if we like to set up a contraction mapping argument, the map $G(s)$ will not be well defined with values in Y (see for example our choice of nonlinearity in the proof of lemma 12). As a consequence, we would like to view the integral (12) as an element in $\tilde{X}$. For this reason we regard the integral from now on as a *weak* integral*, which we define now and collect some properties. Let us choose some element

$$(\eta,\psi) \in \tilde{Y} := \mathbb{C}^N \times L^1([-M,M],\mathbb{C}^N)$$

and note that

$$s \mapsto \big\langle \Phi^s_+(t,s)G(s), (\eta,\psi)\big\rangle \in L^1([0,t],\mathbb{C}), \tag{13}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $Z^\infty = \mathbb{C}^N \times L^\infty([-M,M],\mathbb{C}^N)$ and $\tilde{Y}$; that is

$$\langle(\xi,\phi),(\eta,\psi)\rangle = \xi\cdot\eta + \int_{-M}^{M}\phi(\theta)\psi(\theta)d\theta$$

for $(\xi, \phi) \in Z^\infty$ and $(\eta, \psi) \in \tilde{Y}$. Here, Z^∞ can be identified with the dual space of $\tilde{Y}$. Hence, there exists a unique $Q \in Z^\infty$, such that

$$\langle Q, (\eta, \psi) \rangle = \int_0^t \left\langle \Phi_+^s(t,s)G(s), (\eta, \psi) \right\rangle ds \tag{14}$$

for every $(\eta, \psi) \in \tilde{Y}$; see the appendix of [19].

Definition 4
We set $\int_0^t \Phi_+^s(t,s)G(s)ds := Q$ and call Q the weak* integral.

From now on we view the integral term in (12) as a weak* integral, which is an element of $\tilde{Y}^* = Z^\infty$ by definition. Note that if $s \mapsto G(s)$ is continuous and takes values in X, then the weak* integral coincides with the usual Riemann integral. Let us now prove that the integral is actually an element of $\tilde{X} = \{(\xi, \phi) \in \mathbb{C}^N \times C^0([-M, M], \mathbb{C}^N) : \phi(0) = \xi\}$. The next lemma has been proved in [6].

Lemma 3
For each fixed $t \geq 0$ we have $\int_0^t \Phi_+^s(t,s)G(s)ds \in \tilde{X}$.

The weak integral actually depends continuously on t:

Lemma 4
The function $v : t \to \int_0^t \Phi_+^s(t,s)G(s)ds$ is continuous as a function from $[0, \infty)$ to $\tilde{X}$ and

$$\|v(t)\|_{\tilde{X}} \leqslant \int_0^t Me^{-\alpha(t-s)}ds \cdot \sup_{0 \leqslant s \leqslant t} \|G(s)\|_{Z^\infty}.$$

Note that $\Phi_+^s(t,s)$ satisfies the estimate $\|\Phi_+^s(t,s)\|_{L(Z^\infty, Z^\infty)} \leqslant Me^{-\alpha(t-s)}$ for $t \geq s \geq 0$ and some $\alpha \in \mathbb{R}^+$.

Of course, similar results are true if we consider integral terms such as $\int_\infty^t \Phi_+^u(t,s)G(s)ds$ for some fixed $t \geq 0$.

4.3 Reversible equations

Often lattice differential equations are *time-reversible*. This has a consequence for the traveling wave equation and we will therefore define the notion of reversibility for our abstract equation (6) in this section. More precisely, if

$$\mathcal{R}\mathcal{F}(U) = -\mathcal{F}(\mathcal{R}U) \tag{15}$$

for any $U = (\xi, \phi) \in X$, where the linear map $\mathcal{R} : Y \to Y$ is defined by

$$\mathcal{R}(\xi, \phi(\theta)) := (R\xi, R[\mathcal{S}\phi(\cdot)]) = (R\xi, R\phi(-\theta))$$

and $(\mathcal{S}\phi)(\theta) := \phi(-\theta)$ for any $\phi \in C^0([-M, M], \mathbb{R}^N)$, we call the abstract equation (6) reversible. Here, we assume that $R \in L(\mathbb{R}^N)$ can be represented in the form

$$R = P_{i_1} \circ P_{i_2} \circ \ldots \circ P_{i_n}, \tag{16}$$

where the reflection P_i, $1 \leqslant i \leqslant N$, is defined by

$$P_i(x^1, \dots, x^N) \mapsto (x^1, \dots, x^{i-1}, -x^i, x^{i+1}, \dots, x^N).$$

The next example provides a well-known lattice differential equation, where the corresponding abstract equation satisfies (15).

Example
Let us consider the Klein-Gordon equation

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} + V'(u_n), \qquad n \in \mathbb{Z}$$

for some on site potential V with $V(0) = V'(0) = 0$. A traveling wave ansatz leads to the abstract equation

$$\begin{pmatrix} \partial_t x(t) \\ \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} \xi(t) \\ \phi^1(t,1) + \phi^1(t,-1) - 2\phi^1(t,0) + V'(\phi^1(t,0)) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix},$$

where ϕ^1 denotes the first component of $\phi = (\phi^1, \phi^2) : [-1,1] \to \mathbb{R}^2$ and $(x, \xi, \phi) \in X = \{(x, \xi, (\phi^1, \phi^2)) \in \mathbb{R}^2 \times H^1([-1,1], \mathbb{R}^2) : \phi(0) = (x, \xi)\}$. Then $\mathcal{R}$ is given by

$$\mathcal{R}(x, \xi, \phi(\theta)) \mapsto (x, -\xi, \phi^1(-\theta), -\phi^2(-\theta)), \tag{17}$$

which has the upper form (15).

Since we will be particularly interested in the dynamical properties near a homoclinic solution of (6) that intersects $\mathrm{Fix}(\mathcal{R})$, we will make the next definition.

Definition 5
We call a globally solution U of (6) **symmetric***, if U is not a steady state and $U(\tau) \in Fix(\mathcal{R}) = \{V \in Y : \mathcal{R}V = V\}$ for some $\tau \in \mathbb{R}$. Similarly, we call a solution ψ of (2)* **symmetric***, if $U(t) = (\psi(t), \psi_t)$ is a symmetric solution of (6).*

Finally, we will need a result which allows us to construct suitable $\mathcal{R}$-invariant Poincaré-sections near $U(\tau)$, if U denotes a homoclinic solution. The proof of the next result can be found in [8].

Lemma 5 (Poincaré sections)
There exist subspaces $\tilde{\mathcal{E}}^s_+(0) \subset Rg(\Phi^s_+(0,0))$ and $\tilde{\mathcal{E}}^u_-(0) \subset Rg(\Phi^u_-(0,0))$ which are complementary to span$\langle \mathcal{F}(H(0)) \rangle$ and that are closed with respect to the $\tilde{X}$-norm. Moreover, for any finite-dimensional complement $\tilde{Z}$ of the sum $\tilde{\mathcal{E}}^s_+(0) + \tilde{\mathcal{E}}^u_-(0) + $ span $\langle \mathcal{F}(H(0)) \rangle$, the space

$$\tilde{\Gamma} := \tilde{Z} \oplus \tilde{\mathcal{E}}^s_+(0) \oplus \tilde{\mathcal{E}}^u_-(0)$$

is closed with respect to the $\tilde{X}$-norm and defines a Poincaré-section at $U(\tau)$ via $\tilde{\Sigma} := U(\tau) + \tilde{\Gamma}$. If $\tilde{Z}$ is chosen to be $\mathcal{R}$-invariant, then also the space $\tilde{\Gamma}$ is $\mathcal{R}$-invariant.

Let us comment on this result. First let us note that we can always find a subspace $\tilde{Z} \subset \mathrm{Fix}(\mathcal{R})$ or $\tilde{Z} \subset \mathrm{Fix}(-\mathcal{R})$ complementary to the space $\tilde{\mathcal{E}}^s_+(0) + \tilde{\mathcal{E}}^u_+(0) + \mathrm{span}\,\langle \mathcal{F}(H(0))\rangle$ which possesses finite codimension. This follows from the fact that $\mathrm{Fix}(\mathcal{R}\big|_{\tilde{X}}) \oplus \mathrm{Fix}(-\mathcal{R}\big|_{\tilde{X}}) = \tilde{X}$. Secondly, we want to point out that the construction of such a Poincaré-section in $\tilde{X}$ is not at all trivial. In fact, since $\tilde{X}$ is a Banach space we still can construct a Poincaré-section by applying Hahn-Banachs theorem. However, by doing so we cannot guarantee that the constructed section $\tilde{\Gamma}$ is actually $\mathcal{R}$-invariant.

5 Set up

From now on we consider the equation

$$\begin{aligned} \dot{U}(t) &= \mathcal{F}((\xi(t), \phi(t,\cdot)), c) \\ &= \begin{pmatrix} \frac{1}{c} F(\phi(t,-M), \ldots, \phi(t,0), \ldots, \phi(t,M))) \\ \partial_\theta \phi(t,\theta) \end{pmatrix} \end{aligned} \tag{18}$$

and make the following hypothesis.

Hypothesis 2
Equation (18) possesses a homoclinic solution $H(t) = (\psi(t), \psi_t)$ to the hyperbolic steady state zero for $c = c_*$.

Linearizing along the homoclinic solutions $H(t)$ leads to the equation

$$\partial_t V(t) = \mathcal{A}(t) V(t) := \begin{pmatrix} \frac{1}{c_*} D_1 F(\psi_t, 0)\phi(t,\cdot) \\ \partial_\theta \phi(t,\theta) \end{pmatrix}, \tag{19}$$

where $V(t) = (\eta(t), \phi(t,\cdot))$. Throughout this section we want to assume that hypothesis 1 is satisfied.

Exponential dichotomies
On account of theorem 4 the equation (19) possesses exponential dichotomies in $\mathbb{R}_\pm$ with solution operators $\Phi^{s/u}_+$ on $\mathbb{R}_+$ and $\Phi^{s/u}_-$ on $\mathbb{R}_-$ together with a family of projections $P^{s/u}_+(t) : Y \to Y$, $t \geq 0$, and $P^{s/u}_- : Y \to Y$, $t \leqslant 0$, respectively. Moreover, there exist $\alpha, K > 0$ such that

$$\begin{aligned} &\|\Phi^s_+(s,t)\|_{L(Y,Y)} \leqslant K e^{-\alpha|t-s|}, && \|\Phi^u_+(t,s)\|_{L(Y,Y)} \leqslant K e^{-\alpha|t-s|}, && t \geq s \geq 0, \\ &\|\Phi^s_+(s,t)\|_{L(Y,Y)} \leqslant K e^{-\alpha|t-s|}, && \|\Phi^u_+(t,s)\|_{L(Y,Y)} \leqslant K e^{-\alpha|t-s|} && t \leqslant s \leq 0. \end{aligned}$$

We now fix a complement $\tilde{Z} \subset \tilde{X}$ of $\mathrm{Rg}(P^s_+(0)) + \mathrm{Rg}(P^u_-(0))$. Let us make the following generic assumption.

Hypothesis 3
H is non degenerate, i.e. $Rg(P^s_+(0)) \cap Rg(P^u_-(0)) = span\,\langle \mathcal{F}(H(0), c_*)\rangle$.

As a consequence, stable and unstable manifold of zero intersect only along H and the space $\tilde{Z}$ (which appears in the definition of the constructed Poincaré-section) is in fact one-dimensional. Upon choosing new projections we can find an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$, such that $\mathrm{Rg}(P^u_+(0)) := \mathcal{E}^u_-(0) + \tilde{Z}$ and $\mathrm{Rg}(P^s_-(0)) := \mathcal{E}^s_+(0) + \tilde{Z}$, see the introduction of [30].

6 The adaption of Lin's method to advance delay equations

In this section we want to prove our main result. Let us first formulate this result.

Theorem 5
Assume that the hypotheses 1, 2 and 3 are satisfied and fix a complement $\tilde{Z} \subset \tilde{X}$ *of* $Rg(P^s_+(0)) + Rg(P^u_-(0))$. *Then there exist positive constants* $\omega_*, \delta_*, \varepsilon_*$ *such that for all* $\{\omega_i\}_{i\in\mathbb{Z}}$ *with* $\omega_i > \omega_*$ *for all* $i \in \mathbb{Z}$ *and all* $\lambda \in B_{\delta_*}(c_*)$ *there exist unique continuous functions*

$$\begin{aligned} U^i_+ &: [0, \omega_i] \to \tilde{X} \\ U^i_- &: [-\omega_{i-1}, 0] \to \tilde{X} \end{aligned}$$

which have the following properties.

a) U^i_+ *and* U^i_- *are classical solutions of (5) on* $(0, \omega_i)$ *and* $(-\omega_{i-1}, 0)$, *respectively.*

b) $U^i_+(0) \in \tilde{\Sigma}$ *and* $U^i_-(0) \in \tilde{\Sigma}$ *and* $U^i_+(\omega_i) = U^{i+1}_-(-\omega_i)$.

c) $\xi_i(\omega_{i-1}, \omega_i) := U^i_+(0) - U^i_-(0) \in \tilde{Z}$.

d)

$$\begin{aligned} \sup_{-\omega_{i-1} \le t \le 0} \|U^i_-(t) - H(t)\|_{\tilde{X}} &\le \varepsilon_* \\ \sup_{0 \le t \le \omega_i} \|U^i_+(t) - H(t)\|_{\tilde{X}} &\le \varepsilon_*. \end{aligned}$$

e) *Finally, the mapping* $\xi : (\omega_*, \infty) \times (\omega_*, \infty) \times B_{\delta_*}(c_*) \to \tilde{Z}$ *is* C^0 *and* C^2 *with respect to the parameter* c. *Moreover,*

$$\begin{aligned} \lim_{\omega_* \to \infty} \sup_{-\omega_{i-1} \le t \le 0} \|U^i_-(t) - H(t)\|_{\tilde{X}} &= 0 \\ \lim_{\omega_* \to \infty} \sup_{0 \le t \le \omega_i} \|U^i_+(t) - H(t)\|_{\tilde{X}} &= 0 \end{aligned}$$

Moreover, letting $U^{\pm}_i = H + V^{\pm}_i$ *we have the explicit expression*

$$\begin{aligned} \langle \xi_i(\omega_{i-1}, \omega_i), \Psi_0 \rangle_Y &= \left\langle H(\omega_{i-1}), \tilde{\Psi}(-\omega_{i-1}) \right\rangle_Y - \left\langle H(\omega_i), \tilde{\Psi}(-\omega_i) \right\rangle_Y \\ &+ \int_{-\omega_{i-1}}^{\omega_i} \left\langle \mathcal{G}(s, V^{\pm}_i(s)), \tilde{\Psi}(s) \right\rangle_Y ds, \end{aligned} \tag{20}$$

where $\tilde{\Psi}$ denotes the unique nontrivial bounded solution of the adjoint equation $\dot{V}(t) = -\mathcal{A}(t)^* V(t)$ (see also (54)) and where $\Psi_0 \in Y$ is orthogonal to $Rg(P_+^s(0)) + Rg(P_-^u(0))$.

Remark
Before we are going to proceed, let us comment on the last statement of the theorem. On account of technical issues we have to choose $\tilde{Z} \subset \tilde{X}$, and as a consequence $\xi_i \in \tilde{X}$. Since Ψ_0 is not necessarily an element of $\tilde{X}$, we have to project the value of ξ_i onto span$\langle \Psi_0 \rangle$ in the expression (59). However, since $\tilde{X}$ is dense in Y with respect to the Y-norm, we can achieve that given any $\varepsilon > 0$, the right hand side of (59) multiplied by Ψ_0 differs from ξ_i at least by ε (where the distance is measured with respect to the Y-norm).

Examples
a) Multiround homoclinic solutions: As a short example, we would like to illustrate how one could try to detect 2-homoclinic solutions near H using this result. Here, a 2-homoclinic solution H^2 of (18) is a homoclinic solution which intersects a given local Poincaré-section near $H(0)$ exactly two times (provided H intersects this local section exactly one time). We claim that a 2-homoclinic solution corresponds to a zero ω_1 of the jump-functions

$$\xi_1(\infty, \omega_1) = 0, \quad \xi_2(\omega_1, \infty) = 0$$

(and all the other ξ_j vanish with the choice $\omega_{j-1} = \omega_j = \infty$). In fact, let ω_1 be such a zero. Then we can obtain a 2-homoclinic via

$$H^2(t) := \begin{cases} H(t) & \text{on} \quad (-\infty, 0), \\ U_+^1(t) & \text{on} \quad [0, \omega_1] \\ U_-^2(t - 2\omega_1) & \text{on} \quad (\omega_1, 2\omega_1) \\ H(t - 2\omega_1) & \text{on} \quad [2\omega_1, \infty) \end{cases}$$

Note that here $H(t) = U_-^1(t)$ on $(-\infty, 0]$. In this sense we see that the part of the solution on $[0, 2\omega_1]$ is responsible for the "extra" loop.

b) Periodic solutions: If we want to focus on the existence of periodic orbits near H then we have to look for zeros ω_1 of $\xi_1(\omega_1, \omega_1) = 0$ (in this scenario all ξ_i and ω_i are identical). Obviously, such a zero induces a periodic solution of (18).

Instead of proving the upper theorem directly we will show the next lemma.

Lemma 6
There exist unique weak solutions $V_i^+ : [0, \omega_i] \to \tilde{X}$, $V_i^- : [-\omega_{i-1}, 0] \to \tilde{X}$ of

$$V'(t) = \mathcal{A}(t)V(t) + \mathcal{G}(t, V(t)), \tag{21}$$

where for $V \in \tilde{X}$ we have set $\mathcal{G}(t, V) := \mathcal{F}(H(t) + V) - \mathcal{F}(H(t)) - \mathcal{A}(t)V$. Moreover,

I) for $b_i(\omega_i) := H(-\omega_i) - H(\omega_i)$ it is true that $V_i^+(\omega_i) - V_i^-(-\omega_i) = b_i(\omega_i)$.

II) $V_i^\pm(0) \in \tilde{\Gamma}$ and $V_i^+(0) - V_i^-(0) \in \tilde{Z}$.

Since the proof of the lemma is rather technical, we first will give an outline of how we proceed.

6.1 Outline of the proof of lemma 6

We follow closely the presentation in the appendix of [14], see also [3]. The next four steps are the main ingredients in proving lemma 6.

A) For each $j \in \mathbb{Z}$ we will construct a unique solution in $V_j = (V_j^-, V_j^+) \in X^0_{j-1,j}$ of the linear equation

$$\dot{V}(t) = \mathcal{A}(t)V(t) + h_j(t)$$

for any function $h_j \in X_{j-1,j}$. Moreover, we construct V_j in such a way that it satisfies the boundary conditions

$$\begin{aligned} P_+^u V_j^+(\omega_j) &= a_j^+ \\ P_-^s V_j^-(-\omega_{j-1}) &= a_j^- \end{aligned} \tag{22}$$

for any given $(a_j^-, a_j^+) \in \mathrm{Rg}(P_-^s(-\omega_{j-1})\big|_{\tilde{X}}) \times \mathrm{Rg}(P_+^u(\omega_j)\big|_{\tilde{X}})$ (see lemma 8).

B) By establishing a linear relation between h_j, a_j and $b_j = \psi(-\omega_j) - \psi(\omega_j)$ we can provide that V_j solves condition I) if and only if $V_j = V_j(a_j, h_j)$ satisfies this linear relation; i.e. we will translate the dependence of V_j of b_j in a dependence of the terms a_j, h_j instead. This will result in the fact that for large $\omega_j > 0$ and for each h_j, b_j the linear inhomogeneous equation $\dot{V}(t) = \mathcal{A}(t)V(t) + h_j(t)$ possesses a unique solution $V_j \in X^0_{j-1,j}$ satisfying the boundary conditions (22) (see lemma 10).

C) Then we obtain an integral equation, whose fixed points induce solutions of the nonlinear equation (21). More precisely, we will set up a fixed-point equation, whose solutions are in one-to-one correspondence to solutions $V_j = (V_j^-, V_j^+)$ in $X^0_{j-1,j}$ for all j solving the boundary condition $V_j^+(\omega_j) - V_j^-(-\omega_j) = b_j(\omega_j)$. This will be done by the contraction mapping theorem (see lemma 12). As a technical issue, we will strongly make use of the results of section 4.2, which allow us to handle integral terms arising in the fixed-point equation.

D) As a last step, we will address the question of continuity of the so constructed jump functions ξ_j (see condition II) in the upper lemma 6). We will prove that the ξ_j's depend continuously on the flight times ω_j (see lemma 13). In fact, this result may not be optimal and we expect C^1-smoothness in general (a fact, which remains open until now). However, we expect that the explicit calculation of the leading order terms of the jump functions will also imply the C^1-dependence on the flight-times ω_j.

6.2 The proof of lemma 6

We are now going to prove lemma 6.

Notation
We denote by $\pi^{s/u}$ the projection onto the stable and unstable subspace of the

linearisation $\mathcal{A} := \lim_{t\to\infty} \mathcal{A}(t)$.

We begin with the next lemma:

Lemma 7
There exist constants $\omega_{\#} > 0$ and $C > 0$ such that for all $w_i > w_{\#}$ we have

$$\tilde{X} = Rg(P_-^s(-w_i)\big|_{\tilde{X}}) \oplus Rg(P_+^u(w_i)\big|_{\tilde{X}})$$

and $\|\hat{P}_i\| \leq C$, where $\hat{P}_i : \tilde{X} \to \tilde{X}$ denotes the bounded projection which projects onto $Rg(P_-^s(-w_i)\big|_{\tilde{X}})$ along $Rg(P_+^u(w_i)\big|_{\tilde{X}})$.

Proof:
On account of lemma 2 in [9] we have that

$$\begin{aligned} \lim_{t\to\infty} \|P_+^s(t) - \pi^s\|_{L(E,E)} &= 0, \\ \lim_{t\to-\infty} \|P_-^u(t) - \pi^u\|_{L(E,E)} &= 0 \end{aligned}$$

for both choices of spaces $E = Y$ and $E = \tilde{X}$ and therefore

$$\begin{aligned} \lim_{t\to\infty} \|P_+^u(t) - \pi^u\|_{L(E,E)} &= \lim_{t\to\infty} \|(id - P_+^s(t)) - (id - \pi^s)\|_{L(E,E)} = 0, \\ \lim_{t\to-\infty} \|P_-^s(t) - \pi^s\|_{L(E,E)} &= \lim_{t\to-\infty} \|id - P_-^u(t) - (id - \pi^u)\|_{L(E,E)} = 0. \end{aligned}$$

Let us now define

$$S_{i-1} = P_-^s(-w_{i-1}) \circ \pi^s + P_+^u(w_{i-1}) \circ \pi^u,$$

then

$$\begin{aligned} S_{i-1} &= \left(\pi^s + P_-^s(-w_{i-1}) - \pi^s\right)\pi^s + \left(\pi^u + P_+^u(w_{i-1}) - \pi^u\right)\pi^u \\ &= \left(P_-^s(-w_{i-1}) - \pi^s\right)\pi^s + \pi^s + \pi^u + \left(P_+^u(w_{i-1}) - \pi^u\right)\pi^u \\ &= id_E + (P_-^s(-w_{i-1}) - \pi^s)\pi^s + (P_+^u(w_{i-1}) - \pi^u)\pi^u. \end{aligned}$$

We can now choose $\omega_{\#} > 0$ large enough such that for all $\omega_{i-1} > \omega_{\#}$

$$\|(P_-^s(-w_{i-1}) - \pi^s)\pi^s + (P_+^u(w_{i-1}) - \pi^u)\pi^u\|_{L(E,E)} \leq \frac{1}{3};$$

hence $S_i \in L(E,E)$ is invertible with $\|S_i\| \geq \frac{1}{3}$ and $\|(S_i)^{-1}\| \leq 3$. Finally we define the projection $\hat{P}_i = \hat{P}_{i,E} \in L(E,E)$ by

$$\hat{P}_{i,E} := S_{i-1}\pi^s(S_{i-1})^{-1}.$$

We claim that

$$\text{Rg}(\hat{P}_{i,Y}) = \text{Rg}(P_-^s(-\omega_{i-1})\big|_Y) \quad \ker(\hat{P}_{i,Y}) = \text{Rg}(P_+^u(\omega_{i-1})\big|_Y) \tag{23}$$

if ω_{i-1} is large enough, which then shows the claim of the lemma. In order to show (23) let us first note that

$$\hat{P}_{i,Y}S_{i-1} = S_{i-1}\pi^s = P_-^s(-\omega_{i-1})\pi^s,$$

since $\pi^u \circ \pi^s = 0$. We proceed by contradiction and assume that for every $\omega_{i-1} > 0$ large enough there exists a non-trivial unit vector $\psi \in \mathrm{Rg}(P^s_-(-\omega_{i-1}))$ such that

$$\psi \perp \mathcal{K}, \quad \text{where} \quad \mathcal{K} := \mathrm{Rg}\left(P^s_-(-\omega_{i-1}) \circ \pi^s\right).$$

Now

$$\|P^s_-(-\omega_{i-1})\left[P^s_-(-\omega_{i-1}) - \pi^s\right]\psi\|^2 = \|\psi - \phi\|^2$$

for some vector ϕ with $\|\phi\|_Y \leq 1$. Let us write $\phi = \alpha \cdot \tilde{\phi}$ for some scalar $0 \leq \alpha \leq 1$ and some unit vector $\tilde{\phi}$. Then we have

$$\|\psi - \phi\|^2 = \langle \psi, \psi \rangle - 2\langle \psi, \phi \rangle + \alpha^2 \left\langle \tilde{\phi}, \tilde{\phi} \right\rangle = 1 + \alpha^2 \geq 1$$

on account of $\langle \psi, \phi \rangle = 0$, since $\phi \in \mathcal{K}$. This is impossible however, since on the other hand

$$\begin{aligned}
\|P^s_-(-\omega_{i-1})\left[P^s_-(-\omega_{i-1}) - \pi^s\right]\psi\| &\leq \|P^s_-(-\omega_{i-1})\|_{L(Y,Y)}\|P^s_-(-\omega_{i-1}) - \pi^s\|_{L(Y,Y)} \\
&\leq C \cdot \|P^s_-(-\omega_{i-1}) - \pi^s\|_{L(Y,Y)} \leq \frac{1}{5}
\end{aligned}$$

for $\omega_{i-1} > 0$ large enough. This shows that $\mathrm{Rg}(\hat{P}_{i,Y}) = \mathrm{Rg}(P^s_-(-\omega_{i-1}))$ and similarly we can prove that $\ker(\hat{P}_{i,Y}) = \mathrm{Rg}(P^u_+(\omega_{i-1}))$ if ω_{i-1} is large enough. From this observation we can deduce the statements of the lemma. □

We now consider the equation

$$\dot{V}_i^{\pm}(t) = \mathcal{A}(t)V_i^{\pm} + h_i^{\pm}(t) \tag{24}$$

for $h_i = (h_i^-, h_i^+) \in X_{i-1,i}$.

Lemma 8
Let $\omega_i > 0$, $h_i \in X_{i-1,i}$ and $a_i = (a_i^-, a_i^+) \in F_i := Rg(P^s_-(-\omega_{i-1})) \times Rg(P^u_+(\omega_i))$ for all $i \in \mathbb{Z}$. Then (24) has a unique weak solution $V_i = \bar{V}_i(h_i, a_i) \in X^0_{i-1,i}$ satisfying the boundary conditions

$$\begin{aligned}
P^u_+(\omega_i)V_i^+(\omega_i) &= a_i^+ \\
P^s_-(-\omega_{i-1})V_i^-(-\omega_{i-1}) &= a_i^-
\end{aligned}$$

Proof
Consider a classical solution V_i^- of (24) on $-\omega_{i-1} \leq t \leq 0$; then by integration

$$\begin{aligned}
V_i^-(t) &= \Phi^u_-(t,0)V^{u,0} + \int_0^t \Phi^u_-(t,s)h_i^-(s)ds \\
&+ \Phi^s_-(t,-\omega_{i-1})V^{s,\omega_{i-1}} + \int_{-\omega_{i-1}}^t \Phi^s_-(t,s)h_i^-(s)ds
\end{aligned} \tag{25}$$

for some $V^{u,0} \in \mathrm{Rg}(P^u_-(0))$ and $V^{s,\omega_{i-1}} \in \mathrm{Rg}(P^s_-(-\omega_{i-1}))$. Let us point out that the operator $\Phi^s_-(t,-\omega_{i-1})$ is in fact well defined on the interval $-\omega_{i-1} \leq t \leq 0$.

On the interval $0 \le t \le \omega_i$ we have

$$V_i^+(t) = \Phi_+^s(t,0)V^{s,0} + \int_0^t \Phi_+^s(t,s)h_i^+(s)ds + \Phi_+^u(t,\omega_i)V^{u,\omega_i} + \int_{\omega_i}^t \Phi_+^u(t,s)h_i^+(s)ds \tag{26}$$

for $V^{s,0} \in \mathrm{Rg}(P_+^s(0))$ and $V_{u,\omega_i} \in \mathrm{Rg}(P_+^u(\omega_i))$. Setting $t = -\omega_{i-1}$ in (25) we end up with

$$V_i^-(-\omega_{i-1}) = \Phi_-^u(-\omega_{i-1},0)V^{u,0} + \int_0^{-\omega_{i-1}} \Phi_-^u(-\omega_{i-1},s)h_i^-(s)ds + P_-^s(-\omega_{i-1})V^{s,\omega_{i-1}}. \tag{27}$$

and therefore

$$P_-^s(-\omega_{i-1})V_i^-(-\omega_{i-1}) = V^{s,\omega_{i-1}} := a_i^-.$$

This observation shows that with the choices $V^{s,\omega_{i-1}} = a_i^-$ and $V^{u,\omega_i} = a_i^+$ in (25) and (26), respectively, we obtain the existence of weak solutions $V^\pm$ solving (24) and satisfying the boundary conditions in the statement of the lemma.

Let us now construct $V^{u,0}$ and $V^{s,0}$ in such a way that

$$V_i^+(0) - V_i^-(0) = V_i^+(0, V^{s,0}) - V_i^-(0, V^{u,0}) \in \tilde{Z}. \tag{28}$$

In order to do this note that

$$V_i^+(0) = V^{s,0} + \Phi_+^u(0,\omega_i)a_i^+ + \int_{\omega_i}^0 \Phi_+^u(0,s)h_i^+ds$$

$$V_i^-(0) = V^{u,0} + \Phi_-^s(0,-\omega_{i-1})a_i^- + \int_{-\omega_{i-1}}^0 \Phi_-^s(0,s)h_i^-ds.$$

Let us denote by $\mathcal{P}^{\tilde{E}} : \tilde{X} \to \tilde{X}$ the bounded projection onto the space $\tilde{E}$, where $\tilde{E} = \tilde{\mathcal{E}}_+^s(0)$ or $\tilde{E} = \tilde{\mathcal{E}}_-^u(0)$, with respect to the decomposition

$$\tilde{X} = \tilde{\mathcal{E}}_+^s(0) \oplus \tilde{\mathcal{E}}_-^u(0) \oplus \tilde{Z} \oplus \operatorname{span}\langle \mathcal{F}(H(0), c_*)\rangle.$$

Hence, in order to satisfy (28) we define

$$\begin{aligned} V^{u,0} &:= \mathcal{P}^{\tilde{\mathcal{E}}_-^u(0)}\left[\Phi_+^u(0,\omega_i)a_i^+ + \int_{\omega_i}^0 \Phi_+^u(0,s)h_i^+ds\right] \\ V^{s,0} &:= \mathcal{P}^{\tilde{\mathcal{E}}_+^s(0)}\left[\Phi_-^s(0,-\omega_{i-1})a_i^- + \int_{-\omega_{i-1}}^0 \Phi_-^s(0,s)h_i^-ds\right]. \end{aligned} \tag{29}$$

With this definition we now easily conclude that $V_i^\pm(0) \in \tilde{\Gamma}$ and (28) is satisfied. □

Remark (*Differentiability of the solutions*)
Let us make the following observation, which shows that we can in fact provide that the initial values $V_i^{\pm}(0)$ lie in the smooth space $\tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M]))$, where from now on $\tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M]))$ is equipped with the canonical $\mathbb{R}^N \times C^1([-M, M])$-topology. First of all note that we are actually free to chose $\tilde{Z} \subset \tilde{X} \cap (\mathbb{R}^N \times C^2([-M, M]))$, see lemma 5. Now

$$\Phi_+^u(0, \omega_i)a_i^+ + \int_{\omega_i}^0 \Phi_+^u(0, s)h_i^+ ds \in \tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M])),$$

$$\Phi_-^s(0, -\omega_{i-1})a_i^- + \int_{-\omega_{i-1}}^0 \Phi_-^s(0, s)h_i^- ds \in \tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M]))$$

if ω_{i-1} is large enough, we can also choose

$$\begin{aligned} V^{u,-} &:= P^u \left[\Phi_+^u(0, \omega_i)a_i^+ + \int_{\omega_i}^0 \Phi_+^u(0, s)h_i^+ ds \right] \\ V^{s,0} &:= P^s \left[\Phi_-^s(0, -\omega_{i-1})a_i^- + \int_{-\omega_{i-1}}^0 \Phi_-^s(0, s)h_i^- ds \right]. \end{aligned} \tag{30}$$

Here $P^u : \tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M])) \to \tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M]))$ (respectively P^s) denotes the bounded projection onto $\tilde{E}_-^u(0) \cap (\mathbb{R}^N \times C^1([-M, M])$ (respectively $\tilde{E}_+^s(0) \cap (\mathbb{R}^N \times C^1([-M, M]))$ with respect to the decomposition

$$\begin{aligned} \tilde{X} \cap (\mathbb{R}^N \times C^1([-M, M])) \;&=\; \Big(\tilde{E}_-^u(0) \cap (\mathbb{R}^N \times C^1([-M, M]))\Big) \oplus \tilde{Z} \\ &\oplus \Big(\tilde{E}_+^s(0) \cap (\mathbb{R}^N \times C^1([-M, M]))\Big) \oplus \operatorname{span}\langle \mathcal{F}(H(0))\rangle . \end{aligned}$$

This observation in fact shows that $V_i^{\pm}(0) \in \mathbb{R}^N \times C^1([-M, M])$, if $\tilde{Z}$ is chosen to lie in a smooth space.

Notation
We set

$$\bar{V}_i = \bar{V}_i(t, \omega_{i-1}, \omega_i, a_i, h_i) := \begin{cases} V_i^+(t), & t > 0 \\ V_i^-(t), & t \le 0 \end{cases}$$

Lemma 9
There exists positive constants independent of ω_i and t such that for all t

$$\|\bar{V}_i(h_i, a_i)(t)\|_{\tilde{X} \times \tilde{X}} \le C_i \left(\|\{h_i\}_{i\in\mathbb{Z}}\| + \|\{a_i\}\|\right) \tag{31}$$

and

$$\begin{aligned} \|P_-^u(-\omega_{i-1})\bar{V}_i^-(a_i)(-\omega_{i-1})\|_{\tilde{X}} + \|P_+^s(\omega_{i-1})\bar{V}_{i-1}^+(a_{i-1})(\omega_{i-1})\|_{\tilde{X}} \\ \le C_i e^{-2\alpha_i \cdot \min\{\omega_{i-1}, \omega_i\}} \left(\|\{h_i\}_{i\in\mathbb{Z}}\| + \|\{a_i\}\|\right) \end{aligned} \tag{32}$$

for some $\alpha_i > 0$.

Proof
For $\bar{V} = (V^+, V^-)$ let us write

$$V^\pm(0) = \sigma + Z^\pm,$$

where $\sigma \in \tilde{\mathcal{E}}^s_+(0) + \tilde{\mathcal{E}}^u_-(0)$ and $Z^\pm \in \tilde{Z}$. We now want to consider the linear operator

$$\begin{aligned} L(\sigma, Z^+, Z^-) &:= \begin{pmatrix} P^u_+(0)[\sigma + Z^+] \\ P^s_-(0)[\sigma + Z^-] \end{pmatrix} \\ &=: \begin{pmatrix} \tilde{\sigma}^+ + Z^+ \\ \tilde{\sigma}^- + Z^- \end{pmatrix} \end{aligned}$$

where $\sigma = \tilde{\sigma}^- + \tilde{\sigma}^+$. Then

$$L : \left[\tilde{\mathcal{E}}^s_+(0) \oplus \tilde{\mathcal{E}}^u_-(0)\right] \times \tilde{Z} \times \tilde{Z} \to \left(\tilde{\mathcal{E}}^u_-(0) \times \tilde{Z}\right) \times \left(\tilde{\mathcal{E}}^s_+(0) \times \tilde{Z}\right)$$

is a bounded linear operator.

Surjectivity
Let $z^\pm \in \tilde{Z}$ and $\tilde{\sigma}^+ \in \tilde{\mathcal{E}}^u_-(0)$ and $\tilde{\sigma}^- \in \tilde{\mathcal{E}}^s_+(0)$. Then we define

$$\sigma := \tilde{\sigma}^+ + \tilde{\sigma}^-, \qquad Z^\pm := z^\pm$$

and we see that $L(\sigma, Z^+, Z^-) = (\tilde{\sigma}^+ + z^+, \tilde{\sigma}^- + z^-)$.

Injectivity
Let $V^\pm(0) = \sigma + Z^\pm$ and assume that

$$\begin{aligned} P^u_+(0)\left[\sigma + Z^+\right] &= 0 \\ P^s_-(0)\left[\sigma + Z^-\right] &= 0 \end{aligned} \tag{33}$$

and therefore

$$\sigma + Z^+ \in \mathrm{Rg}(P^s_+(0)), \qquad \sigma + Z^- \in \mathrm{Rg}(P^u_-(0)).$$

Hence,

$$V^+ - V^u = Z^+ - Z^- \in \tilde{Z} \cap \left(\mathrm{Rg}(P^s_+(0)) \cap \mathrm{Rg}(P^u_-(0)\right)$$

which means that $Z^+ = Z^-$ and in fact $Z^+ = Z^- = 0$ due to (33) and the definition of $P^u_-(0), P^s_+(0)$. Hence

$$\sigma \in \mathrm{Rg}(P^s_+(0)) \cap \mathrm{Rg}(P^u_-(0)) \cap \Sigma$$

we conclude that $\sigma = 0$ which proves injectivity.

Hence, L_i is boundedly invertible. We now apply the projections $P^s_-(0)$ and $P^u_+(0)$ to the equations (25) and (26), respectively, evaluated at $t = 0$. Hence, we can estimate

$$
\begin{aligned}
\|(V_i^-(0), V_i^+(0))\|_{\tilde{X}\times\tilde{X}} &\leq \|L^{-1}\|_{L(\tilde{X},\tilde{X})}\left(K_i e^{-\alpha\omega_i}\|a_i^+\| + K_i e^{-\alpha\omega_{i-1}}\|a_i^-\|\right) \\
&+ \|L^{-1}\|_{L(\tilde{X},\tilde{X})}\int_0^{\omega_i} K_i e^{-\alpha|s|}\|h_i^+\|_{\tilde{X}} ds \qquad (34) \\
&+ \|L^{-1}\|_{L(\tilde{X},\tilde{X})}\int_0^{-\omega_{i-1}} K_i e^{-\alpha|s|}\|h_i^-\|_{\tilde{X}} ds \\
&\leq \|L^{-1}\|_{L(\tilde{X},\tilde{X})} K_i\left(e^{-\alpha\omega_i}\|a_i^+\| + e^{-\alpha\omega_{i-1}}\|a_i^-\| + \|h_i^+\| + \|h_i^-\|\right).
\end{aligned}
$$

Furthermore, from (26) and (25) we conclude

$$
\begin{aligned}
P_+^s(t)V_i^+(t) &= \Phi_+^s(t,0)V_i^+(0) + \int_0^t \Phi_+^s(t,s)h_i^+(s)ds, \qquad \omega_i \geq t \geq 0 \\
P_-^u(t)V_i^-(t) &= \Phi_-^u(t,0)V_i^-(0) + \int_0^t \Phi_-^u(t,s)h_i^-(s)ds, \qquad -\omega_{i-1} \leq t \leq 0.
\end{aligned}
$$

and

$$
\begin{aligned}
P_+^u(t)V_i^+(t) &= \Phi_+^u(t,\omega_i)a_i^+ + \int_{\omega_i}^t \Phi_+^u(t,s)h_i^+(s)ds, \quad \omega_i \geq t \geq 0 \\
P_-^s(t)V_i^-(t) &= \Phi_-^s(t,-\omega_{i-1})a_i^- + \int_{-\omega_{i-1}}^t \Phi_-^s(t,s)h_i^-(s)ds, \quad -\omega_{i-1} \leq t \leq 0.
\end{aligned}
$$

Using (34) we end up with

$$
\begin{aligned}
\|P_+^s(t)V_i^+(t)\| &\leq K\cdot\left(\|a_i^-\| + \|a_i^+\| + \|h_i^-\| + \|h_i^+\|\right) \\
\|P_+^u(t)V_i^+(t)\| &\leq K\cdot\left(\|a_i^-\| + \|a_i^+\| + \|h_i^-\| + \|h_i^+\|\right)
\end{aligned}
$$

and analogously

$$
\begin{aligned}
\|P_-^s(t)V_i^-(t)\| &\leq K\cdot\left(\|a_i^-\| + \|a_i^+\| + \|h_i^-\| + \|h_i^+\|\right) \\
\|P_-^u(t)V_i^-(t)\| &\leq K\cdot\left(\|a_i^-\| + \|a_i^+\| + \|h_i^-\| + \|h_i^+\|\right),
\end{aligned}
$$

which proves the estimate (31) of the lemma. Similarly the other estimate can be proved and we omit the details, see the appendix of [14]. □

6.2.1 The gluing procedure

In this section we construct the boundary terms $a_j^\pm$ in such a way that the solution V_j satisfies the boundary condition $V_j^+(\omega_j) - V_{j+1}^-(-\omega_j) = H(-\omega_j) - H(\omega_j)$. The next lemma is the key step in this direction.

Lemma 10
There exists some $\omega_* > 0$ such that for all $\omega_i > \omega_*$ and for all $(h_i, b_i) \in X_{i-1,i} \times \tilde{X}$ the linear equation

$$\dot{V}_i^\pm = \mathcal{A}(t)V_i^\pm(t) + h_i^\pm(t)$$

possesses a unique solution $V_i = \bar{V}_i(h_i, b_i) \in X^0_{i-1,i}$ *such that*

$$\begin{aligned} a^+_{i-1} - a^-_i &= b_i + P^u_-(-\omega_{i-1})\bar{V}^-_i(h^-_i, a^-_i)(-\omega_{i-1}) \\ &- P^s_+(\omega_{i-1})\bar{V}^+_{i-1}(h^+_{i-1}, a^+_{i-1})(\omega_{i-1}). \end{aligned} \tag{35}$$

Moreover, there exists a constant $C_i > 0$ *independent of* $\omega_i, \omega_{i-1}, h_i, b_i$ *with*

$$\|\bar{V}_i(h^+_{i-1}, h^-_i, b_i)\| \leq C_i(\|\{h_i\}\| + \|\{b_i\}\|)$$

Proof
We now want to look for unique $(a^+_i, a^-_i) \in \mathrm{Rg}(P^u_+(\omega_i)) \times \mathrm{Rg}(P^s_-(-\omega_{i-1}))$ for all i and $\omega_i >> 0$ such that

$$\bar{U}^+_i(\omega_i) = \bar{U}^-_{i+1}(-\omega_i),$$

where $\bar{U}^\pm_i = H + \bar{V}^\pm_i$. For this purpose let us look only at the important choice

$$b_i = b_i(\omega_i) := H(-\omega_i) - H(\omega_i)$$

but in general the b_i could be chosen arbitrarily. We look for $a_i = (a^+_i, a^-_i)$ such that

$$\bar{V}^+_i(t; \omega_i, a_i, h_i)\big|_{t=\omega_i} - \bar{V}^-_{i+1}(t; \omega_i, a_{i+1}, h_{i+1})\big|_{t=-\omega_i} = b_i,$$

where $h_i = (h^+_i, h^-_i) \in X_{i-1,i}$. This condition is equivalent to

$$a^+_i - a^-_{i+1} = b_i + P^u_-(-\omega_i)\bar{V}^-_{i+1}(-\omega_i) - P^s_+(\omega_i)\bar{V}^+_i(\omega_i). \tag{36}$$

For $\omega_i > 0$ large enough this is the same as

$$\begin{aligned} a^-_{i+1} &= \hat{P}_i\left(b_i + P^u_-(-\omega_i)\bar{V}^-_{i+1}(-\omega_i) - P^s_+(\omega_i)\bar{V}^+_i(\omega_i)\right) \qquad (37) \\ a^+_i &= (id - \hat{P}_i)\left(b_i + P^u_-(-\omega_i)\bar{V}^-_{i+1}(-\omega_i) - P^s_+(\omega_i)\bar{V}^+_i(\omega_i)\right). \end{aligned}$$

Note that the separation of the terms a^+_i, a^-_{i+1} was actually the reason to define the projection $\hat{P}_i$. The solutions $\bar{V}^\pm_i$ depend on h_i, a_i in a linear way, so we can write the last two equations in the form

$$\begin{pmatrix} a^-_{i+1} \\ a^+_i \end{pmatrix} = \mathcal{L}_1 \begin{pmatrix} a^-_{i+1} \\ a^+_i \end{pmatrix} + \mathcal{L}_2 b_i + \mathcal{L}_3 \begin{pmatrix} h_{i+1} \\ h_i \end{pmatrix}, \tag{38}$$

where $\mathcal{L}_1 \in L(\mathrm{Rg}(P^s_-(-\omega_i)) \times \mathrm{Rg}(P^u_+(\omega_i))) =: L(E_i)$ is bounded. On account of lemma 9 and in particular estimate (32) we conclude that $\|\mathcal{L}_1\| \to 0$ as $\omega_i \to \infty$. Hence, $(id - \mathcal{L}_1) \in L(E_i)$ is invertible if $\omega_i > 0$ is large enough. We can therefore solve for

$$\begin{pmatrix} a^-_{i+1} \\ a^+_i \end{pmatrix}$$

in equation (38) and we can define

$$\bar{V}_i(\cdot) = \bar{V}_i(\cdot, h_i, h_{i-1}, h_{i+1}, b_{i-1}, b_i, \omega_{i-1}, \omega_i)$$

with possible jumps at $t = 0$. □

We now want to solve the nonlinear problem

$$V'(t) = \mathcal{A}(t)V(t) + \mathcal{G}(t, V(t)) \tag{39}$$

for solutions V satisfying lemma 6. In order to do that we define the superposition operator $\mathcal{G}_i : X_{i-1,i} \to X_{i-1,i}$ by

$$\begin{aligned} \mathcal{G}_i^-(V)(t) &= \mathcal{G}(t, V_i^-(t)), \qquad -\omega_{i-1} \le t \le 0, \\ \mathcal{G}_i^+(V)(t) &= \mathcal{G}(t, V_i^+(t)), \qquad 0 \le t \le \omega_i. \end{aligned}$$

We need the next lemma, which implies that the super-position map is C^k and Lipschitz in a neighborhood of zero, which will be important in order to apply the contraction mapping theorem in the sequel.

Lemma 11
For $\omega_i, \omega_{i-1} > 0$ the mapping $\mathcal{G}_i : X_{i-1,i} \to X_{i-1,i}$ is of class C^k. Moreover, for each $\gamma > 0$ there are constants $\varepsilon, C > 0$ which do not depend on ω_i, ω_{i-1} such that

$$\begin{aligned} \|\mathcal{G}_i(V_i)\| &\le \gamma\|V_i\| + C|\mu|, \\ \|D_V\mathcal{G}_i(V_i)\| &\le \gamma \end{aligned}$$

for all $\omega_i, \omega_{i-1} > 0$ and for all $V_i \in X_{i-1,i} \cap B_\varepsilon(0)$.

The proof of the lemma can found in [3] or the appendix of [14]. The next lemma finally deals with the nonlinear equation (39).

Lemma 12
There exist constants $\omega_*, \varepsilon_0, \delta_0 > 0$ such that for all $\omega_i > \omega_*$ and for all μ with $|\mu| < \delta_0$ the equation

$$\dot{V}(t) = \mathcal{A}(t)V(t) + \mathcal{G}(t, V(t)) \tag{40}$$

with boundary condition

$$V_i^+(\omega_i) - V_{i+1}^-(-\omega_i) = H(-\omega_i) - H(\omega_i) \tag{41}$$

has unique solutions V_i^+, V_{i+1}^- with $V_i = (V_i^+, V_i^-) \in X_{i-1,i}^0$ and $\|V_i\| \le \varepsilon_0$.

Proof
Let us set

$$\hat{V}_i = \hat{V}_i(h_i, h_{i+1}, b_i, \omega_i) = (\bar{V}_i^+, \bar{V}_{i+1}^-) \in X_i,$$

where $X_i := C^0([0, \omega_i], \mathbb{R}^N \times C^0) \times C^0([-\omega_i, 0], \mathbb{R}^N \times C^0)$. We have then already shown that

$$\|\hat{V}_i(h_i^\pm, h_{i+1}^-, b_i)\| \le C_i \left(\|h_i^-\| + \|h_i^+\| + \|h_{i+1}^-\| + \|b_i\|\right)$$

if ω_i is large enough and $(h_i, b_i) \in X_{i-1,i} \times \tilde{X}$. From this estimate we readily conclude that

$$\|\hat{V}_i(\mathcal{G}_i^-(V_i^-), \mathcal{G}_i^+(V_i^+), \mathcal{G}_{i+1}^-(V_{i+1}^-), b_i(\omega_i))\| \leq \varepsilon_*$$

if $V_i \in X_{i-1,i}$ for all i satisfies $\|V_i\| \leq \varepsilon$, $\varepsilon < 0$ is small enough and $\omega_i >> 0$ is large enough. Moreover, abbreviating

$$\hat{V}_i(V_i^-, V_i^+, V_{i+1}^-) = \hat{V}_i(\mathcal{G}_i^-(V_i^-), \mathcal{G}_i^+(V_i^+), \mathcal{G}_{i+1}^-(V_{i+1}^-), b_i(\omega_i))$$

we have

$$\begin{gathered} \|\hat{V}_i(V_i^-, V_i^+, V_{i+1}^-) - \hat{V}_i(\tilde{V}_i^-, \tilde{V}_i^+, \tilde{V}_{i+1}^-)\| \leq \\ \varepsilon_\# \left(\|\tilde{V}_i^- - V_i^-\| + \|\tilde{V}_i^+ - V_i^+\| + \|\tilde{V}_{i+1}^- - V_{i+1}^-\|\right) \end{gathered}$$

if $\tilde{V}_j, V_j \in X_{i-1,i}^0$, $j = i-1, i, i+1$, are close enough to zero. Note that on account of (38) the term b_i drops out in the difference

$$\hat{V}_i(V_i^-, V_i^+, V_{i+1}^-) - \hat{V}_i(\tilde{V}_i^-, \tilde{V}_i^+, \tilde{V}_{i+1}^-).$$

Now consider the map

$$\begin{gathered} \mathcal{L} : \oplus_{i\in\mathbb{Z}} X_i \longrightarrow \oplus_{i\in\mathbb{Z}} X_i \\ \{V_i\}_i \mapsto \left\{\hat{V}_i(\mathcal{G}_i^-(V_i^-), \mathcal{G}_i^+(V_i^+), \mathcal{G}_{i+1}^-(V_{i+1}^-), b_i(\omega_i))\right\}. \end{gathered}$$

Then $\mathcal{L}$ induces a Lipschitz map with small Lipschitz constant if ε is chosen small enough; hence the equation

$$\{V_j\}_j = \mathcal{L}\left[\{V_j\}_j\right]$$

possesses a fixed point $\{V_j\} \in \oplus_{j\in\mathbb{Z}} X_j$ and for convenience we write $V_j = (V_j^+, V_{j+1}^-)$ for each $j \in \mathbb{Z}$. By construction, $(V_j^+, V_j^-) \in X_{j-1,j}^0$ and every (V_j^+, V_{j+1}^-) satisfies the conditions (40), (41) of the lemma. □

We have therefore proved all statements of our main result, theorem 5, except continuity of the jump-functions

$$\begin{aligned} \xi_i : (\omega_*, \infty) \times (\omega_*, \infty) &\rightarrow \tilde{Z} \\ \xi_i : (\omega_{i-1}, \omega_i) &\mapsto U_i^-(0) - U_i^+(0), \end{aligned} \tag{42}$$

where $U_i^\pm = V_i^\pm + H$. This will be done in the next lemma. We should point out that this is not an easy task, since the proof strongly relies on time-rescalings which contrary to the ODE-case lead to remarkable technical problems.

Lemma 13
For all i the function ξ_i is C^0.

Proof
Fix some vector $\{\omega_i\}_i$ with $\omega_j > \omega_*$. Now choose a vector $\{\beta_i^\pm\}_i \in \mathbb{R}^{\mathbb{Z}}$ with $|\beta_i^\pm|$ small enough for all i such that $\omega_i(1+\beta_i^\pm) > \omega_*$ for all i. Now consider

$$\begin{aligned} J_i^+(\omega_i,\beta_i)(t) &:= V_i^+((1+\beta_i^+)\omega_i)((1+\beta_i^+)t), \qquad 0 \le t \le \omega_i \\ J_i^-(\omega_i,\beta_i)(t) &:= V_i^-((1+\beta_i^-)\omega_{i-1})((1+\beta_i^-)t), \qquad -\omega_{i-1} \le t \le 0 \end{aligned} \tag{43}$$

for $\beta_i = (\beta_i^+, \beta_i^-)$. Note that the $J_j^\pm$ are well-defined. Then J_i^+ solves

$$\dot{J}(t) = (1+\beta_i^+)\mathcal{A}((1+\beta_i^+)t)J(t) + (1+\beta_i^+)\mathcal{G}((1+\beta_i^+)t, J(t)) \tag{44}$$

and in order to prove the lemma it suffices to show that $J_i^\pm$ depends C^1 on $\beta_i^\pm$ near $\beta_i^\pm \approx 0$. In order to do that we want to identify again the $J_i^\pm$ as fixed points of a certain integral equation. We will show then, that the fixed point equation (and therefore its fixed points) depend continuously on $\beta_i^\pm$. Let us begin by analysing the linear part of (44).

The linear rescaled equation
Let us look at the equation

$$\dot{J}(t) = (1+\beta_i^+)\mathcal{A}((1+\beta_i^+)t)J(t).$$

This equation possesses an exponential dichotomy on $\mathbb{R}_+$ with projection $P^{\beta_i}(t)$ and solution operators $\Phi_+^{\beta_i,s}, \Phi_+^{\beta_i,u}$ given by

$$\begin{aligned} \Phi_+^{\beta_i,s}(t,s) &= \Phi_+^s(at,as), \quad \Phi_+^{\beta_i,u}(s,t) = \Phi_+^u(as,at), \qquad t \ge s \ge 0, \\ P^{\beta_i}(t) &= \Phi_+^{\beta_i,s}(t,t) = \Phi_+^s(at,as), \end{aligned}$$

where $a := 1+\beta_i^+$. In fact, let $U \in \mathrm{Rg}P^{\beta_i}(t) \cap X$ then $J(t) := \Phi_+^{\beta_i,s}(t,s)U$ solves

$$\dot{J}(t) = (1+\beta_i^+)\left(\mathcal{A}((1+\beta_i^+)t)J(t)\right), \qquad J(s) = U.$$

In the next step we will consider the rescaled "variation of constants-formula".

The rescaled variation-of-constants formula
Since J^+ solves (44) we have to look for fixed points $J = J^+ \in BC^0 := BC^0([0,\omega_i],\tilde{X})$ of

$$\begin{aligned} J(t) &= \Phi_+^s(at,0)J_0^s + a\cdot\int_0^t \Phi_+^s(at,as)\mathcal{G}(as,J(s))ds \\ &+ \Phi_+^u(at,a\cdot\omega_i)J_0^u + a\cdot\int_{\omega_i}^t \Phi_+^u(at,as)\mathcal{G}(as,J(s))ds. \end{aligned} \tag{45}$$

Here, $J_0^s, J_0^u \in \tilde{X} \cap (\mathbb{R}^N \times C^1([-M,M],\mathbb{R}^N))$ on account of the specific choice (29) and the remark after the proof of lemma 8. Let us now show, that the right hand side of (45) defines a map from BC^0 into itself which depends C^0 on the parameter a. In order to show this we let $J \in BC^0$ be arbitrary and consider the term

$$a\cdot\int_0^t \Phi_+^s(at,as)\mathcal{G}(as,J(s))ds + a\cdot\int_{\omega_i}^t \Phi_+^u(at,as)\mathcal{G}(as,J(s))ds. \tag{46}$$

Continuity with respect to time t

We want to prove that the term depends continuous on t for fixed a (note that in the expression (46) the a-dependence coincides basically with the t-dependence in a lot of places). Instead of considering (46) we study the expression

$$\begin{aligned} F^\delta(t) &= a\cdot\int_0^t \Phi_+^s(at,as)\begin{pmatrix} g(s,a) \\ g(s,a)\cdot l(\delta)(\cdot)\end{pmatrix} ds \\ &+ a\cdot\int_{\omega_i}^t \Phi_+^u(at,as)\begin{pmatrix} g(s,a) \\ g(s,a)\cdot l(\delta)(\cdot)\end{pmatrix} ds, \end{aligned} \tag{47}$$

where $g(s,a)$ denotes the $\mathbb{R}^N$-component of $\mathcal{G}(s,J(s))$ and where

$$l(\delta)(\theta) := \begin{cases} 2\cdot 2^{\frac{1}{(\theta/\delta)^2-1}} & \theta\in(-\delta,\delta) \\ 0 & \text{else} \end{cases}$$

for $\theta\in[-M,M]$ and $|\delta|<M$. Hence, for fixed $\delta>0$, (47) defines an element in X for each fixed t. Moreover, the integral can be regarded as the usual Riemann integral since the integrand is continuous when considered as a map with values in X. We can now differentiate $F^\delta(\cdot):\mathbb{R}_+\to Y$ and obtain

$$\begin{aligned} \partial_t F^\delta(t) = \begin{pmatrix} \partial_t f^\delta(t) \\ \partial_t \xi^\delta(t,\cdot)\end{pmatrix} &= a\begin{pmatrix} g(t,a) \\ g(t,a)l(\delta)(\cdot)\end{pmatrix} + a\mathcal{A}(at)F^\delta(t) \\ &= a\begin{pmatrix} g(t,a) \\ g(t,a)l(\delta)(\cdot)\end{pmatrix} + \begin{pmatrix} aL(at)[\xi^\delta(t,\cdot)] \\ a\partial_\theta\xi^\delta(t,\cdot)\end{pmatrix}. \end{aligned} \tag{48}$$

Let us take a closer look at the second component of (48) first. Since $F^\delta(t)\in X$ for each fixed t,δ and therefore $\xi^\delta(t,0)=f^\delta(t)$, we obtain from

$$\partial_t\xi^\delta(t,\theta) = a\cdot\partial_\theta\xi^\delta(t,\theta) + a\cdot g(t,a)l(\delta)(\theta)$$

via the method of characteristics the identity

$$\xi^\delta(t,\theta) = \begin{cases} f^\delta(at+\theta) + \int_0^\theta ag(t+\theta-\eta,a)l(\delta)(\eta)d\eta & t+\theta\geqslant 0 \\ \xi^\delta(0,\theta+t) + \int_0^t ag(t,a)l(\delta)(\theta+t-\eta)d\eta, & -M\leqslant t+\theta<0. \end{cases} \tag{49}$$

Note that $g(t,a)l(\delta)(\theta)\to 0$ in $L^2([-M,M],\mathbb{R}^N)$ as $\delta\searrow 0$. Moreover, the integral in (47) converges with respect to the Y-norm to the value

$$F^0(t) = a\cdot\int_0^t \Phi_+^s(at,as)\mathcal{G}(as,J(s))ds + a\cdot\int_{\omega_i}^t \Phi_+^u(at,as)\mathcal{G}(as,J(s))ds$$

as $\delta\searrow 0$. Let us write $F^0(t)=(f(t),\xi(t,\cdot))$. Convergence of (47) in Y implies by definition that $f^\delta(t)\to f(t)$ for fixed t as $\delta\searrow 0$. Therefore, we can pass to the limit $\delta\searrow 0$ in (49) and get

$$\xi(t,\theta) = f(at+\theta) \tag{50}$$

as long as $t + \theta \geqslant 0$ and we extend f by $f(t) := \xi(0, t)$ for $-M \le t \le 0$. Let us now look at the (integrated) first equation of (48) in the limit $\delta \searrow 0$; this equation reads

$$f(t) = f(0) + a\left(\int_0^t g(s, a) + L(as)f(as + \cdot)ds\right) \tag{51}$$

in view of (50). We can see in particular that f is a C^1 function with respect to t and fixed a, since g is continuous (but in general not C^1). As a consequence, $F^0(t)$ (and therefore the right hand side of (47)) is continuous with respect to t for every fixed a and $J \in BC^0$.

Continuity with respect to a
Now let us write (46) in a slightly different form and consider

$$\int_0^{at} \Phi_+^s(at, s)\mathcal{G}\left(s, J\left(\frac{s}{\tilde{a}}\right)\right) ds + \int_{b\omega_i}^{at} \Phi_+^u(at, s)\mathcal{G}\left(s, J\left(\frac{s}{\tilde{a}}\right)\right) ds. \tag{52}$$

In fact, (52) coincides with (46) for the choices $\tilde{a} = a$ and $b = a$. Let us therefore argue that (52) is continuous with respect to $a, b, \tilde{a}$. We first observe that (52) is continuous with respect to a, since it is continuous with respect to t. Arguing as before, we can also see that (52) is continuous with respect to $(a, b) = (a, a)$ (i.e. b is replaced by a in the upper expression (52)). This can again be seen by replacing the nonlinearity $\mathcal{G}$ by the modified one (as in (47)) and arguing as before. Also the term (52) depends C^0 on $\tilde{a}$ on account of $J \in BC^0$. Putting things together and replacing $b, \tilde{a}$ by a, we finally have proved that (52) depends C^0 on a for fixed $t \geq 0$ and therefore also (46). The fact that $F^0(\cdot) \in BC^0$ now depends C^0 on a is standard and we omit it.
We have therefore proved the lemma, since also the remaining terms in (45) are C^0-with respect to a. □

Remark
At this stage we are only able to prove $\xi_i \in C^0$ although we expect the jump functions to be at least C^1 with respect to ω_i. In fact, by calculating the leading order terms of the jump functions explicitly (which will be done in a subsequent article) we can show even differentiability of the jump functions.

Finally, we want to find an explicit expression for the jump functions ξ_i.

Lemma 14
Letting $U_i^\pm = H + V_i^\pm$ we have the explicit expression

$$\begin{aligned} \langle \xi_i(\omega_{i-1}, \omega_i), \Psi_0 \rangle_Y &= \left\langle H(\omega_{i-1}), \tilde{\Psi}(-\omega_{i-1}) \right\rangle_Y - \left\langle H(\omega_i), \tilde{\Psi}(-\omega_i) \right\rangle_Y \\ &\quad + \int_{-\omega_{i-1}}^{\omega_i} \left\langle \mathcal{G}(s, V_i^\pm(s)), \tilde{\Psi}(s) \right\rangle_Y ds, \end{aligned} \tag{53}$$

where $\tilde{\Psi}$ denotes the unique nontrivial bounded solution of the adjoint equation $\dot{V}(t) = -\mathcal{A}(t)^* V(t)$ and where $\Psi_0 \in Y$ is orthogonal to $Rg(P_+^s(0)) + Rg(P_-^u(0))$.

Proof
By definition we have

$$\begin{aligned}\xi_i &= V_i^+(0) - V_i^-(0) = V^{s,0} + \Phi_+^u(0,\omega_i)a_i^+ + \int_{\omega_i}^0 \Phi_+^u(0,s)\mathcal{G}(s,V_i^+(s))ds \\ &- V^{u,0} - \Phi_-^s(0,-\omega_{i-1})a_i^- - \int_{-\omega_{i-1}}^0 \Phi_-^s(0,s)\mathcal{G}(s,V_i^-(s))ds,\end{aligned}$$

see also the equations (25) and (26). Projecting this value onto span$\langle\Psi_0\rangle$ gives

$$\begin{aligned}\langle\xi_i,\Psi_0\rangle_Y &= \left\langle V^{s,0} - V^{u,0},\Psi_0\right\rangle_Y + \left\langle\Phi_+^u(0,\omega_i)a_i^+,\Psi_0\right\rangle_Y - \left\langle\Phi_-^s(0,-\omega_{i-1})a_i^-,\Psi_0\right\rangle_Y \\ &- \int_{\omega_{i-1}}^{\omega_i} \left\langle\mathcal{G}(s,V_i^\pm(s)),\tilde{\Psi}(s)\right\rangle_Y ds,\end{aligned}$$

where explicitly

$$\tilde{\Psi}(t) = \begin{pmatrix} \tilde{\psi}(t) \\ \tilde{\phi}(t,\cdot) \end{pmatrix} = \begin{cases} \Phi_+^u(0,t)^*\Psi_0 & : \quad t \geq 0 \\ \Phi_-^s(0,t)^*\Psi_0 & : \quad 0 > t \end{cases} \tag{54}$$

Now since Ψ_0 is perpendicular to $\mathrm{Rg}(P_+^s(0)) + \mathrm{Rg}(P_-^u(0))$ with respect to the Y-scalar product, we obtain

$$\langle\xi_i,\Psi_0\rangle_Y = \left\langle a_i^+,\tilde{\Psi}(\omega_i)\right\rangle_Y - \left\langle a_i^-,\tilde{\Psi}(-\omega_{i-1})\right\rangle_Y - \int_{\omega_{i-1}}^{\omega_i} \left\langle\mathcal{G}(s,V_i^\pm(s)),\tilde{\Psi}(s)\right\rangle - Y ds.$$

Now recall the definition of $a_i^\pm$ in (36) and (35); as a consequence we obtain

$$\begin{aligned}\langle\xi_i,\Psi_0\rangle_Y &= \left\langle H(\omega_i),\tilde{\Psi}(\omega_i)\right\rangle_Y - \left\langle H(-\omega_{i-1}),\tilde{\Psi}(-\omega_{i-1})\right\rangle_Y \\ &- \int_{\omega_{i-1}}^{\omega_i} \left\langle\mathcal{G}(s,V_i^\pm(s)),\tilde{\Psi}(s)\right\rangle ds,\end{aligned}$$

which proves the lemma and therefore the last statement of theorem 5. □

7 Application: The existence of periodic solutions

As an application of Lin's method we want to prove the existence of periodic solutions of the abstract equation (6) for $c \approx c_*$ near the primary homoclinic solution, whose period become unbounded. Note that such solutions induce periodic waves

$$u^j(t) = p(j + ct), \qquad j \in \mathbb{Z}$$

of the LDE (1) for some periodic function p. We make an assumption which we assume to be satisfied for the rest of this section.

Hypothesis 4
Let $H(t) = (\psi(t),\psi_t)$ be a homoclinic solution of (18) for $c = c_*$ to the hyperbolic equilibrium zero and

$$codim_Y(E) := codim_Y\left(Rg(\Phi_+^s(0,0)) + Rg(\Phi_-^u(0,0))\right) = 1$$

where Φ_+^s, Φ_-^u denote the stable and unstable solution operators with respect to an exponential dichotomy with respect to H.

Remark
i) The last assumption is generic and implies that stable and unstable manifold intersect only along the homoclinic solution near $H(0)$.

We now consider two distinct scenarios. In the first case we investigate the situation that the abstract equation is additional reversible. Afterwards, we look at a general equation without symmetries.

7.1 Homoclinic period blow up

Let us make an additional assumption.

Hypothesis 5
Assume that the abstract equation (18) is reversible (in the sense explained in section 4.3). Finally, assume that the homoclinic solution is symmetric and without loss of generality let $H(0) \in \mathrm{Fix}(\mathcal{R})$.

As a consequence, there exists an involution $R \in L(\mathbb{R}^N)$ which defines the reverser $\mathcal{R}$ via $\mathcal{R}(\eta, \phi(\theta)) = (R\eta, R\phi(-\theta))$. In order to detect periodic solutions near the primary homoclinic orbit, we will apply theorem 1 and look for solutions $\omega > \omega_*$ of

$$\xi_1(\omega, \omega) = U_+^1(0) - U_-^1(0) = 0.$$

In fact, any such zero ω induces a periodic solution of (2) of period ω. Note that we can choose ξ_1 to take values in an $\mathcal{R}$-invariant, one-dimensional complement $\tilde{Z}$ to $\tilde{E} := E \cap \tilde{X}$, where E is defined in hypothesis 5. We now want to assume further that there actually exists an element $\kappa \in \mathrm{Fix}(\mathcal{R})$ which spans a complement of $\tilde{E}$; so we can choose

$$\tilde{Z} := \mathrm{span}\, \langle \kappa \rangle \,. \tag{55}$$

This would be a consequence in ordinary differential equations when the upper hypothesis 4 are satisfied. In fact, if we could not find such an element κ then the intersection of the tangent spaces of stable and unstable manifold would be higher dimensional than 1 (see lemma 4 in [3]); contradicting the upper hypothesis 4. Here, in the context of infinite dimensions, such an implication is no longer obvious and we have to assume the existence of such an element $\kappa \in \mathrm{Fix}(\mathcal{R})$. In order to see that $\xi_1(\omega, \omega) = 0$ we define a function U on $[-\omega, \omega]$ via $U(t) := U_+^1(t)$ on $[0, \omega]$ and $U(t) := U_-^1(t)$ on $[-\omega, 0]$ (since actually $\xi_j \equiv \xi_1$ for all j we also have $U_-^j(t) = U(t)$ on $[-\omega, 0]$ and $U_+^j(t) = U(t)$ on $[0, \omega]$). Then U is continuous. Let us now consider the function

$$\tilde{U}(t) = \mathcal{R}U(-t), \qquad t \in [-\omega, \omega].$$

Then $\tilde{U}_+^1(t) := \tilde{U}\big|_{[0,\omega]}$, $\tilde{U}_-^1(t) := \tilde{U}\big|_{[-\omega,0]}$ satisfy all the conditions a)– e) of theorem 5; hence, $\tilde{U} = U$ by uniqueness. It follows that

$$\begin{aligned} \xi_1(\omega, \omega) &= U_+^1(0) - U_-^1(0) = \mathcal{R}U_-^1(0) - \mathcal{R}U_+^1(0) \\ &= -\mathcal{R}\xi_1(\omega, \omega) = -\xi_1(\omega, \omega) \end{aligned}$$

on account of $\tilde{Z} \subset \mathrm{Fix}(\mathcal{R})$. We therefore conclude that $\xi_1 \equiv 0$ for all $\omega_1 > \omega_*$. Let us summarize this observation in the next theorem.

Theorem 6
Assume that the upper hypotheses 1 and 4,5 are satisfied. If there exists a nontrivial element $\kappa \in Fix(\mathcal{R})$, which is not contained in the subspace

$$Rg(\Phi_+^s(0,0)\big|_{\tilde{X}}) + Rg(\Phi_-^u(0,0)\big|_{\tilde{X}})$$

of $\tilde{X}$, then there exists a $\omega_* > 0$ and a family of periodic waves $u_i^\delta(t) = p^\delta(i + c_* t)$ of (1), where p^δ is a periodic solution of (2) for all $\delta > 0$. Moreover, the orbit of p^δ is close to the orbit of ψ and for $\delta \searrow 0$ the period of p^δ becomes unbounded.

Remark
For ordinary differential equations this result is known as the blue-sky catastrophe.

7.2 Periodic solutions

In this section we want to address the existence of periodic solutions near the homolinic solution H, when no underlying symmetries of the lattice are present (which then would induce additional symmetries of the abstract equation (18)). We assume the following.

Hypothesis 6
The element zero is an algebraically simple eigenvalue of the operator

$$\begin{aligned} L : H^1(\mathbb{R}, \mathbb{R}^N) &\rightarrow L^2(\mathbb{R}, \mathbb{R}^N) \\ (Lu)(t) := c_* \cdot \partial_t u(t) &- \sum_{j=-M}^{M} D_j F(\psi(t-M), \ldots, \psi(t+M)) u(t+j). \end{aligned}$$

That is, we assume that $\psi \notin Rg(L)$.

We can now prove the following theorem, which is originally due to John Mallet-Paret, see theorem C in [22]. However, since the result is probably not well-known to the general audience and its proof provides a nice example how to apply Lin's method in a non-trivial way, we have decided to include the result.

Theorem 7
If the hypotheses 1 and 4,6 are true, then for any $\omega > \omega_*$ there exists a traveling wave speed $c = c(\omega)$ with $\lim_{\omega\to\infty} c = c_*$ such that the equation (1) possesses a periodic wave $u^j(t) = p(j + ct)$, where $p = p^c$ depends on c. $p = p^c$ is periodic with period ω and its orbit is close to the orbit of the primary homoclinic solution ψ.

Proof
Let us first note that the proof of the existence of the jump functions translates verbatim to the case where equation (18) depends on a parameter. The

construction then shows that the jump functions depend C^2 on the parameter. Let us now study the zero-set of the jump-function $\xi_1 = \xi_1(\omega, \omega, c)$ for $c \approx c_*$. Note that

$$\lim_{\omega\to\infty} \xi_1(\omega, \omega, c_*) = 0.$$

Let us first show that $\partial_c \xi \neq 0$ if $\omega > \omega_*$ is large enough. In order to prove this it suffices to show that

$$\left\langle \partial_c V_+^1(0) - \partial_c V_-^1(0), \Psi_0 \right\rangle_Y \neq 0, \tag{56}$$

where $\Psi_0 \in Y$ spans the orthogonal complement of $\mathrm{Rg}(\Phi_+^s(0,0))+\mathrm{Rg}(\Phi_-^u(0,0))$ in Y. Indeed, if (56) is true, then

$$\partial_c V_+^1(0) - \partial_c V_-^1(0) \notin \mathrm{Rg}(\Phi_+^s(0,0)) + \mathrm{Rg}(\Phi_-^u(0,0))$$

and therefore $\partial_c \xi\big|_{c=c_*} \neq 0$. In order to prove (56) note that by definition

$$\begin{aligned} V_+^1(0) \quad - \quad & V_-^1(0) = V^{s,+} + \Phi_+^u(0,\omega)a_1^+ + \int_\omega^0 \Phi_+^u(0,s)\mathcal{G}(s, V_+^1(s), c)ds \\ - \quad & V^{u,-} - \Phi_-^s(0,-\omega)a_1^- - \int_{-\omega}^0 \Phi_-^s(0,s)\mathcal{G}(s, V_-^1(s), c)ds, \end{aligned} \tag{57}$$

where a_1^- and a_1^+ are chosen to satisfy the boundary condition (41). Moreover, by construction $V^{s,+}$ and $V^{u,-}$ have been determined in such a way to ensure $V_+^1(0) - V_-^1(0) \in \tilde{Z}$. More precisely, we recall that

$$\begin{aligned} V_+^1(0) - V_-^1(0) \quad = \quad & \left(id - \mathcal{P}^{\tilde{\mathcal{E}}_+^s(0)+\tilde{\mathcal{E}}_-^u(0)}\right)\Phi_+^u(0,\omega)a_1^+ \\ + \quad & \left(id - \mathcal{P}^{\tilde{\mathcal{E}}_+^s(0)+\tilde{\mathcal{E}}_-^u(0)}\right)\left[\int_\omega^0 \Phi_+^u(0,s)\mathcal{G}(s, V_+^1(s), c)ds\right] \\ - \quad & \left(id - \mathcal{P}^{\tilde{\mathcal{E}}_+^s(0)+\tilde{\mathcal{E}}_-^u(0)}\right)\Phi_-^s(0,-\omega)a_1^- \\ - \quad & \left(id - \mathcal{P}^{\tilde{\mathcal{E}}_+^s(0)+\tilde{\mathcal{E}}_-^u(0)}\right)\left(\int_{-\omega}^0 \Phi_-^s(0,s)\mathcal{G}(s, V_-^1(s), c)ds\right), \end{aligned}$$

where $\mathcal{P}^{\tilde{\mathcal{E}}_+^s(0)+\tilde{\mathcal{E}}_-^u(0)} : \tilde{X} \to \tilde{X}$ projects onto the closed space $\tilde{\mathcal{E}}_+^s(0)+\tilde{\mathcal{E}}_-^u(0)$ along $\tilde{Z} + \mathrm{span}\,\langle \mathcal{F}(H(0))\rangle$. Let us make the following observations:

- We point out that also a_1^+, a_1^- depend on c; fortunately we can neglect this in our computation since one can show that

$$|\partial_c a_1^+| + |\partial_c a_1^-|$$

  remains bounded as $\omega \to \infty$. As a consequence, the terms $\Phi_+^u(0,\omega)\partial_c a_1^+$ and $\Phi_-^s(0,-\omega)\partial_c a_1^-$, which arise in (57) by differentiating with respect to c then become arbitrarily small when $\omega \to \infty$. [One can check this claim by considering the equation (37) (where one now has to take into account that the $V_i^\pm$ are solutions of a nonlinear equation rather than a non-homogeneous linear equation) and solving for the $a_1^\pm$ using (38)]. Hence, we can neglect the terms involving a_1^+, a_1^-.

- The next term we are interested in is the term

$$\int_{\omega}^{0} \Phi_{+}^{u}(0,s) \left\{ \partial_2 \mathcal{G}(s, V_{+}^{1}(s), c_*) \partial_c (V_{+}^{1}(s))\big|_{c=c_*} \right\} ds, \tag{58}$$

  where ∂_2 denotes the derivative with respect to the V-component of $\mathcal{G}$. Again, since $\|\partial_c(V_{+}^{1}(s))\big|_{c=c_*}\|_{\tilde{X}}$ stays bounded as $\omega \to \infty$, we observe that (58) approaches zero as $\omega \to \infty$. Why? On account of property e) in theorem 5 we know that $V_{+}^{1}(s)$ approaches 0 uniformly on $[0,\omega]$ as $\omega \to \infty$ (since $V_{+}^{1}(s) + H(s) = U_{+}^{1}(s)$). As a consequence

$$\partial_2 \mathcal{G}(s, V_{+}(s), c_*) \to \partial_2 \mathcal{G}(s, 0, c_*) = 0$$

  uniformly with respect to $s \in [0,\omega]$.

Finally, we can then compute

$$\begin{aligned}
\left\langle \partial_c (V_{+}^{1}(0) - V_{-}^{1}(0)), \Psi_0 \right\rangle_Y &\approx \left\langle \int_{\omega}^{0} \Phi_{+}^{u}(0,s) \partial_3 \mathcal{G}(s, V_{+}^{1}(s,c_*), c)\big|_{c=c_*} ds, \Psi_0 \right\rangle_Y \\
&- \left\langle \int_{-\omega}^{0} \Phi_{-}^{s}(0,s) \partial_3 \mathcal{G}(s, V_{+}^{1}(s,c_*), c)\big|_{c=c_*} ds, \Psi_0 \right\rangle_Y \\
&= -\int_{-\omega}^{\omega} \left\langle \partial_3 \mathcal{G}(s, V_{+}^{1}(s,c_*), c)\big|_{c=c_*}, \tilde{\Psi} \right\rangle_Y ds
\end{aligned}$$

where we have set

$$\tilde{\Psi}(t) = \begin{pmatrix} \tilde{\psi}(t) \\ \tilde{\phi}(t,\cdot) \end{pmatrix} = \begin{cases} \Phi_{+}^{u}(0,t)^* \Psi_0 & : \quad t \geq 0 \\ \Phi_{-}^{s}(0,t)^* \Psi_0 & : \quad 0 > t \end{cases}$$

Now as $\omega \to \infty$ we have

$$\int_{-\omega}^{\omega} \left\langle \partial_3 \mathcal{G}(s, V_{+}^{1}(s,c_*), c)\big|_{c=c_*}, \tilde{\Psi} \right\rangle_Y ds \to \int_{-\infty}^{\infty} \left\langle \partial_3 \mathcal{G}(s, 0, c)\big|_{c=c_*}, \tilde{\Psi} \right\rangle_Y ds \neq 0$$

on account of

$$\int_{-\infty}^{\infty} \left\langle \partial_3 \mathcal{G}(s, 0, c)\big|_{c=c_*}, \tilde{\Psi} \right\rangle_Y ds = \left\langle \partial_c g(t, \psi(t+\cdot), 0, c)\big|_{c=c_*}, \tilde{\psi}(t) \right\rangle_{L^2(\mathbb{R},\mathbb{R}^N)} \neq 0.$$

The latter is true on account of the hypothesis 6 and $\partial_c g(t, \psi_t, 0, c)\big|_{c=c_*} = C\partial_t \psi(t)$ for some $C \neq 0$. This shows that $\partial_c \xi(\omega, c)\big|_{c=c_*} \neq 0$ if ω is large enough. We can therefore apply the implicit function theorem of the appendix to the function $\xi(\omega, c) =: \partial_c \xi(\omega, c_*)[c - c_*] + h(\omega, c - c_*)$ in order to conclude all the assertions of the theorem (we apply the theorem with $x := c$, $y = 1/\omega$). □

8 Discussion

In this section we discuss some generalizations of our results and provide an outlook.

The importance of geometrical informations
From the last statement of theorem 5 we recall that after prescribed a small error term γ, we can can choose the one-dimensional complement $\tilde{Z} = \text{span} < \hat{\phi} >$ is such a way that

$$\begin{aligned} \tilde{\xi}_i(\omega_{i-1},\omega_i) &= \left\langle H(\omega_{i-1}), \tilde{\Psi}(-\omega_{i-1}) \right\rangle_Y - \left\langle H(\omega_i), \tilde{\Psi}(-\omega_i) \right\rangle_Y \\ &+ \int_{-\omega_{i-1}}^{\omega_i} \left\langle \mathcal{G}(s, V_i^{\pm}(s)), \tilde{\Psi}(s) \right\rangle_Y ds + \gamma, \end{aligned} \tag{59}$$

where $\tilde{\xi}_i = \xi_i \cdot \hat{\phi}$. Note that the small error stems from the fact that we cannot necessarily choose $\hat{\phi} := \Psi_0$, since Ψ_0 may not lie in $\tilde{X}$. However, following the ideas in the case of ordinary differential equations [14], we actually expect that we can define jump functions in such a way such that

$$\begin{aligned} \tilde{\xi}_i(\omega_{i-1},\omega_i) &= \left\langle H(\omega_{i-1}), \tilde{\Psi}(-\omega_{i-1}) \right\rangle_Y - \left\langle H(\omega_i), \tilde{\Psi}(-\omega_i) \right\rangle_Y \\ &+ o(e^{-\omega_{i-1}a}) + o(e^{-\omega_i b}) \end{aligned} \tag{60}$$

for some suitable $a, b \in \mathbb{R}$. This result is true in ordinary differential equations, if the homoclinic solution approaches the steady state zero along its leading eigendirections and the corresponding eigenvalues $-\eta^s, \eta^u > 0$ are simple and real. In this case, in fact $a = 2\eta^s$ and $b = 2\eta^u$. Note that a homoclinic solution $(\psi(t), \psi_t)$ of (6) approaches the steady in forward direction $t \to \infty$ always along an eigendirection, since it lies on the stable manifold of zero and a result of Mallet Paret applies, see proposition 7.2 in [21]. Hence, the information of the way how the homoclinic solution ψ, and therefore H, approaches the steady state as $t \to \pm\infty$ plays a crucial role for the determination of the zero set of the jump-functions. In the case where the leading eigenvalues are real, the zeros of ξ_i can in fact be approximated by the zeros of the truncated bifurcation function ξ_i (where the remainder on the right hand side of (60) are truncated), see section 2.4 in [14]. The validation of these results in the case of advance delay equations will be the contens of a subsequent paper.

Heteroclinic chains
In fact, our main theorem 5 is true for more general scenarios. For example, we may consider a heteroclinic chain H^j of heteroclinic solutions of (18) where $\{H^j\}_j$ connects the steady states Q_j and Q_{j+1}. By applying the same procedure we can then generically find solutions $U_j = (U_j^-, U_j^+) \in X^0_{j-1,j}$ which satisfy all the conditions $a) - e)$ in theorem 5 with respect to the heteroclinic solution H_j and a j-dependent Poincaré-section $\tilde{\Sigma}_j$. In particular, the dimension of the space $\tilde{Z} = \tilde{Z}_j$ now depends on the heteroclinic solution H_j and may differ for different j. For precise statements of these results in the case of ordinary differential equations see [14, 3] and [26, 27]. We have restricted our

attention in this article to the case of a homoclinic "chain" $H_j := H$ for all j for the sake of presentation.
In applications, another trivial case may arise if we consider a reversible LDE like the Klein Gordon lattice, which possesses one heteroclinic solution H connecting two steady states $Q_1, Q_2 \in \mathrm{Fix}(\mathcal{R})$. In this case the reverser $\mathcal{R}$ and H induce a second heteroclinic solution $\widetilde{H}(t) = \mathcal{R}H(-t)$ providing a simple heteroclinic chain. An application of theorem 5 to such a scenario may now provide the existence of large periodic solutions in some cases by adapting ideas from the proof of theorem 7. In fact, results in this spirit have been achieved in [22] and [11].

9 Appendix: A variant of the implicit function theorem

In this section we recall and a variant of the implicit function theorem that is useful in the case where a base point is missing.

Theorem 8
Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces and $U \subset \mathcal{X}$, $V \subset \mathcal{Y}$ open subsets with $0 \in U$. Assume that $f : U \times V \to \mathcal{Z}$ is given by $f(x,y) = \Lambda(y)x - h(x,y)$, where $\Lambda(y) \in L(\mathcal{X}, \mathcal{Z})$ for all $y \in V$. Moreover, assume the following

i) The linear map $\Lambda(y)$ is invertible for all $y \in V$.

ii) There exists a constant K_1, such that $\|\Lambda(y)^{-1}\|_{L(\mathcal{Z},\mathcal{X})} \leq K_1$ for all $y \in V$.

iii) There is a positive constant R, such that for all $x, \bar{x} \in U$ with $\|x - \bar{x}\| \leq R$ and for each $y \in V$ it is true that

$$\|h(x,y) - h(\bar{x},y)\| \leq \frac{1}{2K_1}\|x - \bar{x}\|.$$

iv) There is a positive constant $R_1 > 0$ such that $\|h(0,y)\| \leq \frac{R}{2K_1}$ for all $y \in V$ with $\|y\| \leq R_1$.

Then there is a map $x_* : \{y \in V : \|y\| \leq R_1\} \to \{x \in U : \|x\| \leq R\}$ with $f(x_*(y), y) = 0$. Moreover, for $\|x\| \leq R$ and $\|y\| \leq R_1$ the equation $f(x,y) = 0$ is true if and only if $x = x_*(y)$.

Proof
Consider the map $g : U \times V \to \mathcal{Z}$ defined by

$$g(x,y) := \Lambda(y)^{-1}h(x,y)$$

and note that $g(x,y) = 0$ exactly if $h(x,y) = 0$. For $\|x - \hat{x}\| \leq R$ we compute that

$$\|g(x,y) - g(\hat{x},y)\| \leq \|\Lambda(y)^{-1}(h(x,y) - h(\hat{x},y))\| \leq K_1 \cdot \frac{1}{2K_1}\|x - \hat{x}\|$$

by *ii*) and *iii*). Thus in order to apply the contraction-mapping theorem we have to show that $g(\cdot, y)$ maps the ball $\{x \in U : \|x\| \leq R\}$ into itself for all $y \in V$ with $\|y\| \leq R_1$. Indeed, we have for any x with $\|x\| \leq R$

$$\|g(x,y)\| \leq \|g(x,y) - g(0,y)\| + \|g(0,y)\| \leq K_1 \cdot \frac{\|x\|}{2K_1} + K_1 \frac{R}{2K_1} \leq R$$

Hence, we can apply the uniform contraction mapping theorem to conclude the theorem. □

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