EXISTENCE OF ROLLWAVES IN A VISCOUS SHALLOW WATER EQUATION

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In the present paper we consider a shallow water equation with Chezy friction term. It is well-known that this hyperbolic PDE admits a one-parameter family of discontinuous periodic roll wave solutions parametrized by their wavelength as well as a discontinuous homoclinic wave.
We show that analogous traveling waves exist when small viscous terms of size ε are added to the equation and determine how the velocity of the viscous homoclinic waves differs from the velocity of the inviscid waves. The correponding traveling wave equation leads to a singularly perturbed problem involving points on the slow manifold which are not normally hyperbolic. The periodic roll waves follow stable and unstable parts of the slow manifold and are therefore of "canard" type.

1. Introduction

In 1949, Dressler [1] used the shallow water equations with a nonlinear friction term to give a mathematical explanation for the occurence of periodic roll waves in an open narrow channel with a constant slope. He showed that no continuous periodic solutions are possible, although periodic waves were observed. However, there are periodic discontinuous entropy solutions for any given wave length if the Froude number is larger than 2.

Kranenburg [3] 1992 derived a nonlinear viscosity term and studied the instability of the constant flow which leads to the evolution of roll waves. Recently, Noble and Travadel [5] studied this physically motivated nonlinear viscosity and the corresponding viscous roll wave solutions.

With a different approach using geometric singular perturbation theory and invariant manifolds we are able to prove that periodic and homoclinic traveling waves of the hyperbolic equation persists for small viscosity and that the velocity is only slightly perturbed.

2. The inviscid equation

To describe the flow in an open rectangular channel with constant slope one can use the shallow water equations with Chezy friction

$$\left.\begin{aligned} h_t + (hu)_x &= 0 \\ u_t + \left(\tfrac{1}{2}u^2 + gh\cos\theta\right)_x &= g\sin\theta - C_f\frac{u^2}{h} \end{aligned}\right\} \tag{1}$$

where C_f denotes the friction coefficient and θ is the inclination angle.

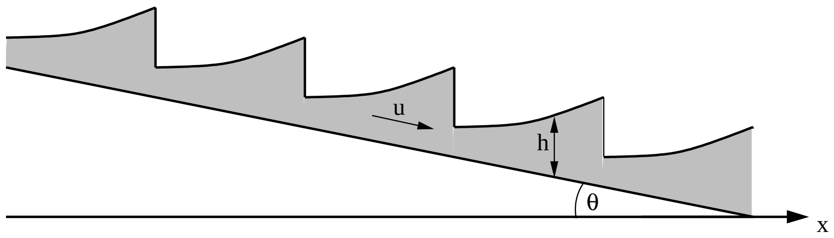

Figure 1. Water waves in an inclined channel.

We want to study traveling waves with speed s.

One may integrate the first equation via $(s-u)h = K$ and scale the variables according to

$$H = \frac{h(g\cos\theta)^{1/3}}{K^{2/3}}, \quad S = \frac{s}{(Kg\cos\theta)^{1/3}\,(1+F)}, \quad \tau = \frac{t(g\cos\theta)^{1/3}}{K^{2/3}}. \tag{2}$$

to eliminate $g\cos\theta$ and K from the traveling wave equation. This leads to

$$\frac{dH}{d\tau} = \frac{dh}{dt} = C_f\,\frac{F^2H^3 - (SH(1+F)-1)^2}{H^3-1} \tag{3}$$

where $F := \sqrt{\frac{\tan\theta}{C_f}}$ denotes the Froude number. The numerator possesses three zeroes $H_0(S)$ with $H_0(1) = 1$ and $H = H_\pm(S)$ with $0 < H_- < H_+ < 1$ provided that $F > 2$. Due to the zero of the denominator there is a singularity at $H = 1$. Trajectories will typically reach $H = 1$ in finite (forward or backward) time and cannot be continued. However, for $H = S = 1$ the numerator and denominator vanish simultaneously allowing trajectories to pass continuously through $H = 1$.

Together with the Rankine-Hugoniot jump condition this can be used to show the following existence result:

Proposition 2.1. *At $S = 1$, equation (3) possesses a one-parameter family of discontinuous periodic solutions ("roll waves") parametrized by their* *wavelength and a unique (up to translation) discontinuous homoclinic traveling wave with $H(\tau) \to H_+$ for $\tau \to \pm\infty$.*

3. The viscous equation

Kranenburg in 1992 [3] derived a one-dimensional model which takes into account the viscosity. He obtains the parabolic-hyperbolic system

$$\left\{\begin{aligned} h_t + (hu)_x &= 0 \\ u_t + \left(\tfrac{1}{2}u^2 + g\cos\theta\, h\right)_x &= C_f\, g\cos\theta\, F^2 - C_f\frac{u^2}{h} + \varepsilon\frac{(hu_x)_x}{h}. \end{aligned}\right. \tag{4}$$

Using the same scaling (2) as for the inviscid equation we arrive at

$$\left(H - \frac{1}{H^2}\right)H' - C_f\, F^2 H + C_f\left(S(1+F) - \frac{1}{H}\right)^2 = \delta\left(\frac{H'}{H}\right)' \tag{5}$$

where $\delta = \frac{\varepsilon}{K^2}$ and the prime now denotes differentiation with respect to τ. Our goal consists of finding homoclinic and periodic traveling waves with speed close to $S = 1$ analogous to the waves in the inviscid equation.

3.1. *Bifurcation of periodic and homoclinic orbits*

For $\delta > 0$ (continuous) periodic roll waves are created via Hopf bifurcation if the Froude number is sufficiently large. They disappears via a homoclinic bifurcation when their period becomes unbounded.

Theorem 3.1. *For $\delta > 0$ sufficiently small, the constant state $H = 1$ undergoes a supercritical Hopf bifurcation at $S = 1$. The branch of periodic orbits terminates in a homoclinic bifurcation when it collides with a homoclinic orbit asymptotic to a stationary point E_+. This homoclinic orbit exists at $S = S_{hom}(\delta)$ with*

$$S_{hom}(\delta) = 1 + \frac{C_f(F-2)(2F-1)}{18(F+1)}\delta + \mathcal{O}(\delta^{3/2}).$$

For the proof we make use of some coordinates which are adapted to the slow-fast structure of the traveling wave problem:
Setting $P = \ln H \Rightarrow P' = \frac{H'}{H}$, we can rewrite (5) as

$$\left.\begin{aligned} \delta P' &= -Q + e^{-P} + \tfrac{1}{2}e^{2P} - \tfrac{3}{2} \\ Q' &= C_f F^2 e^P - C_f(S(1+F) - e^{-P})^2 \end{aligned}\right\} \tag{6}$$

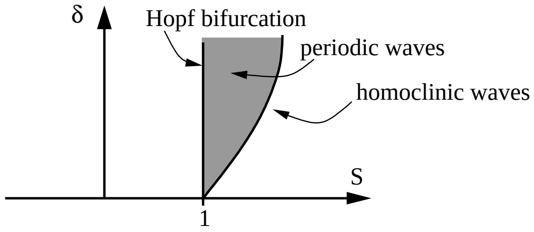

Figure 2. The bifurcation diagram.

In these coordinates the "conserved" quantities are separated from the source term. In terms of fast–slow dynamical systems the conserved quantities correspond to the fast dynamics while the source terms play a role only for the slow dynamics.

Following the usual strategy in singular perturbation theory, system (6) determines for $\delta \to 0$ two limiting systems which both display a part of the behavior which is present for δ small but nonzero. By setting $\delta = 0$ one obtains the slow system

$$0 = -Q + e^{-P} + \tfrac{1}{2}e^{2P} - \tfrac{3}{2}$$

$$Q' = C_f F^2 e^P - C_f(S(1+F) - e^{-P})^2$$

defining the *slow manifold* $\mathcal{C} = \{(P,Q);\ Q = e^{-P} + \frac{1}{2}e^{2P} - \frac{3}{2}\}$, a strictly convex curve with a minimum at $P = Q = 0$. The second equation describes the slow flow on $\mathcal{C}$ by projection. The slow flow possesses three equilibria E_-, E_+ and $P = Q = 0$.
Setting $\delta = 0$ and rescaling the time one arrives at the fast system

$$\dot{P} = -Q + e^{-P} + \frac{1}{2}e^{2P} - \frac{3}{2}, \qquad \dot{Q} = 0.$$

In this system, the complete curve $\mathcal{C}$ consists of equilibria. For $P < 0$ these stationary points are attracting while for $P > 0$ they are repelling.

We know already that for $\delta > 0$ and S close to 1 there are three equilibria: an attracting node E_-, a saddle point E_+ and the stationary point $P = Q = 0$ whose stability depends on the sign of $S - 1$.

According to Fenichels geometric singular perturbation theory [2] a compact piece of the attracting branch of $\mathcal{C}$ will persist as an invariant manifold $\mathcal{A}_\delta$ for $\delta > 0$ sufficiently small. The flow on this invariant manifold is $\mathcal{O}(\delta)$–close to the flow on $\mathcal{C}$ in the slow system. Similarly, a compact normally hyperbolic piece of the repelling branch of $\mathcal{C}$ perturbs to an invariant manifold $\mathcal{R}_\delta$.

Trajectories can only pass from $\mathcal{A}_\delta$ to $\mathcal{R}_\delta$ if some equilibrium is close to the fold point. This situation was recently analyzed by Krupa and Szmolyan. Applying Theorem 3.1 in Krupa and Szmolyan [4] to our setting we get an intersection of the intersection of the two branches $\mathcal{A}_\delta$ and $\mathcal{R}_\delta$ of the slow manifold.

Proposition 3.1. *Let π be the transition map between two Poincaré sections Σ_{in} to Σ_{out} as in figure 3.1. Let $(P_{in}(\delta)$ be the intersection of the slow manifold with Σ_{in} and $(P_{out}(\delta)$ the intersection of the slow manifold with Σ_{out}.*

Then there exists a smooth function $S_c(\sqrt{\delta})$ such that $\pi(P_{in}) = P_{out}$ if and only if

$$S = S_c(\delta) = 1 + \frac{C_f(F-2)(2F-1)}{18(1+F)}\delta + \mathcal{O}(\delta^{3/2}).$$

The transition map π is defined only for S in an interval of length $\mathcal{O}(e^{-\kappa/\delta})$ around $S_c(\sqrt{\delta})$ where $\kappa > 0$ is some constant.

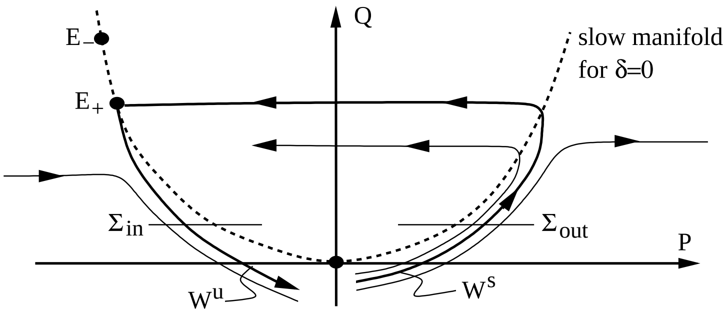

Figure 3. The geometric situation.

Proof of theorem 3.1:

By standard bifurcation theory one can show that for fixed $\delta > 0$ a unique family of periodic orbits is created in a supercritical Hopf bifurcation at $S = 1$. Since the equilibrium $(0,0)$ which is surrounded by the periodic orbits does not have purely imaginary eigenvalues for $S \neq 1$, the family of periodic orbits cannot terminate at another Hopf point. Note that no period–doubling may occur either, because we are in two dimensions only. Concerning the global fate of the family of periodic orbits, there remain only three other possibilities:

(i) the family of periodic orbits exists for all values $S > 1$,
(ii) the family of periodic orbits becomes unbounded in phase space or
(iii) the minimal period becomes unbounded.

Using the fact from proposition 3.1 that the transition map π is not defined for $S-1$ of order $\mathcal{O}(1)$ one can rule out the first two possibilities. We show that indeed (iii) occurs and the family of periodic orbits terminates at a homoclinic orbit.

Since the invariant manifold $\mathcal{A}_\delta$ contains the stationary point E_+, the unstable manifold W^u of E_+ coincides with the slow manifold $\mathcal{A}_\delta$ and therefore passes close to the fold point $(0,0)$. A branch W^s of the stable manifold of E_+ follows the fast direction backward and approaches $\mathcal{R}_\delta$. It then follows the slow manifold down to a vicinity of the fold point. Due to the strong attraction W^s and $\mathcal{R}_\delta$ are $\mathcal{O}(e^{-c/\delta})$-close to each other at a Poincaré section Σ_{out}. A homoclinic orbit exists iff W^u and W^s coincide. Because of the exponential closeness of these manifolds to the two branches $\mathcal{A}_\delta$ and $\mathcal{R}_\delta$ of the slow manifold this intersection happens for a parameter value $S_{hom}(\delta)$ which is exponentially close to $S_c(\delta)$. It therefore possesses the same expansion in δ. Moreover, the homoclinic orbit is a *canard trajectory* since it follows the unstable branch of the slow manifold for a time of order $\mathcal{O}(1)$.

Acknowledgments

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