Travelling Waves
in Systems of Hyperbolic Balance Laws

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1 Introduction

The influence of source terms on the structure of solutions to hyperbolic conservation laws recently has attracted much attention. While the only travelling-wave solutions of hyperbolic conservation laws are single shock waves, systems of balance laws may possess a variety of different continuous and discontinuous travelling waves. In this paper we concentrate on continuous travelling waves and study two different types of interplay between the flux of the conservation law and the dynamics due to the source term.

In both situations we encounter the presence of manifolds consisting of equilibria. Although this looks rather special and non-generic, it will turn out that manifolds of equilibria occur naturally in the travelling-wave problem of conservation laws with source terms. In section 2, we focus on a combination of conservation laws and balance laws which gives rise to subspaces of equilibria. Here the existence of a large number of small heteroclinic waves can be proved. Section 3 deals with the bifurcation of large heteroclinic waves. Here the manifold of equilibria is obtained from a rescaling of the travelling-wave system.

Our viewpoint is from dynamical systems and bifurcation theory. Local normal forms at singularities are used and the dynamics is described with the help of blow-up transformations and invariant manifolds.

2 Oscillatory profiles of stiff balance laws

This section is devoted to a phenomenon in hyperbolic balance laws, first described by Fiedler and Liebscher [FL00], which is similar in spirit to the Turing instability. The combination of two individually stabilising effects can lead to quite rich dynamical behaviour, like instabilities, oscillations, or pattern formation.

Our problem is composed of two ingredients. First, we have a strictly hyperbolic conservation law. The second part is a source term which, alone, would describe a simple, stable kinetic behaviour: all trajectories eventually converge monotonically to some equilibrium. The balance law, constructed of these two parts, however, can support profiles with oscillatory tails. They emerge from singularities in the associated travelling-wave system.

2.1 Travelling waves

We are interested in profiles of balance laws of the form

$$u_t + f(u)_x = \frac{1}{\varepsilon} g(u), \tag{2.1}$$

with $x \in \mathbb{R}$ and $u \in \mathbb{R}^N$. Travelling-wave solutions

$$u(t,x) = u\left(\frac{x - st}{\varepsilon}\right), \qquad \lim_{\xi \to \pm\infty} u(\xi) = u_\pm \in \mathbb{R}^N \tag{2.2}$$

are heteroclinic orbits of the dynamical system

$$(\mathrm{D}f(u) - s \cdot \mathrm{id})\, u' = g(u). \tag{2.3}$$

In particular, the asymptotic states have to be zeros of the source term:

$$g(u_\pm) = 0. \tag{2.4}$$

Choose a fixed wave speed s. As long as the speed of the wave does not coincide with one of the characteristic speeds, i.e. $s \notin \operatorname{spec} \mathrm{D}f(u)$, the travelling-wave equation (2.3) yield a system of ordinary differential equations:

$$u' = (\mathrm{D}f(u) - s \cdot \mathrm{id})^{-1}\, g(u). \tag{2.5}$$

2.2 Manifold of equilibria

Usually one expects the zeros of a generic function g to form a set of isolated points. However, typical systems of balance laws are combinations of pure conservation laws and balance laws. For example, conservation of mass and momentum are often assumed to hold strictly. Let us consider a system of K pure conservation laws with $N - K$ balance laws. Then, generic source terms g give rise to K-dimensional zero-sets.

Near points of maximal rank of $\mathrm{D}g$, the zeros of g and the equilibria of (2.3) form a K-dimensional manifold. In the following sections, we shall investigate the dynamics near such a manifold of equilibria and describe the resulting structure of travelling waves of the system of balance laws (2.1).

The asymptotic behaviour of profiles $u(\xi)$ of (2.3) for $\xi \to \pm\infty$ depends on the linearisation

$$L = (\mathrm{D}f(u) - s \cdot \mathrm{id})^{-1}\, \mathrm{D}g(u). \tag{2.6}$$

of the vectorfield (2.5). Of particular interest are fixed points where the stability of L changes. At these points the linearisation L has non-maximal rank less than $N - K$ and

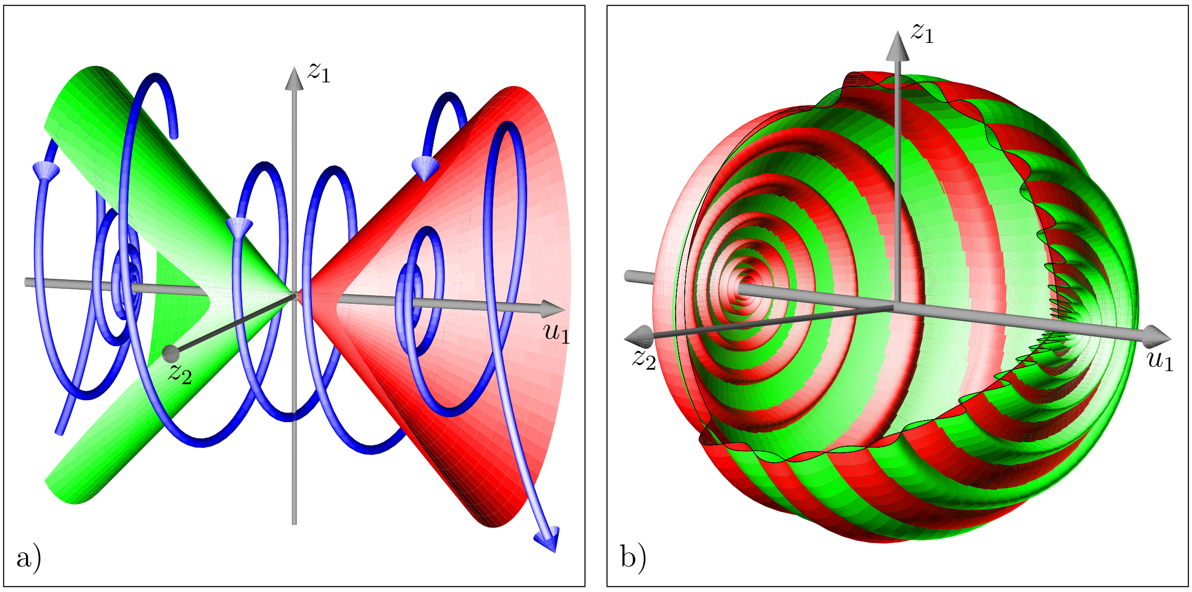


Figure 2.1: Dynamics near a Hopf point along a line of equilibria: a) hyperbolic, $\eta = +1$, b) elliptic, $\eta = -1$.

normal hyperbolicity of the equilibrium manifold breaks down. Depending on the type of the singularity, a very rich set of local heteroclinic connections can emerge and leads to small-amplitude travelling waves of (2.1). We call this phenomenon "bifurcation without parameters", because is does not depend on the variation of some additional parameter.

2.3 Hopf point

Let us start with the case $K = 1$ of a one-dimensional curve of equilibria. Bifurcations without parameters along lines of equilibria have been studied in [FLA00, FL00, Lie00]. Typical singularities are of codimension one. They are characterised by a simple eigenvalue of (2.6) crossing zero (simple-zero point) or by a pair of conjugate complex eigenvalues crossing the imaginary axis (Hopf point). The more interesting Hopf point is described as follows.

Theorem 2.1 *[FLA00] Let $F : I\!R^N \to I\!R^N$ be a C^5-vectorfield with a line of fixed points along the u_1-axis, $F(u_1, 0, \ldots, 0) \equiv 0$. At $u_1 = 0$, we assume the Jacobi matrix $\mathrm{D}F(u_1, 0, \ldots, 0)$ to be hyperbolic, except for a trivial kernel vector along the u_1-axis and a complex conjugate pair of simple, purely imaginary, nonzero eigenvalues $\mu(u_1), \overline{\mu(u_1)}$ crossing the imaginary axis transversely as u_1 increases through $u_1 = 0$:*

$$\begin{aligned} \mu(0) &= \mathrm{i}\omega(0), \qquad \omega(0) > 0, \\ \mathrm{Re}\, \mu'(0) &\neq 0. \end{aligned} \tag{2.7}$$

Let Z be the two-dimensional real eigenspace of $F'(0)$ associated to $\pm \mathrm{i}\omega(0)$. By Δ_Z we denote the Laplacian with respect to variations of u in the eigenspace Z. Coordinates in Z are chosen as coefficients of the real and imaginary parts of the complex eigenvector associated to $\mathrm{i}\omega(0)$. Note that the linearisation acts as a rotation with respect to these not necessarily orthogonal coordinates. Let P_0 be the one-dimensional eigenprojection onto the trivial kernel along the u_1-axis. Our final nondegeneracy assumption then reads

$$\Delta_Z P_0 F(0) \neq 0. \tag{2.8}$$

Fixing orientation along the positive u_0-axis, we can consider $\Delta_Z P_0 F(0)$ as a real number. Depending on the sign

$$\eta := \mathrm{sign}\ (\mathrm{Re}\ \mu'(0))\ \cdot \mathrm{sign}\ (\Delta_Z P_0 F(0)), \tag{2.9}$$

we call the Hopf point $u = 0$ elliptic *if $\eta = -1$ and* hyperbolic *for $\eta = +1$.*

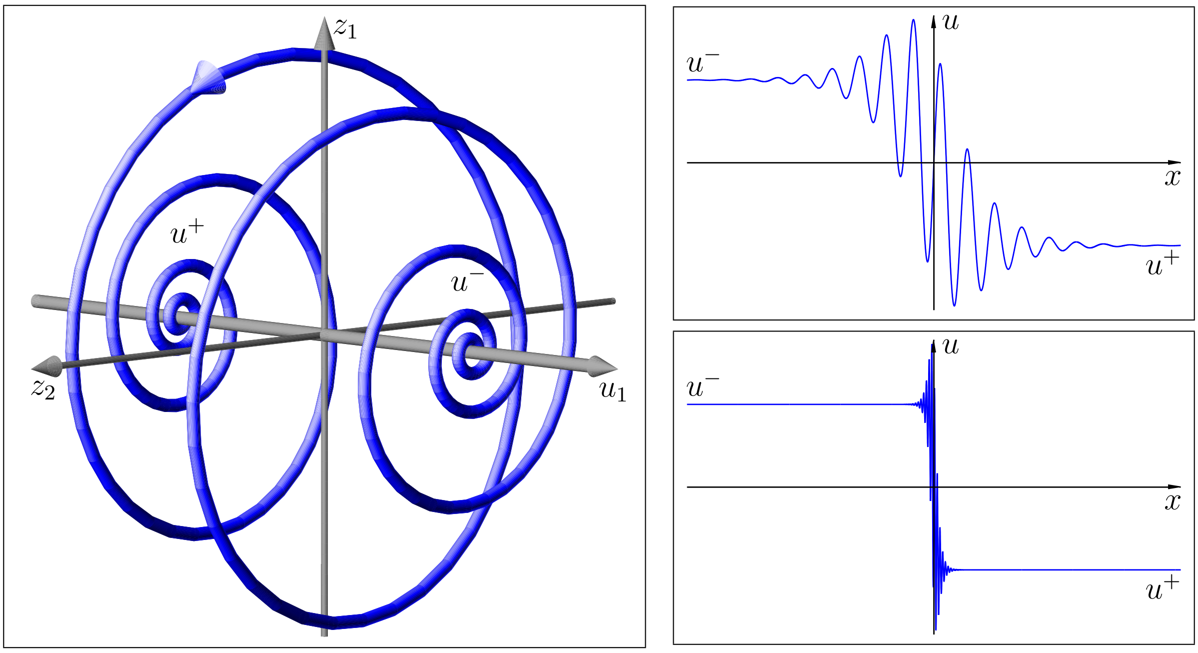
Then the following holds true in a neighbourhood U of $u = 0$ within a three-dimensional centre manifold to $u = 0$.

In the hyperbolic case, $\eta = +1$, all non-equilibrium trajectories leave the neighbourhood U in positive or negative time direction (possibly both). The stable and unstable sets of $u = 0$, respectively, form cones around the positive/negative u_1-axis, with asymptotically elliptic cross section near their tips at $u = 0$. These cones separate regions with different convergence behaviour. See Fig. 2.1(a).

In the elliptic case all non-equilibrium trajectories starting in U are heteroclinic between equilibria $u^{\pm} = (u_1^{\pm}, 0, \ldots, 0)$ on opposite sides of the Hopf point $u = 0$. If $F(u)$ is real analytic near $u = 0$, then the two-dimensional strong stable and strong unstable manifolds of $u^{\pm}$ within the centre manifold intersect at an angle which possesses an exponentially small upper bound in terms of $|u^{\pm}|$. See Fig. 2.1(b).

Note that the heteroclinic connections which fill an entire neighbourhood in the centre manifold of an elliptic Hopf point then lead to travelling waves of the balance law (2.1). In Fig. 2.2, such a wave is shown, and a generic projection of the n-dimensional space of u-values onto the real line was used. For stiff source terms, $\varepsilon \searrow 0$, the oscillations imposed by the purely imaginary eigenvalues now look like a Gibbs phenomenon. But here, they are an intrinsic property of the analytically derived solution.

In [FL00, Lie00] simple examples of Hopf points in systems of viscous balance laws have been provided. The following result goes beyond these examples and emphasises the possibility of Hopf points in systems with *arbitrary* flux functions when combined with a *stabilising* source term.



Heteroclinic orbit near the Hopf point. Profile for two values of ε in (2.1).

Figure 2.2: Oscillatory travelling wave emerging from an elliptic Hopf point.

Theorem 2.2 *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a generic C^6-vectorfield such that $\mathrm{D}f(u)$ has only real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \lambda_3(u)$ for all u in a neighbourhood of the origin $u = 0$.*

Then, for every value $s \notin \{\lambda_1(0), \lambda_2(0), \lambda_3(0)\}$ there exists a C^5-vectorfield

$$g : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\} \tag{2.10}$$

such that

1. *the kinetic part g stabilises the line of equilibria near the origin, i.e. the linearisation $\mathrm{D}g(0)$ has one (trivial) zero eigenvalue and two negative real eigenvalues,*

2. *the travelling-wave equation (2.5) admits a Hopf point in the sense of Theorem 2.1.*

Proof. Without loss of generality, we choose $s = 0$ and require the eigenvalues of $\mathrm{D}f(0)$ to be nonzero. We shall provide a particular source g with a straight line of equilibria.

First, we construct a suitable linearisation $\mathrm{D}g(0)$ that creates the purely imaginary eigenvalues of $\mathrm{D}f(0)^{-1}\mathrm{D}g(0)$. Secondly, we continue this linearisation along the line of equilibria such that the transversality (2.7) holds. Finally, we use genericity to satisfy the nondegeneracy condition (2.8). The main problem of the construction is the constraint (2.10) imposed by the structure of one conservation law and two balance laws.

Let S be the transformation of $\mathrm{D}f(0)$ into diagonal form:

$$\mathrm{D}f(0) \;=\; S\,\Lambda\,S^{-1}, \qquad \Lambda \;=\; \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3). \tag{2.11}$$

A Hopf point of system (2.5) at the origin requires the existence of a transformation $T \in GL(3)$, such that

$$\mathrm{D}g(0) \;=\; S\,\Lambda\,S^{-1}\,T^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} T. \tag{2.12}$$

Here we have normalised the imaginary part of the Hopf eigenvalue to one. On this linear level, the constraint (2.10) yields $0 = e_3^{\mathrm{T}}\,\mathrm{D}g(0)$ which is equivalent to

$$\Lambda\,S^{\mathrm{T}}\,e_3 \;\perp\; S^{-1}\,T^{-1}\left(\{0\}\times I\!R^2\right), \tag{2.13}$$

where $e_3 = (0,0,1)^{\mathrm{T}}$ denotes the third standard unit vector.

Aside from (2.13) we can define T arbitrarily in order to construct the two negative eigenvalues of $\mathrm{D}g(0)$ defined by (2.12). This is done as follows. We start with two arbitrary, linearly independent vectors $(a_1,a_3),(a_2,a_4) \in I\!R^2$. (The actual choice will be made later on.) As a first genericity condition of f we require

$$e_3^{\mathrm{T}} S e_k \neq 0, \qquad k = 1,2,3. \tag{2.14}$$

Then we can obtain a basis of $I\!R^3$ by:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} * \\ a_1 \\ a_3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} * \\ a_2 \\ a_4 \end{pmatrix}, \qquad v_2, v_3 \in (\Lambda S^{\mathrm{T}} e_3)^{\perp}. \tag{2.15}$$

We define T by the equation

$$(T\,S)^{-1} \;=\; \left(v_1\;v_2\;v_3\right) \;=\; \begin{pmatrix} 1 & * & * \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}. \tag{2.16}$$

and insert it into (2.12) to obtain

$$\begin{aligned}
&T\,\mathrm{D}g(0)\,T^{-1}\\
&\quad = T\,S\,\Lambda\,S^{-1}\,T^{-1}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}\\
&\quad = \frac{1}{a_1a_4-a_2a_3}\begin{pmatrix} 1 & * & *\\ 0 & a_4 & -a_2\\ 0 & -a_3 & a_1\end{pmatrix}\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}\begin{pmatrix} 0 & 0 & 0\\ 0 & -a_2 & a_1\\ 0 & -a_4 & a_3\end{pmatrix} \qquad (2.17)\\
&\quad = \frac{1}{a_1a_4-a_2a_3}\begin{pmatrix} 0 & * & *\\ 0 & (\lambda_3-\lambda_2)a_2a_4 & \lambda_2a_1a_4-\lambda_3a_2a_3\\ 0 & \lambda_2a_2a_3-\lambda_3a_1a_4 & (\lambda_3-\lambda_2)a_1a_3\end{pmatrix}
\end{aligned}$$

The lower right (2×2)-block has trace $(\lambda_3-\lambda_2)(a_1a_3+a_2a_4)/(a_1a_4-a_2a_3)$ and determinant $\lambda_2\lambda_3$. The trace can be made negative of arbitrary size regardless of λ_2, λ_3 by choice of $a_1, ..., a_4$. We conclude: if $\lambda_2\lambda_3 > 0$ then we can find parameters $a_1, ..., a_4$ in (2.15) such that the resulting matrix $\mathrm{D}g(0)$ has two negative real eigenvalues. In fact, there is an open region of admissible parameters. The requirement $\lambda_2\lambda_3 > 0$ can be fulfilled without loss of generality by a permutation of $\lambda_1, \lambda_2, \lambda_3$, since at least two of them must have the same sign.

From now on, let v_1, v_2, v_3, T be fixed according to the above considerations. Then we continue

$$\ker \mathrm{D}g(0) \;=\; \mathrm{span}\left\{S\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} \;=\; \mathrm{span}\left\{T^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} \qquad (2.18)$$

to a straight line of equilibria by the definition

$$\left(g\circ T^{-1}\right)\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} \;=\; S\,\Lambda\,S^{-1}\,T^{-1}\begin{pmatrix} 0 & 0 & 0\\ 0 & cw_1 & 1\\ 0 & -1 & cw_1\end{pmatrix}\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} \qquad (2.19)$$

The transversality of the Hopf eigenvalue (2.7) can be achieved by an appropriate choice of the parameter c dependent on the higher order terms of f. Again, there is in fact an open region of admissible parameter values.

The required nondegeneracy of the Hopf point (2.8) is the second nondegeneracy condition needed for f. That finishes the proof.

Note that (2.10) yields

$$\mathrm{D}f(0)^{-1}\Delta_Z g(u)\Big|_{u=0} \in Z = \mathrm{span}\,\{Sv_2, Sv_3\} \qquad (2.20)$$

and does not contribute to (2.8). Therefore, the inclusion of higher order terms in (2.19) would not enter the nondegeneracy condition. The value of $P_0\Delta_Z\mathrm{D}f(0)^{-1}g(0)$ is specified by first-order terms of g (that have been defined only using $\mathrm{D}f(0)$) and second-order terms of f. Indeed, (2.8) is a genericity condition on f. ⋈

Remark 2.3 *The nondegeneracy condition (2.8) is equivalent to the requirement, that every flow-invariant foliation transverse to the line of equilibria breaks down at the Hopf point already to second order. In terms of our system of conservation laws and balance laws, it requires in particular that the flux couples the component with source terms back to the pure conservation law. Without such a coupling, the conservation law gives rise to a foliation, such that in each fibre only finitely many of the equilibria remain. That happens for instance in the systems of extended thermodynamics that are one of the motivating examples of the second part of this article, see section 3.5.*

2.4 Takens-Bogdanov points

Along two-dimensional surfaces of equilibria, we expect singularities of codimension two to occur. The possible cases are characterised by the critical eigenvalues of the linearisation in directions transverse to the surface of equilibria: a geometrically simple and algebraically double eigenvalue zero (Takens-Bogdanov point), a pair of purely imaginary eigenvalues accompanied by a simple eigenvalue zero (Hopf-zero point), or two non-resonant pairs of purely imaginary eigenvalues (double Hopf point). Additionally, simple-zero points and Hopf points with a degeneracy in the higher order terms are possible.

Takens-Bogdanov points have been studied in [FL01].

Theorem 2.4 *[FL01] Let $F : \mathbb{R}^4 \to \mathbb{R}^4$ be a vectorfield with a plane of fixed points, $F(0,0,u_3,u_4) \equiv 0$. At $u = 0$, we assume the Jacobi matrix to be nilpotent.*

Then, for generic F, the vectorfield can be transformed to the normal form

$$\begin{aligned} \dot{u}_1 &= a u_1(-u_3 + u_4) - u_2 u_3 + a b u_2^2, \\ \dot{u}_2 &= u_1, \\ \dot{u}_3 &= u_2 + u_1(c_1 u_3 + c_2 u_4), \\ \dot{u}_4 &= c_3 u_1(c_1 u_3 + c_2 u_4), \end{aligned} \tag{2.21}$$

written up to second order terms.

Depending on the value of b three qualitatively different cases occur. The structure of the set of heteroclinic connections between different equilibria near the Takens-Bogdanov point is depicted in Fig. 2.3. In each case, the Takens-Bogdanov point is the intersection

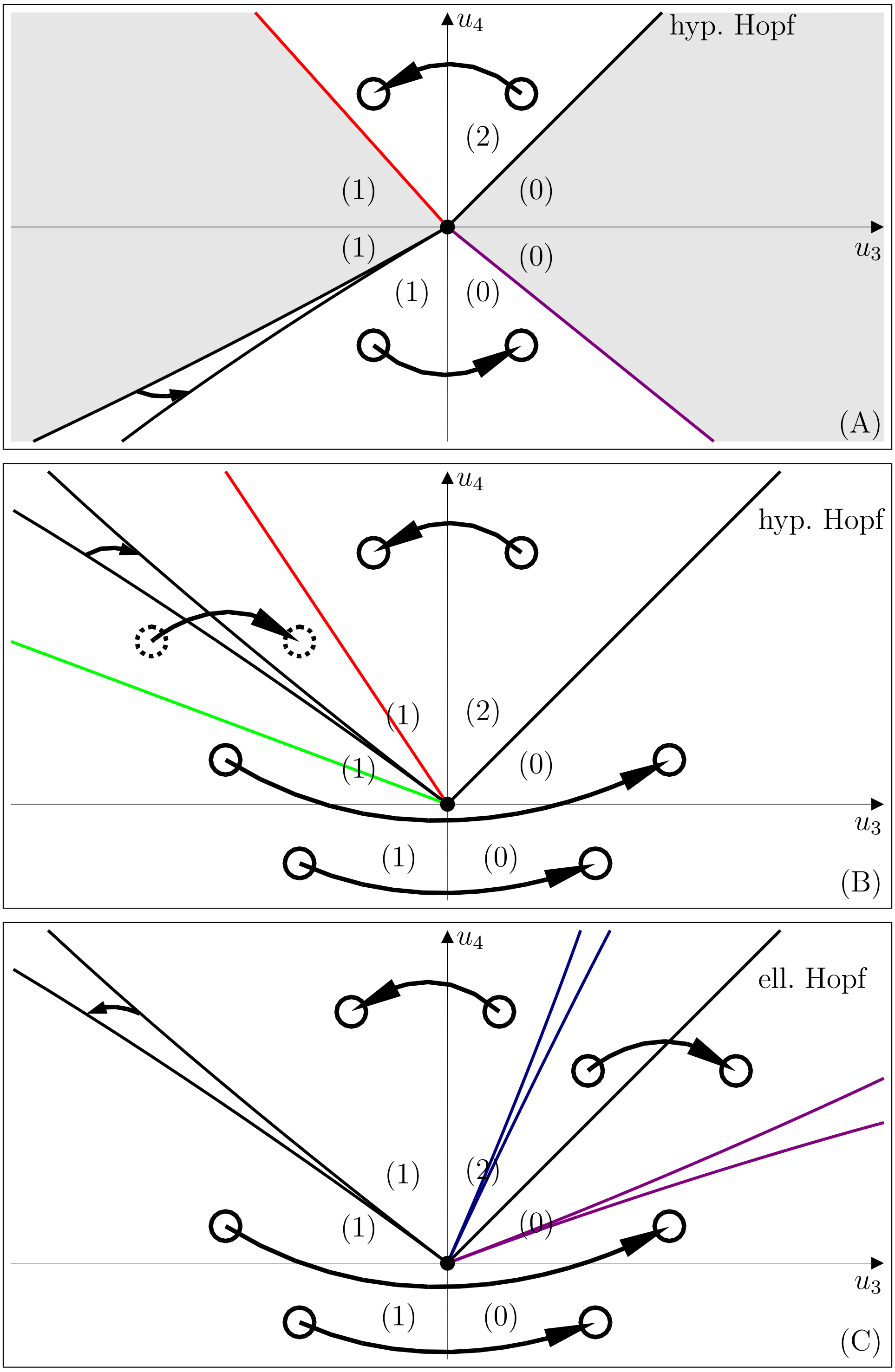


Figure 2.3: Three cases of Takens-Bogdanov bifurcations without parameters, see (2.21). (A) $b < -17/12$; (B) $-17/12 < b < -1$; (C) $-1 < b$. Unstable dimensions i of trivial equilibria $(0, \mathbf{y})$ are denoted by (i). Arrows indicate heteroclinic connections between different regions of the manifold of equilibria.

of the line $\{u_3 = 0\}$ *of simple-zero points and the line* $\{u_3 = u_4 > 0\}$ *of Hopf points of either hyperbolic or elliptic type.*

Similar to Theorem 2.2, Takens-Bogdanov points can occur in systems with at least two conservation laws and two balance laws. Note that travelling waves corresponding to heteroclinic orbits starting or ending near the Hopf line have oscillatory tails.

2.5 Discussion

In summary, Hopf point as well as Takens-Bogdanov points are possible in systems of stiff hyperbolic balance laws. For all generic strictly hyperbolic flux functions and a suitable number of pure conservation laws and balance laws there exist appropriate source terms such that these bifurcations occur in a structurally stable fashion. The bifurcations are generated by the interaction of flux and source. In particular, Hopf points can be constructed for generic fluxes and stabilising sources. For Takens-Bogdanov points at least one example is given in [FL01].

This holds true under small perturbations of the system, for instance in numerical calculations. In particular, an additional viscous regularisation

$$u_t + f(u)_x = g(u) + \delta u_{xx} \tag{2.22}$$

still yields the bifurcation scenario for small positive δ. In [FL00, Lie00, FL01] viscous oscillatory profiles are constructed for specific examples of f, g. The treatment of the viscous terms is still applicable in the general case presented here.

In particular, the proof of convective stability of the oscillatory profiles near an elliptic Hopf point in [Lie00] is applicable for systems given by Theorem 2.2 with additional viscosity. For numerical calculations on bounded intervals in co-moving coordinates this implies nonlinear stability of the corresponding oscillatory travelling waves.

For hyperbolic conservation laws one usually expects viscous shock profiles to be monotone. In particular, in numerical simulations small oscillations near the shock layer are regarded as numerical artefacts due to grid phenomena or unstable numerical schemes. In many schemes "artificial viscosity" is used to automatically suppress such oscillations as "spurious". Near elliptic Hopf as well as near the elliptic Hopf line of Takens-Bogdanov points, in contrast, all heteroclinic orbits correspond to travelling waves with necessarily oscillatory tails. Numerical schemes should therefore resolve this "overshoot" rather than suppress it.

3 Bifurcation of heteroclinic waves

In this section, we study heteroclinic travelling waves of (2.1) with $\varepsilon = 1$, i.e.

$$u_t + f(u)_x = g(u), \qquad x \in \mathbb{R},\ u \in \mathbb{R}^N. \tag{3.1}$$

Travelling-wave solutions

$$u(t,x) = u\left(x - st\right), \qquad \lim_{\xi \to \pm\infty} u(\xi) = u_\pm \in \mathbb{R}^N, \tag{3.2}$$

are again orbits of the dynamical system

$$A(u,s)u' := \left(\mathrm{D}f(u) - s \cdot \mathrm{id}\right) u' = g(u). \tag{3.3}$$

As before we concentrate on heteroclinic waves connecting two equilibria of the reaction dynamics.

In contrast to section 2 we will now consider the wave speed s as a bifurcation parameter and study the bifurcation of heteroclinic waves.

As long as s does not coincide with one of the characteristic speeds for all u on the heteroclinic orbit, (3.3) is equivalent to the explicit ordinary differential equation (2.5).

However, in general there are also orbits containing points where s coincides with one of the characteristic speeds such that $\det(\mathrm{D}f(u) - s \cdot \mathrm{id})$ vanishes at some point on the heteroclinic profile.

3.1 Quasilinear implicit DAEs

In general, the travelling-wave equation (3.3) is a differential-algebraic equation. In contrast to the common setting in differential-algebraic equations the rank of the matrix $\mathrm{D}f(u) - s \cdot \mathrm{id}$ is non-maximal only on a codimension-one surface

$$\Sigma_s := \{u \in \mathbb{R}^N;\ \det A(u,s) = 0\} \tag{3.4}$$

of the phase space. One might suspect that solutions can never cross this surface. Rabier and Rheinboldt [RR94] have studied solutions in a neighbourhood of Σ_s and shown that they typically reach Σ_s in finite (forward or backward) time and cannot be continued. For this reason Σ_s is often referred to as the *impasse surface*. However, there may exist parts of Σ_s where it is possible to cross from one side to the other. Heteroclinic solutions passing through Σ_s have been found in applications [MNP00, Wei95] and, as will be shown below, their behaviour differs from that of heteroclinic orbits in ordinary differential equations.

To identify points on Σ_s where crossing is possible, one needs to desingularise the vector field near the hypersurface Σ_s. To this end one uses the adjugate matrix adj $A(u,s)$ which is defined as the transpose of the matrix of cofactors of $A(u,s)$ and which satisfies the identity

$$(\text{adj}\, A(u,s))A(u,s) = A(u,s)(\text{adj}\, A(u,s)) = \det A(u,s) \cdot \text{id}. \tag{3.5}$$

Solution curves of (3.3) coincide outside Σ_s with trajectories of the *desingularised system*

$$u' = \text{adj}\, A(u,s)g(u) = \text{adj}\, \left(\mathrm{D}f(u) - s \cdot \text{id}\right) g(u). \tag{3.6}$$

This system is obtained by multiplying (3.3) from the left with adj $A(u,s)$ and rescaling time by using $\det A(u,s)$ as an Euler multiplier. Also, the direction of the orbits is reversed in that part of the phase space where $\det A(u,s) < 0$.

In addition to the equilibria of (3.3), equation (3.6) may possess additional fixed points on Σ_s. Since they are not equilibria of the original system they are called *pseudo-equilibria*. As we will see, they play an important role in the bifurcations. The time rescaling is singular at the impasse surface Σ_s, so trajectories of (3.6) that need an infinite time to reach a pseudo-equilibrium correspond to solutions of the original system (3.3) which reach the pseudo-equilibrium in finite time. A solution of (3.3) may therefore consist of a concatenation of several orbits of (3.6).

The dynamics near the impasse surface is strongly affected by the interaction between "true" equilibria and pseudo-equilibria, when equilibria cross the impasse surface as s is varied. In the context of differential-algebraic equations, such a passage of a non-degenerate equilibrium U_0 through the impasse surface was first studied by Venkatasubramanian et al. in [VSZ95]. Their *Singularity-Induced Bifurcation Theorem* states that under certain non-degeneracy conditions one eigenvalue of the linearisation of (3.3) at U_0 moves from the left complex half plane to the right complex half plane or vice versa by diverging through infinity, while all other eigenvalues remain bounded and stay away from the origin.

3.2 Scalar balance laws

Let us very briefly consider the simplest situation of a scalar balance law to describe some of the features that show up in larger systems, too. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex flux function with $f'(0) = 0$ and $g : \mathbb{R} \to \mathbb{R}$ be a nonlinear source term with three simple zeroes $u_\ell < u_m < u_r$ and the sign condition $g(u) \cdot u < 0$ outside $[u_\ell, u_r]$. Looking for travelling waves of (3.3) with speed s then leads to the scalar equation

$$(\partial_u f(u) - s)u' = g(u). \tag{3.7}$$

It is easy to check that the "impasse surface" consists here of a single point u_s where $\partial_u f(u_s) = s$. No trajectory can pass through this impasse point except when $u_s = u_m$, i.e. $s = \partial_u f(u_m)$. For this exceptional wave speed there is a heteroclinic orbit from u_ℓ to u_r which consists of the concatenation of two heteroclinic orbits of the desingularised system

$$u' = g(u). \tag{3.8}$$

Note that the flow has to be reversed for $u > u_m$ such that the two heteroclinic orbits of (3.8) from u_ℓ to u_m and from u_r to u_m can indeed be combined to yield a single heteroclinic orbit of (3.7).

3.3 The p-system with source

While in scalar balance laws heteroclinic waves crossing Σ_s occur only for isolated values of s, already in (2×2)-systems of balance laws such heteroclinic waves may occur for a open set of wave speeds.

Instead of studying general (2×2)-systems with arbitrary source terms we are going to illustrate our results for this case using the well-known p-system. This does not change the results in an essential way, however, it has the advantage that the impasse surface Σ_s is a straight line $u = \text{const}$.

Consider therefore the system

$$\begin{aligned} u_t + v_x &= g_1(u, v) \\ v_t + p(u)_x &= g_2(u, v). \end{aligned} \tag{3.9}$$

We assume that $p'(u) > 0$ such that the conservation-law part is strictly hyperbolic. Moreover we require that there exists an non-degenerate equilibrium, i.e. a point (u_0, v_0) with $g_1(u_0, v_0) = g_2(u_0, v_0) = 0$ and $\det \mathrm{D}g(u_0, v_0) \neq 0$.

The travelling-wave equation corresponding to this balance law is

$$\begin{pmatrix} -s & 1 \\ p'(u) & -s \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix} \tag{3.10}$$

such that for fixed s the impasse set Σ_s is either empty or consists of the line $\Sigma_s := \{(u, v);\ p'(u) = s^2\}$. While orbits which do not cross this line can be treated by standard methods, some care is needed for orbits which reach the line Σ_s.

The Singularity-Induced Bifurcation Theorem tells that the stability type of the equilibrium (u_0, v_0) changes when it crosses the impasse surface at $s = s_0 = \sqrt{p'(u_0)}$. To

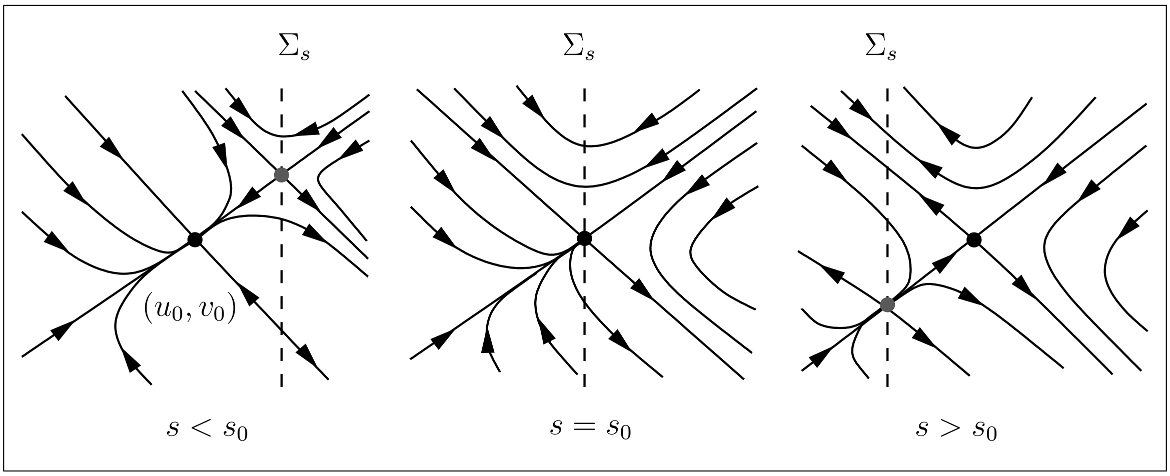


Figure 3.1: A Singularity Induced Bifurcation occurs when a non-degenerate equilibrium crosses the impasse surface (dotted line). The pseudo-equilibrium involved in the transcritical bifurcation of the desingularised system is drawn in grey. For $s > s_0$ there exist orbits which pass through Σ_s.

describe more precisely what happens at this bifurcation, we perform the desingularisation via the adjugate matrix. This leads to the desingularised system

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -sg_1(u,v) - g_2(u,v) \\ -p'(u)g_1(u,v) - sg_2(u,v) \end{pmatrix} \tag{3.11}$$

The implicit-function theorem can now be applied to this equation restricted to Σ_s to find for $|s - s_0|$ small a branch of pseudo-equilibria $(\tilde{u}(s), \tilde{v}(s))$ with $\tilde{u}(s_0) = 0$ and $\tilde{v}(s_0) = v_0$ if

$$s_0 \partial_v g_1(u_0, v_0) + \partial_v g_2(u_0, v_0) \neq 0. \tag{3.12}$$

Assuming that this condition holds, one can describe the dynamics close to (u_0, v_0) for $|s - s_0|$ sufficiently small by using classical bifurcation theory for system (3.11) and translating the results back to the original system (3.10).

Lemma 3.1 *Consider the p-system with a source term which possesses a non-degenerate equilibrium at (u_0, v_0) for all wave speeds s.*

Then the desingularised travelling-wave system (3.11) undergoes a transcritical bifurcation at $s = s_0$. The trivial branch of equilibria crosses a branch $(\tilde{u}(s), \tilde{v}(s))$ of equilibria which are pseudo-equilibria of system (3.10). For $|s - s_0|$ sufficiently small the pseudoequilibrium $(\tilde{u}(s), \tilde{v}(s))$ and the equilibrium (u_0, v_0) are connected by a heteroclinic orbit.

There are different cases depending on the eigenvalue structure at the equilibria. One of them is depicted in Fig. 3.1.

Remark 3.2 *Recall that system (3.11) and system (3.10) are related via a rescaling of time with the factor* $\det A(u, v, s)$ *which is singular at the impasse surface* Σ_s*. For this reason the trajectory of (3.10) corresponding to the heteroclinic orbit between* (u_0, v_0) *and* $(\tilde{u}(s), \tilde{v}(s))$ *needs only a finite time to reach the pseudo-equilibrium* $(\tilde{u}(s), \tilde{v}(s))$*.*

3.4 Heteroclinic waves in the p-system

Since we are basically interested in heteroclinic travelling waves, we will now assume that there exists some heteroclinic orbit of (3.10) asymptotic to the equilibrium (u_0, v_0) at $s = s_0$.

We restrict our attention to heteroclinic orbits which connect some equilibrium (u_-, v_-) to (u_0, v_0) and which are structurally stable.

There are three cases which may occur:

- **Case I:** (u_-, v_-) is of source type while (u_+, v_+) is a saddle equilibrium of (3.3)
- **Case II:** (u_-, v_-) is a saddle while (u_+, v_+) is a sink.
- **Case III:** (u_-, v_-) is of source type while (u_+, v_+) is a sink.

In the first two cases we may think of the heteroclinic orbit for instance as coming from a saddle-node bifurcation.

As the parameter s is varied across s_0 the stationary point (u_0, v_0) moves through Σ_s. The following lemma tells what happens to the heteroclinic connection for $s > s_0$ when the two equilibria lie on different sides of Σ_s.

Theorem 3.3 *Assume that (3.10) possesses two stationary points* (u_-, v_-) *and* (u_0, v_0) *which are on the same side of* Σ_s *for* $s < s_0$*. Assume furthermore that there is a heteroclinic connection from* (u_-, v_-) *to* (u_0, v_0) *at* $s = s_0$ *and that the tangent vector to this heteroclinic orbit at* (u_0, v_0) *is transverse to* Σ_{s_0}*. Then for* $s - s_0 > 0$ *sufficiently small the following holds:*

(i) In case I the desingularised system (3.11) possesses a unique heteroclinic orbit from the pseudo-equilibrium $(\tilde{u}(s), \tilde{v}(s))$ *to the saddle* (u_-, v_-) *and a unique heteroclinic orbit from* $(\tilde{u}(s), \tilde{v}(s))$ *to the saddle* (u_0, v_0)*. The concatenation of these two orbits provides a heteroclinic orbit from* (u_-, v_-) *and* (u_0, v_0) *in the original system (3.3). See Fig. 3.2(a).*

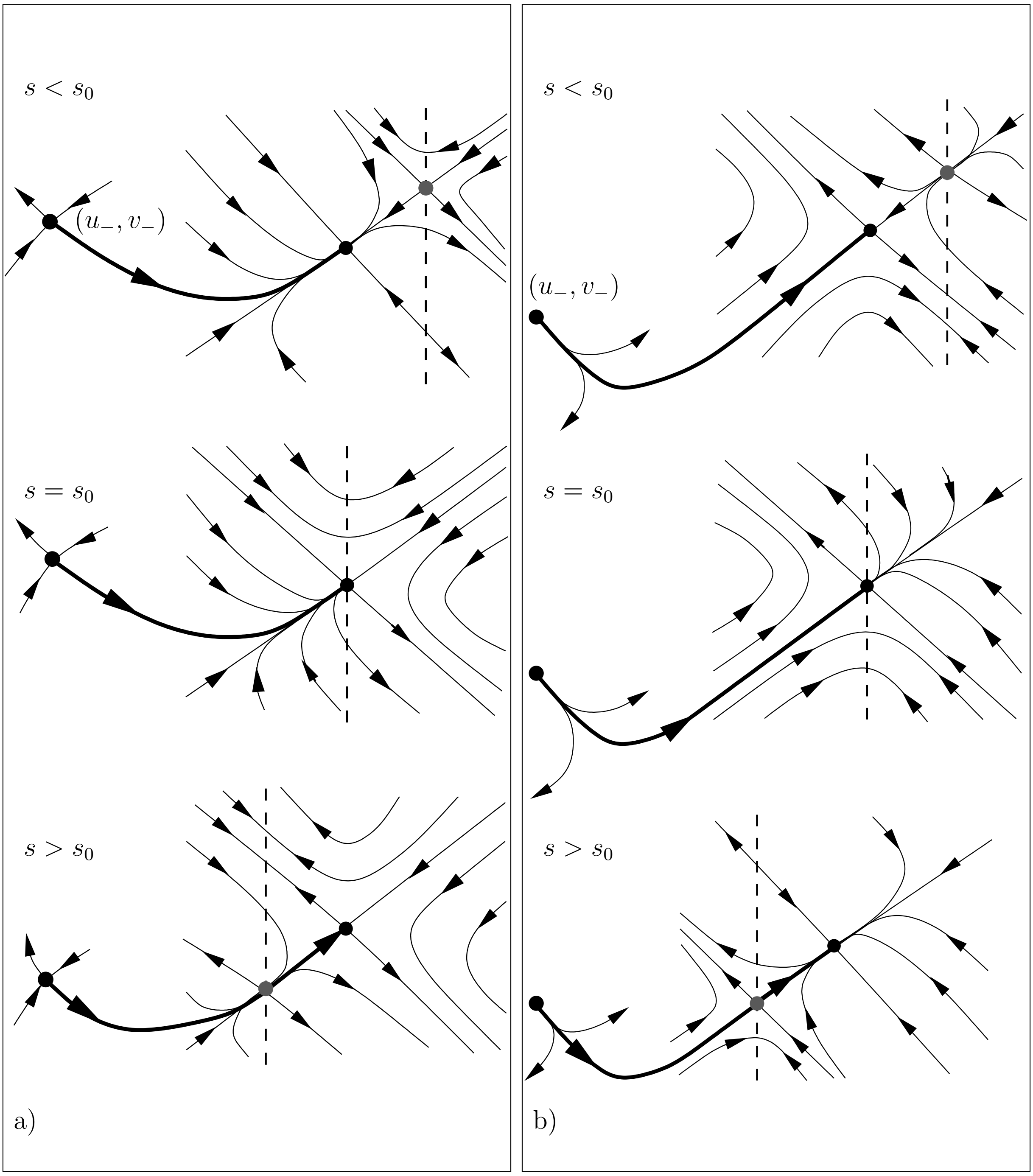


Figure 3.2: Continuation of heteroclinic orbits in the p-system: a) Case I, b) Case II The impasse surface is depicted as a dotted line the pseudo-equilibrium is shown in grey, the heteroclinic connection from (u_-, v_-) to (u_0, v_0) is the bold curve.

(ii) In case II the pseudo-equilibrium $(\tilde{u}(s), \tilde{v}(s))$ is of saddle-type and possesses unique heteroclinic connections to (u_-, v_-) and (u_0, v_0). A heteroclinic orbit of the original system (3.3) is obtained by piecing these two orbits together. See Fig. 3.2(b).

(iii) Case III is similar to case I with the difference that there exist infinitely many heteroclinic orbits from the pseudo-equilibrium sink $(\tilde{u}(s), \tilde{v}(s))$ to the source (u_-, v_-). This in turn yields infinitely many heteroclinic orbits of (3.3) from (u_-, v_-) and (u_0, v_0).

Remark 3.4 *When the pseudo-equilibrium is of saddle type (case II), the heteroclinic orbit between the source and a sink is as smooth as the vector field. In contrast, if the pseudo-equilibrium is of source/sink-type and the heteroclinic wave connects two saddle equilibria, then the heteroclinic orbit has in general only a finite degree of smoothness which depends on the ratio of the eigenvalues at the pseudo-equilibrium.*

An analogous result for general (2×2)-systems can be obtained using Lyapunov-Schmidt reduction. Under a certain non-degeneracy condition the passage of a non-degenerate equilibrium through the impasse surface corresponds to a transcritical bifurcation of the desingularised system. Close to the bifurcation point there exist solutions which pass through the impasse surface and converge to the equilibrium.

It turns out that the situation is similar for N-dimensional systems (3.6) associated with $(N \times N)$-systems of hyperbolic balance laws. Here, the impasse surface Σ_s is of codimension one and generically within Σ_s there is a codimension one set of pseudo-equilibria. The linearisation of (3.6) in such a pseudo-equilibrium possesses 0 as an eigenvalue of multiplicity at least $N-2$ corresponding to the $(N-2)$–dimensional set of pseudo-equilibria.

3.5 Shock profiles in extended thermodynamics

Extended thermodynamics comprises a class of systems of hyperbolic balance laws which describe for instance the thermodynamics of rarefied gases under the physical assumption that the propagation speed of heat flux and shear stress is finite. We concentrate on one specific model, the 14-moment system, as described in [Wei95], [MR98].

It is one in a hierarchy of models based on the kinetic theory of gases. In particular, they are used to get a better resolution of the internal structure of shock waves in rarefied gases if more moments are taken into account. For brevity, we do not write down the full system consisting (in one space dimension) of three conservation laws for mass, momentum, and energy and of three balance laws. Since it is invariant under Galilei transformations it suffices to look for stationary solutions instead of travelling waves with arbitrary speed. Integrating the three conservation laws allows to eliminate three variables and to replace them by integration constants. Moreover, by scaling the variables suitably, it is possible to reduce the system to a DAE in the three variables v, p and Δ with a single real parameter α which can be related to the Mach number. In quasilinear implicit form the travelling-wave equation then reads

$$A(v,p,\Delta,\alpha)\begin{pmatrix} v' \\ p' \\ \Delta' \end{pmatrix} = \begin{pmatrix} -(1-v-p) \\ -(4\alpha+2v(1-6p-2v))/3 \\ -(4v^3+4v^2+36pv^2-16\alpha v-2\Delta)/3 \end{pmatrix}. \tag{3.13}$$

where $A(v,p,\Delta,\alpha)$ is a polynomial matrix function. We omit here most of the (lengthy) calculations and formulas and concentrate on the geometric situation. A more detailed treatment will be performed elsewhere. The impasse surface

$$\Sigma_\alpha = \{(v,p,\Delta);\ \det A(v,p,\Delta,\alpha) = 0\}$$

is a graph over the v-p-plane. For any $\alpha < 25/32$, there are precisely two equilibria

$$E_{1,2} = \left(\frac{5 \mp \sqrt{25-32\alpha}}{8}, \frac{3 \pm \sqrt{25-32\alpha}}{8}, 0\right)$$

which bifurcate at $\alpha = 25/32$ in a subcritical saddle-node bifurcation. The main object of interest are continuous heteroclinic orbits from E_2 to E_1 alias shock profiles. It is clear that for α close to the bifurcation value there exists a unique heteroclinic connection between E_2 and E_1.

It has been observed numerically by Weiss [Wei95] that in the 14-moment system this heteroclinic orbit can be continued to values of α where the shock profiles has to cross the impasse surface Σ_α because E_1 and E_2 lie on different sides of Σ_α. However, in this parameter regime, the dimension of the unstable manifold of E_1 is one while the stable manifold of E_2 is two-dimensional. Without some additional structure one cannot explain that a heteroclinic connection between these two saddle-type equilibria persists for a whole range of α.

In the following we propose a scenario how a one-dimensional manifold $\mathcal{E}$ of pseudo-equilibria can be responsible for a structurally stable heteroclinic connection between E_1 and E_2 in a way similar to case I in the p-system with source. Let α_1 be the parameter value where E_1 lies in Σ_α.

Proposition 3.5 *For $\alpha < \alpha_1$ the one-dimensional stable manifold of E_1 connects to some pseudo-equilibrium $E_{pseudo}(\alpha)$ on $\mathcal{E}$. The two-dimensional unstable manifold of E_2 connects to a whole interval of points on $\mathcal{E}$ containing $E_{pseudo}(\alpha)$. The concatenation of the two heteroclinic orbits of the desingularised system involving $E_{pseudo}(\alpha)$ yields a heteroclinic orbit from E_2 to E_1 in the original system (3.13).*

The scenario is in accordance with numerical calculations performed for the 14-moment system, although we do not have an analytic proof that the heteroclinic orbit created in the saddle-node bifurcation at $\alpha = 25/32$ can be continued down to $\alpha = \alpha_1$ without intersecting the impasse surface Σ_α. However, assuming the existence of such a heteroclinic profile at $\alpha = \alpha_1$ proposition 3.5 is able to explain why the heteroclinic shock profile persists for $\alpha < \alpha_1$.

Let us remark that the bifurcation is connected to a change of stability along the line $\mathcal{E}$ of pseudo-equilibria, similar to the situation considered in section 2.

3.6 Viscous profiles

In many situations systems of balance laws include a small viscous term:

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u), \qquad x \in \mathbb{R},\ u \in \mathbb{R}^N. \tag{3.14}$$

The travelling-wave equation now becomes a singularly perturbed equation of the form

$$\varepsilon u'' = (\mathrm{D}f(u) - s \cdot \mathrm{id})u' - g(u) \tag{3.15}$$

where the prime denotes differentiation with respect to the comoving coordinate $\xi := x - st$. Note that, unlike in viscous conservation laws, the viscosity ε is still present in the travelling-wave equation.

For scalar balance laws, the travelling-wave equation is a planar system with one fast and one slow variable involving the small parameter ε and the wave speed s as an additional parameter. Returning to the setting of section 3.2 where the flux was convex and the source term had three simple zeroes $u_\ell < u_m < u_r$ one might ask whether (3.14) possesses a travelling wave close to the monotone solution of (3.1) that connects u_ℓ to u_r.

However, it turns out that such a solution necessarily has to pass close to a non-hyperbolic point on the slow manifold such that standard techniques in geometric singular perturbation theory can give no answer. For this reason, recent blow-up techniques [KS01] have to be used to establish the following existence result:

Theorem 3.6 *[Här03] Consider a scalar viscous balance law (3.14) with a convex flux $f : \mathbb{R} \to \mathbb{R}$ and a source term $g : \mathbb{R} \to \mathbb{R}$ which possess three simple zeroes $u_\ell < u_m < u_r$. Let $s_0 = f'(u_m)$ be the velocity of the heteroclinic wave that connects u_ℓ to u_r for $\varepsilon = 0$.*

Then for $\varepsilon > 0$ sufficiently small there is a unique velocity $s(\varepsilon)$ such that a unique monotone heteroclinic wave u_ε of (3.15) connects u_ℓ to u_r. To first order the wave speed

$s(\varepsilon)$ depends linearly on the viscosity ε:

$$s(\varepsilon) = s_0 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{g'(u)}{f''(u)}\right)\Bigg|_{u=u_m} \varepsilon \; + \; \mathcal{O}(\varepsilon^{3/2}).$$

Since the heteroclinic travelling wave u_ε follows both stable and unstable parts of the slow manifold, it is a so-called *canard trajectory.*

For larger systems the viscous travelling-wave equation (3.15) can be written as a fast-slow-system with N slow and N fast variables:

$$\begin{aligned} \varepsilon u' &= w + f(u) - su \\ w' &= -g(u) \end{aligned}$$

The N-dimensional slow manifold $\{(u, w) \in \mathbb{R}^{2N};\; w + f(u) - su = 0\}$ is a graph over the subspace $\{w = 0\}$ spanned by the variables of the hyperbolic balance laws. A short calculation shows that points on the slow manifold where normal hyperbolicity fails correspond exactly to the impasse surface Σ_s. This implies that the problem of finding heteroclinic travelling waves of the viscous system which are close to travelling waves of the hyperbolic system intersecting Σ_s will necessarily lead to a rather difficult singularly perturbed problem involving Canard solutions.

An interesting and completely open question is the stability of such viscous travelling waves. In particular, as we have seen in the p-system with source, “ordinary” heteroclinic waves can become rather singular when one of the asymptotic states crosses the impasse surface Σ_s as s is varied. In the viscous setting this would correspond to a transition from a “ordinary” heteroclinic orbit to a canard orbit. It is not clear whether this transition affects the stability of heteroclinic waves.

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