Bifurcations from localized steady states to generalized breather solutions in the Klein-Gordon lattice

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Abstract

We consider an infinite chain of particles linearly coupled to their nearest neighbors and look for time-periodic, spatially almost localized solutions which are called generalized discrete breather solutions. As a starting point, we consider a time-independent breather solution that induces a transverse homoclinic solution of a two-dimensional recurrence relation. By imposing suitable conditions on the leading order coefficients of the potential, we can then prove the existence of any finite number of (generalized) breather solutions which bifurcate from the time-independent breather solution at low frequency.

Since we address the case where the equation is not close to the uncoupled limit and the obtained generalized breather solutions do not lie on a center manifold, our results are complementary to the results in [MA94, Jam03, JSRC07]. However, one of the main motivations of this article is to provide a set up, where the existence of chaotic behavior near (generalized) breather solutions becomes accessible to analytical methods. In fact, regarding a breather solution as a homoclinic solution of a suitable recurrence relation (which is ill-posed), we typically expect to encounter chaotic behavior near this solution.

Keywords: *lattice differential equation, ill-posed recurrence relation, discrete breather, center stable manifold, chaotic behavior*

1 Introduction

We are interested in the existence of time-periodic and spatially localized solutions of the Klein Gordon equation

$$\ddot{u}_n(t) + \mathcal{W}'(u_n(t)) = u_{n+1}(t) - 2u_n(t) + u_{n-1}(t), \qquad u \in \mathbb{R} \tag{1}$$

where the on-site potential $\mathcal{W}$ satisfies $\mathcal{W}'(0) = 0, \mathcal{W}''(0) =: \beta^2 > 0$. The kind of solutions we are interested in are commonly referred to as *discrete breather*

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solutions (DB). Discrete breathers and their generalizations such as travelling discrete breathers, see [MS02], play an important role in physical systems [Mac00]. Existence proofs go back to Aubry and MacKay who considered Hamiltonian lattices close to the uncoupled case [MA96, MA94]. In the limit of uncoupling (which refers to the equation $\ddot{u}_n(t) + \mathcal{W}'(u_n(t)) = 0$) a breather solution consists of a single oscillating particle while the others are at rest. Under a non-degeneracy condition this special solution can then be continued to the case of small coupling using the implicit function theorem, see also [SM97].

Recently, James et al used center manifold theory to prove the existence of DB's in a broad class of Hamiltonian lattice differential equations including equation (1), [JSRC07], and the Fermi-Pasta-Ulam lattice [Jam03]. This approach reduces the problem to studying only a finite dimensional recurrence relation. On the other hand, all solutions are by nature of small amplitude, i.e. all such solutions are close to the steady state zero in an appropriate norm. The method used in [Jam03, JSRC07] relies on *spatial dynamics* and it is this approach we want to adapt and generalize in this work. In order to outline the main idea, let us rewrite equation (1) in the form

$$u_{n+1} = 2u_n - u_{n-1} + \partial_t^2 u_n + \mathcal{W}'(u_n). \tag{2}$$

Restricting our attention to time-periodic solutions $u_n(t)$ with period $\frac{2\pi}{\omega}$, we can cast this equation as a first order recurrence relation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} b_n \\ 2b_n - a_n + \omega^2 \partial_t^2 b_n + \mathcal{W}'(b_n) \end{pmatrix}, \tag{3}$$

where the relation between a solution $u_n = u_n(t)$ of (2) and $\{(a_n, b_n)(t)\}$ of (3) is given by $\{(a_n, b_n)(t)\} = \{(u_n(t \cdot \omega), u_{n+1}(t \cdot \omega))\}$. Hence, we arrive at equation (3) by interchanging the role of time t and "space" n. A natural choice of a state space for equation (3) is

$$X = H^2_{per}((0, 2\pi)) \times L^2_{per}((0, 2\pi)),$$

which contains only periodic functions (although we will later restrict our attention to the subspace X_e of even functions). An important feature of equation (3) is its reversible structure. Indeed, by defining a map R by $R(x, y) = (y, x)$ for any $(x, y) \in X$ we can easily verify that $(F \circ R)^2 = id$ in X, whenever the value $(F \circ R)^2(x, y)$ is well-defined. This structure stems from the invariance $n \mapsto -n$ in the original equation (1) and will play an important role in this work.

Note that time-periodic and localized solutions u_n of our original equation (1) now correspond to globally defined solutions $\{U_n\} = \{(a_n, b_n)\}$ of (3), which approach the trivial fixed point zero in forward and backward direction $n \to \pm\infty$. In particular, a nontrivial breather solution now induces a non-trivial homoclinic solution $\{H_n\}$ of (3). By the correspondence with finite dimensional recurrence relations, we therefore expect the existence of chaotic behavior near $\{H_n\}$ once this solutions approaches a hyperbolic steady state. More precisely, with "chaotic behaviour" we mean the existence of a compact invariant neighborhood of H_0 with respect to X, such that the dynamics of (3) restricted to this neighborhood is topologically conjugated to the Bernoulli shift on two symbols, see also [AP90] for a definition in the framework of ordinary differential equations. As a consequence, infinitely many homoclinic solutions of (3) different from $\{H_n\}$ exist, which again induce breather solutions of the original lattice equation (1). We should point out, that this (expected) behavior cannot be validated for small breathers using center manifold techniques: The reason is that in any relevant example, the map which is defined by the recurrence relation (3) is conjugated on a center manifold to the time-one-map of some flow up to any order, see [Jam03], section 6.2.3. However, in this article we also incorporate methods other than center manifolds techniques, which enable us to provide all necessary ingredients in showing the existence of chaotic behaviour near generalized breather solutions existing near time-independent breather solutions, see section 7.

We now want to explain our approach and to relate our results to previous work in more detail. Let us comment on the spatial dynamics approach first. The idea of this method in the case of lattice differential equations has first been used by James [Jam01, Jam03, IK00, Ioo00]. Originally, the idea goes back to Kirchgässner [Kir82, Kir92] and has been generalized by Scheel and Sandstede in the stability-analysis of modulated travelling waves in the framework of semilinear parabolic equations [HSS02, SS99]. Let us now explain a few typical properties of this approach. Linearizing equation (3) at the fixed point zero induces a densely defined unbounded operator L. Although the spectrum of L is unbounded, it consists of isolated simple eigenvalues, see section 2.2. Moreover, L possesses the property of *spectral separation*, see [Jam03], which allows us to apply the idea of exponential dichotomies (see section 5.1 for a definition). As a consequence, the existence of finite dimensional center manifolds of equation (3) near zero can be proved [Jam03, JSRC07]. With its help the essential dynamics of the recurrence relation (3) near the trivial fixed point can be reduced to a finite dimensional recurrence relation, see also section 4. It is one of the advantages of the spatial dynamics approach that center manifold reduction can be applied at all. In fact, let us linearize the lattice differential equation (1) at the fixed point zero, which reads

$$\ddot{y}_n + \beta^2 y_n = y_{n+1} - 2y_n + y_{n-1}. \tag{4}$$

Restricting to linear solutions y_n of the form $y_n(t) = e^{i(n\eta - \omega_\eta t)}$ for some $\eta \in \mathbb{R}$ we recover the dispersion relation

$$\omega_\eta^2 = \beta^2 + 2(1 - \cos(\eta)), \tag{5}$$

where the frequencies ω_η lie in the bounded interval $[\beta, \sqrt{\beta^2 + 4}]$. As a consequence, a straightforward application of center manifold theory or a Lyapunov-Schmidt reduction in the space $l^\infty(\mathbb{Z}, \mathbb{R}^2)$ fails.

However, there is yet another important feature of the spatial dynamics formulation (3), which has not been exploited in the previous works [Jam03, JSRC07]: Let us make the easy observation that the subspace $\mathcal{V}_0 \subset X$ of time-independent functions defines an invariant subspace with respect to the dynamic of the recurrence relation (3). In this linear subspace the dynamics reduces to

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} q_n \\ 2q_n - p_n + \mathcal{W}'(q_n) \end{pmatrix}, \tag{6}$$

where $(p_n, q_n) \in \mathbb{R}^2$. The fixed point zero in this reversible recurrence relation is a saddle (see section 2), i.e. the linearization of the right hand side of (6) at zero possesses an eigenvalue of modulus greater and one smaller than 1. Moreover, the involution R restricted to $\mathcal{V}_0$ possesses a one-dimensional fixed point space. We will then show in section 3 that if the derivative $\mathcal{W}'(x)$ of the on-site potential is sufficiently negative for some $x > 0$ (which does not necessarily imply that $\mathcal{W}$ is negative somewhere), the unstable manifold of the trivial steady state zero of (6) intersects $\mathrm{Fix}(R)$ in a point H_0. Hence, H_0 induces a *symmetric* homoclinic solution $\{H_n\} = \{(h_n, h_{n+1})\}$ of (6), that is $H_n \to 0$ as $n \to \pm\infty$ and $H_0 \in \mathrm{Fix}(R)$. Alternatively, we can also view this homoclinic solution as a localized steady state solution or a time-independent breather solution of the Klein Gordon equation (1). We now want to investigate the set of solutions of (3) near $\{H_n\}$ upon varying the value $\beta = \mathcal{W}''(0)$ for fixed frequency ω. More precisely, we are interested in the existence of bounded solutions $\{\tilde{H}_n\}$ of (3) near $\{H_n\}$ which approach an orbit on the center manifold asymptotically as $n \to \pm\infty$ and for which $\tilde{H}_0$ depends non trivially on time t. We have depicted this scenario in figure 1.

Under appropriate assumptions on the on-site potential $\mathcal{W}$ and if the frequency ω is near the lower edge of the phonon band (see also section 4), the behavior on the center manifold has already been clarified in [JSRC07]: In a sufficiently small neighborhood of zero the solutions are periodic and the set of all such solutions is confined by two non trivial fixed points and two branches joining them. Initial values in either of these branches induce heteroclinic solutions of (3) connecting the nontrivial fixed points. Such solutions in fact induce so called dark breathers, since the amplitude at infinity is larger than at the center. Keeping ω fixed and varying β slightly, we will show that typically nontrivial generalized breather solutions exist near the time-independent solution $\{H_n\}$. These solutions stay uniformly close to the primary solution $\{H_n\}$ of (1) and are spatially almost localized (with respect to n) with constant or oscillating amplitude in the asymptotic limits $n \to \pm\infty$. We now state our main result (see also theorem 7 for a more detailed formulation of the assumptions and additional properties of the solutions).

Theorem 1

Assume that there exists a symmetric homoclinic solution $\{H_n\}$ of (6), alias a symmetric time-independent breather solution, and consider the case $\omega = \beta_0$ for some $\beta_0 > 2/\sqrt{3}$. Then if $\beta < \beta_0$, and under suitable sign conditions on the first derivatives of the on-site potential $\mathcal{W}$, the following is generically true. There exists a one-parameter family of solutions $\{h_n^{sym,\beta,\kappa}\}$ of (1), such that for each $\kappa \approx 0$ the element $h_n^{sym,\beta,\kappa}$ is non trivially time-periodic of period $T = (2\pi/\omega)$. Moreover, $\{h_n^{sym,\beta,\kappa}\}$ has the following properties.

i) $\{h_n^{sym,\beta,0}\}$ is a time-independent breather solution to the steady state zero, which is contained in $\mathcal{V}_0$.

ii) For each $\kappa \approx 0$, $\kappa \neq 0$ the symmetric generalized breather solution $\{h_n^{sym,\beta,\kappa}\}$ approaches time-periodic solutions $\{s_n^{\pm,\beta}\}$ with exponential rate asymptotically in spatial direction $n \to \pm\infty$. These have the property that $0 < \|s_n^{\pm,\beta}\|_{H^2_{per}} \le r$ for all $n \in \mathbb{Z}$ and some small $r > 0$ such that $r \to 0$ as $\beta \to \beta_0$. More precisely, it is true that $h_{n+1}^{sym,\beta,\kappa} = h_{-n}^{sym,\beta,\kappa}$ for all n and

$$|s_n^{+,\beta} - h_n^{sym,\beta,\kappa}|_{H^2_{per}} \to 0, \qquad n \to \infty. \tag{7}$$

An analogous statement holds for $s_n^{-,\beta}$ and $n \to -\infty$. Moreover, the solutions $s_n^{\pm,\beta}(\cdot)$ are non trivially periodic in time and

$$s_n^{\pm}(t \cdot \omega) = \alpha_n^{\pm} \cos(t) + \phi(\alpha_n^{\pm}, \alpha_{n+1}^{\pm}, \mu)(t)$$

for some suitable $\alpha_n^{\pm} \in \mathbb{R}$, $\mu = \omega^2 - \beta^2$, and some map $\phi : (\mathbb{R}^2 \cap B_\varepsilon(0)) \times B_\varepsilon(0) \to X$ with $\phi(0,0,\mu) = 0$ and $D\phi(0,0,0) = 0$ (see also theorem 7).

iii) There exists a discrete generalized breather solution $\{h_n^{db,\beta}\}$ of (1) satisfying $h_{n+1}^{db,\beta} = h_{-n}^{db,\beta}$ for all $n \in \mathbb{Z}$. $\{h_n^{db,\beta}\}$ approaches a specific n-independent, nontrivial time-periodic solution $\tilde{\alpha}$ asymptotically in spatial direction $n \to \pm\infty$, that is

$$|\tilde{\alpha} - h_n^{db,\beta}|_{H^2_{per}} \to 0, \qquad n \to \pm\infty \tag{8}$$

and the values $h_n^{db,\beta}$ are uniformly close to the values h_n for all $n \in \mathbb{Z}$ with respect to the $H^2([0, 2\pi/\omega], \mathbb{R})$-norm.

Remark
Note that the solutions obtained on $i)$ simply correspond to the persisting homoclinic solution in the invariant, two-dimensional subspace $\mathcal{V}_0$; their existence is therefore trivial and we have stated case $i)$ only for the sake of completeness.

Remark
The theorem covers only one special case of equation (1), where the leading order terms of the Taylor expansion of $\mathcal{W}$ at zero have to satisfy suitable sign conditions (see the statement of theorem 7) and the frequency is near the lower edge of the phonon band. However, the case where the frequency ω lies near the upper edge of the phonon band $[\beta, \sqrt{(4+\beta^2)}]$ could be treated similarly.

Theorem 1 guarantees the existence of generalized *discrete breather* solutions $\{h_n^{db}\}$ of (1), where we suppress the β-dependence in the notation for the rest of the introduction. We call these solutions breather solutions, since they are time periodic and the amplitude of h_n^{db} as $n \to \pm\infty$ is very small compared to the values h_n^{db} at the "center". Moreover, the solution $\{h_n^{db}\}$ is close to the time-independent breather solution $\{h_n\}$. In other words, the values of the bifurcating breather solutions are comparable in amplitude to the values of the original solution h_n. In this context we can prove the existence of generalized breather solutions of equation (1) without restricting to equations close to the uncoupled limit as in [MA96]. Moreover, since $|h_0^{db}|_{H^2_{per}}$ can be very large, the solution cannot be detected using center manifold theory. As a consequence,

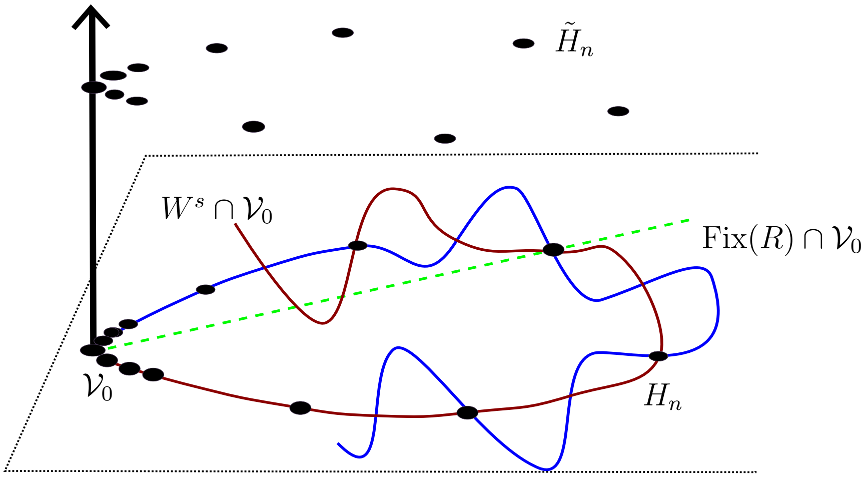

Figure 1: A schematic plot of the bifurcation scenario in equation (3). For $\omega \approx \beta$, $\omega > \beta$, the homoclinic solution $\{H_n\}$ induces nearby solutions $\{\tilde{H}_n\}$ which do not lie in $\mathcal{V}_0$.

the existence of the generalized breather solutions $\{h_n^{db}\}$ and $\{h_n^{sym,\kappa}\}$ does not follow from the earlier works [Jam03, JSRC07, MA96, MA94, SM97, MS02].
In order to prove our main result, we first prove the existence of invariant manifolds of the ill-posed recurrence relation (3) near the homoclinic solution $\{H_n\}$. From this point of view, the symmetric solution $\{H_n\}$ of (3) is induced by an intersection point H_0 in the intersection of the center stable manifold W^{cs} of zero and the fixed point space Fix(R). As a consequence, H_0 also lies in the intersection of W^{cu} and W^{cs}, where W^{cu} denotes the center unstable manifold of (3) of zero, and these manifolds typically intersect transversely along a two-dimensional surface. Hence, we conclude the existence of a two-dimensional family of homoclinic solutions to the center manifold, see section 5.
We will argue in section 6 that after slightly varying the parameter β for fixed frequency ω, we obtain an intersection point of Fix(R) and W^{cs} which induces a solution $\{H_n^{db}\} = \{(h_n^{db}, h_{n+1}^{db})\}$ of (3) homoclinic to a steady state different from zero. Standard techniques [Pal88b, PSS97, SW89] involving exponential dichotomies now imply the existence of chaotic behavior in the neighborhood of $\{H_n^{db}\}$. That is, there exists a neighborhood of H_0^{db} with respect to X where the dynamic of (3) is conjugated to the Bernoulli shift on two symbols, see section 7 for more details. In fact, for the sake of brevity and since our framework does not coincide directly the set up of [Pal88b] or [PSS97, SW89] we have not proved this fact but have restricted ourselves to collect all necessary ingredients for the proof (such as the existence of exponential dichotomies of the variational equation); see for example [Pal88b, PSS97, SW89] for a rigorous proof in general though slightly different contexts.
The methods to construct invariant manifolds in the framework of equation (3) rely on (center-) dichotomies for linear, autonomous recurrence relations (see [Hen81] and section 5.1) and do also apply to more general lattice differential equations other than (1). However, for the sake of clarity and presentation we

have restricted our attention to the Klein-Gordon lattice.
We remark that our bifurcation scenario is very reminiscent of the scenario studied in [SS99, SS01], where the authors study essential instabilities of pulses. In this respect we also would like to mention another aspect of the spatial dynamics approach which is important, namely, the aspect of stability (linear or nonlinear) of discrete breather solutions. In fact, the method introduced by Scheel and Sandstede [SS99, SS01] in the context of semilinear parabolic equations may also be useful to contribute to the theory of stability-analysis of discrete breathers in lattice differential equations.
The work is divided in the following sections. In the next section we introduce the spatial dynamics formulation of (1) and discuss a few properties of this approach. In section 4 we discuss the local behavior of the spatial dynamics equation (3) near the trivial fixed point. The discussion of the global bifurcation scenario as well as the construction of the center-stable manifold is addressed in section 5. The main results of this work are stated in section 6, and in section 7 we also address the question of complicated behavior near the obtained breather solutions. In section 8 we finally deal with the validation of an important assumption which enters the statement of our main result.

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2 The spatial dynamics approach

Let us consider the Klein-Gordon equation

$$\ddot{u}_n(t) + \mathcal{W}'(u_n(t)) = u_{n+1}(t) - 2u_n(t) + u_{n-1}(t) \tag{9}$$

for $n \in \mathbb{N}$, $u \in \mathbb{R}$ and where $\mathcal{W} : \mathbb{R} \to \mathbb{R}$ is a C^3-function with $\mathcal{W}'(0) = 0, \mathcal{W}''(0) =: \beta^2 > 0$. Written as a first order equation, (9) possesses the formal Hamiltonian H

$$E(\{(u_n, v_n)\}) = \sum_{k=-\infty}^{\infty} \frac{1}{2}v_k^2 + \mathcal{W}(u_k) + (u_{k+1} - u_k)^2,$$

where the sequence $\{(u_n, v_n)\}$ has to decay fast enough as $|n| \to \infty$. Casting equation (9) in a spatial dynamics formulation we get

$$u_{n+1}(t) = 2u_n(t) - u_{n-1}(t) + \left(\partial_t^2 u_n(t) + \mathcal{W}'(u_n(t))\right). \tag{10}$$

Let us write this as recurrence relation

$$\begin{aligned} \begin{pmatrix} a_{n+1}(t) \\ b_{n+1}(t) \end{pmatrix} &= \begin{pmatrix} b_n(t) \\ 2b_n(t) - a_n(t) + (\omega^2 \partial_t^2 b_n(t) + \mathcal{W}'(b_n(t))) \end{pmatrix} \\ &= F((a_n(t), b_n(t)), \omega, \beta) \end{aligned} \tag{11}$$

in the phase space $X = H^2_{per}((0,2\pi)) \times L^2_{per}((0,2\pi))$ where the parameter β takes into account variations in $\mathcal{W}''(0)$. Note that after setting $a_n(t) = u_n(t{\cdot}\omega)$, where $u_n(t)$ solves (10), we obtain a solution $\{(a_n, b_n)\} = \{(a_n, a_{n+1})\}$ of (11).

2.1 Symmetries and reversibility

2.1.1 The time translation

Since solutions $u_n(t)$ of (9) can be shifted in time, the map F commutes with the linear map

$$S_c \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a(t+c) \\ b(t+c) \end{pmatrix}$$

for any $(a,b) \in X$ and $c \in (0,2\pi)$. To ease the subsequent analysis, we will restrict our attention from now on to the subspace of even functions

$$X_e = H^2_{per,e}((0,2\pi)) \times L^2_{per,e}((0,2\pi)),$$

(which have by definition a Fourier representation that only involves the functions $\cos(k\bullet)$). In this subspace only S_π is well defined on X_e and satisfies $F \circ S_\pi = S_\pi \circ F$.

2.1.2 Reversibility

System (11) is reversible with the reversibility map defined by $R(x,y) = (y,x)$. In fact, one easily checks that $(F \circ R)^2 = id$ and $\mathrm{Fix}(R) = \{(z,z) : z \in H^2_{per}((0,2\pi))\}$. Let us point out some consequence of the reversibility, namely, that whenever $\{U_n\}$ is a globally defined solution of (11) then also $V_n := RU_{-n}$ defines a solution of (11). In fact, let us suppress the ω-dependence of F for a moment. We then compute

$$\begin{aligned} V_{n+1} &= (F \circ R \circ F \circ R)[RU_{-(n+1)}] = F \circ R \circ F(U_{-n-1}) \\ &= F(RU_{-n}) = F(V_n), \end{aligned}$$

since by assumption $U_{-n} = F(U_{-n-1})$.

2.2 The linear equation

The (densely defined) linear part $L : D \subset X_e \to X_e$ of (11) reads

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 2y - x + (\omega^2 \partial_t^2 y + \beta^2 y) \end{pmatrix},$$

where $D := \big(H^2_{per}((0,2\pi)) \times H^2_{per}((0,2\pi))\big) \cap X_e$. Note that L admits 2-dimensional invariant subspaces $\mathcal{V}_k = \{(a \cdot \cos(kt), b \cdot \cos(kt)) : a,b \in \mathbb{R}\}$ for each $k \in \mathbb{N}$.
Let us compute the spectrum of $L\big|_{\mathcal{V}_k}$. More precisely, we get

$$L\big|_{\mathcal{V}_k} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & (2 - \omega^2 k^2 + \beta^2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, the characteristic equation is

$$\lambda^2 + \lambda(-2 + \omega^2 k^2 - \beta^2) + 1 = 0. \tag{12}$$

We denote by λ_k the solution of this equation with $|\lambda_k| \geq 1$ and $\mathrm{Im}(\lambda_k) \leq 0$. If ω is sufficiently large, the spectrum is real negative and lies strictly off the unit circle. When ω decreases λ_1 and $1/\lambda_1$ approach the unit circle and collide at $\lambda = -1$, yielding a non-semi-simple eigenvalue. The exact value of ω_* is

$$(-\beta^2 + \omega_*^2) = 4 \qquad \Longleftrightarrow \qquad \omega_* := \sqrt{4 + \beta^2}.$$

Decreasing $\omega > \omega_*$ further now leads to a rotation of $\lambda_1, 1/\lambda_1$ along the unit circle; therefore inducing a simple eigenvalue λ^1 on the imaginary axis. More generally, if $\omega = \omega_{k,*} := \sqrt{4+\beta^2}/k$ two eigenvalues $\lambda_k, (\lambda_k)^{-1}$ collide at $\lambda = -1$ and rotate along the unit circle if $\omega > \omega_{k,*}$ is decreased further. Hence, if we choose ω with the property

$$\omega_{2,*} < \omega < \omega_* \tag{13}$$

then L possesses exactly two simple eigenvalues on the unit circle.

Remark
The eigenvalues $\lambda_k, \lambda_k^{-1}$ lie on the unit circle if $\beta \leq k\omega \leq \sqrt{(4+\beta^2)}$. The relation of this observation to the dispersion relation is the following. If we multiply (12) by λ^{-1}, set $\lambda = e^{i\eta}$ and $\omega_\eta = \omega k$, we end up with the dispersion relation $\omega_\eta^2 = \beta^2 + 2(1 - \cos(\eta))$. Hence, if ωk lies in the phonon band for some $k \in \mathbb{N}$ then L possesses a pair of eigenvalues $e^{\pm i\eta}$, where η is defined by the dispersion relation.

Let us now compute the eigenvalues explicitly:

$$\lambda_k^{\pm} = \frac{2 - \omega^2 k^2 + \beta^2}{2} \pm \sqrt{\left(\frac{2 - \omega^2 k^2 + \beta^2}{2}\right)^2 - 1} \tag{14}$$

and for $k = 0$ we have

$$\lambda_0^{\pm} = \frac{2 + \beta^2}{2} \pm \sqrt{\left(\frac{2 + \beta^2}{2}\right)^2 - 1}.$$

Hence, if $\beta^2 \neq 0$ then $\lambda_0^+ > 1$ and $\lambda_0^- < 1$, i.e. 0 is a saddle in the invariant subspace $\mathcal{V}_0$.

3 The invariant subspace of time-independent functions

In the subspace $\mathcal{V}_0 \subset X_e$ of time-independent functions the recurrence relation reduces to

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} q_n \\ 2q_n - p_n + \mathcal{W}'(q_n) \end{pmatrix} \tag{15}$$

for $(p_n, q_n) \in \mathbb{R}^2$. Note that equation (15) is reversible and 0 is a saddle. We make the following assumption.

Hypothesis 1
System (9) possesses a time-independent breather solution $x_n(t) \equiv h_n$ for some $\{h_n\} \in l^\infty(\mathbb{Z}, \mathbb{R})$, such that $h_n \to 0$ as $n \to \pm\infty$ and $(h_n, h_{n+1}) \in Fix(R)$ for some n.

In other words, we want to assume that the recurrence relation (15) possesses a symmetric homoclinic solution $\{(h_n, h_{n+1})\}$.
Let us argue how one can construct a broad class of potentials $\mathcal{W}$ satisfying hypothesis 1. More precisely, we want to show that the class of admissible $\mathcal{W}$, where $\mathcal{W}$ is positive, $\mathcal{W}(0) = \mathcal{W}'(0) = 0$, $\mathcal{W}''(0) = \beta$ satisfying hypothesis 1 is dense with respect to the Lebesgue-norm: In fact, let us consider any potential $\mathcal{W}_0 \in C^3$ such that $\mathcal{W}_0 \geq 0$ and $\mathcal{W}_0(0) = \mathcal{W}_0'(0) = 0$, $\mathcal{W}_0''(0) = \beta$. First of all, since $\mathcal{W}_0$ is defined in a small neighborhood of zero; it is true that $\mathcal{W}_0(x) \approx \beta^2 x^2$ for $x \approx 0$. It is now a straightforward computation to determine the unstable eigenvector of the linearization at $(p,q) = 0$ of (15) which is $(1, \lambda_0^+)$. This eigenvector also coincides (up to linear order) with the unstable manifold of (15) of zero, see [AP90]. Hence, starting with a point (p,q) on the unstable manifold, we will approximately leave the steady state along the unstable eigenvector when iterating F. Let us now denote $(p_n^*, q_n^*) := F^n(p^*, q^*)$ for some $n > 0$ such that $\mathcal{W}_0'$ is defined during the iterations and note that $q_k^* \to 0$ as $k \to -\infty$. We then have $p_n^* < q_n^*$ (otherwise the unstable manifold has already intersected $\mathrm{Fix}(R)$ inducing a homoclinic solution). In this case we can now redefine $\mathcal{W}_0'(q)$ in a small neighborhood of q_n^* without changing $\mathcal{W}_0$ on the values q_k^*, $k < n$ to achieve that $F(p_n^*, q_n^*)$ lies in the sector $\{(p,q) \in \mathbb{R}^2 : p > q\}$ and therefore the unstable manifold has to cross $\mathrm{Fix}(R)$. In fact, the modified value $\mathcal{W}_0'(q_n^*)$ has to satisfy

$$\mathcal{W}_0'(q_n^*) < p_n^* - q_n^* < 0. \tag{16}$$

Let us point out that with this procedure the defined potential satisfies the hypothesis 1. Moreover, condition (16) does not necessarily enforce $\mathcal{W}_0$ to be negative somewhere; in fact, if necessary we can redefine $\mathcal{W}_0$ on intervals which do not contain the values q_m^*, $m < n$ (which does not alter our argumentation). Finally, the modified potential is close to the original $\mathcal{W}_0$ with respect to $L^2([-M, M], \mathbb{R})$ for some appropriate $M > 0$ (which is large enough such that it contains the values where $\mathcal{W}_0$ has been modified). This shows that hypothesis 1 is true for a large set of potentials.
The existence of a symmetric homoclinic solution corresponds to an intersection of the unstable manifold with the fix point space $\mathrm{Fix}(R)$ within the space $\mathcal{V}_0$ and these intersections are typically transverse. We also would like to point out that we typically expect the existence of infinitely many additional symmetric homoclinic solutions besides the homoclinic solution $\{h_n\}$ (see also section 7). The typical scenario is expected to look like in figure 1.

4 Local dynamics

In this section we consider the new parameter $\mu := (-\beta^2 + \omega^2)$. The reasoning for considering the parameter μ in this section is the fact that the following

results only depend on the variation of $\omega^2 - \beta^2$ and hence both parameters ω, β have the same effect.
Under the assumption that the frequency ω satisfies inequality (13) we now conclude the existence of a two-dimensional center manifold near the steady state by the results of James [Jam03]. More precisely, for μ in a small neighborhood of zero, the solutions of (11), which remain in a neighborhood of $Y = 0$ for all $n \in \mathbb{Z}$, belong to a two-dimensional, locally invariant manifold $\mathcal{M}_\mu$. This manifold can be represented as a graph over the center eigenspace

$$E^c := \operatorname{span}\left\langle \begin{pmatrix} 0 \\ \cos t \end{pmatrix}, \begin{pmatrix} \cos t \\ 0 \end{pmatrix} \right\rangle$$

and is commonly referred to as *center manifold*. The reduced dynamics on the center manifold $\mathcal{M}_\mu$ has been studied in great detail by [Jam03, JSRC07]. Theorem 3 of [JSRC07] now takes the following form:

Theorem 2 (James et al)
Fix $\omega^2 = \omega_c^2 + \mu$, where $\omega_c = \beta$ or $w_c = \sqrt{4+\beta^2}$ (in the former case we additionally assume that $\beta > 2/\sqrt{3}$). There exist neighborhoods $\mathcal{U}, \mathcal{V}$ of 0 in $\mathbb{R}^2$ and $\mathbb{R}$, respectively, and a C^k-map $\phi : (\mathbb{R}^2 \cap \mathcal{U}) \times \mathcal{V} \to H^2_{per,e}$ with $\phi(0, \mu) = 0$ $D\phi(0, 0) = 0$ such that the following holds. Let $\{x_n\}$ be a solution of (10) which satisfies $x_n \in \mathcal{V}$ for all n. Then

$$x_n(t) = \alpha_n \cos t + \phi(\alpha_{n-1}, \alpha_n, \mu),$$

where (α^x, α^y) denote the coordinates with respect to the vectors $(\cos(t), 0)$ and $(0, \cos(t))$ in the center eigenspace of L.

i) For $\omega_c = \beta^2$, α_n satisfies the recurrence relation

$$\alpha_{n+1} - 2\alpha_n + \alpha_{n-1} = \mathcal{K}(\alpha_{n-1}, \alpha_n, \mu) \tag{17}$$

where $\mathcal{K} : \mathcal{U} \times (\mathbb{R} \cap \mathcal{V}) \to H^2_{per,e}$ is a C^k-map. The normal form of the iteration (17) is given by

$$\alpha_{n+1} - 2\alpha_n + \alpha_{n-1} = -\mu\alpha_n + B\alpha_n^3 + h.o.t. =: r(\alpha_{n-1}, \alpha_n, \mu), \tag{18}$$

where $B := \frac{\beta}{8}(\mathcal{W}^{(3)}(0) - \frac{5}{3}(\mathcal{W}^{(2)}(0))^2)$.

ii) For $\omega_c = \sqrt{4+\beta^2}$, α_n satisfies the recurrence relation

$$\alpha_{n+1} + 2\alpha_n + \alpha_{n-1} = \mathcal{K}(\alpha_{n-1}, \alpha_n, \mu) \tag{19}$$

where $\mathcal{K} : \mathcal{U} \times (\mathbb{R} \cap \mathcal{V}) \to H^2_{per,e}$ is a C^k-map. The normal form of the iteration (17) is given by

$$\alpha_{n+1} - 2\alpha_n + \alpha_{n-1} = -\mu\alpha_n + \tilde{B}\alpha_n^3 + h.o.t. =: \tilde{r}(\alpha_{n-1}, \alpha_n, \mu), \tag{20}$$

where $\tilde{B} := \frac{\beta}{8}(\mathcal{W}^{(3)}(0) + (\mathcal{W}^{(2)}(0)))^2 \cdot \left(\frac{\beta^2}{16+3\beta^2} - 2\right)$.

iii) The functions ϕ and $\mathcal{K}$ have the following symmetries

$$\phi(-a,-b,\mu)(\cdot) = \mathcal{S}\phi(a,b,\mu)(\cdot), \qquad \mathcal{K}(-a,-b,\mu) = -\mathcal{K}(a,b,\mu),$$

where $\mathcal{S}$ denotes the time-shift operator $(\mathcal{S}\eta)(t) = \eta(t+\pi)$.

The proof of this result can be found in [JSRC07]. We now restrict to the case that the frequency ω is near the lower end of the phonon band, that is $\omega \approx \beta$.

Lemma 1
Assume that $\omega = \omega_c = \beta$ and $\beta > 2/\sqrt{3}$. For $\mu \approx 0$ the iteration (18) has the following properties.

a) For $\mu < 0$ and $B < 0$ there exist two homoclinic solutions q_n^1, q_n^2, such that $\lim_{n\to\pm\infty} q_n^i = 0$. These solutions have the symmetries $q_n^1 = q_{-n+1}^1$, $q_n^2 = q_{-n}^2$ and satisfy $0 < q_n^i \le C|\mu|^{1/2}(1+\mathcal{O}(|\mu|^{1/2}))^{-|n|}$, where $1+\mathcal{O}(|\mu|^{1/2}) > 1$.

b) If μ and B have the same sign, (18) has two symmetric fixed points $\pm\alpha^* = \mathcal{O}(|\mu|^{1/2})$.

c) For $\mu > 0$ and $B > 0$ there exist two heteroclinic solutions q_n^3, q_n^4, such that $\lim_{n\to\pm\infty} q_n^i = \pm\alpha^*$. Moreover, q_n^3, q_n^4 have the symmetries $q_n^3 = q_{-n+1}^3$, $q_n^4 = q_{-n}^4$. q_n^3, q_n^4 are $\mathcal{O}(|\mu|^{1/2})$ as $n \to \pm\infty$ and $\mathcal{O}(\mu)$ for bounded n.

Notation
The fixed points $(\alpha_n, \alpha_{n+1}) = \pm(\alpha^*, \alpha^*)$ of the reduced recurrence relation (18) induce fixed points $\pm A^*$ of the main equation (11), where $A^* := (\alpha^*, \alpha^*) + (\phi((\alpha^*, \alpha^*, \mu)), \phi(\alpha^*, \alpha^*, \mu))$.

The claims are proved in Lemma 2 of [JSRC07], but let us comment on the statements. The solutions $\{q_n^1\}$ and $\{q_n^2\}$ correspond to discrete breather solutions, since these are localized in space n and periodic in time (note that $q_n^i \in X_e$ for each n and that the center manifold $\mathcal{M}_\mu$ is locally a graph over the center-eigenspace E^c). Hence, $q_n^i = b_n^i \cos(t) + \mathcal{O}(|\mu|)$ for $i = 1,2$ and suitable constants b_n^i. The solutions $\{q_n^3\}$ and $\{q_n^4\}$ can be regarded as *dark breather* solutions, whose oscillations have an amplitude $\mathcal{O}(|\mu|^{1/2})$ as $n \to \pm\infty$ (where the heteroclinic solutions $\{q_n^j\}$, $j = 3,4$ approach one of the symmetric steady states $\pm\alpha^*$ of amplitude $\mathcal{O}(|\mu|^{1/2})$; certificate point b) of the lemma) and a smaller amplitude $\mathcal{O}(\mu)$ in the "center". All these facts have already been observed in [Jam03, JSRC07]. Our aim in this paper is to obtain *large* (with respect to n) generalized breather solutions, which exhibit the same asymptotic behavior as the breather solutions corresponding to the $\{q_n^j\}$ for $j = 3,4$.

5 The global bifurcation scenario

In the previous section we have clarified the local behavior near the fix point zero for small $\mu \approx 0$. In particular, small variations of μ lead to the existence of either small heteroclinic or homoclinic solutions for suitable conditions on the coefficients of the reduced Taylor expansion of $\mathcal{W}$. Moreover, as we have already argued, homoclinic solutions of the recurrence relation

$$Y_{n+1} = LY_n + N(Y_n, \mu) = F(Y_n, \mu) \tag{21}$$

induce discrete breather solutions of the Klein-Gordon equation (9). These breather solutions are trivial with respect to time t (i.e. time-independent) if and only if the associated homoclinic solution lies in the invariant subspace $\mathcal{V}_0$. In this section we now want to analyse in which way the occurrence of the small amplitude solutions on the center manifold $\mathcal{M}_\mu$ interact with the primary homoclinic solution $\{H_n\} = \{(h_n, h_{n+1})\}$ of (21). In order to this, we need to construct invariant manifolds of equation (21) which capture bifurcations near $\{H_n\}$. Since we want to conclude the existence of solutions to the center manifold $\mathcal{M}_\mu$, important manifolds are the center-stable and center-unstable manifold W^{cs}, W^{cu} of zero, respectively. As suggested by dynamical systems theory, solutions starting in the center-stable manifold W^{cs} approach an orbit on the center manifold in forward direction $n \to \infty$. Similarly, solutions with initial values in the center-unstable manifold W^{cu} approach an orbit on the center manifold in backward direction $n \to -\infty$. Once we can prove that W^{cs} and W^{cu} intersect non trivially for $\mu \neq 0$, we immediately conclude the existence of globally defined solutions $\{\tilde{H}_n\}$ converging to the center manifold $\mathcal{M}_\mu$, see figure 1. In particular, if $\{\tilde{H}_n\}$ does not approach the fixed point zero in both, forward and backward direction, with respect to n, then the resulting solution $\{\tilde{H}_n\}$ of (21) induces a *non*-trivial time-periodic solution of the Klein-Gordon equation (9). The rigorous construction of the manifolds W^{cs}, W^{cu} relies strongly on the existence of (center-) dichotomies for the linear equation $V_{n+1} = L_n V_n$, which we will address in the next section.

5.1 Center-dichotomies

Let us linearize equation (11) along the homoclinic solution:

$$\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} w_n \\ 2w_n - v_n + \tilde{\omega}^2 \partial_t^2 w_n + \mathcal{W}''(h_n) w_n \end{pmatrix} = L_n \begin{pmatrix} v_n \\ w_n \end{pmatrix}$$

for some frequency $\tilde{\omega}$. We recall the definition of an exponential dichotomy, see also [Hen81] for the case of bounded operators.

Definition 1
The family $\{L_n\}_{n\in\mathbb{Z}}$, $L_n : D \subset X_e \to X_e$, has a discrete exponential dichotomy if there exist positive constants $M, \theta < 1$ and a family of projections $\{P_n\}_{n=-\infty}^{\infty}$, $P_n \in \mathcal{L}(X_e, X_e)$, such that the following holds:

a) The range of the densely defined operator $L_n \circ P_n : D \subset X \to X$ is contained in $Rg(P_{n+1})$ for all n.

b) The densely defined, closed operator $L_n|_{Rg(P_n)}$ is a bijective map from $Rg(P_n)$ to $Rg(P_{n+1})$ and possesses a bounded inverse

$$\left(L_n|_{Rg(P_n)}\right)^{-1} : Rg(P_{n+1}) \to Rg(P_n),$$

where both spaces are equipped with the X_e-norm.

c) *If $L_{n,m} = L_{n-1} \circ \ldots \circ L_{m+1} \circ L_m$ for $n > m$, then*

$$\|L_{n,m}(id - P_m)\|_{L(X_e,X_e)} < M\theta^{n-m}, \qquad n > m.$$

In particular, $L_{n,m}(id - P_m) : X_e \to X_e$ is a bounded operator.

d) $\|L_{n,m}P_m\|_{\mathcal{L}(X_e,X_e)} < M\theta^{m-n}$ *if $n < m$, where $L_{n,m}P_m x = y \in Rg(P_n)$ if and only if $P_m x = L_{m,n}y$, which is well-defined by b).*

Let us note that L_n (defined in (22)) can be written in the form

$$L_n = \tilde{L} + C_n, \tag{22}$$

where $\tilde{L}$ is defined by the right hand side of (22) if h_n is replaced by 0 and the bounded operators $C_n : X_e \to X_e$ are actually compact, since $h_n \to 0$ for $n \to \pm\infty$. On account of at most finitely many eigenvalues of $\tilde{L}$ on the unit circle we cannot expect that the family $\{L_n\}_n$ induces an exponential dichotomy. This suggests considering the modified operator

$$L_n^\eta = \eta \cdot L_n$$

for some $\eta < 1$ sufficiently close to 1. Now, if V_n is a solution of the linear iteration $V_{n+1} = L_n V_n$ then $Z_n := \eta^n V_n$ solves the equation

$$Z_{n+1} = \eta^{n+1} V_{n+1} = \eta^{n+1}(L_n V_n) = \eta L_n Z_n = L_n^\eta Z_n.$$

We observe that for η close to 1 the linear operator $\eta \cdot \tilde{L} =: L_{\eta,\infty}$ does not possess any eigenvalues on the unit circle. Indeed, this can easily be seen by computing the spectrum of $L_{\eta,\infty}$ restricted to the invariant subspaces $\mathcal{V}_k$ for each k. Hence, the map $L_{\eta,\infty}$ admits an exponential dichotomy in the sense of the definition above. The projection $P \equiv P_n$ in this context is defined by the spectral projection onto the eigenspace of $L_{\eta,\infty}$ associated to all unstable eigenvalues, see Kato [Kat95] for a definition. In this case, no eigenvalues of $L_{\eta,\infty}$ lie on the unit circle and all stable eigenvalues have at least some positive (η-dependent) distance to the unit circle. Note that

$$L_{\eta,\infty}\big|_{\mathrm{Rg}(id-P)} : \mathrm{Rg}(id - P) \to X_e$$

is a bounded operator (where $\mathrm{Rg}(id - P)$ is regarded as a subspace of X_e), since the norms $\|\cdot\|_D$ and $\|\cdot\|_{X_e}$ are equivalent on $\mathrm{Rg}(id - P)$. Hence, the spectral radius of $L_{\eta,\infty}(id - P)$ is strictly less than one and we conclude

$$\|(L_{\eta,\infty}(id - P))^n\|_{L(X_e,X_e)} < Cr^n, \qquad n > 0$$

for some $0 < r < 1$ and an appropriate $C > 0$. Similarly, we have

$$\|(L_{\eta,\infty} \circ P)^{-n}\|_{L(X_e,X_e)} < Cr^n, \qquad n > 0$$

and hence the operator $L_{\eta,\infty}$ induces an exponential dichotomy. By theorem 7.6.9 of [Hen81] the following cases can occur.

Theorem 3 (Henry)
If the operator $L_{\eta,\infty} : D \subset X_e \to X_e$ admits an exponential dichotomy, then one of the following conditions is satisfied:

I) *Either the family $\{L_n^\eta\}$ possesses an exponential dichotomy or*

II) *there exists a nontrivial bounded solution $\{V_n\} \in l^\infty(\mathbb{Z}, D)$ of the equation $V_{n+1} = L_n^\eta V_n$.*

We make the following generic assumption, which excludes possibility II).

Hypothesis 2
The linear equation $V_{n+1} = L_n V_n$ does not possess a nontrivial bounded solution $\{V_n\} \in l^\infty(\mathbb{Z}, D)$, which decays exponentially for at least one asymptotic direction $n \to \infty$ or $n \to -\infty$.

We will comment on the validation of this hypothesis separately in section 8. If hypothesis 2 is satisfied, however, we have the following corollary which essentially follows from the results in [Hen81]:

Corollary 1 (Center-stable dichotomy)
Assume that hypothesis 2 is true. Then there exists a number $0 < r < 1$, a family of projections $Q_n^{cs} \in \mathcal{L}(X_e, X_e)$, $Q_n^u := id_{X_e} - Q_n^{cs}$, such that for any $\varepsilon > 1$ there is a constant $M > 0$ and:

a) *The range of $L_n \circ Q_n^u : D \subset X_e \to X_e$ is contained in $Rg(Q_{n+1}^u)$ for all n.*

b) *The densely defined, closed operator $L_n\big|_{\mathrm{Rg}(Q_n^u)}$ is bijective as a map from $Rg(Q_n^u)$ to $Rg(Q_{n+1}^u)$ and possesses a bounded inverse.*

c) *If $L_{n,m} = L_{n-1} \circ \ldots \circ L_{m+1} \circ L_m$ for $n > m$, then*

$$\|L_{n,m}(id - Q_m^u)\|_{\mathcal{L}(X_e,X_e)} < M(1+\varepsilon)^{n-m}, \qquad n > m.$$

In particular, $L_{n,m}(id - Q_m^u) : X_e \to X_e$ is a bounded operator.

d) *$\|L_{n,m} Q_m^u\|_{\mathcal{L}(X_e,X_e)} < M r^{m-n}$ if $n < m$, where $L_{n,m} Q_m^u x = y \in Rg(Q_n^u)$ if and only if $Q_m^u x = L_{m,n} y$, which is well-defined by b).*

Proof
Since hypothesis 2 is true, the condition II) of theorem 3 is satisfied for the family $\{L_n^\eta\}$. Hence, the family $\{L_n^\eta\}_{n\in\mathbb{Z}}$ possesses an exponential dichotomy and using the relation $L_n^\eta = \eta \cdot L_n$ it is now straightforward to conclude the claims of corollary 1. □

Analogously, we also obtain a center-unstable dichotomy on $\mathbb{Z}$, i.e. there exist a family of projections $\{P_n^{cu}\}_{n\in\mathbb{Z}}$, $P_n^s := id - P_n^{cu}$ and constants $M > 0$, $1 > \beta > 0$ with the following properties: Fix an $\varepsilon > 0$; then

- The densely defined, closed operator $L_n\big|_{\mathrm{Rg}(P_n^{cu})}$ is bijective as a map from $\mathrm{Rg}(P_n^{cu})$ to $\mathrm{Rg}(P_{n+1}^{cu})$ and possesses a bounded inverse.

- If $L_{n,m} = L_{n-1} \circ \ldots \circ L_{m+1} \circ L_m$ for $n > m \geq 0$, then

$$\|L_{n,m} \circ P_m^s\|_{\mathcal{L}(X_e,X_e)} < M\beta^{n-m}, \qquad n > m \geq 0.$$

  In particular, $L_{n,m} \circ P_m^s : X_e \to X_e$ is a bounded operator.

- $\|L_{n,m}(P_n^{cu})\|_{\mathcal{L}(X_e,X_e)} < M(1+\varepsilon)^{m-n}$ if $0 \leq n < m$, where $L_{n,m}P_m^{cu}x = y \in \mathrm{Rg}(P_n^{cu})$ if and only if $P_n^{cu}x = L_{m,n}y$.

5.2 Invariant manifolds near the homoclinic orbit

We now want to prove the existence of a center-stable manifold W^{cs} and a center-unstable manifold W^{cu} near a point of the homoclinic solution $\{H_n\}$. Contrary to the previous section we will only consider variations of $\beta = \mathcal{W}''(0)$ in this chapter while thinking of the frequency as fixed (the reason for this will be explained below (24) in the proof of the next theorem). We can now state the main result:

Theorem 4 (Center-stable manifold)
Assume that hypothesis 2 is satisfied and fix some frequency $\tilde{\omega}$ close to some β_0 with $\beta_0 > 2/\sqrt{3}$. Then there exists a neighborhood Ω of the point H_0 in X_e, a neighborhood Λ of $(0, \beta_0)$ in $E^{cs} \times \mathbb{R}$, where $E^{cs} := Rg(Q_0^{cs})$, and a differentiable map $\psi \in C_b^3(\Lambda, E^u)$, where $E^u := Rg(Q_0^u)$ with $\psi(0,\beta) = 0$, $D\psi(0,\beta_0) = 0$. Moreover, for all $\beta \approx \beta_0$ the manifold

$$W_\beta^{cs} = \{H_0 + V + \psi(V,\beta) : (V,\beta) \in \Lambda\}$$

has the following properties.

A) W_β^{cs} is locally invariant under $F(\cdot, \tilde{w}, \beta)$, i.e. if $V \in W_\beta^{cs}$ and $F(V, \tilde{w}, \beta) \in \Omega$ (which is well-defined) then $F(V, \tilde{w}, \beta) \in W_\beta^{cs}$.

B) If $\{V_n\}$ is a solution of (11) such that $\|V_n - H_n\|_{X_e} \leq \varepsilon$ for all $n \geq 0$ and $\varepsilon > 0$ is small enough, then $V_0 \in W_\beta^{cs}$.

C) Let $\gamma, \tilde{\varepsilon} > 0$ be positive constants and assume that all solutions of (11) with initial value on the center manifold $\mathcal{M}_\mu$, $\mu = \omega^2 - \beta^2$, within a suitable small neighborhood remain in this neighborhood of zero for all $n \in \mathbb{Z}$. Then there exists a $\delta > 0$ and $0 < \eta < 1$ such that if $Y \in W_\beta^{cs}$ with $\|Y - H_0\|_{X_e} \leq \delta$, then there exists a solution $\{Y_n\}, n \geq 0$, of (11) with $Y_0 = Y$, $\sup_{n\geq 0} \|\eta^n(Y_n - H_n)\|_X \leq \gamma$. Furthermore, there exists a $n_* \in \mathbb{N}$, a $0 < \alpha < 1$ and a unique solution $\{S_n\}, n \geq 0$, on the center manifold $\mathcal{M}_\mu$ with $S_n \in B_{\tilde{\varepsilon}}(0)$ for all $n \geq 0$, such that

$$\sup_{n \geq n_*} \alpha^{-n} \|Y_n + S_n\|_{X_e} < \tilde{\varepsilon}.$$

D) Consider the map $l^\beta : W_\beta^{cs} \to X_e$ defined by

$$l^\beta(Y) \mapsto S_0,$$

where the sequence $\{S_n\}, n \geq 0$, has been defined in C). Then l^β is continuous.

Remark
a) The value $0 < \alpha < 1$ is defined by the spectral gap of the operator $\tilde{L}$. Hence, α in case C) in the upper theorem cannot be chosen too close to one, since otherwise the sequence $\{S_n\}, n \geq 0$, with the property as in C) is not uniquely defined.
b) In fact, the theorem is valid for any $\tilde{\omega} > 0$. In particular, if $\tilde{\omega}$ is such that the center eigenspace is trivial, the manifold W^{cs} coincides with the strong stable manifold W^s and all solutions approach zero in forward time with exponential rate; hence $S_n^{\pm} = 0$

Proof
Let $\{Y_n\} = \{(u_n, v_n)\}$ be a solution of (11) and let us consider the coordinates $Z_n = Y_n - H_n$. Then, Z_n solves the equation

$$Z_{n+1} = L_n Z_n + \tilde{\mathcal{H}}(Z_n, n, \beta), \tag{23}$$

where $\tilde{\mathcal{H}}(Z, n, \beta) := F(Z + H_n, \tilde{w}, \beta) - L_n Z - F(H_n, \tilde{w}, \beta)$ and F has been defined in (11). Let us make the important observation that we can regard $\tilde{H}(\cdot, n, \beta)$ as a map from X_e to X_e; in particular $\tilde{H}(Z, n, \beta)$ is well defined even for $Z \in X_e$. In fact $\tilde{H}$ can be written in the form:

$$\tilde{H}(Z, n, \beta) = -C_n Z + \mathcal{G}(Z + H_n, \beta) - \mathcal{G}(H_n, \beta) \tag{24}$$

for $L_n = \tilde{L} + C_n$, where $\tilde{L}$ is given in (22), and $\mathcal{G}(Z, \beta) := F(Z, \tilde{w}, \beta) - \tilde{L}Z$ can be regarded as a C^3-map from X_e to X_e. Note that here we use the fact that we only allow for variations in β for fixed $\tilde{w}$!
By corollary 1 the family $\{L_n\}$ admits a center-stable dichotomy with associated projections Q_n^{cs}. Let us choose a suitable cut-off function $\chi_\rho : [-\rho, \rho] \to \mathbb{R}_+$ with compact support in $[-\rho, \rho]$. We are now looking for fixed points $\{Z_n\}$, where $n \geq 0$, $Z_n \in X_e$, of the equation

$$\begin{aligned} Z_n &= L_{n,0}(id - Q_0^u) Z^{cs} + \sum_{k=0}^{n} L_{n,k+1}(id - Q_{k+1}^u) \tilde{\mathcal{H}}_{mod}(Z_k, k, \beta) \\ &- \sum_{k=n+1}^{\infty} L_{k+1,n} Q_{k+1}^u \tilde{\mathcal{H}}_{mod}(Z_k, k, \beta) \end{aligned} \tag{25}$$

for $Z^{cs} \in E^{cs}$. Note that equation (25) is well-defined if we look for fixed-points in the space $\{V_n\} \in l^{\infty,\eta}(\mathbb{N})$, where

$$l^{\infty,\eta}(\mathbb{N}) := \{\{V_n\} : n \geq 0, V_n \in X_e, \sup_{n \geq 0} \|\eta^n V_n\|_X < \infty\}$$

(for some suitable $0 < \eta < 1$) and any fixed point induces a solution of (23) (with $\tilde{\mathcal{H}}$ replaced by $\tilde{\mathcal{H}}_{mod}$). Moreover,

$$\tilde{\mathcal{H}}_{mod}((a, b), k, \beta) := \chi_\rho(\|b\|_{L^2}) \cdot \tilde{\mathcal{H}}((a, b), k, \beta)$$

for $(a, b) \in X_e$ Choosing $\varepsilon > 0$ small enough and $0 < \eta < \frac{1}{1+\varepsilon}$ we can now prove the existence of a unique fixed point $\{Z_n^*\} \in l^{\infty,\eta}$ for every $V^{cs} \in E^{cs}$ and fixed β. Hence, we can define a map $\psi : E^{cs} \times \mathbb{R} \to E^u$ by

$$\psi(Z^{cs}, \beta) := Q_0^u [Z_0^*]$$

and set $W_\beta^{cs} := \mathrm{graph}(\psi) + H_0$. By the results of [VI92] the claims A) and B) of the theorem now follow. Indeed, also the parameter dependence of the map ψ can be deduced analogously as in [VI92] or as in [Jam03].
Let us now show how one proves claim C). The proof follows along the lines of [Van89], where the case of ordinary differential equations has been addressed. Since we want to study the asymptotic behavior of solutions $\{Y_n\}, n \geq 0$, with initial value in W^{cs}, we have to work with a (not yet constructed) local center-stable manifold $\mathcal{M}_{loc}^{cs}$ near the fix point 0. Let us therefore consider a solution $\{Y_n\} = \{Z_n\} + \{H_n\}$, where $\{Z_n\}$ solves (25). Then Y_n solves for $n \geq 0$

$$\begin{aligned} Y_{n+1} &= \tilde{L}Y_n + [C_n Y_n - C_n H_n \\ &+ \mathcal{G}(H_n, \beta) + \tilde{H}_{mod}(Y_n - H_n, n, \beta)] \\ &=: \tilde{L}Y_n + \mathcal{C}(Y_n, n, \beta). \end{aligned} \tag{26}$$

One can now check that $\mathcal{C}(0, n, \beta) = 0$ and $\mathcal{C}(\cdot, n, \beta) : X_e \to X_e$ is well-defined and has a uniform small Lipschitz constant if $n \geq n_*$ and $n_* >> 0$ is sufficiently large. Let us now choose a new cut-off-function $\tilde{\chi}$ by

$$\tilde{\chi}((a,b), n) := \begin{cases} 1 & n \geq n_*, \\ \chi_\rho(\|b\|_{L^2}) & n < n_* \end{cases} \tag{27}$$

for $(a, b) \in X_e$ and consider the modified equation

$$Y_{n+1} = L_* Y_n + \tilde{\chi}(Y_n, n) \cdot \mathcal{C}(Y_n, n, \beta) =: L_* Y_n + \mathcal{C}^{mod}(Y_n, n, \beta). \tag{28}$$

Note that for $n \geq n_*$ the modified nonlinearity $\mathcal{C}^{mod}$ coincides with the original one. As above, we can now prove that (28) possesses a local center stable manifold $\mathcal{M}_{loc}^{cs}$ near 0. More precisely, $\mathcal{M}_{loc}^{cs}$ has the property that for every $Y \in \mathcal{M}_{loc}^{cs}$ there exists a solution $\{Y_n\}$ of (28) for $n \geq n_*$ which is bounded in a space $l_{n_*}^{\infty,\eta} := \{\{Y_n\} : n \geq n_*, Y_n \in X_e \sup_{n \geq n_*} \eta^n \|Y_n\|_X < \infty\}$ for some $0 < \eta < 1$. Moreover, Y_n can be written in the form $Y_n = V_n + \phi(V_n, \beta)$ for some map $\phi(\cdot, \beta) : \mathrm{Rg}(\pi^{cs}) \to \mathrm{Rg}(id - \pi^{cs})$, $V_n \in \mathrm{Rg}(\pi^{cs})$ and $\{V_n\}$ solves the recurrence relation

$$V_{n+1} = L_* \pi^{cs} V_n + \pi^{cs} \mathcal{C}^{mod}(V_n + \phi(V_n, \beta), n, \beta) =: \mathcal{K}(V_n, n, \beta) \tag{29}$$

for $n \geq n_*$, where $\pi^{cs} : X_e \to X_e$ denotes the spectral projection associated to the center-stable eigenspace of $\tilde{L}$. Let us denote by $W_{loc}^c \subset X_e$ the center manifold associated to equation (29). In order to prove claim C) it suffices to show that every solution $\{V_n\}$ of (29) for $n \geq n_*$ approaches a solution $\{S_n\}$ on the manifold W_{loc}^c, if V_0 is close enough to zero. More precisely, we show that if V_{n_*} is sufficiently close to zero, then there exists a globally small solution $\{S_n\}_{n \in \mathbb{Z}}, S_n \in W_{loc}^c$ for all n, and an exponentially decaying sequence $\{T_n\}, n \geq n_*$, with a sufficiently large exponential rate, such that

$$V_n = S_n + T_n, \qquad n \geq n_*.$$

In order to prove this we need the next lemma, see [Van89], which we reformulate in terms of equation (29).

Lemma
Let $\Gamma : \mathbb{Z} \times Rg(\pi^{cs}) \to Rg(\pi^{cs})$ be a map with the following properties.

i) $\Gamma(n, V) = V_n$ *for all* $n \geq n_*$, *where* $\{V_n\}$ *denotes the solution of (29),* $V_{n_*} = V$, *with* $\{V_n\} \in l^{\infty,\eta}_{n_*}$.

ii) *If* $n < n_*$ *then* $\Gamma(\cdot, V) \in l^{\infty,\eta}_{n_*,-} := \{\{U_n\} : n < n_*, \sup_{n<n_*} |\eta^{-n} U_n|_X < \infty\}$ *for some* $0 < \eta < 1$.

Let now $V \in Rg(\pi^{cs})$. *If there exists* $\{T_n\}_{n\in\mathbb{Z}}$ *with* $\sup_{n\in\mathbb{Z}} \alpha^{-n}|T_n|_X < \infty$ *for some* $0 < \alpha < 1$ *such that* $\Gamma(n, V) + T_n$ *is a solution of (29), then there exists a unique solution* $\{S_n\}$ *of (29) in* W^c_{loc} *with*

$$\sup_{n\geq n_*} \alpha^{-n}\|S_n - V_n\|_{X_e} < \infty.$$

The proof of the lemma and the existence of a map Γ satisfying the properties above follow analogously to [Van89]. For example, we can define $\Gamma(V, n)$ in backward direction $n \leq n_*$ to be the solution of the recurrence relation $U_{n+1} = \pi_c\mathcal{K}(U_n, n, \beta)$, subject to the initial value $U_{n_*} = V$; that is $\Gamma(V, n_*) = U_{n_*} = V$. We now have to construct a sequence $\{T_n\}_{n\in\mathbb{Z}}$ with $\sup_{n\in\mathbb{Z}} \alpha^{-n}|T_n|_{X_e} < \infty$ for some suitable $0 < \alpha < 1$, such that $\{\Gamma(n, V) + T_n\}$ is a solution of (29). For given $V \in \mathrm{Rg}(\pi^{cs})$ we consider the fixed point formulation

$$\begin{aligned} T_n &= -\pi^s\Gamma(n, V) + \sum_{k=-\infty}^{n} L_*^{n-k}\pi^s[\mathcal{C}^{mod}(k-1, \Gamma(k-1, V) + T_{k-1})] \\ &- \sum_{k=n+1}^{\infty} L_*^{n-k}\pi^c[\mathcal{C}^{mod}(k-1, \Gamma(k-1, V) + T_{k-1})] \\ &+ \sum_{k=n+1}^{\infty} L_*^{n-k}\pi^c[\mathcal{C}^{mod}(k-1, \Gamma(k-1, V))]. \end{aligned} \tag{30}$$

Here, π^s, π^c denote the projections onto the stable and center eigenspace of $\tilde{L}$, respectively, and we look for a fixed point of (30) in the space of all $\{T_n\}_{n\in\mathbb{Z}}$ satisfying $\sup_{n\in\mathbb{Z}} \alpha^{-n}\|T_n\|_X < \infty$ for some suitable $0 < \alpha < 1$. Moreover, if $\{T_n\}$ is a solution of (30) it can be easily verified that $\{\Gamma(n, V) + T_n\}$ is a solution of (29). One can now check that the right hand side of (30) defines a contraction in the space of all $\{T_n\}$ satisfying $\sup_n \alpha^{-n}|T_n|_X < \infty$ for some suitable $0 < \alpha < 1$; for details we refer to [VI92]. This completes our proof of case C) and case D) can also be deduced from the above integral representation, see again [VI92].

For the sake of clarity, let us finally put things together: Given a point Y_0 in the center-stable manifold W^{cs}_β sufficiently close to H_0 the associated solution $\{Y_n\}$ has the property that Y_{n_*} lies in a small neighborhood of zero if $n_* > 0$ is large enough, and $\{Y_n\}, n \geq n_*$, satisfies the recurrence relation (28). Hence, Y_n is contained in the *local* center-stable manifold $\mathcal{M}^{cs}_{loc}$ for all $n \geq n_*$ and therefore $Y_n = V_n + \phi(V_n, \beta)$, where V_n solves (29) and V_{n_*} is close to zero. We have now proved that $\{V_n\}, n \geq n_*$, approaches a solution on the center manifold W^c_{loc} in the sense of the lemma above. Note that all globally small solutions of the original center manifold $\mathcal{M}_\mu$ also induce globally small solutions in the manifold W^c_{loc} associated to (29). This shows that Y_n is *uniformly* small for all $n \geq n_*$ and therefore $\{Y_n\}, n \geq 0$, solves the equation (28) with cut-off function identical to one. Hence, every point in W^{cs} induces a solution $\{Y_n\}, n \geq 0$, of (11) which stays near the solution $\{H_n\}$ for *all* $n \geq 0$ and which approaches a globally small solution on the center manifold $\mathcal{M}_\mu$, $\mu = \tilde{\omega}^2 - \beta^2$. Note that we assumed initially that all solutions within $\mathcal{M}_\mu$ starting in a sufficiently small neighborhood of zero remain close to zero for all $n \geq 0$. This proves the theorem. $\square$

Similarly, we can prove the existence of an unstable manifold W^u near H_0.

Theorem 5 (Unstable manifold)
Let $\tilde{\omega}$, β_0 be like in the statement of theorem 4. There exists a neighborhood Ω of 0 in X_e and a neighborhood Λ of (H_0, β_0) in $E^u \times \mathbb{R}$ and a differentiable map $\psi \in C^3_b(\Lambda, Rg(Q^{cs}_0))$ with $\psi(0, \beta) = 0$, $D\psi(0, \beta_0) = 0$. Moreover, for all $\beta \approx \beta_0$ the manifold

$$W^u_\beta = \{H_0 + V + \psi(V, \beta) : (V, \beta) \in \Lambda\}$$

has the following properties.

A) *There exists an $0 < \alpha < 1$ such that if $\{Y_n\}$ is a solution of (11) for $n \leq 0$ and $\sup_{n \leq 0} \alpha^n \|Y_n - H_n\|_{X_e} < \varepsilon$ for all $n \leq 0$ and $\varepsilon > 0$ is small enough, then $Y_0 \in W^u_\beta$.*

B) *On the other hand, if $\tilde{Y}_0 \in W^u_\beta$ then there exists a solution $\{Y_n\}, n \leq 0$, with $Y_0 = \tilde{Y}_0$ which converges to zero with exponential rate almost α as $n \to -\infty$. Hence $(\alpha + \varepsilon)^n Y_n \to 0$ for any $\varepsilon > 0$ small enough and $n \to -\infty$.*

C) *W^u_β is invariant, i.e. if $\tilde{Y}_0 \in W^u_\beta$ and $\{Y_n\}, n \leq 0$, denotes the corresponding solution then $F(Y_n, \tilde{w}, \beta) \in W^u_\beta$ for all $n \leq -1$.*

Finally, let us denote by W^{cu}_β and W^s_β the center-unstable and stable manifold of zero, respectively. Their existence can be proved analogously to the existence of W^{cs}_β and W^u_β, respectively, by using the center-unstable dichotomy with projections $\{P^{cu}_n\}^\infty_{n=-\infty}$, see the end of the last section.

Remark
Let us note that under the upper assumptions the manifolds satisfy the relation $R[W^{cs}_\beta] \subset W^{cu}_\beta$. Indeed, fixing a point $Y \in W^{cs}$ there exists a global solution $\{Y_n\}, n \geq 0$, of (11) near $\{H_n\}$, such that the difference $H_n - Y_n$ stays uniformly small for $n \geq 0$. Hence, if we define $\tilde{Y}_n := RY_{-n}$ for $n \leq 0$ then $\{\tilde{Y}_n\}$ solves (11) for $n \leq 0$, stays uniformly close to $H_n = RH_{-n}$ for $n \leq 0$ and therefore $\tilde{Y}_0 = RY_0 \in W^{cu}_\beta$. Indeed, any solution in W^{cs} whose initial value is close enough to H_0 already solves the original equation (11) for $n \geq 0$ exactly (i.e. without cut-off function).
However, in the case $\mu = 0$, that is $\beta = \tilde{\omega}$, we have not clarified the situation on the center manifold. In particular, we have not shown that solutions with initial value in a sufficiently small neighborhood of zero actually stay within

this neighborhood for all n (which is true for small $\mu > 0$ by lemma 1, since solutions of sufficiently small initial data are confined by the stable manifold of the two nontrivial, symmetric steady states). But let us observe that due to our special choice of cut-off-function, see (27), the modified recurrence relation still respects the reversibility (see [Jam03] for more details). Hence, we still have the property that $R[W^{cs}_\beta] \subset W^{cu}_\beta$ even for β and $\tilde{\omega}$ chosen such that $\mu = 0$.

5.3 Relative positions of the invariant manifolds with respect to Fix(R)

Let us now choose some $\beta_0 > 2/\sqrt{3}$ and set $\tilde{\omega} := \beta_0$. We now want to clarify the relative positions of the manifolds $W^{cu/cs}_\beta$, $W^{s/u}_\beta$ and Fix(R) for the specific choice $\beta = \beta_0$ and hence $\mu = 0$. Our hypothesis 2 states that the tangent spaces $T_{H_0}W^{cs}$ and $T_{H_0}W^u$ have trivial intersection; i.e.

$$T_{H_0}W^{cs} \cap T_{H_0}W^u = \{0\}. \tag{31}$$

Indeed, any nontrivial element in the intersection induces a nontrivial solution $\{V_n\}$ of $V_{n+1} = L_n V_n$ with $\{\eta^{-n}V_n\} \in l^\infty(\mathbb{Z}, D)$ for some $\eta > 1$ close to one, which contradicts hypothesis 2. Counting dimensions, we conclude that

a) $T_{H_0}W^{cs} + T_{H_0}W^u = X_e$ and

b) $T_{H_0}W^{cs} + T_{H_0}W^{cu} = X_e$, $\qquad \dim(T_{H_0}W^{cs} \cap T_{H_0}W^{cu}) = 2.$

In fact, let us observe first $T_{H_0}W^{cs} \cap T_{H_0}W^{cu}$ coincides with the space $\mathcal{V}_1$; in particular $\dim(T_{H_0}W^{cs} \cap T_{H_0}W^{cu}) = 2$. We also would like to stress that the existence of a first integral in general imposes further restrictions on the relative positions of W^{cs} and W^{cu} for $\mu = 0$. Next, let us clarify the relative position between Fix(R) and W^{cs} for $\beta = \tilde{\omega}$:

Lemma 2
If hypothesis 2 is satisfied then

i) $Fix(R) + T_{H_0}W^{cs} = X_e$ *and*

ii) $dim(Fix(R) \cap T_{H_0}W^{cs}) = 1$.

Note that if the first condition $i)$, i.e. Fix$(R) + T_{H_0}W^{cs} = X_e$, is satisfied then also the second condition $ii)$ is true: The one-dimensional intersection of Fix(R) and $T_{H_0}W^{cs}$ lies in the space $\mathcal{V}_1$ and coincides with the linear span of $\langle(\cos(\cdot), \cos(\cdot))\rangle$. In all other subspaces $\mathcal{V}_k$, $k \neq 1$, the spaces Fix(R) and $T_{H_0}W^{cs}$ then have trivial intersection; otherwise, the first condition in lemma 2 would be violated. Similarly one can show that $i)$ can be deduced from $ii)$.

Proof of the lemma
Let us assume that hypothesis 2 is true. Then, in the invariant subspace $\mathcal{V}_1$ we have $\dim(\text{Fix}(R) \cap T_{H_0}W^{cs}) = 1$. In any other $\mathcal{V}_k$, $k \neq 1$, it is true

that $\mathcal{V}_k \cap T_{H_0}W^{cs} = \mathcal{V}_k \cap T_{H_0}W^s$. Hence, any nontrivial intersection point in $\mathcal{V}_k \cap T_{H_0}W^{cs}$ leads to a nontrivial intersection point in

$$T_{H_0}W^s \cap T_{H_0}W^u$$

on account of $R[T_{H_0}W^s] \subset T_{H_0}W^u$. This clearly would contradict the hypothesis 2. $\square$

5.4 The bifurcation map

Let us now study solutions which are induced by intersection points of W^{cs} and W^{cu}. We already know that $H_0 \in W^{cs} \cap W^u$ for $\beta = \beta_0 = \tilde{\omega}$ and we are looking for intersections $W^{cs}_\beta \cap W^{cu}_\beta$ and $W^{cs}_\beta \cap \mathrm{Fix}(R)$ near H_0 for $\beta \neq \tilde{\omega}$. In particular, intersection points in $W^{cs} \cap \mathrm{Fix}(R)$ give rise to symmetric homoclinic solutions to the center manifold. In order to obtain these solutions, let us consider the map

$$\begin{aligned} \Gamma(V^s, Z^u, V^c, Z^c, \beta) &= (V^s + V^c) - (Z^u + Z^c) \\ &+ \psi^{cs}(V^s, V^c, \beta) - \psi^{cu}(Z^u, Z^c, \beta), \end{aligned}$$

where $V^s \in E^s$, $Z^u \in E^u$ and $\psi^{cs}(V^s, V^c, \beta) \in \mathrm{Rg}(Q^u_0)$. Moreover, we let $V^c = (V^c_+, V^c_-)$, where $V^c_+ \in \mathrm{Fix}(R) \cap \mathcal{V}_1$ and $V^c_- \in \mathrm{Fix}(-R) \cap \mathcal{V}_1$. Similarly, we write Z^c in the form $Z^c = (Z^c_+, Z^c_-)$. The set

$$\{H_0 + V^c + V^s + \psi^{cs}(V^s, V^c, \beta) : V^c \in \mathcal{V}_1, V^s \in E^s\}$$

denotes the center-stable manifold W^{cs} and $\{H_0 + Z^c + Z^u + \psi^{cs}(Z^u, Z^c, \beta)\}$ denotes the center-unstable manifold W^{cu}. Let us consider Γ as a map, with

$$\Gamma : E^s \times E^u \times \mathcal{V}_1 \times \mathcal{V}_1 \times (\mathbb{R} \cap B_\varepsilon(\beta_0)) \longrightarrow E^u \oplus E^s \oplus \mathcal{V}_1$$

for some $\varepsilon > 0$ small enough such that ψ^{cs}, ψ^{cu} are well-defined. Note that $\mathcal{V}_1$ is a two-dimensional complement of the codimension-2-space $E^u \oplus E^s = \oplus_{k\neq 1}\mathcal{V}_k$ and any zero of Γ induces an intersection point of W^{cs}_β and W^{cu}_β. In fact, under the hypotheses 1, 2 of the previous sections we immediately conclude the existence of nontrivial zeros of Γ and furthermore obtain precise informations concerning the asymptotic behavior of the induced solutions. We start with the following lemma.

Lemma 3
Let hypothesis 1 and 2 be satisfied. Then there exists an $\gamma > 0$ such that the following properties hold:

a) *For each $\beta_0 > \frac{2}{\sqrt{3}}$, $0 < \beta_0 - \beta < \gamma$, and frequency $\tilde{\omega} = \beta_0$ there exists a one-parameter family $(H^{sym,\beta,\kappa}_n)_n$, $\kappa \in B_\gamma(0) \subset \mathbb{R}$, of symmetric homoclinic solutions to the center manifold. More precisely,*

$$H^{sym,\beta,\kappa}_0 \in \mathit{Fix}(R) \cap W^{cs}_\beta$$

for all $|\kappa|$ small.

b) Moreover, for $\beta_0, \beta, \tilde{\omega}$ as above and $\varepsilon > 0$ small enough there exists a two-parameter family $(\tilde{H}_n^{\beta,\lambda})_n$, $\lambda \in (\mathbb{R}^2 \cap B_\varepsilon(0))$, of homoclinic solutions to the center manifold, i.e.

$$\tilde{H}_0^{\beta,\lambda} \in W_\beta^{cs} \cap W_\beta^{cu}.$$

These solutions satisfy $\tilde{H}_0^{\beta,\lambda} \in Fix(R)$ if and only if $\tilde{H}_0^{\beta,\lambda} = H_0^{sym,\beta,\kappa}$ for some κ and λ. Finally, for each $0 < \eta < 1$ the map

$$\begin{aligned}
\iota &: B_\gamma(0) \times B_\varepsilon(\beta_0) \longrightarrow \{\{X_n\}_{n\in\mathbb{Z}} : X_n \in X, \sup_n \eta^{|n|} \|X_n\|_X < \infty\} \\
\iota &: (\lambda, \beta) \longmapsto \{\tilde{H}_n^{\beta,\lambda}\}
\end{aligned}$$

is well-defined and continuously differentiable.

Proof

Let us study the zero-set of Γ near H_0. We want to point out the following symmetry, namely

$$\Gamma(RZ^u, RV^s, R(V_+^c, V_-^c), R(Z_+^c, Z_-^c), \beta) = -R\Gamma(V^s, Z^u, (V_+^c, -V_-^c), (Z_+^c, -Z_-^c), \beta) \tag{32}$$

which follows directly from the definition of the map Γ and the remark after theorem 5. In particular, (32) is true even in the special case $\beta = \beta_0 = \tilde{\omega}$. Moreover, we have

$$\Gamma(0, 0, 0, 0, \beta_0) = 0.$$

We now apply the implicit function theorem by observing that

$$\begin{aligned}
\mathrm{Rg}(D_{1,2}\Gamma(0,0,0,0,0)) &= E^u \oplus E^s, \\
\ker(D_{1,2}\Gamma(0,0,0,0,0)) &= \{0\},
\end{aligned}$$

where D_j, $j = 1, 2, 3, 4, 5$, denotes the derivative with respect to the j-th component of Γ. Hence, there exist C^2-maps $\tilde{V}^s, \tilde{Z}^u$, such that zeros of Γ near $(0, 0, 0, 0, \beta_0)$ are equivalent to zeros of $\tilde{\Gamma}$, where

$$\begin{aligned}
\tilde{\Gamma} &: \mathcal{V}_1 \times \mathcal{V}_1 \times \mathbb{R} \longrightarrow \mathcal{V}_1, \\
\tilde{\Gamma}(V^c, Z^c, \beta) &= \pi_{\mathcal{V}_1}\Gamma(\tilde{V}^s(V^c, Z^c, \beta), \tilde{Z}^u(V^c, Z^c, \beta), V^c, Z^c, \beta) \\
&= (V_+^c - Z_+^c) + (V_-^c - Z_-^c).
\end{aligned}$$

Here, $\pi_{\mathcal{V}_1} : X_e \to X_e$ denotes the projection with range $\mathcal{V}_1$ and kernel $\oplus_{j\neq 1}\mathcal{V}_j$. On account of (32) the functions $\tilde{V}^{s/u}$ satisfy the relations $R\tilde{V}^s(V^c, Z^c) = \tilde{V}^u(RV^c, RZ^c)$ and $R\tilde{V}^u(V^c, Z^c) = \tilde{V}^s(RV^c, RZ^c)$ which can be concluded by the fact that the solutions $\tilde{V}^s, \tilde{Z}^u$ obtained by the implicit function theorem are unique. Moreover, $\tilde{\Gamma} = 0$ can be solved if $V_+^c = Z_+^c$ and $V_-^c = Z_-^c$, which induces a two-parameter family of intersection points $H^{V_+^c, V_-^c}$ in $W_\beta^{cs} \cap W_\beta^{cu}$ given by

$$\begin{aligned}
H^{V_+^c, V_-^c} &= V_+^c - V_-^c + \tilde{V}^s(V_+^c, V_-^c, \beta) + \psi^{cs}(\tilde{V}^s(V_+^c, V_-^c), V_+^c, V_-^c, \beta) \\
&= V_+^c - V_-^c + \tilde{V}^u(V_+^c, V_-^c, \beta) + \psi^{cu}(\tilde{V}^u(V_+^c, V_-^c), V_+^c, V_-^c, \beta).
\end{aligned}$$

Now fix some $\beta \approx \beta_0$. Then exactly the points $H^{V_+^c,0}$ with $V_-^c = 0$ induce symmetric solutions to the center-manifold, i.e. correspond to intersections of W_β^{cs} and $\mathrm{Fix}(R)$ near H_0. In order to prove this claim, we have to show that

$$H^{V_+^c,0} = V_+^c + \tilde{V}^s(V_+^c, 0, \beta) + \psi^{cs}(\tilde{V}^s(V_+^c, 0), V_+^c, 0, \beta) \in \mathrm{Fix}(R).$$

Indeed, we calculate

$$\begin{aligned}
RH^{V_+^c,0} &= R\left(V_+^c + \tilde{V}^s(V_+^c, 0, \beta) + \psi^{cs}(\tilde{V}^s(V_+^c, 0), V_+^c, 0, \beta)\right) \\
&= RV_+^c + \tilde{V}^u(RV_+^c, 0, \beta) + \psi^{cu}(R\tilde{V}^s(V_+^c, 0, \beta), RV_+^c, 0, \beta) \\
&= RV_+^c + \tilde{V}^u(RV_+^c, 0, \beta) + \psi^{cu}(\tilde{V}^u(RV_+^c, 0, \beta), RV_+^c, 0, \beta) \\
&= V_+^c + \tilde{V}^u(V_+^c, 0, \beta) + \psi^{cu}(\tilde{V}^u(V_+^c, 0, \beta), V_+^c, 0, \beta) = H^{V_+^c,0}.
\end{aligned}$$

Similarly, we observe that $H^{V_+^c,V_-^c}$ does not lie in $\mathrm{Fix}(R)$, if $V_-^c \neq 0$. Now writing $\lambda = (\lambda_1, \lambda_2)$ instead of (V_+^c, V_-^c) the claims of the lemma follow except the claim concerning the smoothness of ι. To see the latter, note the functions $H^{V_+^c,V_-^c}$ depend C^1 on V_+^c, V_-^c, β on account of the implicit function theorem. Moreover, also the map

$$(V_+^c, V_-^c, \beta) \mapsto H^{V_+^c,V_-^c} \mapsto \left\{H_n^{V_+^c,V_-^c}\right\}, \tag{33}$$

is C^1, where here the last term, regarded as an element in $\{\{X_n\}_{n\in\mathbb{Z}} : X_n \in X, \sup_n \eta^{|n|}\|X_n\|_X < \infty\}$ for any fixed $0 < \eta < 1$, denotes the associated global solution to the initial value $H^{V_+^c,V_-^c}$. In fact, the claim concerning C^1-smoothness of the map in (33) is true on account of the results in Vanderbauwhede [Vl92]. $\square$

6 The main result

We can now state the main results of this work. Let us recall that we call a solution $\{Y_n\}$ of (11) symmetric if $Y_n \in \mathrm{Fix}(R)$ for some n.

Theorem 6
Assume that hypothesis 1 and 2 are satisfied. Choose some $\beta_0 > 2/\sqrt{3}$ and make the choice $\tilde{\omega} = \beta_0$. Then for each $\beta < \beta_0$, $\beta \approx \beta_0$, there exists a two-parameter family

$$\{H_n^{\beta,\lambda}\} = \{(h_n^{\beta,\lambda}, h_{n+1}^{\beta,\lambda})\}, \quad n \in \mathbb{Z}$$

of globally defined solutions of equation (11) for $\lambda \in (\mathbb{R}^2 \cap B_\varepsilon(0))$ (for some sufficiently small $\varepsilon > 0$ with $\varepsilon \searrow 0$ as $\beta \nearrow \beta_0$). The solutions satisfy

$$\|(H_n^{\beta,\lambda})_n - (H_n)_n\|_{l_{loc}^\infty(\mathbb{Z},X_e)} \to 0, \qquad \text{as} \quad (\lambda, \beta) \to (0, \beta_0).$$

For each $\lambda \approx 0$ and $\beta < \beta_0$ (and hence $\tilde{\omega} > \beta$), the solution $\{H_n^{\beta,\lambda}\}$ approaches a solution on the center manifold in forward and backward direction $n \to \pm\infty$.

More precisely, there exist globally small solutions $\{S_n^{\pm,\beta}\}_{n\in\mathbb{Z}}$, $\|S_n^{\pm,\beta}\|_X > 0$ for all n, which are contained in $\mathcal{M}_\mu \cap B_\delta(0)$ for all $n \in \mathbb{Z}$, $\mu = \tilde{\omega}^2 - \beta^2$ and an appropriate $\delta = \delta(\mu)$. Moreover,

$$\begin{array}{rclcl} \alpha^n \|H_n^{\beta,\lambda} - S_n^{+,\beta}\|_{X_e} & \to & 0, & \quad as & n \to \infty, \\ \alpha^{-n} \|H_n^{\beta,\lambda} - S_n^{-,\beta}\|_{X_e} & \to & 0, & \quad as & n \to -\infty \end{array}$$

for some suitable $\alpha > 1$ if $\lambda = \lambda(\delta)$ is close enough to zero. Moreover, $H_0^{\beta,\lambda} \in Fix(R)$ if and only if $\lambda = (\kappa, 0)$, where $\kappa \in \mathbb{R}$ is sufficiently close to zero. In particular, there exists exactly a one-parameter family of symmetric homoclinic solutions to the center manifold in the family $\{H_n^{\beta,\lambda}\}$.

Proof
Let us point out that when $\beta < \tilde{\omega}$ and hence $\mu > 0$ is close enough to zero, then the dynamics on the center manifold $\mathcal{M}_\mu$ is clarified by lemma 1. In particular there exist two symmetric fixed points $\pm A^* \in \mathrm{Fix}(R)$ which are of the order $\mathcal{O}(|\mu|^{1/2})$ and two heteroclinic orbits joining them. Hence, any solution on the center manifold with sufficiently small initial value is globally contained in a small neighborhood of zero for all n. In particular, the assumptions in case C) of theorem 4 are satisfied. The claims of the upper theorem now follow by lemma 3, theorem 4 and the analogous results for the center unstable manifold.

Let us point out that we comment on the validation of hypothesis 2 separately in the last section. We now have the following result, which implies the existence of (possibly large amplitude) breather solutions of the Klein-Gordon lattice for frequencies ω near the lower edge β_0 of the phonon band $[\beta_0, \sqrt{(4+\beta_0^2)}]$.

Theorem 7
Assume that all the assumptions of the previous theorem are satisfied. Choose some $\beta_0 > 2/\sqrt{3}$ and consider the case $B > 0$, $\beta < \beta_0$ and $\omega := \beta_0$, where we recall that

$$B := \frac{\beta_0}{8}(\mathcal{W}^{(3)}(0) - \frac{5}{3}(\mathcal{W}^{(2)}(0))^2).$$

Then there exists a $\gamma = \gamma(\beta) > 0$, $\gamma \to 0$ as $\beta \nearrow \beta_0$, and a one-parameter family of solutions $\{h_n^{sym,\beta,\kappa}\}$ of (1), $|\kappa| < \gamma$, such that each $\{h_n^{sym,\beta,\kappa}\}$ is non trivially time-periodic with period $2\pi/\omega$ and has the following properties:

i) $\{h_n^{sym,\beta,0}\}$ is a time-independent breather solution to the steady state zero.

ii) The solutions $\{h_n^{sym,\beta,\kappa}\}$ asymptotically approach non trivially time-periodic solutions $\{s_n^{\pm,\beta}\}$ with exponential rate in spatial direction $n \to \pm\infty$. Moreover, $s_n^{\pm,\beta} \in H^2_{loc}(\mathbb{R})$ has period $2\pi/\omega$ satisfying $0 < |s_n^{\pm,\beta}| < r$ for all $n \in \mathbb{Z}$ and some sufficiently small $r > 0$. Finally,

$$s_n^{\pm,\beta}(t \cdot \omega) = \alpha_n^{\pm} \cos(t) + \phi(\alpha_n^{\pm}, \alpha_{n+1}^{\pm}, \mu)(t)$$

for some suitable $\alpha_n^{\pm} \in \mathbb{R}$, where the map ϕ has been defined in theorem 2 and $\mu = \omega^2 - \beta^2$. If we choose a $\kappa = \kappa(\beta)$ for each $\beta < \beta_0$ such that

$\kappa \to 0$ as $\beta \nearrow \beta_0$, then $\{h_n^{sym,\beta,\kappa}\}$ converges to $\{h_n\}$ with respect to the locally uniform convergence: That is, given $n_* \in \mathbb{N}$ arbitrarily, then

$$\sup_{|k|\le n_*} |h_k^{sym,\beta,\kappa} - h_k|_{H^2_{per}} \to 0$$

as $\beta \nearrow \beta_0$. Finally, $h_{n+1}^{sym,\beta,\kappa} = h_{-n}^{sym,\beta,\kappa}$ for all $n \in \mathbb{Z}$ and all $\kappa \approx 0$.

iii) For every $\beta < \beta_0$, $|\beta - \beta_0|$ small enough, there exists a $\kappa_+ \approx 0$, such that $h_{n+1}^{sym,\beta,\kappa_+} = h_{-n}^{sym,\beta,\kappa_+}$ for all $n \in \mathbb{Z}$ and

$$|\tilde{q}_n^1 - h_n^{sym,\beta,\kappa_+}|_{H^2_{per}} \to 0, \qquad n \to \infty, \tag{34}$$

where $\{(\tilde{q}_n^1, \tilde{q}_{n+1}^1)\}$ denotes a solution of the recurrence relation on the center manifold approaching A^* as $n \to \infty$ (see lemma 1 and the paragraph "Notation" afterward for the definition of A^*). In particular, A^* is non trivially time-periodic with period $2\pi/\omega$ and $\{h_n^{sym,\beta,\kappa_+}\}$ itself approaches the fix point $\alpha^* + \phi(\alpha^*, \alpha^*, \mu)$ (i.e. the first component of A^*) as $n \to \pm\infty$, where $\mu = \omega^2 - \beta^2$. Similarly,

$$|(\alpha^* + \phi(\alpha^*, \alpha^*, \mu) - h_n^{sym,\beta,\kappa_+}|_{H^2_{per}} \to 0, \qquad n \to -\infty. \tag{35}$$

Finally, there exists a solution $\{h_n^{sym,\beta,\kappa_-}\}$ with similar properties, such that h_n^{sym,β,κ_-} approaches the steady state $-A^*$ as $n \to \pm\infty$. The solutions $\{h_n^{sym,\beta,\kappa_+}\}$, $\{h_n^{sym,\beta,\kappa_-}\}$ satisfy

$$\{h_n^{sym,\beta,\kappa_+}\} \to \{h_n\}, \qquad \{h_n^{sym,\beta,\kappa_-}\} \to \{h_n\}$$

with respect to the locally uniform convergence as $\beta \nearrow \beta_0$.

Let us note that the time-independent breather solutions $\{h_n^{sym,\beta,0}\}$ are induced by the transversal intersection of $\mathrm{Fix}(R) \cap \mathcal{V}_0$ and $W^s \cap \mathcal{V}_0$ within the two-dimensional subspace $\mathcal{V}_0$. Their existence is therefore trivial.

Proof of the theorem

Recall that $\beta_0 > \beta$ means $\mu = \omega^2 - \beta^2 = \beta_0^2 - \beta^2 > 0$. On account of theorem 6, there exists a one-parameter family

$$\{H_n^{sym,\beta,\kappa}\} = \{(\tilde{h}_n^{sym,\beta,\kappa}, \tilde{h}_{n+1}^{sym,\beta,\kappa})\}$$

of symmetric solutions of (11) (for the choice $\lambda = (\kappa, 0)$ in theorem 6). If we denote by $l^\beta : \kappa \mapsto H_0^{sym,\beta,\kappa} \mapsto S_0^{+,\beta,\kappa}$ the well-defined map, which associates to each value $\kappa \approx 0$ the unique point $S_0^{+,\beta,\kappa} \in \mathcal{M}_\mu$, such that $\|H_n^{sym,\beta,\kappa} - S_n^{+,\beta,\kappa}\|_{X_e} \to 0$ with exponential rate α as $n \to \infty$, then l^β is continuous, see theorem 4. Hence, $\mathrm{Rg}(l^\beta)$ defines a continuous curve on the center manifold, such that $0 \in \mathrm{Rg}(l^0)$. In fact, $\{H_n\}$ persists as a symmetric homoclinic solution in the subspace $\mathcal{V}_0$ after varying the parameter β slightly. In particular, this solution is contained in the intersection of W^s_β and W^u_β and is therefore an element of the family $\{H_n^{sym,\beta,\kappa}\}$ for some $\kappa \approx 0$. Hence, $0 \in \mathrm{Rg}(l^\beta)$ for all $\beta \approx \beta_0$, $\beta < \beta_0$. Moreover, for $|\mu|$ small enough, the union of the stable and unstable manifold of $\pm A^*$ within the two-dimensional center manifold $\mathcal{M}_\mu$ provides a closed curve of amplitude $\mathcal{O}(|\mu|^{1/2})$ encircling the fix point zero (see case c) of lemma 1). Let us denote this circle by $\mathcal{N}^\beta$. Hence,

$$\mathrm{Rg}(l^\beta) \cap \mathcal{N}^\beta \neq \{\}$$

for $\beta < \beta_0$ and therefore the curve $\mathrm{Rg}(l^\beta)$ will also intersect the stable manifold of A^* of the recurrence relation (11) restricted to the center manifold. This proves part *iii*) of the corollary. The other cases are consequences of theorem 6. $\square$

7 Discussion: Chaotic behavior

The solutions

$$\{H_n^{sym,\beta,\kappa_+}\} = \{(h_n^{sym,\beta,\kappa_+}, h_{n+1}^{sym,\beta,\kappa_+})\}$$

in case iii) correspond to generalized breather solutions of the original lattice differential equation (9). Let us recall that the primary homoclinic solution $\{H_n\}$ for $\beta = \beta_0 = \omega$ induces a whole family of (symmetric) homoclinic solutions $\{\tilde{Q}_n\}$, $\tilde{Q}_n \in \mathcal{V}_0$, making several loops before converging to zero as $n \to \pm\infty$. If the assumptions in theorem 6 and 7 hold for $\{H_n\}$ replaced by $\{\tilde{Q}_n\}$ (which is generically satisfied) then we conclude the existence of a whole family of generalized breather solutions satisfying iii) in the upper theorem. More precisely, given any given number $n_* \in \mathcal{N}$ there exists a γ small enough, such that for all $\beta < \beta_0$, $|\beta - \beta_0| < \gamma$ there exist at least n_* distinct generalized breather solutions of (9), which correspond to homoclinic solutions to the steady state A^* of (11). These solutions reflect the chaotic behavior near the primary homoclinic solution $\{H_n\}$ (in the subspace of time-independent functions $\mathcal{V}_0$), but now within the set of non trivial generalized breather solutions. However, we can also proceed differently by observing that $\{H_n^{sym,\beta,\kappa_+}\}$ is a homoclinic solution of the abstract equation (11), which possesses a *transverse* intersection of stable and unstable manifold. Indeed, note that the asymptotic symmetric steady state A_*, which is approached by the homoclinic solution, is actually hyperbolic on the center manifold (certificate James [Jam03]). We observe that for each n the value $H_n^{sym,\beta,\kappa_+} \in X_e$ is a nontrivial time-periodic solution and therefore $n \mapsto \partial_t H_n^{sym,\beta,\kappa_+}$ solves the variational equation with respect to the linearization along $\{H_n^{sym,\beta,\kappa_+}\}$, which seems to prevent the existence of exponential dichotomies. However, since we restricted our analysis to the phase space X_e of *even* functions, the function $\partial_t H_n^{sym,\kappa_+}$ is an odd function for each n and therefore not contained in X_e. Hence, arguments as in the work of Palmer [Pal88b, PSS97] or [SW89] become applicable and should actually prove the existence of a compact subset of X_e near H_0^{sym,β,κ_+}, where the dynamics of (11) is conjugated to the Bernoulli shift on two symbols. The rigorous validation of this fact will appear elsewhere.

8 Genericity of hypothesis 2

Finally, let us comment on the validation of hypothesis 2 in this section. First of all we note that this assumption is equivalent to the assumption that

$$V_{m+1} = L_m V_m \tag{36}$$

does not possess a solution $\{V_n\}$ such that

$$\sup_{n\in\mathbb{Z}} \eta^n |V_n|_{X_e} < \infty \tag{37}$$

if $\eta \neq 1$ is sufficiently close to one. Indeed, let us consider $\eta > 1$ for the sake of clarity. Note that every $\mathcal{V}_k$ is an invariant subspace with respect to the linear recurrence relation $V_{n+1} = L_n V_n$ and in fact the restriction of this recurrence relation to the subspace $\mathcal{V}_k$, $k \neq 1$, possesses an *exponential* dichotomy with associated stable and unstable subspace $E^s_{+,k}$, $E^u_{-,k}$, respectively, see [Pal88b, Pal88a, Cop65]. More precisely, the spaces $E^s_{+,k}$ and $E^u_{-,k}$ are characterized by the fact that for every point in $E^s_{+,k}$ there exists a solution $\{V_n\}, n \geq 0$ of the linear recurrence relation (36) restricted to $\mathcal{V}_k$, which decays exponentially for $n \to \infty$. Similarly, for every point in $E^u_{-,k}$ there exists a solution $\{V_n\}, n \leq 0$, which decays exponentially for $n \to -\infty$. These spaces exist, since the fixed point zero is hyperbolic in the space $\mathcal{V}_k$ for $k \neq 1$, see [Cop65].
In the space $\mathcal{V}_1$ the linearization at zero possesses exactly two simple critical eigenvalues on the unit circle which implies that every solution $\{V_n\}_{n\in\mathbb{Z}}$ of $V_{n+1} = L_n V_n$ restricted to $\mathcal{V}_1$ is actually bounded. Hence, if a globally defined solution $\{V_n\}$ satisfies (37) for some $\eta > 1$, then V_n approaches zero exponentially for $n \to +\infty$ and necessarily $V_n \in \oplus_{k\in I}\mathcal{V}_k$ for every n and some index set $I \subset \mathbb{Z}$, which does not contain the value $k = 1$. As a consequence, such a sequence $\{V_n\}$ also approaches zero exponentially in backward direction $n \to -\infty$.
The question whether or not there exists a nontrivial bounded solution $\{V_n\}$ of the recurrence relation (36) which decays exponentially as $n \to \pm\infty$, is equivalent to the question whether the one-dimensional spaces $E^s_{+,k}$ and $E^u_{-,k}$ coincide for some $k \in \mathbb{Z}$ or not. We certainly know that $E^s_{+,k}$ and $E^u_{-,k}$ have trivial intersection if

$$|k| > k_0 \tag{38}$$

and k_0 is large enough: In this case the recurrence relation

$$V_{n+1} = L_n\big|_{\mathcal{V}_k} V_n \tag{39}$$

can be viewed as a small bounded perturbation of the autonomous recurrence relation $V_{n+1} = \tilde{L}\big|_{\mathcal{V}_k} V_n$, which possesses an exponential dichotomy that persists for (39) if $|k|$ is large, see [Cop65, Pal88b, Pal88a] (in fact, this can be seen from the explicit representation of the eigenvalues $\lambda^k_\pm$ in (14)). Hence $E^s_{+,k}$ and $E^u_{-,k}$ have transverse intersection in $\mathcal{V}_k$ if $|k| > k_0$ and $k_0 >> 0$ is large enough.
If in one $\mathcal{V}_{\tilde{k}}$, $|\tilde{k}| \leq k_0$ and $\tilde{k} \neq 1$, this assumption is not satisfied, we will now show that we can perturb the nonlinearity $\mathcal{W}''$ slightly in order to obtain a

transverse intersection of $E^s_{+,\tilde{k}}$ and $E^u_{-,\tilde{k}}$ in $\mathcal{V}_{\tilde{k}}$. Note that small perturbations of the potential $\mathcal{W}''$ do not destroy the existence of a homoclinic solution $\{H_n\}$ in $\mathcal{V}_0$, since in the reversible recurrence relation (15) small perturbations lead again to the existence of homoclinic solutions.

We now want to be more explicit and assume that the nonlinearity $\mathcal{W} \in BC^3(\mathbb{R})$ depends smoothly on a parameter $\varepsilon \approx 0$; hence $\mathcal{W}(\cdot) = \mathcal{W}(\cdot, \varepsilon)$. Upon varying ε the recurrence relation (15) then possesses a homoclinic solution $\{h_n\} = \{h^\varepsilon_n\}$ due to the transverse intersection of stable and unstable manifold within $\mathcal{V}_0$. It is now our goal to provide an explicit condition which assures that hypothesis 2 is satisfied with respect to the perturbed linear equation (40) below. So let us assume that hypothesis 2 is violated for the potential $\mathcal{W}(\cdot) = \mathcal{W}(\cdot, 0)$ for $\varepsilon = 0$ and there exists a exponentially decaying solution $\{U^*_n\} \subset \mathcal{V}_{\tilde{k}}$ of

$$U_{n+1} = L^\eta_n U_n,$$

where $L^\eta_n := \eta L_n$ for some $|\tilde{k}| < k_0$, $\tilde{k} \neq 1$ and $\eta > 1$ close enough to 1. We now want to show that under a suitable condition on the family of potentials $\mathcal{W}(\cdot, \varepsilon)$ the perturbed linear equation

$$\begin{aligned} \begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} &= L^{\eta,\varepsilon}_n \begin{pmatrix} v_n \\ w_n \end{pmatrix} - \begin{pmatrix} 0 \\ \varepsilon \cdot \eta \cdot \tilde{W}''(h^\varepsilon_n, \varepsilon) w_n \end{pmatrix} \\ &=: L^{\eta,\varepsilon}_n \begin{pmatrix} v_n \\ w_n \end{pmatrix} - \varepsilon A^\varepsilon_n \begin{pmatrix} v_n \\ w_n \end{pmatrix} \end{aligned} \tag{40}$$

does not possess a bounded solution. Here, we have defined $\tilde{W}''(x, \varepsilon)$ via $\varepsilon \cdot \tilde{W}''(x, \varepsilon) := -\mathcal{W}''(x, \varepsilon) + \mathcal{W}''(x, 0)$ (note that $-\mathcal{W}''(x, \varepsilon) + \mathcal{W}''(x, 0) \in \mathcal{O}(\varepsilon)$) and

$$L^{\eta,\varepsilon}_n \begin{pmatrix} v_n \\ w_n \end{pmatrix} := \eta \cdot \begin{pmatrix} w_n \\ 2w_n - v_n + \omega_c^2 \partial_t^2 w_n + \mathcal{W}''(h^\varepsilon_n, 0) w_n \end{pmatrix}$$

for $\{(v_n, w_n)\} \in l^2_{\tilde{k}} := l^2(\mathbb{Z}, \mathcal{V}_{\tilde{k}})$. Note, that $L^{\eta,\varepsilon}_n$ is indeed a bounded operator from $l^2_{\tilde{k}}$ to itself. Let us begin by assuming that there actually exists a bounded solution

$$U^\varepsilon_n = U^*_n + \tilde{U}_n, \tag{41}$$

$\tilde{U}_n = \mathcal{O}(\varepsilon)$, of (40) for $\varepsilon \approx 0$. Then there exits a bounded sequences $\{\Psi^\varepsilon_n\} \in l^\infty(\mathbb{Z}, \mathcal{V}_{\tilde{k}})$ such that

$$\{\tilde{Z}_n\} \in \mathrm{Rg}\,(Y_n \mapsto Y_{n+1} - L^{\eta,\varepsilon}_n Y_n)$$

if and only if

$$\{\tilde{Z}_n\} \perp \{\Psi^\varepsilon_n\}$$

with respect to the $l^2_{\tilde{k}}$ scalar product, where the map $Y_n \mapsto Y_{n+1} - L^{\eta,\varepsilon}_n Y_n$ is considered as a map from $l^2_{\tilde{k}}$ into $l^2_{\tilde{k}}$. Indeed, this is a consequence of the fact that $Y_n \mapsto Y_{n+1} - L^{\eta,\varepsilon}_n Y_n$ is a Fredholm operator of index zero for $|\varepsilon|$ small enough. Let us now write Ψ^ε_n in the form $\Psi^\varepsilon_n = \Psi^0_n + \Gamma_n$, where $\{\Gamma_n\} = \mathcal{O}(\varepsilon)$ with respect to $l^2_{\tilde{k}}$. Considering the ansatz (41), we see that U^ε_n satisfies the recurrence relation

$$U^\varepsilon_{n+1} = L^{\eta,\varepsilon}_n U^\varepsilon_n - \varepsilon A^\varepsilon_n (U^*_n + \tilde{U}_n).$$

As a consequence, we see that

$$\begin{aligned} 0 &= \left\langle \varepsilon A_n^\varepsilon (U_n^* + \tilde{U}_n), \Psi_n^\varepsilon \right\rangle \\ &= \left\langle \varepsilon A_n^\varepsilon U_n^*, \Psi_n^0 \right\rangle + \left\langle \varepsilon A_n^\varepsilon \tilde{U}_n, \Psi_n^0 \right\rangle + \left\langle \varepsilon A_n^\varepsilon (U_n^* + \tilde{U}_n), \Gamma_n^\varepsilon \right\rangle \\ &= \varepsilon \left\langle -A_n^\varepsilon U_n^*, \Psi_n \right\rangle + o(\varepsilon), \end{aligned} \tag{42}$$

where the term $o(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$ and originates from the fact that $\tilde{U}_n \in \mathcal{O}(\varepsilon)$ and $\Gamma_n \in \mathcal{O}(\varepsilon)$. Hence, if we postulate that

$$\left\langle A_n^0 U_n^*, \Psi_n \right\rangle \neq 0 \tag{43}$$

we see that (42) cannot be satisfied for $|\varepsilon| \approx 0$ small enough. Note that it cannot happen that the second component of $\psi_n = (\psi_n^1, \psi_n^2)$ vanishes for all n (which would imply that (43) could never be true), since if $\psi_{n_*}^1 \neq 0$ and $\psi_{n_*}^2 = 0$ for some n_* then a straightforward calculation of the adjoint operator of $(u_n, v_n) \mapsto (u_{n+1}, v_{n+1}) - L_n(u_n, v_n)$ with respect to $l_{\tilde{k}}^2$ shows that then $\psi_{n_*+1}^2 \neq 0$.
We want to point out that (43) is a condition on the family $\mathcal{W}(\cdot,\varepsilon)$ *only* (i.e. does not involve variations of h_n^ε with respect to ε etc.) and shows, that we can always guarantee that hypothesis 2 holds upon slightly varying the potential $\mathcal{W}$ with respect to the BC^3-norm, if necessary. Note, that once hypothesis 2 is true, it remains true even if we allow $\mathcal{W}$ to change slightly with respect to the BC^3-norm. As a consequence, hypothesis 2 is in fact generically satisfied with respect to $\mathcal{W} \in BC^3$, which was claimed in theorem 1. Let us formulate the main results of this section in the next lemma.

Lemma 4
Assume that hypothesis 2 is violated in any subspace $\mathcal{V}_{\tilde{k}}$ for some $|\tilde{k}| < k_0$, $\tilde{k} \neq 1$ (see (38) for the definition of k_0). Let us then consider a family of potentials $\mathcal{W}(\cdot,\varepsilon) \in BC^3(J,\mathbb{R})$ such that $\varepsilon \mapsto \mathcal{W}(x,\varepsilon)$ is C^1 for every $x \in J$, where $J \subset \mathbb{R}$ denotes an open interval which contains all values of the homoclinic solution $h_n \in \mathbb{R}$. If then

$$\left\langle \begin{pmatrix} 0 \\ \partial_\varepsilon \mathcal{W}''(h_n,\varepsilon)\big|_{\varepsilon=0} w_n^* \end{pmatrix}, \psi_n \right\rangle \neq 0$$

with respect to the $l^2(\mathbb{Z},\mathcal{V}_{\tilde{k}})$ scalar product, where $U_n^* = (v_n^*, w_n^*)$ denotes the bounded solution of the recurrence relation $U_{n+1} = L_n^\eta U_n$, then hypothesis 2 is satisfied for $\varepsilon \neq 0$ small enough with respect to the perturbed linear equation (i.e. the linear equation (40)). In particular, hypothesis 2 is generically satisfied with respect to $\mathcal{W} \in BC^3(J,\mathbb{R})$.

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