Interaction of Homoclinic Solutions and Hopf Points in Functional Differential Equations of Mixed Type

Marc Georgi *

Abstract

In this article we study a homoclinic bifurcation in a general functional differential equation of mixed type. More precisely, we investigate the case when the asymptotic steady state of a homoclinic solution undergoes a Hopf bifurcation. Bifurcations of this kind are hard to analyse due to the lack of Fredholm properties. In particular, a straightforward application of a Lyapunov-Schmidt reduction is not possible.

As one of the main results we prove the existence of center stable and center unstable manifolds of steady states near homoclinic orbits. With their help, we can analyse the bifurcation scenario similar to the ODE-case and can show the existence of solutions which bifurcate near the homoclinic orbit, are decaying in one direction and oscillatory in the other direction. These solutions can be visualized as an interaction of the homoclinic orbit and small periodic solutions, which exist on account of the Hopf bifurcation, for exactly one asymptotic direction $t \to \infty$ or $t \to -\infty$.

1 Introduction

Functional differential equations of mixed type are equations of the form

$$\dot{x}(t) = f(x_t), \tag{1}$$

where $f : C^0([-a, b], \mathbb{R}^N) \to \mathbb{R}^N$, $a \geq 0$, $b \geq 0$ and $x_t \in C^0([-a, b], \mathbb{R}^N)$ denotes the "window" $x_t(\theta) := x(t+\theta)$. The case $a > 0$ and $b = 0$ corresponds to a pure delay differential equation.

Mixed type equations, both linear and nonlinear, occur naturally in problems of traveling waves in discrete spatial media such as lattices, see, for example [4, 5, 9, 10, 15, 16]. Often mixed type equations arise as traveling wave equations of spatially nonlocal equations of convolution type [1, 2, 13]. Traveling waves then appear as homoclinic or heteroclinic solutions of the corresponding traveling wave equation. A better understanding of homoclinic and heteroclinic bifurcations is therefore a crucial step in the understanding of traveling

*Institute of Mathematics I, Freie Universität Berlin, email: georgi@mi.fu-berlin.de

waves of the original equation, which may be a lattice differential equation for example.

Let us now assume that equation (1) is equipped with two real parameters λ, c and possesses a homoclinic solution h for $(\lambda, c) = (\lambda_*, c_*)$. Thinking of the special case of an ordinary differential equation (1) for the moment, the assumption of a hyperbolic steady state will then generically lead to the existence of a curve HOM in the two-dimensional parameter plane with the following property: For every parameter point on HOM, equation (1) possesses a homoclinic solution. It is now natural to ask what happens if the steady state becomes non-hyperbolic. More specifically, we are interested in the case where the linearization at the asymptotic steady state of equation (1) has exactly two purely imaginary eigenvalues $\pm i\omega$ for some real number $\omega \neq 0$. This would be a consequence of a Hopf bifurcation, which may occur at the steady state. From a technical point of view, such a bifurcation scenario is not easy to handle, since the linearization of (1) along the homoclinic solution h does not induce a Fredholm operator. Therefore, Lyapunov-Schmidt reductions to track down bifurcating solutions near the homoclinic orbit are not possible. It is one of the aims of this article to provide tools for studying these kind of bifurcations in the framework of general advance-delay equations.

Let us now illustrate the main results of this work for an ordinary differential equation (1), where we assume that f depends on two real parameters, hence $f(\cdot) = f(\cdot, \lambda, c)$, and $f(0, \lambda, c) = 0$ for all λ, c. We are interested in the interaction of a homoclinic solution h of (1) with the property

$$\lim_{t\to\infty} h(t) = 0$$

and a Hopf bifurcation, which occurs at the steady state. Let us furthermore assume that zero is (nonlinearly) stable with respect to the dynamics on the center manifold. As a consequence, h approaches zero for $t \to \infty$ along the center direction, generically. We conclude that the bifurcation has codimension two, which justifies the introduction of the parameters λ, c. From now on, we want to think of λ as the parameter, which induces the Hopf bifurcation. Thus, varying λ near some critical parameter-vector (λ_*, c_*), nontrivial periodic orbits arise near the steady state. Assuming that the Hopf bifurcation is supercritical (meaning that the periodic orbits are stable for $\lambda > \lambda_*$ with respect to the dynamics on the center manifold), the equilibrium becomes linearly unstable when increasing λ. For simplicity, we assume that nontrivial periodic orbits exist exactly for the parameter values (λ, c) and $\lambda > \lambda_*$. But in which way does the existence of the periodic orbits near zero influence the homoclinic solution h when varying (λ, c) near (λ_*, c_*)?

Our assumptions imply that the orbit of h lies in the intersection of the unstable and center stable manifold W^u and W^{cs}, respectively, of the steady state zero. Generically, these manifolds will not intersect transversely but with codimension one in the ambient space (which is $\mathbb{R}^N$ in the case of an ODE). However, if we supply (1) with $\dot{c} = 0$ and consider the *extended* center stable and unstable manifolds $\hat{W}^{cs}$, $\hat{W}^u$, respectively, in the extended phase space $\mathbb{R}^N \times \mathbb{R}$, we expect a transverse intersection along the homoclinic solution

$(h(t), c_*)$. Thus, we conclude the existence of an intersection point of W^u and W^{cs} for parameter values on some specific parameter curve $(\lambda, c) = (\lambda, c(\lambda))$ near (λ_*, c_*). Each intersection point induces a solution h^λ of (1). What can we say about the asymptotic behaviour of h^λ?

Let us consider a point $(\lambda, c(\lambda))$ on the curve with $\lambda > \lambda_*$. Then the periodic orbit is stable on the center manifold, and we expect h^λ to converge towards the periodic orbit for $t \to \infty$. What happens in backward time? Since h^λ approaches the equilibrium zero in backward time with exponential rate, we can actually think of two possibilities: Either h^λ approaches the steady state for $t \to -\infty$ as in figure 1, a) or converges to the periodic orbit for $t \to -\infty$, see figure 1, b). On the other hand, if $(\lambda, c) = (\lambda, c(\lambda))$ and $\lambda < \lambda_*$ then the steady state zero is linearly stable with respect to the dynamics on the center manifold. As a consequence, h^λ is a homoclinic solution to zero in this case.

Let us now summarize these observations in the next theorem, which is the main result of this article. For the moment, the reader should again think of (1) as an ordinary differential equation and we refer to theorem 6.1 in section 6 for a statement of this theorem in the general case.

Theorem 1.1

Consider the system

$$\dot{x}(t) = f(x_t, \lambda, c), \tag{2}$$

where $f(0, \lambda, c) = 0$ for all (λ, c). Assume that the steady state zero undergoes a supercritical Hopf bifurcation for $(\lambda, c) = (\lambda_*, c_*)$ when varying λ. Moreover, let $h(t)$ be a homoclinic solution of (2) for (λ_*, c_*) which approaches zero in forward time along the center direction (that is, not exponentially). If the extended center stable and strong unstable manifolds

$$\hat{W}^{cs} = \{(x, c) \in W^{cs} \times \mathbb{R} : |c - c_*| < \delta\}, \hat{W}^u = \{(x, c) \in W^u \times \mathbb{R} : |c - c_*| < \delta\}$$

intersect transversely at $(h(0), c_*)$ for $\lambda = \lambda_*$, the following is true. There exist a continuous function $c(\lambda)$ with $c(\lambda_*) = c_*$ and a family of functions $h^\lambda = h^{\lambda,1} : \mathbb{R} \to \mathbb{R}^N$ for $\lambda \approx \lambda_*$, such that

i) $h^{0,1} = h$ and each $h^{\lambda,1}$ is a solution of (2) on $\mathbb{R}$ for the parameters $(\lambda, c) = (\lambda, c(\lambda))$.

ii) Let $\lambda > \lambda_*$. Then $h^{\lambda,1}$ approaches a periodic orbit in forward time and converges to the steady state zero in backward time, see figure 1,a).

iii) Fix a $\lambda_+ \approx \lambda_*$. Then $h^{\lambda,1} \to h^{\lambda_+,1}$ uniformly on compact intervals as $\lambda \to \lambda_+$

The discussion of the existence of solutions $h^{\lambda,2}$, which are depicted in figure 1,b), is postponed to section 7.

We should point out that we assumed the existence of a homoclinic solution of (1) as a starting point. This is a nontrivial assumption in the case of a general advance-delay equation. However, by using center manifold theory [15, 16], continuation methods [1, 2, 22] or variational methods [7, 8, 6], there has been

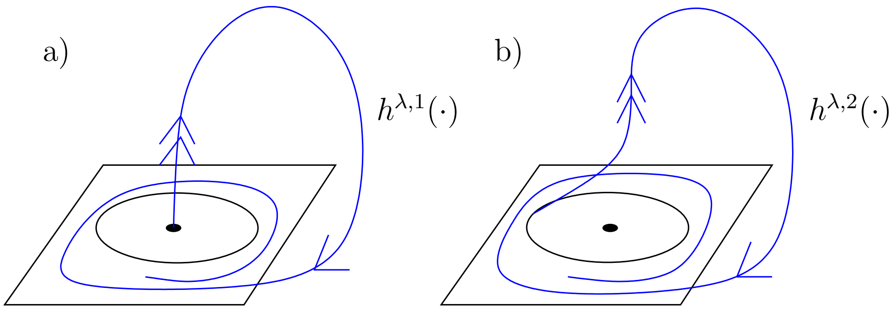

Figure 1: The solution $h^{\lambda,1}$ converges in backward time to zero, while the solution $h^{\lambda,2}$ approaches a periodic orbit with exponential rate in backward time.

some recent progress concerning the existence of homoclinic and heteroclinic solutions in advance delay equations.

Let us caution the reader that most of the above arguments are still formal for general functional differential equations of mixed type (1) so far. Neither the existence of a Hopf bifurcation (with the exception of a recent result of Lunel [17]) nor the existence of a center stable manifold for equations of the form (1) near the homoclinic orbit has been proved up to now. It is the goal of this paper to make the above picture rigorous for general functional differential equations.

Let us point out some difficulties which arise when studying equations of the form (1) with $a, b > 0$ (i.e. nontrivial advance-delay). First of all it is well known that (1) is ill-posed and will *not* generate a semiflow (see for example [24]). Therefore standard techniques, such as Poincaré maps, to analyse homoclinic bifurcations are not available. Thinking of a Lyapunov-Schmidt reduction instead, the linearization of (1) along the homoclinic solution h, namely the equation

$$\dot{y}(t) = D_1 f(h_t, \lambda_*, c_*) y_t, \tag{3}$$

becomes important. This equation induces a linear operator

$$\begin{aligned} \mathcal{L} : H^1(\mathbb{R}, \mathbb{R}^N) &\to L^2(\mathbb{R}, \mathbb{R}^N) \\ (\mathcal{L}y)(t) &= \dot{y}(t) - D_1 f(h_t, \lambda_*, c_*) y_t. \end{aligned}$$

Since the seminal work [20] of Mallet-Paret, the question under which conditions this operator is a Fredholm operator has been answered. Namely, $\mathcal{L}$ is a Fredholm operator if the limiting equation

$$\dot{y}(t) = D_1 f(0, \lambda_*, c_*) \tag{4}$$

is hyperbolic. This means in our case that equation (4) does not possess solutions of the form $y(t) = e^{i\beta t} y_*$ for any real number β. Unfortunately, on account of the Hopf bifurcation occurring at (λ_*, c_*), we cannot expect $\mathcal{L}$ to be a Fredholm operator. A straightforward application of the Lyapunov-Schmidt

method therefore fails and we have to proceed differently. Instead, we will show that equation (3) possesses center-dichotomies: There exist closed subspaces on which we can solve (3) in forward and backward time, respectively. On the contrary to *exponential* dichotomies, solutions do not decay exponentially, but may even grow algebraically. We will use these center-dichotomies to construct invariant manifolds along the homoclinic orbit in section 5. The existence of solutions, which behave as in figure 1, a), are addressed in section 6. We conclude with a discussion in section 7. Finally, we investigate a nontrivial toy-example in section 8, for which all hypotheses of theorem 1.1 can be verified explicitly.
We remind the reader that stable and unstable manifolds of steady states near heteroclinic orbits have already been constructed in [13]. In our situation hyperbolicity of the steady state fails. However, we are still able to construct center stable and center unstable manifolds of the steady state near the homoclinic orbit. It should be pointed out that the existence of these invariant manifolds can be used to analyse *general* homoclinc or heteroclinic bifurcations arising in equations of the form (1). In this respect our bifurcation scenario may be seen as a first example of a complicated bifurcation, where standard techniques such as a Lyapunov-Schmidt reductions fail. Furthermore, we introduce important and powerful tools from the theory of dynamical systems in the framework of general advance-delay equations, which will prove very helpful in analysing other bifurcations as well.

2 The framework

In the following we want to consider the system

$$\dot{x}(t) = f(x_t, \lambda, c), \tag{5}$$

where, for some $a, b > 0$, the function $f : C^0([-a, b], \mathbb{R}^N) \times \mathbb{R}^2 \to \mathbb{R}^N$ satisfies $f(0, \lambda, c) = 0$ for all λ, c. Furthermore, we want to assume that $f \in C^2$. Instead of working with (5) directly we prefer to study the related abstract equation

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} f(\phi(t, \cdot), \lambda, c) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix} =: F((\xi(t), \phi(t, \cdot)), \lambda, c). \tag{6}$$

Here $F : X \to Y$

$$\begin{aligned} Y &:= \mathbb{R}^N \times L^2([-a, b], \mathbb{R}^N), \\ X &:= \{(\xi, \varphi) \in Y \mid \varphi \in H^1([-a, b], \mathbb{R}^N) \text{ and } \varphi(0) = \xi\}. \end{aligned}$$

Let us note that this set-up has first been used in [15, 16] and is reminiscent of the "sun-star"-formulation of delay differential equations introduced by Lunel et al [17]. The next lemma clarifies the connection between solutions of (6) and our original equation (5). We first specify the notion of a solution of (6):

Definition 2.1
We call a continuous function $U(t) : [t_1, t_2) \to Y$ *a solution of (6) on* (t_1, t_2)*, where* $-\infty < t_1 < t_2 \leq \infty$*, if* $t \to U(t)$ *is continuous regarded as a map on* (t_1, t_2) *with values in* X *and differentiable regarded as a map on* (t_1, t_2) *with values in* Y *and (6) is satisfied on* (t_1, t_2)*.*
We call a continuous function $U(t) : [t_1, t_2) \to Y$ *a solution of (6) on* $(-\infty, t_2)$ *and* $t_2 \in \mathbb{R}$*, if* $t \to U(t)$ *is continuous regarded as a map on* $(-\infty, t_2)$ *with values in* X *and differentiable regarded as a map on* $(-\infty, t_2)$ *with values in* Y *and (6) is satisfied on* $(-\infty, t_2)$*.*

We can now state the next lemma:

Lemma 2.1
Let

$$U(t) = \begin{pmatrix} \xi(t) \\ \varphi(t)(\cdot) \end{pmatrix}$$

be a solution of (15) on $(t_1 - a, t_2 + b)$*. Then* $\varphi(t)(\theta) = \xi(t + \theta)$ *for all* $t \in (t_1 - a, b + t_2)$ *and* $\theta \in [-a, b]$ *with* $t + \theta \in (t_1 - a, t_2 + b)$*. Furthermore* $\xi(t)$ *solves (5) on the interval* (t_1, t_2)*.*

Proof
In order to show the lemma it suffices to prove

$$\varphi(t + \theta)(0) = \varphi(t)(\theta)$$

for all $t \in (t_1 - a, t_2 + b)$ and $\theta \in [-a, b]$ with $t + \theta \in (t_1 - a, t_2 + b)$, since $\varphi(t)(0) = \xi(t)$ for all t. For $t \in (t_1 - a, t_2 + b)$ we introduce the coordinates $(\tau, \theta) = (t + \theta, \theta)$ and consider

$$[\tilde{\varphi}(\tau)](\theta) := [\varphi(\tau - \theta)](\theta).$$

Let now $t \in (t_1 - a, t_2 + b)$ and $t + \theta \in (t_1 - a, t_2 + b)$ then we have $\tau \in (t_1 - a, t_2 + b)$ and $\tau - \theta \in (t_1 - a, t_2 + b)$. Since by assumption $\partial_t \varphi = \partial_\theta \varphi$ holds on the interval $(t_1 - a, t_2 + b)$ with respect to the coordinates (t, θ), we can deduce the identity $\tilde{\varphi}(\tau, \theta) = \tilde{\varphi}(\tau, 0)$ with respect to (τ, θ) for almost every τ. Since $\tilde{\varphi}(\tau, 0) = [\varphi(\tau)](0)$ and $[\varphi(\tau)](0) = \xi(\tau)$ depends continuously on τ, we have $\tilde{\varphi}(\tau, \zeta) = \tilde{\varphi}(\tau, 0)$ for every τ. This shows $\varphi(\tau - \theta)(0) = \varphi(\tau)(0)$ for all τ and θ and we have $\varphi(t + \theta)(0) = \varphi(t)(\theta)$. □

3 The Hopf bifurcation

In this section we want to prove a theorem which assures the existence of periodic solutions of equation (5) near zero. In the spirit of a Hopf bifurcation, we therefore assume the existence of purely imaginary eigenvalues $\pm i\omega$: More precisely, let us consider the linearization of (5) at the steady state, which is

$$\dot{y}(t) = D_1 f(0, \lambda_*, c_*) y_t. \tag{7}$$

Here, $D_1 f(0,\lambda,c) \in L(C^0([-a,b],\mathbb{C}^N),\mathbb{C}^N)$ and we can therefore find a function $\zeta^{\lambda,c} : \mathbb{C} \to \mathbb{C}^{N\times N}$ of bounded variation, such that $D_1 f(0,\lambda_*,c_*)\phi(\cdot) = \int_{-a}^{b} \phi(\theta) d\zeta^{\lambda_*,c_*}(\theta)$, see the appendix of [17]. We make the following hypothesis.

Hypothesis 1 (Hopf eigenvalues)
For $\eta \in \mathbb{C}$ consider the $\mathbb{C}^{N\times N}$-valued function

$$\triangle(\eta,\lambda,c) \;=\; \eta \cdot (id)_{N\times N} - \int_{-a}^{b} e^{\eta\theta} \cdot (id)_{N\times N}\, d\zeta^{\lambda,c}(\theta) \tag{8}$$

and let $\pm i\omega$ for some $\omega \neq 0$ be simple zeros of $det(\triangle(\cdot))$. Assume that $det(\triangle(ik)) \neq 0$ for $k \neq \pm\omega$ and $k \in \mathbb{R}$.

Let us note that the function $\triangle$ in (8) appears naturally when looking for solutions of (7), which are of the form $y(t) = e^{\eta t} y_*$ for some $\eta \in \mathbb{C}$, $y_* \in \mathbb{C}^N$. Indeed, fix some $\eta \in \mathbb{C}$. Then there exists a solution $y(t) = e^{\eta t} y_*$ of (7) for some $y_* \in \mathbb{C}^N$ if and only if $\det(\triangle(\eta,\lambda_*,c_*)) = 0$. Moreover, as we will see in lemma 3.1 below, $\det(\triangle(\cdot,\lambda_*,c_*))$ is the characteristic function of the linearization of the abstract equation (6) in $(\xi,\phi) = (0,0)$, which is

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} D_1 f(0,\lambda_*,c_*)\phi(t,\cdot) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix} =: \mathcal{A} \begin{pmatrix} \xi(t) \\ \phi(t,\cdot) \end{pmatrix}.$$

Note that $\mathcal{A}$ has a compact resolvent. Therefore, every $\eta \in \operatorname{spec}(\mathcal{A})$ is an eigenvalue of finite multiplicity (i.e. the generalized eigenspace is finite dimensional).

Lemma 3.1
Let $\eta \in \mathbb{C}$. Then η is an eigenvalue of $\mathcal{A}$ if and only if $det(\triangle(\eta,\lambda_*,c_*)) = 0$. Moreover, the algebraic multiplicity of η as an eigenvalue coincides with the order of η as a zero of $det(\triangle(\cdot,\lambda_*,c_*))$.

For a proof of this theorem we refer to [11]. Alternatively, the results in [17] can easily be adapted to our situation.
Hypothesis 1 states that $\pm i\omega$ are simple eigenvalues of $\mathcal{A}$. Let us denote by $E_c \subset X$ the center-eigenspace with respect to the eigenvaules $\pm i\omega$ and with $P_c : Y \to Y$, $\operatorname{Rg}(P_c) = E_c$, a corresponding projection. Finally, we set $E_h := \operatorname{Rg}(id_Y - P_c)$. In order to prove a Hopf bifurcation result for the abstract equation (15), we use the existence of a *center manifold* near the steady state $0 \in X$. Let us therefore state the next result concerning the existence of such a manifold.

Theorem 3.1 (Center manifold)
Let U denote a sufficiently small neighborhood of zero. Then under the assumptions on the smoothness of f and hypothesis 1, equation (15) possesses a two-dimensional, local invariant manifold $\mathcal{M} \subset X$, which is tangent to E_c at $0 \in \mathcal{M}$. In particular, solutions exist locally in $\mathcal{M}$. Moreover, $\mathcal{M}$ depends two times differentiable on λ, c, i.e. $\mathcal{M}$ can be locally represented as a graph of a function $\Psi^{\lambda,c} : E_c \cap U \to E_h \cap X$ for $(\lambda,c) \approx (\lambda_*,c_*)$, which is two times differentiable with respect to (λ,c).

This theorem has been proved in [11]. Alternatively, the existence of a center manifold can be deduced similarly to the existence of a center stable manifold, see section 5.
We call $\mathcal{M}$ *center manifold* for equation (15). Hypothesis 1 now implies that the linearization of the reduced vector field on $\mathcal{M}$ possesses the eigenvalues $\{\pm i\omega\}$. An additional condition, which guarantees that the eigenvalues cross the imaginary axis with non vanishing speed when varying λ, will now generically assure the existence of a Hopf bifurcation.

Hypothesis 2 (Crossing condition)
We assume the non-degeneracy condition

$$\partial_\lambda Re(\mu(\lambda_*, c_*)) = -Re[\partial_\mu det \triangle (i\omega, \lambda_*, c_*)^{-1}(\partial_\lambda det \triangle (i\omega, \lambda_*, c_*))] < 0.$$

This assumptions implies that the critical Hopf eigenvalues of $\mathcal{A}$ cross the imaginary axis from left to right when increasing $\lambda \approx \lambda_*$. Moreover, this hypothesis implies the existence of a "Hopf-curve" C_H in the (λ, c)-plane near (λ_*, c_*) that has the following property: The linearization of (5) at the steady state 0 for some parameter $(\lambda, c) \approx (\lambda_*, c_*)$ possesses purely imaginary eigenvalues $\pm i\kappa$ if and only if $(\lambda, c) \in C_H$. It is convenient to choose new parameters $(\tilde{\lambda}, c)$, such that C_H coincides with the c-axis in the $(\tilde{\lambda}, c)$-plane; i.e. $(\tilde{\lambda}, c) \in C_H$ if and only if $\tilde{\lambda} = 0$. Indeed, the existence of such parameters can be easily seen by an application of the implicit function theorem. Moreover we get the representation $(\lambda, c) = (\sigma_*(c) + \tilde{\lambda}, c)$ for some differentiable function $\sigma_*(c)$ with $\sigma_*(c_*) = \lambda_*$. From now on we will work with the equation

$$\dot{x}(t) = f(x_t, \tilde{\lambda}, c), \tag{9}$$

where the Hopf-curve C_H locally coincides with the c-axis.
Let us now consider the reduced vector field $F_{red} : \mathbb{R}^2 \to \mathbb{R}^2$ of (6) on the center manifold $\mathcal{M}$. Written in complex coordinates the normal form of the reduced vector field is of the form

$$F_{red}(z, \tilde{\lambda}, c_*) = \Big(A(\tilde{\lambda}, c_*) + i[B(\tilde{\lambda}, c_*)]\Big)\, z + D(\tilde{\lambda}, c_*)z|z|^2 + \text{ h.o.t},$$

where $A(0, c_*) = 0$, $B(0, c_*) = \omega$. Aiming at a supercritical Hopf bifurcation we have to assure that the nontrivial periodic orbits occur for $\tilde{\lambda} > 0$ and are stable with respect to the dynamics of the equation $\dot{z} = F_{red}(z, \tilde{\lambda}, c)$ if $\tilde{\lambda} > 0$ and $c \approx c_*$, which is the content of the next hypothesis.

Hypothesis 3
$Re(D(0, c_*)) < 0$.

Theorem 3.2 (Supercritical Hopf bifurcation)
Suppose that the hypotheses 1,2 and 3 are true. Consider equation (6) with the new parameters $(\tilde{\lambda}, c)$ (i.e. $f(\cdot, \lambda, c)$ is replaced by $f(\cdot, \tilde{\lambda}, c)$). Then (6) possesses nontrivial periodic solutions

$$\Gamma(t) = \Gamma^{\tilde{\lambda}, c}(t) = (\gamma^{\tilde{\lambda}, c}(t), \gamma_t^{\tilde{\lambda}, c})$$

for every $(\tilde{\lambda}, c) \approx (0, c_*)$ *and* $\tilde{\lambda} > 0$, *where* $\gamma^{\tilde{\lambda},c}(t) : \mathbb{R} \to \mathbb{R}^N$. *The solutions* $\Gamma(t)$ *are stable with respect to the dynamics on the center manifold* $\mathcal{M} = \mathcal{M}^{\tilde{\lambda},c}$ *and* $\gamma^{\tilde{\lambda},c}(t)$ *is a periodic solution of the equation (9). Moreover,* $\Gamma^{\tilde{\lambda},c}(t)$ *has the representation*

$$\Gamma^{\tilde{\lambda},c}(t) = A_H \sqrt{\tilde{\lambda}} e^{i\omega t} + \mathcal{O}(|c - c_*|\sqrt{\tilde{\lambda}} + |\tilde{\lambda}|). \tag{10}$$

for some $A_H \in X$ *with* $A_H \neq 0$.

Proof
With the help of theorem 3.2, the proof follows directly by an application of the corresponding version for Hopf bifurcations of ordinary differential equation, see [17]. □

4 Center-dichotomies

The homoclinic solution $h(t)$ of equation (5) induces via $H(t) := (h(t), h_t)$ a solution of the abstract equation (6). Since we are interested in the existence of center stable and unstable manifolds near the homoclinic orbit H, we have to deal with non-autonomous linear equations of the form

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} L(t)\phi(t, \cdot) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix} =: \mathcal{A}(t) \begin{pmatrix} \xi(t) \\ \phi(t, \cdot) \end{pmatrix}, \tag{11}$$

which are *asymptotically constant*, meaning that the limit $\lim_{t\to\pm\infty} L(t) = L_\pm$ exists in the norm $L(C^0([-a, b], \mathbb{C}^N), C^0([-a, b], \mathbb{C}^N))$. Equations of this form arise naturally by linearizing (6) along the homoclinic solution H.

Hypothesis 4
$L(t) : C^0([-a, b], \mathbb{C}^N) \to \mathbb{C}^N$ *can be represented in the form*

$$L(t)\varphi(\cdot) = \int_{-a}^{b} p(t, \theta)\varphi(\theta)d\theta + \sum_{k=1}^{m} A_k(t)\varphi(r_k),$$

where $t \mapsto p(t, \cdot) \in BC^0(\mathbb{R}, C^0([-a, b], \mathbb{C}^{N\times N}))$ *and* $A_k(\cdot)$ *is an element of* $BC^0(\mathbb{R}, \mathbb{C}^{N\times N})$ *for each* k. *We want to assume furthermore that the functions* $A_1(\cdot)$ *and* $A_m(\cdot)$ *do not vanish identically and* $-a = r_1 < \ldots < r_m = b$.

Our main goal in this section is to prove the existence of (time-dependent) closed subspaces of Y, on which we can solve (11). Before we do that we need a further assumption.

Hypothesis 5 (Unique-extension-property)
Let $x \in H^1(\mathbb{R}, \mathbb{C}^N)$ *be a solution of* $\dot{x}(t) = L(t)x_t$ *with* $x_\tau = 0$ *for some* $\tau \in \mathbb{R}$. *Then* $x \equiv 0$.

We can now state the next theorem which is the main result of this section.

Theorem 4.1 (Dichotomy on $\mathbb{R}_+$)
Assume that the hypotheses 4 and 7 are satisfied. Consider an equation of the form (11) and assume that $\mathcal{A}(t)$ is asymptotically constant. Choose $\delta > 0$. Then there exists a $\kappa > 0$, a constant $K > 0$ and a family of strongly continuous projections $Q(t) : Y \to Y$ for $t \in \mathbb{R}_+$, such that the following holds: Let $U \in Y$ and $t_0 \in \mathbb{R}_+$ then

- *there exists a continuous function $V^{cs}(\cdot) : [t_0, \infty) \to Y$ with $V^{cs}(t_0) = Q(t_0)U$. Moreover, $V^{cs}(t) \in Rg(Q(t))$ and $|V^{cs}(t)|_Y \le Ke^{\delta|t-t_0|}|U|_Y$ for all $t \ge t_0$ with $t, t_0 \in \mathbb{R}_+$.*
- *There exists a continuous function $V^u(\cdot) : (0, t_0] \to Y$ with $V^u(t_0) = (id - Q(t_0))U$. Moreover, $V^u(t) \in ker(Q(t))$ and $|V^u(t)|_Y \le Ke^{-\kappa|t-t_0|}|U|_Y$ for all $t_0 \ge t$ with $t, t_0 \in \mathbb{R}_+$.*

Moreover, if $U \in X$ then the functions $V^{cs}(t)$ and $V^u(t)$ are classical solutions of (11). In any case, if $U \in Rg(Q(t_0))$ and $U = (\xi, \phi(\cdot))$ then $V^{cs}(t) = (x(t), x_t)$, where $x : [-a + t_0, \infty) \to \mathbb{R}^N$ denotes the unique solution of $\dot{x}(t) = L(t)x_t$ on (t_0, ∞) with $x_{t_0} = \phi$. A similar statement holds for $V^u(t)$.

Theorem 4.1 has been proved in the case that $\lim_{t\to\pm\infty} \mathcal{A}(t) := \mathcal{A}_\pm$ exist and are *hyperbolic*; see Scheel et al [24] and [12]. Here, hyperbolicity means that the characteristic equations $\det(\triangle_\pm(\cdot))$ (corresponding to the operators $\mathcal{A}_\pm$) do not possess purely imaginary zeros. In this scenario even more is true, namely, the functions $V^{cs}(t)$ additionally converge to zero exponentially for $t \to \infty$ and we say that equation (11) possesses *exponential* dichotomies, see also [12].

Proof of theorem 4.1
Let us consider the equation $\dot{x}(t) = L(t)x_t$, where $L(t)$ satisfies hypothesis 4. Actually, we only need not to assume that $L(t)$ is asymptotically constant, but not necessarily with the same limits and we define by $L^{nh}_\pm := \lim_{t\to\pm\infty} L(t)$ the corresponding limits. Let us denote by $\det\triangle_\pm(\lambda)$ the associated characteristic equations, where

$$\triangle_\pm(\eta) = \eta - L^{nh}_\pm(e^{\eta\cdot}).$$

By assumption at least one of these functions possesses purely imaginary zeros. Let us now choose $\mu > 0$ small enough and consider the function $y(t) := e^{-\mu t}x(t)$ for a solution $x(t)$ of $\partial_t x(t) = L(t)x_t$. Then $y(t)$ solves

$$\partial_t y(t) = -\mu y(t) + L(t)[e^{\mu\cdot} y_t(\cdot)] =: L_{-\mu}(t)y_t. \tag{12}$$

Now $L_{-\mu}(t) \to L^\pm_h$ for $t \to \pm\infty$, where

$$L^\pm_h \varphi = -\mu\varphi(0) + L_{nh}(e^{\mu\cdot}\varphi(\cdot))$$

for $\phi \in C^0([-a, b], \mathbb{C}^N)$. If we denote by $\triangle^\pm_h(\cdot)$ the characteristic equation with respect to $L^\pm_h$, we observe the relation $\triangle^\pm_h(\lambda) = \triangle_\pm(\lambda + \mu)$. Thus, if $\mu > 0$ is small enough, the abstract equation

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} L_{-\mu}(t)\phi(t, \cdot) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix} =: \mathcal{A}^\mu(t) \begin{pmatrix} \xi(t) \\ \phi(t, \cdot) \end{pmatrix} \tag{13}$$

is asymptotically hyperbolic. Theorem 4.1 applies for equation (13) by the results in [24, 12], where we have the stronger estimate $|V^{cs}(t)| \leq Me^{-\sigma|t-t_0|}|V|$ for some arbitrary $\mu > \sigma > 0$ and $M = M(\sigma)$. Let us denote the family of projections corresponding to (13) by $Q^\mu(t)$, such that $\mathrm{Rg}(Q^\mu(t))$ coincides with the stable subspace. Before we proceed with the proof, we need the following definition. Let us denote by $\mathbf{e}^\eta_\mu : Y \to Y$ the bounded linear map

$$\mathbf{e}^\eta_\mu \begin{pmatrix} \xi \\ \varphi(\cdot) \end{pmatrix} := \begin{pmatrix} e^{\eta\mu}\xi \\ e^{\mu(\eta+\cdot)}\varphi(\cdot) \end{pmatrix}. \tag{14}$$

With the help of this map we can now define our desired family of projections by $Q(t_0) := \mathbf{e}^0_\mu Q^\mu(t_0)\mathbf{e}^0_{-\mu}$. Obviously, this defines projections from Y to Y, which are strongly continuous for $t_0 \geq 0$. It is now straightforward to show that initial values in $\mathrm{Rg}(Q(t_0))$ give rise to solutions which behave as stated in theorem 4.1. Similarly, the other claims of the theorem can be proved. □

We want to point out that the idea of the proof was to shift the spectrum of the asymptotic operators $\mathcal{A}_\pm$ to the *left*. In this way we obtained center stable dichotomies on $\mathbb{R}_+$ for equation (11), meaning that solutions do not necessarily decay exponentially in forward time. Analogously, we may shift the spectrum to the right instead. We can then prove that there exist solutions $V^s(t)$, defined for $t > t_0$, which decay exponentially for $t \to \infty$ and there exist solutions $V^{cu}(t)$ for $0 < t < t_0$, which satisfy the estimate $|V^{cu}(t)| \leq Me^{\delta|t-t_0|}$.

5 Invariant manifolds near the homoclinic orbit

Let us now consider the abstract equation

$$\begin{pmatrix} \partial_t\xi(t) \\ \partial_t\phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} f(\phi(t,\cdot),\tilde{\lambda},c) \\ \partial_\theta\phi(t,\cdot) \end{pmatrix} = F((\xi(t),\phi(t,\cdot)),\tilde{\lambda},c). \tag{15}$$

Our starting point is the existence of a homoclinic orbit $H(t)$ of (15). For convenience, we will state all hypotheses in terms of our original equation (9).

Hypothesis 6
The equation $\dot{x}(t) = f(x_t, 0, c_*)$ possesses a homoclinic solution $h(t)$, such that $\lim_{t\to\pm\infty} h(t) = 0$.

Note that due to the nonlinear stability of the steady state zero with respect to the dynamics on the center manifold, $H(t)$ (and therefore also $h(t)$) must approach zero along a strong unstable direction as $t \to -\infty$ and thus with exponential rate. We will make this claim rigorous in the next section, where we introduce and prove the existence of various invariant manifolds.
It is the aim of this chapter to prove the existence of solutions, which are depicted in figure 1 a). In order to achieve that, we will prove the existence of a center stable manifold $W^{cs,+}(0)$ of zero near the homoclinic orbit. This will be done in the next section.

5.1 The center stable manifold

If we parametrize solutions $U(t)$ of equation (15) near $H(t)$ by $U(t) = V(t) + H(t)$ then $V(t)$ solves the equation

$$\dot{V}(t) = \mathcal{A}(t)V(t) + \mathcal{G}(t, V(t), \tilde{\lambda}, c). \tag{16}$$

Here, $\mathcal{A}(t) : X \to Y$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} \xi(t) \\ \phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} D_1 f(h_t, 0, c_*)\phi(t, \cdot) \\ \partial_\theta \phi(t, \cdot) \end{pmatrix}$$

and we have set

$$\mathcal{G}(t, V, \tilde{\lambda}, c) = \mathcal{F}(H(t) + V, \tilde{\lambda}, c) - \mathcal{F}(H(t), 0, c_*) - \mathcal{A}(t)V.$$

More explicitly, $\mathcal{G}$ can be represented in the form

$$\mathcal{G}(t, (\xi, \varphi), \tilde{\lambda}, c) := \begin{pmatrix} f(h_t + \varphi, \tilde{\lambda}, c) - D_1 f(h_t, 0, c_*)\varphi - f(h_t, 0, c_*) \\ 0 \end{pmatrix}.$$

Let us assume from now on that $L(t) := D_1 f(h_t, \tilde{\lambda}, c)$ satisfies hypothesis 4 for all $t, \tilde{\lambda}, c$. On account of theorem 4.1 the system $\dot{V}(t) = \mathcal{A}(t)V(t)$ possesses a center-stable dichotomy on $\mathbb{R}_+$. Let us denote the associated solution operators by $\Phi^{cs}_+(t, s)$ and $\Phi^u_+(t, s)$ for $t \geq s \geq 0$ and $s \geq t \geq 0$, respectively. These are defined by $\Phi^{cs}_+(t, s)V := V^{cs}(t)$ and $\Phi^u_+(t, s)V := V^u(t)$, where the solutions $V^{cs}(t)$,$V^u(t)$ are defined in theorem 4.1. Furthermore, there exist solution operators $\Phi^{cs}_-(t, s)$ and $\Phi^u_-(t, s)$ on $\mathbb{R}_-$, which are defined for $0 \leq t \leq s$ and $0 \leq s \leq t$, respectively, and satisfy the estimates

$$\begin{aligned} \|\Phi^{cs}_-(t, s)\|_{L(\tilde{Z}, \tilde{Z})} &\leq Me^{\delta|t-s|} \\ \|\Phi^u_-(t, s)\|_{L(\tilde{Z}, \tilde{Z})} &\leq Me^{-\kappa|t-s|} \end{aligned} \tag{17}$$

for some $\kappa > 0$ and any small $\delta > 0$, where $M = M(\delta) > 0$ depends on the choice of $\delta > 0$.

Definition 5.1
We define $E^{cs}_\pm(0) := Rg(\Phi^{cs}_\pm(0, 0))$, $E^u_\pm(0) := Rg(\Phi^u_\pm(0, 0))$. Moreover, let us set

$$\hat{X} := \{(\xi, \phi) \in \mathbb{R}^N \times C^0([-a, b], \mathbb{R}^N) : \xi = \phi(0)\},$$

equipped with the norm $\|(\xi, \phi)\|_{\hat{X}} := |\xi|_{\mathbb{R}^N} + |\phi|_{C^0}$. Furthermore, let us set $\hat{E}^{cs}_\pm := E^{cs}_\pm(0) \cap \hat{X}$, $\hat{E}^u_\pm := E^u_\pm(0) \cap \hat{X}$. Both spaces are regarded as closed subspaces of $\hat{X}$. Finally, let $\tilde{Z} := \mathbb{R}^N \times L^\infty([-a, b], \mathbb{R}^N)$.

Theorem 5.1 (Center stable manifold)
Equation (15) possesses a local invariant C^2-manifold $W^{cs,+}(0) = W^{cs,+}_{\tilde{\lambda},c}(0) \subset \hat{X}$ near $H(0)$ which has the following properties:

a) $W^{cs,+}_{0,c_*}(0)$ is tangent to $\hat{E}^{cs}_+$ at $H(0) \in W^{cs,+}_{0,c_*}(0)$.

b) The manifold contains all points U_+, which are close to $H(0)$ with respect to the $\hat{X}$-norm and which admit a solution $U(t)$ of (15) on $(0,\infty)$ that stays uniformly close to $H(t)$ for $t \to \infty$.

c) If $U_+ \in W^{cs,+}(0)$ is close enough to $H(0)$ and $\tilde{\lambda} > 0$, there exists a continuous function $U : [0,\infty) \to \hat{X}$ with $U(t) = (\xi(t), \xi_t)$, such that $U(0) = U_+$ and $\xi(t) \in C^1(\mathbb{R}_+, \mathbb{R}^N)$ solves

$$\dot{\xi}(t) = f(\xi_t, \tilde{\lambda}, c) \tag{18}$$

on $(0,\infty)$. Morever, if $\tilde{\lambda} > 0$ then the ω-limit set of $U(t)$ consists either of the steady state zero or the periodic orbit $\Gamma = \Gamma_{\tilde{\lambda},c}$.

d) $W^{cs,+}(0)$ is continuously differentiable with respect to $\tilde{\lambda}$ and c: If we supply equation (15) with $\dot{\tilde{\lambda}} = 0, \dot{c} = 0$ then the extended system possesses a local invariant C^1-manifold $W^{cs,+}_{ex}(0) \subset \hat{X} \times \mathbb{R}^2$ and $W^{cs}_{\tilde{\lambda},c} := W^{cs,+}_{ex}(0) \cap \left(\hat{X} \times \{\tilde{\lambda}\} \times \{c\}\right)$ satisfies the properties a), b), c).

Remark 5.1
The property that every initial value in $W^{cs,+}(0)$ near $H(0)$ gives rise to a global *defined solution $U(t)$ for $\tilde{\lambda} > 0$ and $t > a$, which additionally stays uniformly close to $H(t)$ for all $t > 0$, cannot be expected in general. As we will see, this is a consequence of the fact that the periodic solutions, which appear due to the Hopf bifurcation, are stable with respect to the dynamics on the center manifold.*

Proof of theorem 5.1
Let us define for $V \in \hat{X}$ the modified nonlinearity by $\mathcal{G}_{mod}(t, V, \tilde{\lambda}, c) := \chi_\sigma(\|V\|_{\hat{X}}) \cdot \mathcal{G}(t, V, \tilde{\lambda}, c)$, where χ_σ is a cut-off-function with compact support in $[-\sigma, \sigma]$. Then for small enough $\delta > 0$

$$\mathcal{G}_{mod}(t, \cdot, \cdot, \cdot) : \hat{X} \times (-\tilde{\lambda}_*, \tilde{\lambda}_*) \times (-\delta + c_*, c_* + \delta) \to \hat{X}$$

is well defined, two times continuously differentiable, and a global Lipschitz map with a small (σ-dependent) Lipschitz constant. Our goal is to construct $V(\cdot)$ as a fixed point of the equation

$$\begin{aligned} V(t) &= \Phi^{cs}_+(t,0)V^{cs}_0 + \int_0^t \Phi^{cs}_+(t,s)\mathcal{G}_{mod}(s, V(s), \tilde{\lambda}, c)ds \\ &+ \int_\infty^t \Phi^u_+(t,s)\mathcal{G}_{mod}(s, V(s), \tilde{\lambda}, c)ds, \end{aligned} \tag{19}$$

where $V^{cs}_0 \in \hat{E}^{cs}_+$. We want to find fixed points $V(\cdot)$ in the space $BC^\gamma := BC^\gamma(\mathbb{R}_+, \hat{X})$ for some $\gamma > 0$, which satisfies $0 < \delta < \gamma < \kappa$ (see theorem 4.1 for the definition of δ, κ). Here, the norm $\|V(\cdot)\|_\gamma$ in BC^γ is defined by

$$\|V(\cdot)\|_\gamma = \sup_{t \geq 0} e^{-t\gamma} |V(t)|_{\hat{X}}.$$

Let us now discuss in which sense the right hand side of (19) is defined for fixed t. Note that the map $s \mapsto \Phi_+^{cs}(t,s)U$, regarded as a map with values in $\tilde{Z}$, is not Lebesgue integrable in general. Therefore the integrals, which appear in (19), cannot be considered in the Lebesgue sense. But they are well defined as weak* integrals as explained in the appendix. Lemma 9.1 states that the integral terms are actually elements of the space $\hat{X}$. Furthermore, lemma 9.2 in the appendix implies that the right hand side of (19) defines a (well defined) contraction $\mathcal{K}$ on the space $BC^{\gamma}(\mathbb{R}_+, \hat{X})$, if the Lipschitz constant δ_0 of the map $V \mapsto \mathcal{G}_{mod}(s, V, \tilde{\lambda}, c)$ and $\gamma > 0$ both are small enough. Indeed, we can estimate the first integral in (19) with the help of lemma 9.2 in the appendix by

$$\begin{aligned} \int_0^t e^{\delta(t-s)} |\mathcal{G}_{mod}(s, V(s), \tilde{\lambda}, c)|_{\tilde{X}} ds &\leq \int_0^t e^{\delta(t-s)} \delta_0 |V(s)|_{\hat{X}} ds \\ &\leq \int_0^t e^{\delta(t-s)} \delta_0 C e^{\gamma s} ds \\ &= \delta_0 C e^{\delta t} \left[e^{(\gamma-\delta)s} / (\gamma - \delta) \right]_0^t = \delta_0 \mathcal{O}(e^{\gamma t}) \end{aligned}$$

and the other integral can be estimated analogously. Thus, if the Lipschitz-constant δ_0 of $\mathcal{G}_{mod}$ is chosen small enough, (19) possesses a unique fixed-point $V_*(\cdot) \in BC^{\gamma}(\mathbb{R}_+, \hat{X})$ for every $V_0^{cs} \in \hat{E}_{cs}$ and $\tilde{\lambda} \approx 0$, $c \approx c_*$. Due to Vanderbauwhede [26], theorem 2, the map

$$\begin{aligned} \Psi(\cdot, \tilde{\lambda}, c) : \hat{E}_+^{cs} &\rightarrow \hat{E}_+^{u} \\ \Psi(\cdot, \tilde{\lambda}, c) : V_0^{cs} &\mapsto \Phi_+^{u}(0,0) V_*(0) \end{aligned}$$

is a global Lipschitz function and we can define our desired manifold by

$$W_{\varepsilon,c}^{cs,+}(0) := \operatorname{graph}(\Psi(\cdot, \tilde{\lambda}, c)) + H(0). \tag{20}$$

This manifold is then locally invariant; see [11] or [26] for more details. We will comment on the smoothness of this manifold in the last step of this proof.
Parameter dependence:
In order to study the parameter dependence of $W_{\varepsilon,c}^{cs,+}(0)$ we consider the extended system

$$\begin{pmatrix} \dot{\tilde{V}}(t) \\ \dot{\tilde{\lambda}} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \mathcal{A}(t)\tilde{V}(t) + D_{\tilde{\lambda},c}\mathcal{G}(t,0,0,c_*)(\tilde{\lambda}, c) + \mathcal{G}_{rest}(t, \tilde{V}(t), \tilde{\lambda}, c) \\ 0 \\ 0 \end{pmatrix}, \tag{21}$$

where $\mathcal{G}_{rest} + D_{\tilde{\lambda},c}\mathcal{G} = \mathcal{G}$ and $\mathcal{G}_{rest}$ is quadratic in $V, c, \tilde{\lambda}$. The geometric multiplicity of the eigenvalue zero of the t-dependent linear part $A^{ex}(t) : X \times \mathbb{R}^2 \rightarrow Y \times \mathbb{R}^2$ of (21) increases by two for every fixed t. Therefore, the equation $\dot{W}(t) = A^{ex}(t)W(t)$ possesses a dichotomy on $\mathbb{R}_+$ with associated solution operators $\tilde{\Phi}_+^{cs}(t,s) : Y \times \mathbb{R}^2 \rightarrow Y \times \mathbb{R}^2$ and $\tilde{\Phi}_+^{u}(s,t) : Y \times \mathbb{R}^2 \rightarrow Y \times \mathbb{R}^2$ for $t \geq s \geq 0$. We observe that the map $(V, \tilde{\lambda}, c) \mapsto \mathcal{G}_{rest}(t, V, \tilde{\lambda}, c)$ with values in $\hat{X}$ is a C^2-function. Arguing as in the step before, we see that equation (21) possesses a Lipschitz manifold. This proves claim d) of theorem 5.1.

Solutions of the fixed-point equation:

Let us now clarify in which sense fixed points $V_* \in BC^\gamma$ of (19) induce solutions of (18). The next claim has been proved in [13]: There exists an $\alpha > 0$, such that the following holds: If $V_*(\cdot) \in BC^\gamma$ is a fixed point of (18) and $\|V_*(t)\|_{\hat{X}} < \alpha$ for all $t > 0$ then $U_*(t) := H(t) + V_*(t)$ has the form $U_*(t) = (\xi^*(t), \xi^*_t)$ for some C^0- function $\xi^* : [-a, \infty) \to \mathbb{R}^N$ and $\xi \in C^1(\mathbb{R}_+, \mathbb{R}^N)$ is a solution of (18) on $(0, \infty)$. So part b) of the theorem is proved, if we verify that the solutions V_* of (19) remain uniformly close to zero for all $t > 0$. This will follow from the next step.

Asymptotic behaviour:

We now want to study the asymptotic behaviour of solutions $U(t)$ with initial values $U_+ \in W^{cs,+}(0)$. Let us consider a point $U_+ \in W^{cs,+}(0) \cap X$; then there exists a classical solution $U(t) = H(t) + V(t)$ of the modified equation

$$\begin{aligned} \dot{U}(t) &= \mathcal{F}_{mod}(U(t), \tilde{\lambda}, c) \qquad (22)\\ &= \mathcal{F}(H(t)) + \mathcal{A}(t)(U(t) - H(t)) + \mathcal{G}_{mod}(U(t) - H(t), \tilde{\lambda}, c). \end{aligned}$$

Let us write $U(t) = (\xi(t), \xi_t)$, suppress the parameter dependence from now on and note that $D_1 f(h_t)\bullet = D_1 f(0) \bullet + D_1^2 f(0)[h_t, \bullet] + \tilde{R}(h_t, \bullet)$, $f(h_t) = f(0) + D_1 f(0) h_t + R(h_t)$. Using (22) we obtain the equation

$$\begin{aligned} \dot{\xi}(t) &= D_1 f(0, 0, c_*)\xi_t + f_{mod}(\xi_t, \tilde{\lambda}, c)\\ &- f_{mod}(h_t, 0, c_*) - D_1 f_{mod}(h_t, 0, c_*)(\xi_t - h_t)\\ &+ R(h_t) + \tilde{R}(h_t, \xi_t - h_t) + D_1^2 f(0)[h_t, \xi_t - h_t]. \end{aligned}$$

This equation defines an abstract differential equation

$$\dot{U}(t) = AU(t) + G(t, U(t), \tilde{\lambda}, c) \qquad (23)$$

for $U = (\xi, \phi) \in X$, where the operator $A : X \to Y$ and $G(t, U, \tilde{\lambda}, c) : \hat{X} \to \hat{X}$ are defined by

$$A \begin{pmatrix} \xi \\ \phi(\cdot) \end{pmatrix} = \begin{pmatrix} D_1 f(0, 0, c_*)\phi(\cdot) \\ \partial_\theta \phi(\cdot) \end{pmatrix}$$

and

$$\begin{aligned} G(t, (\xi, \phi)) &= \begin{pmatrix} f_{mod}(\phi, \tilde{\lambda}, c) - f_{mod}(h_t, 0, c_*) \\ 0 \end{pmatrix}\\ &- \begin{pmatrix} -D_1 f_{mod}(h_t, 0, c_*)(\phi - h_t) + R(h_t) \\ 0 \end{pmatrix}\\ &+ \begin{pmatrix} \tilde{R}(h_t, \phi - h_t) + D_1^2 f(0)[h_t, \phi - h_t] \\ 0 \end{pmatrix}. \end{aligned}$$

Again on account of theorem 4.1, the linear equation $\dot{W} = AW$ possesses a center-stable dichotomy on $\mathbb{R}_+$ (and even on $\mathbb{R}$). If we choose $t_* > 0$ large enough, then for fixed $t \in (t_*, \infty)$ the nonlinearity $G(t, \cdot, \tilde{\lambda}, c)$ is a Lipschitz-function with a *globally* small Lipschitz-constant $\delta_0 = \delta_0(t_*)$. It is

now straightforward to show that (23) possesses a local center stable manifold $\hat{W}^{cs}$ near zero. The existence of $\hat{W}^{cs}$ follows analogously to the construction of $W^{cs,+}(0)$ with the help of an appropriate fixed point equation, which is posed on $BC^{\sigma}([t_*,\infty),\hat{X})$ for some $\sigma > 0$, see [11] for further details. As in theorem 5.5 in [26], one can now prove that every orbit in W^{cs} approaches some orbit on the center manifold. Note that the results in [26] are stated for ordinary differential equations. However, the results only rely on the existence of solution operators of linear equations; the existence of which has also been shown for our equations and we refer the reader to [11] for a detailed proof.
Let us now prove claim c) of the upper theorem and consider $\tilde{\lambda} > 0$ with $\tilde{\lambda} \sim 0$. In this case the picture on the center manifold is the following: there exists a periodic orbit, which is the ω-limit set of every orbit on the center manifold except the steady state. This shows that every orbit in $\hat{W}^{cs}$ stays uniformly close to zero and either the ω-limit set contains zero or a periodic orbit. Since every orbit starting in $W^{cs,+}(0)$ close enough to $H(0)$ finally lies in $\hat{W}^{cs}$ we have shown that orbits starting in $W^{cs,+}(0)$ are uniformly close to the orbit of H for all times.

Smoothness

Finally, we want to discuss the smoothness of the constructed manifold $W^{cs,+}(0)$. Since $\hat{X}$ is not an Hilbert space, the modified nonlinearity $\mathcal{G}_{mod}$ is not differentiable. Indeed, the cut off function is not C^1, because the norm is not necessarily C^1. Still we can prove that $W^{cs,+}(0)$ is of class C^2, where we use the general theory developed for the abstract formulation for retarded functional differential equations in [17], chapter 9. Indeed, the operator $\mathcal{G}_{mod}$ is C^2, when restricted to a sufficiently small neighborhood of zero, since there the cut off function is constant with value one. If we denote, as before, by $V^*(\cdot, V_0^{cs}) \in BC^{\gamma}$ the unique fixed point of the integral equation (19) for $V_0^{cs} \in \hat{E}_+^{cs}$, then the previous step actually shows that $V^*(\cdot, V_0^{cs}) \in BC^0([0,\infty),\hat{X})$ with small supnorm, if $V_0^{cs} \in \hat{E}_+^{cs}$ is sufficiently small. Hence, for all $t \geq 0$ and V_0^{cs} small enough, $V^*(\cdot,\cdot)$ takes values in a set where $\mathcal{G}_{mod}(t,\cdot,\tilde{\lambda},c)$ is C^2. This, however, is the main ingredient of the proof of smoothness and we refer to [17], section 9.7, for further details. $\square$

Analogously, one can show the existence of the following manifolds:

- There is a Lipschitz **center unstable manifold** $W^{cu,-}(0)$ of zero near $H(0)$ that contains all solutions $U(t)$ of (15), which exist for all $t < 0$ and stay uniformly close to the orbit of H. However, in contrast to $W^{cs,+}(0)$ not every solution starting in $W^{cu,-}(0)$ will necessarily stay near $H(t)$ for all negative t, since the periodic orbits Γ are unstable in backward time.

- The local **strong stable manifold** $W^{ss,loc}$ of zero which is of class C^2. This manifold is characterized as follows: Let $U_+ \in W^{ss,loc}$. Then there exists a continuous function $U(t) : [0,\infty) \to \hat{X}$ with $U(0) = U_+$ and $\|U(t)\|_{\hat{X}} \leq Me^{-\kappa|t|}$ for some constants $\kappa, M > 0$ as $t \to \infty$. Moreover, $U(t)$ has the form $U(t) = (\xi(t), \xi_t)$ for some C^0-function $\xi : [-a,\infty) \to \mathbb{R}^N$, $\xi \in C^1((0,\infty),\mathbb{R}^N)$ and $\xi(t)$ solves the equation $\dot{x}(t) = f(x_t,\tilde{\lambda},c)$.

- The **strong unstable manifold** $W^{u,-}(0)$ of zero which is of class C^2. This manifold is tangent to $\hat{E}^u_-(0)$ at $H(0) \in W^{u,-}(0)$ and is characterized as follows: Let $U_- \in W^{u,-}$. Then there exists a continuous function $U(t) : (-\infty, 0] \to \hat{X}$ with $U(0) = U_-$ and $\|U(t)\|_{\hat{X}} \leq Me^{-\kappa|t|}$ for some constants $\kappa, M > 0$ as $t \to -\infty$. Moreover, $U(t)$ has the form $U(t) = (\xi(t), \xi_t)$ for some C^0-function $\xi : (-\infty, b] \to \mathbb{R}^N$, $\xi \in C^1((-\infty, 0), \mathbb{R}^N)$ and $\xi(t)$ solves the equation $\dot{x}(t) = f(x_t, \tilde{\lambda}, c)$.

- The **local center unstable** W^{cu}_{loc} of zero of Lipschitz class. Let $U_- \in W^{cu,loc}$. Then there exists a continuous function $U(t) : (-\infty, 0] \to \hat{X}$ with $U(0) = U_-$ and $\|U(t)\|_{\hat{X}} \leq Me^{\kappa|t|}$ for some constants $\kappa, M > 0$ as $t \to -\infty$. Moreover, $U(t)$ has the form $U(t) = (\xi(t), \xi_t)$ for some C^0-function $\xi : (-\infty, b] \to \mathbb{R}^N$, $\xi \in C^1((-\infty, 0), \mathbb{R}^N)$ and $\xi(t)$ solves the equation $\dot{x}(t) = f(x_t, \tilde{\lambda}, c)$.

6 Solutions connecting the steady state to a periodic orbit

In this section we will construct solutions of the original equation

$$\dot{x}(t) = f(x_t, \tilde{\lambda}, c), \tag{24}$$

which begin to oscillate as $t \to +\infty$ and look like in figure 1 a). For simplicity, let us restrict to the generic case that the homoclinic orbit h of (24) approaches the steady state zero in forward time $t \to \infty$ along the center direction. More precisely, we want to assume that the homoclinic orbit H of the rescaled equation (15) does not lie in the strong stable manifold $W^{ss,loc}$ of zero for any $t > 0$ large enough.
Note that $H(0) \in W^{cs,+}_{0,c_*}(0) \cap W^{u,-}_{0,c_*}(0)$. We now want to prove the existence of an intersection point of these two manifolds as the parameters $\tilde{\lambda}, c$ are varied. Let us make the generic assumption that the manifolds $W^{cs,+}_{0,c_*}(0)$ and $W^{u,-}_{0,c_*}(0)$ intersect only along H. As always, the hypothesis will be stated in terms of our original equation (9).

Hypothesis 7
Fix $\eta > 0$ and consider the linear operator

$$\begin{aligned} \mathcal{L}^\eta(t) : H^{1,\eta}(\mathbb{R}, \mathbb{R}^N) &\to L^{2,\eta}(\mathbb{R}, \mathbb{R}^N) \\ (\mathcal{L}^\eta v)(t) &= \dot{v}(t) - D_1 f(h_t, 0, c_*) v_t, \end{aligned}$$

where the norm $|v|_{L^{2,\eta}}$ is defined by $|v|^2_{L^{2,\eta}} := \int_{\mathbb{R}} e^{-\eta t} |v(t)|^2 dt$ and $|v|_{H^{1,\eta}} = |v|_{L^{2,\eta}} + |v'|_{L^{2,\eta}}$. Assume that there exists a $\eta_* > 0$, such that $\mathcal{L}^\eta$ is a Fredholm operator of index zero with a one-dimensional kernel for every $\eta \in (0, \eta_*)$.

Note that $\mathcal{L}^\eta$ is indeed always a Fredholm operator of index zero, if $\eta > 0$ is chosen small enough; thus the only assumption concerns the dimension of the kernel.

Lemma 6.1
Suppose that hypothesis 7 is satisfied. Then $\Sigma = E^u_-(0) + E^{cs}_+(0)$ *is a closed subspace of* Y*, where* $E^u_-(0) = Rg(\Phi^u_-(0,0)\big|_Y)$ *and* $E^{cs}_+(0) = Rg(\Phi^{cs}_+(0,0)\big|_Y)$ *and* Φ^u_- *is defined in (17). Furthermore,* Σ *has codimension one in* Y*.*

Proof
Let us observe that for $\eta \in (0, \eta_*)$ the "translated" operator

$$\begin{aligned} \mathcal{L}^{trans}_\eta : H^1(\mathbb{R}, \mathbb{R}^N) &\rightarrow L^2(\mathbb{R}, \mathbb{R}^N) \\ (\mathcal{L}^{trans}_\eta w)(t) &= \dot{w}(t) + \eta w(t) - D_1 f(h_t, 0, c_*)(e^{\eta \cdot} w_t(\cdot)) \end{aligned}$$

is a Fredholm operator of index zero with a one-dimensional kernel. Indeed, this is true due to theorem A of Mallet-Paret, see [20]. Now let $v(\cdot) \in \ker(\mathcal{L}^\eta)$. Then $w(t) := e^{-\eta t} v(t \cdot T(0, c_*)) \in H^1(\mathbb{R}, \mathbb{R}^N)$ is a solution of $\mathcal{L}^{trans}_\eta w = 0$. On the other hand, every element in $\ker(\mathcal{L}^{trans}_\eta)$ induces an element in $\ker(\mathcal{L}^\eta)$. Thus, the kernel of $\mathcal{L}^{trans}_\eta$ is one-dimensional. Lemma 4 in [12] now states that the Fredholm index of $\mathcal{L}^{trans}_\eta$ and $\mathcal{T}^\eta : \mathcal{D}(\mathcal{T}^\eta) \subset L^2(\mathbb{R}, Y) \rightarrow L^2(\mathbb{R}, Y)$ and the dimension of their kernels coincide, where

$$\mathcal{T}^\eta : \begin{pmatrix} \xi(t) \\ \Phi(t, \cdot) \end{pmatrix} \mapsto \begin{pmatrix} \partial_t \xi(t) + \eta \xi(t) - D_1 f(h_t, 0, c_*)(e^{\eta \cdot} \Phi(t, \cdot)) \\ \partial_t \Phi(t, \cdot) - \partial_\theta \Phi(t, \cdot) \end{pmatrix}.$$

The domain $\mathcal{D}(\mathcal{T}^\eta)$ is actually independent of η and given by

$$\begin{aligned} \mathcal{D}(\mathcal{T}^\eta) = \{(\xi, \Phi(\cdot,\cdot)) \in L^2(\mathbb{R}, Y) : \quad & (\partial_t - \partial_\theta)\Phi(\cdot,\cdot) \in L^2(\mathbb{R} \times I, \mathbb{C}^N), \\ & \xi \in H^1(\mathbb{R}, \mathbb{C}^N), \Phi(t, 0) = \xi(t) \ \forall t\}, \end{aligned}$$

see [12],[24]. The operator $\mathcal{T}^\eta$ induces the abstract equation

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \Phi(t, \cdot) \end{pmatrix} = \begin{pmatrix} -\eta \xi(t) + D_1 f(h_t, 0, c_*)(e^{\eta \cdot} \Phi(t, \cdot)) \\ \partial_\theta \Phi(t, \cdot). \end{pmatrix}.$$

By theorem 5 in [11] this equation has exponential dichotomies with solution operators $\Phi^{s,+}_\eta, \Phi^{u,+}_\eta$ and $\Phi^{s,-}_\eta, \Phi^{u,-}_\eta$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. By lemma 4 in [13] the codimension of $\Sigma^\eta := \mathrm{Rg}(\Phi^{s,+}_\eta(0,0)) + \mathrm{Rg}(\Phi^{u,-}_\eta(0,0))$ in Y is one. But on account of the proof of theorem 4.1, we observe that $E^u_-(0) = \mathrm{Rg}(\mathbf{e}^0_\eta[\Phi^{u,-}_\eta(0,0)])$ and $E^{cs}_+(0) = \mathrm{Rg}(\mathbf{e}^0_\eta[\Phi^{s,+}_\eta(0,0)])$, see (14) for the definition of $\mathbf{e}^{\mathbf{0}}_\eta$. Thus, also Σ has codimension one in Y and the lemma is proved except the claim concerning the closeness of Σ, which will be addressed in the next lemma. $\square$

Finally, we show that Σ is a closed subspace of Y.

Lemma 6.2
The vector space $E^{cs}_+(0) + E^u_-(0) \subset Y$ *is closed.*

Proof
It is sufficient to show that for suitable small $\eta > 0$ the space $\Sigma^\eta = \mathrm{Rg}(\Phi^{s,+}_\eta(0,0)) + \mathrm{Rg}(\Phi^{u,-}_\eta(0,0))$ is closed (where the operators $\Phi^{s,+}_\eta, \Phi^{u,-}_\eta$ have been introduced in the previous lemma). Indeed, note that $E^u_-(0) = \mathrm{Rg}(\mathbf{e}^0_\eta[\Phi^{u,-}_\eta(0,0)])$ and

$E_+^{cs}(0) = \mathrm{Rg}(\mathbf{e}_\eta^0[\Phi_\eta^{s,+}(0,0)])$, which would then prove the lemma, since $\mathbf{e}_\eta^{\mathbf{0}}$ maps closed sets to closed sets (see (14) for a definition of $\mathbf{e}_\eta^{\mathbf{0}}$). We now eliminate the intersection of the spaces $\mathrm{Rg}(\Phi_\eta^{s,+}(0,0))$, $\mathrm{Rg}(\Phi_\eta^{u,-}(0,0))$ and write

$$\Sigma^\eta =: \mathrm{Rg}(\Phi_\eta^{s,+}(0,0)) \oplus E_\eta^{u,-},$$

where $E_\eta^{u,-}$ is a closed subspace of Y, which has only a nontrivial intersection with $\mathrm{Rg}(\Phi_\eta^{s,+}(0,0))$. Let us now consider a sequence $U^n = V^n + W^n$, where $V^n \in \mathrm{Rg}(\Phi_\eta^{s,+}(0,0))$, $W^n \in E_\eta^{u,-}$, and $U^n \to U$ with respect to the Y-norm. By approximation, if necessary, we can assume that $V^n, W^n \in X$. Let us write $V^n = (\xi^n, \Phi^n)$, $W^n = (\rho^n, \Psi^n)$. Then two cases are possible: Either

i) $\Phi^n(\cdot)$ and $\Psi^n(\cdot)$ are bounded in the $L^2([-a,b],\mathbb{C}^n)$-norm, or

ii) at least one sequence is unbounded.

We consider the first case and define the operator $P_+ : \mathrm{Rg}(\Phi_\eta^{s,+}(0,0)) \to \mathbb{R}^N \times L^2([0,b],\mathbb{R}^N)$ by

$$(\xi, \Phi(\cdot)) \mapsto (\xi, \xi_0(\cdot)\big|_{[0,b]}) = (\xi, \Phi(\cdot)\big|_{[0,b]}), \tag{25}$$

where $\xi(\cdot) \in L^2([-a,\infty),\mathbb{C}^n) \cap H^1(\mathbb{R}_+,\mathbb{C}^n)$ denotes the unique solution of

$$\dot{x}(t) = D_1 f(h_t, 0, c_*)x_t \tag{26}$$

with respect to the initial value $\Phi(\cdot)$. Then P_+ is compact, since $\xi_0(\cdot) \in H^1([0,b],\mathbb{R}^N)$. Analogously, $P_- : \mathrm{Rg}(\Phi_\eta^{u,-}(0,0)) \to \mathbb{R}^N \times L^2([-a,0],\mathbb{R}^N)$, defined by

$$(\rho, \Psi(\cdot)) \mapsto (\rho, \rho_0(\cdot)\big|_{[-a,0]}) = (\rho, \Psi(\cdot)\big|_{[-a,0]}) \tag{27}$$

is compact, where $\rho(\cdot) \in L^2([-\infty,b),\mathbb{C}^n) \cap H^1(\mathbb{R}_-,\mathbb{C}^n)$ denotes the unique solution to (26) with respect to the initial value $\Psi(\cdot)$. Since $\Phi^n(\cdot), \Psi^n(\cdot)$ are bounded, we conclude that the L^2-part of $P_+(V^n)$ converges in $L^2([0,b],\mathbb{C}^n)$. Due to $W^n = U^n - V^n$ also the L^2-part of W^n converges, if restricted to the interval $(0,b)$. Analogously, we can prove that the sequence $\Psi^n(\cdot)$ converges on $(-a,0)$ and therefore also $\Phi^n(\cdot)$ converges on $(-a,0)$. This shows that $\Phi^n(\cdot)$ and $\Psi^n(\cdot)$ converge in $L^2((-a,b),\mathbb{C}^n)$ and possess a limit in $\mathrm{Rg}(\Phi_\eta^{s,+}(0,0))$ and $E_\eta^{u,-}$ since these spaces are closed. This proves the first case i).
The second case ii) can be shown analogously to the proof of lemma 5 in [12]. It is here where we make use of the fact that the intersection of $E_\eta^{u,-}$ and $\mathrm{Rg}(\Phi_\eta^{s,+}(0,0))$ is trivial. □

Remark 6.1
We shall point out that the last two lemmas imply that the codimension of $\hat{E}_+^{cs} + \hat{E}_-^u$ in the space $\hat{X}$ is less or equal to one (see the notation on page 12 *for the definition of $\hat{E}_+^{cs}, \hat{E}_-^u$). Indeed,*

$$\hat{E}_+^{cs} + \hat{E}_-^u = Rg(\Phi_+^{cs}(0,0)\big|_{\hat{X}}) + Rg(\Phi_-^u(0,0)\big|_{\hat{X}}) \tag{28}$$

is dense in a codimension one subspace of $\hat{X}$. Moreover, arguing as in the proof of the previous lemma 6.2 one can show that the space $\hat{E}_+^{cs} + \hat{E}_-^u$ is closed in $\hat{X}$.

6.1 Transversality of the extended manifolds

We now want to look for intersection points of $W^{cs,+}(0)$ and $W^{u,+}(0)$ for $(\tilde\lambda, c) \neq (0, c_*)$. However, since these manifolds do not intersect transversely at $H(0)$ for $(\tilde\lambda, c) = (0, c_*)$, we cannot expect an intersection point for all parameter values as c and $\tilde\lambda$ are varied. It is therefore natural to consider the *extended* manifolds

$$\begin{aligned}\tilde W^{cs}_{\tilde\lambda} &= \{(U, c) \in W^{cs,+}_{\tilde\lambda,c}(0) : |c - c_*| < \delta\},\\ \tilde W^{u}_{\tilde\lambda} &= \{(U, c) \in W^{u,-}_{\tilde\lambda,c}(0) : |c - c_*| < \delta\}\end{aligned}$$

in the extended phase space $\hat X \times \mathbb{R}$, where $\delta > 0$ is some small real number. Generically, these extended manifolds then intersect transversely in $\hat X \times \mathbb{R}$, which is true if hypothesis 8 below is satisfied. As before, we will state the hypothesis in terms of our original equation (9). For the statement of that assumption we need the next lemma:

Lemma 6.3
Assume that the linear map $L(t) := D_1 f(h_t, 0, c_*)$ satisfies hypothesis 4 with some function $p(t, \theta, 0, c_0)$ and matrices $A_k(t, 0, c_0)$, such that $t \mapsto p(t, \cdot, 0, c_0) \in BC^0(\mathbb{R}, C^0([-a, b], \mathbb{R}^{N\times N}))$ and $t \mapsto A_k(t, 0, c_0) \in BC^0(\mathbb{R}, \mathbb{R}^{N\times N})$ for each k. Consider the adjoint equation

$$\partial_t v(t) = -\int_{-a}^{b} p^*(t-\theta, \theta, 0, c_*)v(t-\theta)d\theta - \sum_{k=1}^{m} A_k^*(t - r_k, 0, c_*)v(t - r_k), \quad (29)$$

where $p^*(t-\theta, \theta, 0, c_*)$ denotes the adjoint of the matrix $p(t-\theta, \theta, 0, c_*)$. Then equation (29) possesses a unique bounded solution $\rho(t) : \mathbb{R} \to \mathbb{R}^N$ (up to scalar multiples), which tends to zero exponentially fast as $t \to \infty$.

Proof
Consider the equation $\dot V(t) = \mathcal{A}(t)V(t)$, defined in (11), with $L(t) = D_1 f(h_t, 0, c_*)$. Then this equation possesses a center stable dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ by theorem 4.1 with solution operators Φ^{cs}_+, Φ^u_+ and Φ^{cs}_-, Φ^u_-, respectively. We choose a non-zero vector

$$\Psi^0 \in (\mathrm{Rg}(\Phi^{cs}_+(0,0) + \mathrm{Rg}(\Phi^u_-(0,0)))^\perp,$$

where $\perp$ denotes the orthogonal complement with respect to the Y-scalar-product. It has been proved in [12] (see the proof of theorem 6 there) that the operators $\Phi^{cs,+}_{adj}(t,s) := (\Phi^u_+(s,t))^*$, $\Phi^{u,+}_{adj}(t,s) := (\Phi^{cs}_+(s,t))^*$ and also $\Phi^{cs,-}_{adj}(t,s) := (\Phi^u_-(s,t))^*$, $\Phi^{u,-}_{adj}(t,s) := (\Phi^{cs}_-(s,t))^*$ define dichotomies of the adjoint equation $\dot W(t) = (\mathcal{A}(t))^* W(t)$ on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, see [11]. Here, for fixed t the map $\mathcal{A}(t)^*$ denotes the adjoint operator of the operator $\mathcal{A}(t)$ with respect to the Y-scalar product. Now $\Phi^u_+(0,0)^*\Psi^0 = \Phi^{cs}_-(0,0)^*\Psi^0$ and therefore

$$W(t) = \begin{cases} (\Phi^u_-(0,t))^*\Psi^0 & : \ t \geq 0 \\ (\Phi^{cs}_+(0,t))^*\Psi^0 & : \ 0 \geq t \end{cases} \quad (30)$$

defines a solution of the adjoint equation $\dot{W}(t) = \mathcal{A}(t)^* W(t)$ on $\mathbb{R}$. On account of lemma 3 in [12], $W(t)$ has the form $W(t) = (\rho(t), \Psi(t,\cdot))$ for some function ρ, which solves equation (29) and is an element in H^1_{loc}. Moreover, due to the definition of $W(t)$ in (30), the function $\rho(t)$ decays exponentially as $t \to \infty$. We remark that $\rho(t)$ also remains bounded for $t \to \infty$. This is a consequence of the *simplicity* of the Hopf-eigenvalues and we refer to [11], theorem 3.5, for a proof of the boundedness of ρ in backward time. Let us point out that we won't make use of the boundedness of ρ in backwards time, though. It is sufficient to know that $\rho(t)$ can grow at most with small exponential rate as $t \to \infty$, which can be seen by the definition of $W(t)$ in (30). Since every solution $\rho(t)$ of (29) induces a bounded solution of the abstract adjoint equation $\dot{W}(t) = \mathcal{A}(t)^* W(t)$ via $W(t) = (\rho(t), \Psi(t,\cdot))$ for some appropriate function $\Psi(t,\cdot)$ (see [11]), we know that $W(t)$ can be represented in the form (30) for some appropriate vector $\Psi_* \in Y$, which then also satisfies $\Psi_* \in (\mathrm{Rg}(\Phi^{cs}_+(0,0) + \mathrm{Rg}(\Phi^u_-(0,0)))^\perp$. Therefore, Ψ_* is a scalar multiple of Ψ_0, which proves uniqueness (up to scalar multiples) of $\rho(t)$ as a bounded solution of (29), which decays exponentially in forward time. □

We can now state the next hypothesis.

Hypothesis 8
For $g(t, \bullet, \tilde{\lambda}, c) := f(\bullet, \tilde{\lambda}, c) - D_1 f(h_t, 0, c_*)[\bullet]$ *let*

$$\int_{-\infty}^{\infty} \rho(s) \partial_c g(s, h_s, 0, c)\big|_{c=c_*} ds \neq 0, \tag{31}$$

where $\rho(\cdot) : \mathbb{R} \to \mathbb{R}^N$ *denotes the unique bounded solution of the adjoint equation in the statement of the previous lemma 6.3.*

This hypothesis will now assure that the extended manifolds $\tilde{W}^{cs}_{\tilde{\lambda}}$ and $\tilde{W}^u_{\tilde{\lambda}}$ intersect transversely in $\hat{X} \times \mathbb{R}$ at $H(0)$ for $\tilde{\lambda} = 0$.

Lemma 6.4
Assume that the hypotheses 7 and 8 are satisfied. Then

$$\begin{array}{rcl} T_{(H(0),c_*)}\tilde{W}^{cs}_0 & + & T_{(H(0),c_*)}\tilde{W}^u_0 = \hat{X} \times \mathbb{R} \\ T_{(H(0),c_*)}\tilde{W}^{cs}_0 & \cap & T_{(H(0),c_*)}\tilde{W}^u_0 = \mathit{span}\left\langle (\partial_t H(0), 0) \right\rangle . \end{array}$$

Proof
Let us first observe that

$$\begin{aligned} T_{(H(0),c_*)}\tilde{W}^{cs}_0 &= (\hat{E}^{cs}_+ \times \{0\}) + \mathrm{span}\{(\partial_t H(0), 0)\} + \mathrm{span}\{(\tilde{V}^{cs}_{0,c_*}(0), 1)\}, \\ T_{(H(0),c_*)}\tilde{W}^u_0 &= (\hat{E}^u_- \times \{0\}) + \mathrm{span}\{(\partial_t H(0), 0)\} + \mathrm{span}\{(\tilde{V}^u_{0,c_*}(0), 1)\}, \end{aligned}$$

where $\tilde{V}^{cs}_{0,c_*}(0)$ can be computed by differentiating (19) with respect to c at $c = c_*, V^{cs} = 0, V(\cdot) = 0, \tilde{\lambda} = 0$. Similarly, $\tilde{V}^u_{0,c_*}(0)$ can be computed by differentiating the corresponding fixed point equation of $W^{u,-}_{\tilde{\lambda},c}(0)$ with respect to c at $c = c_*$. More explicitly, we obtain

$$\tilde{V}^{cs}_{0,c_*}(0) = \int_{\infty}^{0} \Phi^u_+(0,\xi)\partial_c\mathcal{G}(\xi,0,0,c_*)d\xi,$$
$$\tilde{V}^{u}_{0,c_*}(0) = \int_{-\infty}^{0} \Phi^{cs}_-(0,\xi)\partial_c\mathcal{G}(\xi,0,0,c_*)d\xi.$$

On account of the remark after lemma 6.2 we know that $\operatorname{codim}(\hat{E}^{cs}_+ + \hat{E}^u_-) = 1$ in $\hat{X}$. Let Ψ_0 denote a vector, which spans the orthogonal complement of the space $\Sigma = E^{cs}_+(0) + E^u_-(0)$ in Y and note that $\hat{E}^{cs}_+ \subset E^{cs}_+(0)$, $\hat{E}^u_- \subset E^u_-(0)$. For the proof of this theorem it is therefore enough to show that

$$\left\langle \Psi^0, \tilde{V}^{cs}_{0,c_*}(0)\right\rangle_Y \neq \left\langle \Psi^0, \tilde{V}^{u}_{0,c_*}(0)\right\rangle_Y. \tag{32}$$

Indeed, this would prove that $\tilde{V}^{cs}_{0,c_*}(0) - \tilde{V}^{u}_{0,c_*}(0)$ does not lie in $E^{cs}_+(0) + E^u_-(0)$ and therefore not in $\hat{E}^{cs}_+ + \hat{E}^u_- \subset E^{cs}_+(0) + E^u_-(0)$. The condition (32) can be represented in the form

$$\left\langle \Psi^0, \int_{-\infty}^{0} \Phi^{cs}_-(0,\xi)\partial_c\mathcal{G}(\xi,0,0,c_*)d\xi - \int_{\infty}^{0} \Phi^u_+(0,\xi)\partial_c\mathcal{G}(\xi,0,0,c_*)d\xi \right\rangle_Y \neq 0,$$

which can be further simplified to

$$\int_{-\infty}^{\infty} \left\langle \tilde{\Psi}(\xi)\Psi^0, \partial_c\mathcal{G}(\xi,0,0,c_*)\right\rangle_Y d\xi \neq 0, \tag{33}$$

where

$$\tilde{\Psi}(t) = \begin{cases} \Phi^u_+(0,t)^* & : \ t \geq 0 \\ \Phi^{cs}_-(0,t)^* & : \ 0 > t \end{cases}$$

By the definition of $\mathcal{G}$ (see the definition below (16)) we have

$$\partial_c\mathcal{G}(t,0,0,c_*) = \begin{pmatrix} \partial_c g(t,h_t,0,c)\big|_{c=c_*} \\ 0 \end{pmatrix},$$

where $g(t,\bullet,\tilde{\lambda},c) := f(\bullet,\tilde{\lambda},c) - D_1 f(h_t,0,c_*)[\bullet]$. As we have already argued in lemma 6.3, the function $\tilde{\Psi}(t)\Psi_0$ can be represented in the form $\tilde{\Psi}(t)\Psi_0 = (\rho(t),\Psi(t,\cdot))$, where $\rho(\cdot) : \mathbb{R} \to \mathbb{R}^N$ solves the adjoint equation

$$\begin{aligned}\partial_t z(t) = &-\int_{-a}^{b} p^*(t-\theta,\theta,0,c_*)z(t-\theta)d\theta \\ &-\sum_{k=1}^{m} A^*_k(t-r_k,0,c_*)z(t-r_k).\end{aligned} \tag{34}$$

Therefore, (33) is equivalent to $\int_{\mathbb{R}} \left\langle \rho(\xi), \partial_c g(\xi,h_\xi,0,c)\big|_{c=c_*}\right\rangle_{\mathbb{R}^N} d\xi \neq 0$, which is satisfied if and only if hypothesis 8 is true. □

An application of the implicit function theorem leads to the following result.

Corollary 6.1
There exists a continuous curve $c = c(\tilde{\lambda})$ in the $(\tilde{\lambda}, c)$-plane, such that the manifolds $W^{cs,+}(0)$ and $W^{u,-}(0)$ have an intersection point $U^{\tilde{\lambda}}$ for $(\tilde{\lambda}, c) = (\tilde{\lambda}, c(\tilde{\lambda}))$. $U^{\tilde{\lambda}}$ induces a classical solution $U^{\tilde{\lambda}}(t) = (h^{\tilde{\lambda},1}(t), h_t^{\tilde{\lambda},1})$ of (15) on $\mathbb{R}$, which satisfies $U^{\tilde{\lambda}}(0) = U^{\tilde{\lambda}}$. Furthermore, $h^{\tilde{\lambda},1}$ solves the equation $\dot{x}(t) = f(x_t, \tilde{\lambda}, c(\tilde{\lambda}))$ on $\mathbb{R}$ and approaches the periodic orbit $\gamma_{\tilde{\lambda},c(\tilde{\lambda})}$ in forward time.

Proof
Fix $\tilde{\lambda} \approx$ and let the functions

$$\begin{aligned}
\Psi_1^{\tilde{\lambda}}(\cdot,\cdot) &: (\hat{E}_+^{cs} \times (-\delta + c_*, c_* + \delta)) \cap \mathcal{U} \to \hat{E}_+^u, \\
\Psi_2^{\tilde{\lambda}}(\cdot,\cdot) &: (\hat{E}_-^{u} \times (-\delta + c_*, c_* + \delta)) \cap \mathcal{U} \to \hat{E}_-^{cs}
\end{aligned}$$

be defined by the property that

$$\begin{aligned}
(\mathrm{graph}[\Psi_1^{\tilde{\lambda}}], \cdot) + (H(0), c_*) &= \tilde{W}_{\tilde{\lambda}}^{cs} \\
(\mathrm{graph}[\Psi_2^{\tilde{\lambda}}], \cdot) + (H(0), c_*) &= \tilde{W}_{\tilde{\lambda}}^{u},
\end{aligned}$$

where $\mathcal{U} \subset \hat{X} \times \mathbb{R}$ denotes a sufficiently small neighborhood of zero and $\delta > 0$ is small enough. Then $\Psi_1^0(0, c_*) = \Psi_2^0(0, c_*) = 0$. Let us now choose a small neighborhood $\hat{\mathcal{U}}$ of $(0, 0, c_*) \in \hat{X} \times \hat{X} \times \mathbb{R}$ and consider the bifurcation map

$$\begin{aligned}
\Gamma(\cdot,\cdot,\cdot,\cdot) &: ((\hat{E}_+^{cs} \times \hat{E}_-^u \times \mathbb{R}) \cap \hat{\mathcal{U}}) \times (-\delta_0, \delta_0) \to \hat{X} \times \mathbb{R} \\
\Gamma(V, \tilde{V}, c, \tilde{\lambda}) &= \Big[(H(0) + V + \Psi_1^{\tilde{\lambda}}((V, c), c)\Big] - \Big[(H(0) + \tilde{V} + \Psi_2^{\tilde{\lambda}}((\tilde{V}, c), c)\Big].
\end{aligned}$$

for $\delta_0 > 0$ small enough. Then $\Gamma((0, 0, c_*), 0) = 0$ and the linearization $D_{(V,\tilde{V},c)}\Gamma((0, 0, c_*), 0)$ of Γ is surjectiv on account of the previous lemma 6.4. The claims of the corollary therefore follow by an application of the implicit function theorem, if we can show that the one-dimensional kernel

$$\mathcal{K} = \mathrm{span}\,\langle \partial_t H(0), \partial_t H(0), 0, 0\rangle$$

of $D_{(V,\tilde{V},c)}\Gamma((0, 0, c_*), 0)$ admits a closed complement in $\hat{E}_+^{cs} \times \hat{E}_-^u \times \mathbb{R}$. In order to show this, we construct a closed complement of $\mathcal{K} \cap (\hat{E}_+^{cs} \times \hat{E}_-^u \times \{0\})$ in the space $\hat{E}_+^{cs} \times \hat{E}_-^u \times \{0\}$. By considering $\mathcal{K} \cap (\hat{E}_+^{cs} \times \hat{E}_-^u \times \{0\})$ as a subspace of $\mathrm{Rg}(\Phi_+^{cs}(0, 0)\big|_X) \times \mathrm{Rg}(\Phi_-^u(0, 0)\big|_X)$ with the X-topology, we may define a complement $\mathcal{C}$ of $\mathcal{K}$ in X via $\mathcal{C} := \mathrm{Rg}(\Phi_+^{cs}(0, 0)) \times \mathrm{Rg}(\Phi_-^u(0, 0)\big|_E)$ first. Here, E denotes a closed subspace of the Hilbert X of codimension one, which satisfies

$$\mathrm{Rg}(\Phi_-^u(0, 0)\big|_E) \perp \mathrm{span}\,\langle \partial_t H(0)\rangle\,,$$

i.e. $\partial_t H(0)$ is perpendicular to $\mathrm{Rg}(\Phi_-^u(0, 0)\big|_E)$ with respect to the X-scalar product. In particular, $\mathrm{Rg}(\Phi_-^u(0, 0)\big|_E) + \mathrm{span}\,\langle \partial_t H(0)\rangle = \mathrm{Rg}(\Phi_-^u(0, 0)\big|_X)$. Note that such a vector space $E \subset X$ is easy to construct, since X is a Hilbert space. Moreover,

$$\mathcal{C} \subset \mathrm{Rg}(\Phi_+^{cs}(0, 0)\big|_Y) + \overline{\mathrm{Rg}(\Phi_-^u(0, 0)\big|_E)} =: \Pi, \tag{35}$$

where the closure of $\mathrm{Rg}(\Phi^u_-(0,0)\big|_E)$ is considered with respect to the Y-topology. The crucial observation is that

$$\partial_t H(0) \notin \overline{\mathrm{Rg}(\Phi^u_-(0,0)\big|_E)},$$

since any nontrivial vector in the orthogonal complement of $\mathrm{Rg}(\Phi^u_-(0,0)\big|_E)$ with respect to the Y-scalar product remains in the orthogonal complement upon considering the closure of $\mathrm{Rg}(\Phi^u_-(0,0)\big|_E)$ in Y. Hence, Π defines a closed complement of $\mathcal{K}$ in Y (in particular, with trivial intersection). Let us finally consider the closure C of $\mathcal{C}$ in $\hat{X}$, i.e. $C = \bar{\mathcal{C}}$ with respect to the $\hat{X}$-norm. Then C defines a closed complement of $\mathcal{K}$ in $\hat{E}^{cs}_+ \times \hat{E}^u_-$: Note first that C intersects only trivially with $\mathcal{K}$ on account of (35). Indeed, the closure C is still contained in Π. Finally, since by construction of C the space $C + \mathcal{K}$ is dense in $\hat{X} \times \hat{X}$ we only have to show that $C + \mathcal{K}$ is closed. But this follows analogously to the proof of lemma 6.2. □

Let us note that we have only shown that initial values in $W^{cs,+}(0)$ give rise to solutions which converge either to a periodic orbit or the steady state, if $(\tilde{\lambda}, c)$ is auch that $\tilde{\lambda} > 0$. But since we have assumed that $H(t)$ does not lie in the strong stable manifold $W^{ss,loc}$ for any $t > 0$ large enough and since $W^{ss,loc}$ depends continuously on the parameters $\tilde{\lambda}, c$ we conclude that $U^{\tilde{\lambda}}(t)$ does not lie in $W^{ss,loc}$. That means that $h^{\tilde{\lambda},1}$ does not approach the steady state zero in forward time. Let us now summarize our results in the next theorem, which is the main result of this section. The bifurcation diagram is discussed in section 7.

Theorem 6.1 (Solutions with one oscillating tail)
Consider the equation

$$\dot{x}(t) = f(x_t, \tilde{\lambda}, c), \tag{36}$$

$x_t \in C^0([-a,b], \mathbb{R}^N)$, *which has been introduced in equation (9). Assume that the hypotheses 1 - 8 are satisfied and that the homoclinic solution $h(t)$ of (36) does not converge to zero exponentially fast for $t \to \infty$. Then there exists a continuous curve $HET(\tilde{\lambda})$ in the $(\tilde{\lambda}, c)$-parameter plane locally near $(\tilde{\lambda}, c) = (0, c_*)$. For every point on the curve there exists a solution $h^{\tilde{\lambda},1}$ of (36), which is defined on $\mathbb{R}$ and has the following properties (see also figure 1 a)):*

- $(0, c_*) = HET(0)$ *and* $h^{0,1} = h$.

- *Let* $\tilde{\lambda} > 0$. *Then there exist* $M, \beta > 0$, *such that* $h^{\tilde{\lambda},1}(t)$ *satisfies* $|h^{\tilde{\lambda},1}(t)| \leq Me^{-\beta|t|}$ *as* $t \to -\infty$ *and* $|h^{\tilde{\lambda},1}(t) - \gamma_{\tilde{\lambda},c(\tilde{\lambda})}(t + \theta_*)| \to 0$ *as* $t \to \infty$ *for some asymptotic phase* θ_*.

- *Fix* $\tilde{\lambda}_* \approx 0$. *Then* $h^{\tilde{\lambda},1} \to h^{\tilde{\lambda}_*,1}$ *uniformly on compact intervals of* $\mathbb{R}$ *as* $\tilde{\lambda} \to \tilde{\lambda}_*$.

7 Discussion

Exponential decay

Theorem 6.1 is also true if the homoclinic solution h approaches zero with exponential rate, though this behaviour is generically not expected. In this case, however, $H(t) \in W^{ss,loc}$ for $t > 0$ large enough and one has to argue why $U^{\tilde{\lambda}}(t)$ does not lie in $W^{ss,loc}$ for $t > 0$ large enough and $\tilde{\lambda} > 0$. Since the construction of the solutions $h^{\tilde{\lambda},1}$ in this case is a little bit more complicated, we refer to the discussion in [25], page 1283, or to [11] for further details.

Solutions with two oscillating tails

Let us now discuss how one can prove the existence of solutions $h^{\tilde{\lambda},2}$, which are depicted in figure 1 b). The existence proof we will present relies strongly on the existence of a C^1 local center unstable manifold W^{cu}_{loc} of the steady state zero. Note that we have only shown that W^{cu}_{loc} is a Lipschitz-manifold, see the end of section 5. If, however, this manifold is of class C^1 we can consider the intersection $W^{cu}_{loc} \cap W^{cs,+}(0)$ for $(\tilde{\lambda}, c) = (0, c_*)$ near $H(\tau)$, $\tau < 0$ small enough. This intersection is then two-dimensional and therefore one-dimensional within a suitable Poincaré-section at $H(\tau)$ (note that we have shown in the proof of corollary 6.1 how to construct Poincaré-sections in the space $\hat{X}$). By the implicit function theorem, the manifolds W^{cu}_{loc} and $W^{cs,+}(0)$ also intersect for $(\tilde{\lambda}, c) \neq (0, c_*)$ and in particular for $(\tilde{\lambda}, c) = (\tilde{\lambda}, c(\tilde{\lambda}))$. Hence, there exist points

$$U^{\tilde{\lambda}}, V^{\tilde{\lambda}} \in \; W^{cu}_{loc} \cap W^{cs,+}(0)$$

with associated solutions $U^{\tilde{\lambda}}(\cdot)$, $V^{\tilde{\lambda}}(\cdot)$ of (6) for $t \geq 0$, which approach the periodic orbit $\Gamma^{\tilde{\lambda},c(\tilde{\lambda})}$ in forward time $t \to \infty$. Moreover, $U^{\tilde{\lambda}}(t)$ is also defined for $t \leq 0$ and converges to the steady state zero as $t \to -\infty$. $V^{\tilde{\lambda}}(t)$, on the other hand, solves the equation (15) for all $t \leq 0$ with a suitable modified nonlinearity f, which appears in the proof of the center-unstable manifold. Since every solution in the center unstable manifold approaches a unique solution $Z(\cdot)$ on the center manifold with exponential rate in backward time, and the periodic orbit is unstable in backward time $t \to -\infty$ with respect to the reduced dynamics, we can choose $V^{\tilde{\lambda}}$ in such a way that $Z(0)$ (i.e. the initial value of the solution on the center manifold, which is approached by $V^{\tilde{\lambda}}(t)$) lies outside the set that is enclosed by the periodic solution on the two-dimensional center manifold. Since the projection $l : V^{cu} \mapsto Z(0)$ is continuous with respect to the $\hat{X}$-norm, where $V^{cu} \in W^{cu}_{loc}$ (see [27]), we conclude that there exists a point $V \in W^{cu}_{loc} \cap W^{cs,+}(0)$ with associated solution $V(\cdot)$ of (15) that satisfies $l(V) = \Gamma^{\tilde{\lambda},c}(0)$. Hence, $V(t) = (h^{\tilde{\lambda},2}(t), h^{\tilde{\lambda},2}_t)$ for some solution $h^{\tilde{\lambda},2}$ of (1), which looks as depicted in figure 1 b).

We should point out that once the existence of the solutions $h^{\tilde{\lambda},2}$ has been proved, we generically expect complicated behaviour near these solutions: Already for ordinary differential equations it is known that there exist complicated behaviour near orbits, which are homoclinic to periodic orbits. However, it is not clear how to detect this behaviour in general functional differential mixed type equations. We remark that for *nonautonomous* functional differential equations the existence of invariant sets, on which the dynamic of (1) is conjugated to a shift on two symbols, has been proved in [12].

The bifurcation diagram
Let us now take a closer look at the bifurcation diagram, figure 2, where the curve HET (see the statement of theorem 6.1) is depicted. First note that for each point $(\tilde{\lambda}, c)$ on HET the center stable manifold $W^{cs,+}(0)$ and unstable manifold $W^{u,-}(0)$ of zero have an intersection point. This intersection point induces a solution $h^{\tilde{\lambda}}(\cdot)$ of equation (9). If $\tilde{\lambda} < 0$, i.e. $(\tilde{\lambda}, c) \in$ HET lies in the half plane $\tilde{\lambda} < 0$, no periodic orbits exist and zero is linearly stable with respect to the dynamics on the center manifold. Thus, $h^{\tilde{\lambda}}(t)$ converges to zero for $t \to \pm\infty$. Since the steady state is hyperbolic, a result of Mallet-Paret [21] implies the existence of periodic solutions with large period near the curve HET in that region under some appropriate non degeneracy conditions. These periodic solutions are indicated with "Per" in figure 2. Let us now consider the other branch of the curve HET, where $\tilde{\lambda} > 0$. For parameters on the curve in this region there exist solutions $h^{\tilde{\lambda},1}$ of (9), which converge to zero in backward time $t \to -\infty$. Since the steady state is linearly stable in backward time and unstable in forward time with respect to the center dynamic, we expect the asymptotic behaviour of $h^{\tilde{\lambda},1}$ to be stable upon variation of the parameters $\tilde{\lambda}, c$. Thus, in the regions which are indicated by "Het", we still obtain solutions which converge towards a periodic orbit in forward time and approach the steady state in backward time. Of course, these solutions are induced by an intersection of the center stable and center unstable manifold of the system (15). Hence, these solutions will generically approach the steady state zero in backward time $t \to -\infty$ along a solution on the center manifold rather than along the strong unstable manifold.

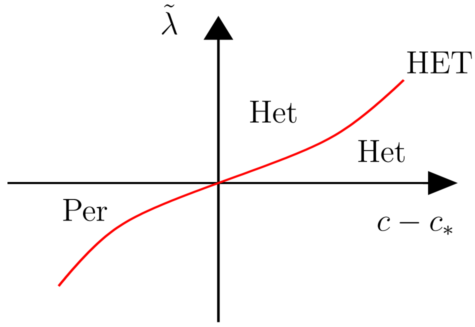

Figure 2: The bifurcation diagram of theorem 6.1.

8 An example

In this section we want to discuss a toy example, where all hypotheses of theorem 6.1 can be verified explicitly. Our main motivation for discussing this

example is that it can serve as a guideline how certain assumptions of theorem 6.1 can be verified in view of of more relevant examples.
The toy example under investigation is

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ x(t) - x(t)^2 + (c-1)x(t) + \gamma x(t-\chi)z(t) \\ \frac{2}{3}(L(z_t) - z(t)) + g(z(t), \tilde{\lambda}, c) + k(x(t))z(t) \end{pmatrix}, \tag{37}$$

where $x, y, z \in \mathbb{R}$ and $k(\cdot), g(\cdot, \tilde{\lambda}, c)$ are real-valued functions, $\chi, \gamma, \zeta, c, \tilde{\lambda} \in \mathbb{R}$ are constants and $L : C^0([-2\pi, 2\pi], \mathbb{R}) \to \mathbb{R}$ is a linear map, which will be specified later.
This toy example has been discussed in [11]. For the sake of completeness we will present most of the relevant calculations. The structure of equation (37) is as follows: If we set $\tilde{\lambda} = 0, z = 0$ in equation (37) and assume that $g(0, 0, c) = 0$ then the dynamics of (37) reduces to the ordinary differential equation

$$\ddot{x}(t) = x - x^2 + (c-1)x.$$

This equation possesses a homoclinic solution $h^1(t)$ for $c = 1$. The $\dot{z}$-equation is a nontrivial functional differential equation of mixed type; posed on the space $C^0([-2\pi, 2\pi], \mathbb{R})$. We now want to show that there exist functions k, g and a constant χ, such that the $\dot{z}$-equation undergoes a supercritical Hopf bifurcation and that the hypotheses 1-8 are satisfied.

Hopf bifurcation
Let us define the linear operator L by

A1) $L\varphi(\cdot) := \frac{1}{20}\varphi(-2\pi) + \frac{1}{5}\varphi(0) + \frac{3}{4}\varphi(2\pi)$,

where we note that $\frac{1}{20} + \frac{1}{5} + \frac{3}{4} = 1$; that is $L(1) = 1$. Furthermore we set $k(0) = 0$. Let us now look at the linearization of (37) at $(x, y, z) = (0, 0, 0)$. The characteristic function $\det\triangle(\mu, c, \tilde{\lambda})$ reads in this case:

$$\det \triangle (\mu, c, \tilde{\lambda}) = (\mu^2 - 1) \cdot (\frac{2}{3}(L(e^{\mu \cdot}) - 1) + g'(0, \tilde{\lambda}, c) - \mu). \tag{38}$$

Here and in the following we denote by ' the derivative with respect to the x-component. Let us now make the ansatz $g(x, \tilde{\lambda}, c) := \tilde{g}(x, \tilde{\lambda}) + (c-1)x$ for a still unspecified function $\tilde{g}(\cdot, \tilde{\lambda})$. We observe that purely imaginary zeros of the characteristic function occur if and only if the factor $(\frac{2}{3}(L(e^{\mu \cdot}) - 1) + \tilde{g}'(0, 0) - \mu)$ vanishes for some $\mu \in i\mathbb{R}$. For $\mu = is$ and $s \in \mathbb{R}$ we obtain from (38):

$$\begin{aligned} \mathrm{Im}(is) &= \frac{2}{3}L(\sin(s\bullet)) - s \\ \mathrm{Re}(is) &= \frac{2}{3}(L(\cos(s\bullet)) - 1) + \tilde{g}'(0, 0). \end{aligned} \tag{39}$$

We observe that there exists a zero $s_N \neq 0$ of the imaginary part on account of $\frac{2}{3}L(\theta) > 1$ and the mean value theorem. Of course $s_N < \frac{2}{3}$, since otherwise $s > \frac{2}{3} \geq \frac{2}{3}L(\sin(s\bullet))$. The next assumption guarantees that s_N is also a zero of the real part:

A2) Let $g(x, \tilde{\lambda}, c) = \tilde{g}(x, \tilde{\lambda}) + (c-1)x$ and $\tilde{g}'(0, 0) := -\frac{2}{3}(L(\cos(s_N\bullet)) - 1)$.

Now let s_* be an arbitrary real number, which satisfies (39). Then

$$\begin{aligned}
\frac{2}{3}\cos(s_* 2\pi) &= \left[\frac{1}{20}+\frac{3}{4}\right]^{-1}\left(\tilde{g}'(0,0)-\frac{2}{3}\cos(0)\frac{1}{5}\right)\\
\frac{2}{3}\sin(s_* 2\pi) &= \left[\frac{3}{4}-\frac{1}{20}\right]^{-1} s_*.
\end{aligned}$$

Squaring and adding both equations leads to a quadratic equation in s_*, which possesses two solutions, namely $\pm s_N$. This shows that there are no other purely imaginary zeros of the characteristic equation $\det\triangle(\cdot,1,0)$. An analogous argument now shows that the zeros $\mu = \pm i s_N$ are simple zeros of the characteristic function. The additional assumption

A3) $\partial_{\tilde{\lambda}}\tilde{g}'(0,\tilde{\lambda})\big|_{\tilde{\lambda}=0} \neq 0$

assures that the real parts of the critical Hopf eigenvalues $\pm i s_N$ cross the imaginary axis with non vanishing speed as $\tilde{\lambda}$ is varied near $\tilde{\lambda}=0$. Indeed, let $\mu(\tilde{\lambda})$ denote the branch of critical eigenvalues with $\mu(0) = i s_N$. Then

$$\partial_{\tilde{\lambda}}\mu(\tilde{\lambda}) = -[\partial_\mu \det(\triangle(\mu(\tilde{\lambda}),1,\tilde{\lambda}))]^{-1}\left[\partial_{\tilde{\lambda}}\tilde{g}'(0,\tilde{\lambda})\right]$$

near $\tilde{\lambda}=0$ and the real part satisfies

$$\partial_{\tilde{\lambda}}\mathrm{Re}\mu(0) = -C(-s_N^2 - f'(0,1))\ \left[\frac{2}{3}L((\bullet)\cos(s_N\bullet))-1\right]\ \partial_{\tilde{\lambda}}\tilde{g}'(0,0)\neq 0,$$

where $C>0$ is a constant. Note that the term $\frac{2}{3}L((\bullet)\cos(s_N\bullet))-1$ does not vanish. The next hypotheses guarantee that the Hopf bifurcation is supercritical. The relevant calculations have been carried out in [11].

A4) Let $\partial_{\tilde{\lambda}}\tilde{g}'(0,\tilde{\lambda})\big|_{\tilde{\lambda}=0}\neq 0$ with sign chosen such that

$$-(-s_N^2-1)\ \left[\frac{2}{3}L((\bullet)\cos(s_N\bullet))-1\right]\ \partial_{\tilde{\lambda}}\tilde{g}'(0,0)>0.$$

A5) Let $\tilde{g}''(0,0)+k''(0)=0$ and $\tilde{g}'''(0,0)+k'''(0)\neq 0$, such that

$$(\tilde{g}'''(0,0)+k'''(0))(-s_N^2-1)(\frac{2}{3}L((\bullet)\cos(s_N\bullet))-1)<0.$$

The Unique-Extension-Property
We consider the linearization of (37) along the homoclinic solution $(h^1(t),h^2(t),0)$ for $c=1$ and $\tilde{\lambda}=0$:

$$\begin{pmatrix}\dot{u}(t)\\ \dot{v}(t)\\ \dot{w}(t)\end{pmatrix} = \begin{pmatrix} v(t)\\ (1-2h^1(t))u(t)+\gamma(h^1(t-\chi))w(t)\\ \frac{2}{3}(L(w_t)-w(t))+\partial_1\tilde{g}(0,0)w(t)+k(h^1(t))w(t)\end{pmatrix}. \tag{40}$$

Let $(u(t),v(t),w(t))$ be a bounded solution of (40), such that $(u_\tau,v_\tau,w_\tau)=0$ for some $\tau\in\mathbb{R}$. On account of section 4.4 in [24] we conclude $w(\cdot)=0$.

Therefore (40) reduces to a system of ordinary differential equation, which shows that $(u(\cdot), v(\cdot), w(\cdot)) = 0$.

The kernel of $\mathcal{L}$
Let us now show that under further conditions we can assure that $\mathcal{L} : H^1(\mathbb{R}, \mathbb{R}^3) \to L^2(\mathbb{R}, \mathbb{R}^3)$ has a one-dimensional kernel, where

$$(\mathcal{L}(u(\cdot), v(\cdot), w(\cdot)))(t) = \partial_t(u(t), v(t), w(t)) - L(t)(u_t, v_t, w_t) \tag{41}$$

and $L(t)(u_t, v_t, w_t)$ is defined by the right hand side of equation (40) for each fixed t. We start with w-component of $\mathcal{L}(u, v, w) = 0$, which is

$$\dot{w}(t) = (\frac{2}{3}(L(w_t) - w(t)) + \tilde{g}'(0,0)w(t) + k(h^1(t))w(t)) =: \Lambda(t)w_t. \tag{42}$$

We want to denote by $W_0^{1,\infty}(-\zeta, \zeta)$ the set of all functions $w(\cdot) \in W^{1,\infty}(-\zeta, \zeta)$, which satisfy $w(\zeta) = w(-\zeta) = 0$. We can now use theorem 7.3 of Mallet-Paret and Lunel in [19].

Theorem (Mallet-Paret)
Consider the equation

$$\dot{w}(t) = a(t)w(t - 2\pi) + b(t)w(t) + c(t)w(t + 2\pi),$$

where $a, b, c : \mathbb{R} \to \mathbb{R}$ are bounded and measurable. Let $a(t) > 0, c(t) > 0$ or $a(t) < 0, c(t) < 0$ for almost every $t \in \mathbb{R}$. Then $\mathrm{kern}(\mathcal{B}^\zeta) = \{0\}$ for every $\zeta > 0$, where $\mathcal{B}^\zeta : W^{1,\infty}(-\zeta, \zeta) \to L^\infty(-\zeta, \zeta)$ is defined by

$$(\mathcal{B}^\zeta w(\cdot))(t) := \partial_t w(t) - (a(t)w(t - 2\pi) + b(t)w(t) + c(t)w(t + 2\pi))$$

and where we have extended $w(\cdot)$ by zero outside $(-\zeta, \zeta)$.

In our case $a(t) = \frac{1}{20}$ and $c(t) = \frac{3}{4}$. With the help of this theorem we can now show that:

a) The kernel of the operator $\mathcal{B}^\zeta$ with $(\mathcal{B}^\zeta w(\cdot))(t) := \partial_t w(t) - \Lambda(t)w_t$ is at most three-dimensional.

b) Let $w(\cdot)$ be a nontrivial solution of (42) on $\mathbb{R}$. Then there exists at most one interval $J \subset \mathbb{R}$ with $|J| = 4\pi$ on which $w(\cdot)$ possesses zeros. If $w(\cdot)$ has more than one zero, then $w(\cdot)$ is identical zero between two successive zeros.

For a proof of these facts let $w^1, w^2, w^3, w^4 \in H^1(\mathbb{R}, \mathbb{R})$ denote four linear independent solutions of (42). Choose any $\zeta > 0$, such that $[-2\pi, 2\pi] \subset (-\zeta, \zeta)$. Then there exist linear combinations v^1 of w^1, w^2 and v^2 of w^3, w^4, respectively, such that $v^2(\zeta) = v^1(\zeta) = 0$. If v^1 and v^2 would be linear independent there would exist a nontrivial linear combination z of v^1, v^2 with $z(-\zeta) = z(\zeta) = 0$. z defines a kernel element of $\mathcal{B}^\zeta$ and $z = 0$ on $[-2\pi, 2\pi]$ and thus $z = 0$. This shows that there are at most three linear independent solutions w^1, w^2, w^2 which proves claim a). With similar arguments one can show claim b).

We note that a nontrivial kernel element of the adjoint operator $\mathcal{L}^* : \mathcal{D}(\mathcal{L}^*) \subset L^2(\mathbb{R}, \mathbb{R}^3) \to L^2(\mathbb{R}, \mathbb{R}^3)$ is given by

$$t \mapsto (-\partial_t h^2(t), \partial_t h^1(t), 0). \tag{43}$$

Let us denote by $\Theta : H^1(\mathbb{R}, \mathbb{R}^2) \subset L^2(\mathbb{R}, \mathbb{R}^2) \to L^2(\mathbb{R}, \mathbb{R}^2)$ the map

$$\Theta(u, v)(t) = (\partial_t u(t), \partial_t v(t)) - (v(t), (1 - 2h^1(t))u(t))$$

We now want to show that there exist a constant $\chi \in \mathbb{R}$, such that we have $(0, h^1(\cdot - \chi)w(\cdot)) \notin \mathrm{Rg}(\Theta)$ for any solution $w(\cdot)$ of (42). This is equivalent to

$$\int_{\mathbb{R}} [h^1(s - \chi)w(s)]\partial_t h^1(s)ds \neq 0, \tag{44}$$

if we show that there is a unique kernel element (up to scalar multiples) of $\mathcal{L}^*$. The function $\partial_t h^1(t) = h^2(t)$ is different from zero for all $t > t_*$ for some appropriate t_* and converges to zero for $|t| \to \infty$ with exponential rate. Moreover, there exists some $M > 0$, such that all $w^i(t)$ have a constant sign on (M, ∞) and we can assume that all functions are positive. With these arguments we can state the next assumption.

A6) Let $\chi \in \mathbb{R}$ be such that (44) is satisfied for $w = w^i$, $i = 1, 2, 3$.

This assumption shows that $\ker(\mathcal{L})$ is one-dimensional, if the kernel of Θ is one-dimensional. But kernel elements of Θ solve an ordinary differential equation on $\mathbb{R}$. The results in [23] now show that the kernels of Θ and Θ^* are in fact one-dimensional.

Transversality
Let us now show that

$$\begin{pmatrix} 0 \\ \partial_t h^1(\cdot) \\ 0 \end{pmatrix} \notin \mathrm{Rg}(\mathcal{L}), \tag{45}$$

which arises by differentiating (37) with respect to c at $(h_1', h_2', 0)$. But (45) is satisfied, since

$$\left\langle \begin{pmatrix} 0 \\ \partial_t h^1(\cdot) \\ 0 \end{pmatrix}, \begin{pmatrix} -\partial_t h^2(\cdot) \\ \partial_t h^1(\cdot) \\ 0 \end{pmatrix} \right\rangle \neq 0,$$

on account of $\int_{\mathbb{R}} \partial_t h^1(s)\partial_t h^1(s)ds \neq 0$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product with respect to $L^2(\mathbb{R}, \mathbb{R}^3)$. Analogously to the calculation of $\ker(\mathcal{L})$ one can now show that hypothesis 7 is satisfied. Thus, if the assumptions **A1)-A6)** are satisfied, we have verified all assumptions of theorem 6.1 (note that theorem 6.1 is true even if the homoclinic solution decays exponentially in *both* asymptotic directions $t \to \pm\infty$; see section 7). Therefore, we have proved the existence of solutions of equation (37) for suitable parameters $(\tilde{\lambda}, c) \approx (0, 1)$ which converge towards a periodic solution in forward time and approach a steady state in backward time.

9 Appendix: The weak* integral

In this section we want to clarify, in which sense the integral

$$\int_0^t T(t,s)G(s)ds \tag{46}$$

is well defined, if $s \to G(s) = (g(s), 0)$ maps continuously into the space $\tilde{X} = \mathbb{C}^N \times C^0([-a,b],\mathbb{C}^N)$ and $a, b \geq 0$. For the case of pure delay differential equations, where $a = 0$, $b > 0$, our results follow from [17]. Without loss of generality, we only consider the case that $T(t,s)$ is the solution operator associated to a dichotomy on $\mathbb{R}_+$. More precisely, let us make the following assumption.

Assumption 1
Let $L(\cdot) \in BC^0(\mathbb{R}, L(C^0([-a,b],\mathbb{C}^N),\mathbb{C}^N))$ and let $L(t) \to L_\pm$ with respect to the operator norm as $t \to \pm\infty$, where $L_\pm \in L(C^0([-a,b],\mathbb{C}^N),\mathbb{C}^N)$. Consider

$$\begin{pmatrix} \partial_t \xi(t) \\ \partial_t \phi(t,\cdot) \end{pmatrix} = \mathcal{A}(t) \begin{pmatrix} \xi(t) \\ \phi(t,\cdot) \end{pmatrix} = \begin{pmatrix} L(t)\phi(t,\cdot) \\ \partial_\theta \phi(t,\cdot) \end{pmatrix}. \tag{47}$$

If the equations $\dot{y}(t) = L_\pm y_t$ are hyperbolic, (47) possesses an exponential dichotomy on $\mathbb{R}_+$ with associated solution operators $\Phi^s_+(\tau,\sigma)$, $\Phi^u_+(\sigma,\tau)$ for $\tau \geq \sigma \geq 0$. Otherwise, equation (47) possesses a (center-) dichotomy on $\mathbb{R}_+$ with solution operators $\Phi^{cs}_+(t,s), \Phi^u_+(s,t)$ or $\Phi^s_+(t,s), \Phi^{cu}_+(s,t)$ for $t \geq s \geq 0$. We now consider the case that $T(t,s)$ is one of these solution operators on $\mathbb{R}_+$.

Let us now choose some element

$$(\eta,\psi) \in \tilde{Y} := \mathbb{C}^N \times L^1([-a,b],\mathbb{C}^N)$$

and note that

$$s \mapsto \langle T(t,s)G(s), (\eta,\psi)\rangle \in L^1([0,t],\mathbb{C}), \tag{48}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $\tilde{Z} = \mathbb{C}^N \times L^\infty([-a,b],\mathbb{C}^N)$ and $\tilde{Y}$; that is

$$\langle(\xi,\phi),(\eta,\psi)\rangle = \xi\cdot\eta + \int_{-a}^b \phi(\theta)\psi(\theta)d\theta$$

for $(\xi,\phi) \in \tilde{Z}$ and $(\eta,\psi) \in \tilde{Y}$. Here, $\tilde{Z}$ can be identified with the dual space of $\tilde{Y}$. Hence, there exists a unique $Q \in \tilde{Z}$, such that

$$\langle Q, (\eta,\psi)\rangle = \int_0^t \langle T(t,s)G(s),(\eta,\psi)\rangle\, ds \tag{49}$$

for every $(\eta,\psi) \in \tilde{Y}$; see the appendix of [17]. Note that if $s \mapsto G(s)$ is continuous and takes values in X, then the weak* integral coincides with the Riemann integral.

Definition 9.1
We set $\int_0^t T(t,s)G(s)ds := Q$ and call Q the weak integral.*

From now on we view the integral term in (46) as a weak* integral, which is an element of $\tilde{Y}^* = \tilde{Z}$ by definition. Let us now prove that the integral is actually an element of $\hat{X} = \{(\xi, \phi) \in \mathbb{C}^N \times C^0([-a,b],\mathbb{C}^N) : \phi(0) = \xi\}$.

Lemma 9.1
For each fixed $t \geq 0$ we have $\int_0^t T(t,s)G(s)ds \in \hat{X}$.

Proof
Consider

$$F^\delta(t) := \int_0^t T(t,s) \begin{pmatrix} g(s) \\ g(s) \cdot l(\delta)(\cdot) \end{pmatrix} ds, \tag{50}$$

where

$$l(\delta)(\theta) := \begin{cases} 2 \cdot 2^{\frac{1}{(\theta/\delta)^2 - 1}} & \theta \in (-\delta, \delta) \\ 0 & \text{else} \end{cases}$$

for $\theta \in [-a,b]$ and $|\delta| < \min\{a,b\}$. Hence, for fixed $\delta > 0$, (50) defines an element in X for each fixed t. Moreover, the integral can be regarded as the usual Riemann integral since the integrand is continuous when considered as a map with values in X. We can now differentiate $F^\delta(\cdot) : \mathbb{R}_+ \to Y$ and obtain

$$\begin{aligned} \partial_t F^\delta(t) = \partial_t \begin{pmatrix} f^\delta(t) \\ \xi^\delta(t,\cdot) \end{pmatrix} &= T(t,t) \begin{pmatrix} g(t) \\ g(t) l(\delta)(\cdot) \end{pmatrix} + \mathcal{A}(t) F^\delta(t) \\ &= \begin{pmatrix} \zeta(t) \\ h(t,\cdot) \end{pmatrix} + \begin{pmatrix} L(t)[\xi^\delta(t,\cdot)] \\ \partial_\theta \xi^\delta(t,\cdot) \end{pmatrix}. \end{aligned} \tag{51}$$

Let us take a closer look at the second component of (51). Since $F^\delta(t) \in X$ for each fixed t, δ and therefore $\xi(t,0) = f(t)$, we obtain from

$$\partial_t \xi^\delta(t,\theta) = \partial_\theta \xi^\delta(t,\theta) + h(t,\theta)$$

via the method of characteristics the identity

$$\xi^\delta(t,\theta) = \begin{cases} f^\delta(t+\theta) + \int_0^\theta h(t+\theta-\eta,\theta)d\eta & t+\theta \geqslant 0 \\ \xi^\delta(0,\theta+t) + \int_0^t h(t,\theta+t-\eta)d\eta, & -a \leqslant t+\theta < 0. \end{cases} \tag{52}$$

Note that $\xi^\delta(0,\cdot) = 0$ and $f^\delta(0) = 0$ for any $\delta > 0$. Since $T(t,t) : Y \to Y$ is a bounded projection for each t, we conclude that $h(t,\cdot) \to \sigma(t,\cdot)$ for some function $\sigma(t,\cdot) \in L^2$ as $\delta \searrow 0$ in $L^2([-a,b],\mathbb{C}^N)$, because $g(t)l(\delta)(\cdot)$ converges in L^2 as $\delta \searrow 0$. Moreover, the integral in (50) converges with respect to the Y-norm to the value

$$F^0(t) = \int_0^t T(t,s) \begin{pmatrix} g(s) \\ 0 \end{pmatrix} ds$$

as $\delta \searrow 0$. Let us write $F^0(t) = (f(t), \xi(t,\cdot))$. Convergence of (50) in Y implies by definition that $f^\delta(t) \to f(t)$ for fixed t as $\delta \searrow 0$. Therefore, we can pass to the limit $\delta \searrow 0$ in (52) and get

$$\xi(t,\theta) = f(t+\theta) + \int_0^\theta \sigma(t+\theta-\eta,\theta)d\eta \tag{53}$$

as long as $t+\theta \geqslant 0$. Hence, $\xi(t,\cdot)$ is continuous if the spatial variable θ satisfies $t+\theta \geq 0$. In particular we conclude that

$$\xi(t,0) = f(t)$$

for all $t \geqslant 0$. Because $\xi^{\delta}(t,\cdot)$ also converges with respect to the sup-norm in the region $t+\theta \leqslant 0$ and fixed t (namely, it converges to zero), we conclude that $\xi(t,\cdot) \in C^0$, which proves that $(f(t),\xi(t,\cdot)) \in \hat{X}$.
Finally, we note that $F^0(t)$ actually coincides with the weak* integral; i.e. $F^0(t) = Q$. Indeed, $F^0(t) \in \hat{X}$. Moreover, for any $(\xi,\psi(\cdot)) \in \mathbb{C}^N \times L^1([-a,b],\mathbb{C}^N)$ the identity

$$\left\langle F^{\delta}(t),(\xi,\psi(\cdot))\right\rangle = \int_0^t \left\langle T(t,s)G^{\delta}(s),(\xi,\psi(\cdot))\right\rangle ds$$

holds. Passing to the limit $\delta \searrow 0$ we obtain

$$\left\langle F^{0}(t),(\xi,\psi(\cdot))\right\rangle = \int_0^t \left\langle T(t,s)G(s),(\xi,\psi(\cdot))\right\rangle ds.$$

By uniqueness this shows that $Q = F^0(t)$. □

The next lemma tells us that the weak integral actually depends continuously on t.

Lemma 9.2
The function $v : t \to \int_0^t T(t,s)\mathcal{G}(s)ds$ is continuous as a function from $[0,\infty)$ to $\tilde{X} := \mathbb{C}^N \times C^0([-a,b],\mathbb{C}^N)$ and

$$\|v(t)\|_{\tilde{X}} \leqslant \int_0^t Me^{\alpha(t-s)}ds \cdot \sup_{0\leqslant s\leqslant t} \|\mathcal{G}(s)\|_{\tilde{X}},$$

if $T(t,s)$ satisfies the estimate $\|T(t,s)\|_{L(\tilde{Z},\tilde{Z})} \leqslant Me^{\alpha(t-s)}$ for $t \geq s \geq 0$ and some $\alpha \in \mathbb{R}$, where, as before, $\tilde{Z} = \mathbb{C}^N \times L^{\infty}([-a,b],\mathbb{C}^N)$.

Proof
Note that the integral is well defined with values in $\tilde{X}$ by the previous lemma. Since the map $t \to \int_0^t T(t,s)\mathcal{G}(s)ds$ is continuous when regarded with values in $\tilde{Y}^* = \tilde{Z}$ (see lemma 2.1, page 54 in [17]) and the norm of $L^{\infty}([-a,b],\mathbb{C}^N)$ of an element in $C^0([-a,b],\mathbb{C}^N)$ coincides with the usual norm in C^0, the claim concerning continuity follows immediately by lemma 2.3 in [17]. □

References

[1] P. Bates, C. Fife, X. Ren, X. Wang *Travelling waves in a convolution model for phase transitions*, Arch. Rat. Mech. Anal. **138**, pp. 105-136, 1997

[2] P. Bates, X. Ren, *Heteroclinic orbits for a higher order phase transition problem*, Euro. J. Appl. Math. **8**, pp. 149-163, 1997

[3] J. Bell, H. Chi, B. Hassard *Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory*, J. Math. Biol. **24**, pp. 583 – 601, 1986

[4] J.W. Cahn, J. Mallet-Paret and E.S. Van Vleck *Traveling wave solutions for systems of ODE's on a two-dimensional spatial lattice*, SIAM J. Appl. Math. **59**, pp. 455 – 493, 1999

[5] J.N. Chow, J. Mallet-Paret and W. Chen *Traveling waves in lattice dynamical systems*, J. Diff. Eq. **149**, pp. 248 – 291, 1998

[6] A. Chmaj, X. Ren *Homoclinic Solutions of an Integral Equation: Existence and Stability*, J. Diff. Eq. **155**, pp. 17 – 43, 1999

[7] G. Friesecke, K. Matthies *Geometric solitary waves in a 2d mass-spring lattice*, Disc. Cont. Dyn. Sys. **3**, pp. 105 – 114, 2003

[8] G. Friesecke, J. Wattis *Existence theorem for solitary waves on a lattice*, Comm. Math. Phys. **161**, pp. 391 – 418, 1994

[9] B. Gavage, *Semi-discrete shock profiles for hyperbolic systems of conservation laws*, Phys. D **115**, pp. 109 – 123, 1998

[10] B. Gavage, P. Huot, F.Rousset *Nonlinear stability of semi-discrete shock waves*, SIAM, J. on Math. Analysis **35**, No. 3, pp. 639 – 707, 2003

[11] M. Georgi *Zentrumsmannigfaltigkeiten und Verzweigungen in Forward-Backward-Delay Gleichungen*, Dissertation Thesis, Freie Universität Berlin, 2007

[12] M. Georgi *Exponential Dichotomies and Transversal Homoclinic Points in Functional Differential Equations of Mixed Type*, J. Diff. Eq., submitted, 2007

[13] M. Georgi *Structural Stability of Travelling Waves in an Integrodifferential Equation*, Dyn. Sys., to appear, 2008

[14] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes **840**, Springer Verlag, 1981

[15] G. Iooss, *Travelling Waves in the Fermi-Pastam-Ulam lattice*, Nonlinearity **13**, pp. 849 – 866, 2000

[16] G. Iooss, K. Kirchgässner *Travelling Waves in a Chain of Coupled Oscillators*, Commun. Math. Phys., pp. 439 – 464, 2001

[17] S.V. Lunel, S.A. van Gils, H.O. Walther, O. Diekman *Delay equations*, Springer Verlag, 1994

[18] S.V. Lunel, H.J. Hupkes *Center Manifold Theory for Functional Differential Equations of Mixed Type*, J. Diff. Eq., to appear

[19] V. Lunel, J. Mallet-Paret *Exponential dichotomies and Wiener-Hopf Factorizations for mixed-type functional differential equations*, J. Diff. Eq., to appear

[20] J. Mallet-Paret *The Fredholm Alternative for functional differential equations of mixed type*, J. Dyn. and Diff. Eq. **11**, pp. $1-47$ 1997

[21] J. Mallet-Paret *Crystallographic Pinning: Direction Dependent Pinning in Lattice Differential Equations*, J. Diff. Eq., to appear

[22] J. Mallet-Paret *The Global Structure of Travelling Waves in Spatially Discrete Dynamical Systems*, J. Dyn. Diff. Eq. **11**, pp. $49-127$, 1999

[23] K. Palmer *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc. **104**, p. $149-156$, 1998

[24] B. Sandstede, A. Scheel, J. Härterich *Exponential dichotomies for linear non-atonomous functional differential equations of mixed type*, Ind. Univ. Math. J. **51**, pp. $1081-1109$, 2002

[25] B. Sandstede, A. Scheel *Essential Instabilities of Pulses and Bifurcations to Modulated Travelling Waves*, Proc. Royal Soc. of Edinb. **129 A**, pp. $1263-1290$, 1999

[26] A. Vanderbauwhede, G. Iooss *Center Manifold Theory in Infinite Dimensions*, Dynamics Reported **I**, 1991

[27] A. Vanderbauwhede *Center manifolds, normal forms and elementary bifurcations*, Dynamics Reported **2**, 1989