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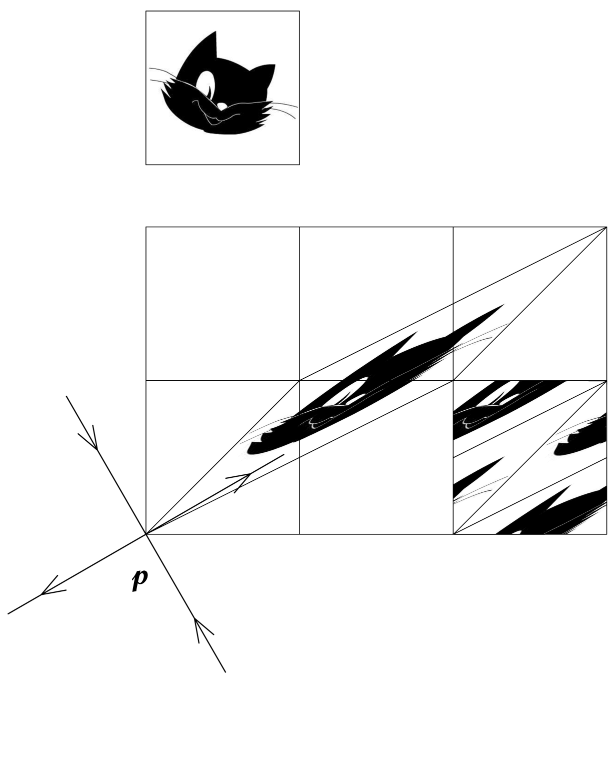
On manifolds admitting Anosov diffeomorphisms



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Movesi l'aria come *fiume* e tira con seco di nuvoli,
si come l'acqua corrente tira tutte le cose
che sopra di lei si sostengano.

Die Luft bewegt sich wie ein *Fluß* und führt Wolken mit sich,
wie das fließende Wasser jedes Ding mit sich führt,
das sich auf ihm hält.

Leonardo da Vinci

Contents

0.1 Introduction and overview

0.1.1 Anosov's question

The problem to classify the manifolds admitting Anosov diffeomorphisms goes back to the question of Dimitrii Anosov in his Congress talk, Moscow 1966 [Smale], whether there exist non-toral examples of Anosov diffeomorphisms. By an Anosov diffeomorphism $f \in C^1$, we mean a diffeomorphism for which the whole compact manifold M without boundary is a hyperbolic set, for exact definition see chapter 2. Id est, there is a continuous Df-invariant splitting of the tangent bundel TM of the manifold M $TM = E^s \oplus E^u$ and for any Riemannian metric on M, there exist constants c, λ, $c > 0$, $0 < \lambda < 1$ with

$$||Df^n(x)|_{E^s_x}|| < c\lambda^n \quad \text{and} \quad ||Df^{-n}(x)|_{E^u_x}|| < c\lambda^n$$

for any $n > 0$ and $x \in M$, [Manning]. In fact, the only known construction of Anosov diffeomorphisms until the year 1967 was on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, which are also called algebraic Anosov diffeomorphisms [Gorbatsevitch]. The reason for this is because the construction of Anosov diffeomorphism on the n-torus is obtained via a hyperbolic automorphism, which is representable as a hyperbolic matrix $A \in \mathcal{SL}(n, \mathbb{R})$, i.e has no eigenvalues modulus 1 (this is the property of hyperbolic) and has the determinant equals ± 1 (corresponds to $A \in \mathcal{SL}(n, \mathbb{R})$). The easiest and the most famous example appears up the dimension 2 on $\mathbb{T}^2$ via

$$A : \mathbb{T}^2 \to \mathbb{T}^2,$$

$$(x, y) \mapsto (2x + y, x + y)(\text{mod } 1),$$

or equivalently $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (mod 1). The matrix A has exactly two eigenvalues which are both not equal to one. Precisely, one of the eigenvalues is smaller than 1 and the other is bigger, so that iteration of the map A gives us a contruction or an expansion in each point $p \in \mathbb{T}^2$, see the figure on the first page. This map is named after the russian mathematician Vladimir I. Arnol'd "Arnold's cat map", who with the aid of a cat head demonstrated the hyperbolic behavior of A on the 2-dimensional torus $\mathbb{T}^2$. This is an example for a system which is extremely sensitive to initial conditions. The eigendirections $E_p^{\pm}$ are the directions in which small perturbations p grow fastest in forward E_p^+/ backward E_p^- iteration direction. This phenomenon is known as "butterfly effect" due to the metheorologist Edward Lorenz, the inventor of the Lorentz attractor standing for chaotic systems. More precisely, he described the sensitive dependence on initial conditions with the following: Weather appears to be a chaotic dynamical system, so it is conceivable that a butterfly that flutters by in Rio may cause a typhoon in Tokyo a few days later [Ha/Kat].

0.1.2 Answer to Anosov's question

The answer to Anosov's question was given by Steve Smale in [Smale] in 1967 which was a construction of an Anosov diffeomorphism on a nilmanifold $\mathcal{N}/\Gamma$, where $\mathcal{N}$ is a nilpotent Lie group and Γ is a lattice in $\mathcal{N}$. $\mathcal{N}/\Gamma$ can be interpreted as the generalization of the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, since $\mathbb{R}^n$ is abelian and $\mathbb{Z}^2$ is a lattice in $\mathbb{R}^n$. The generalized version of an abelian Lie group is a nilpotent Lie group. More precisely, abelian Lie groups are 1-step nilpotent. As we will see, the notion of the nilpotency of a Lie group can be more easily understood with the aid of the nilpotency of a Lie algebra because a Lie group is nilpotent if and only if its Lie algebra is nilpotent, see theorem 1.4.6. A Lie algebra is roughly speaking a tangent space to the corresponding Lie group in the identity. With a Lie bracket, it becomes a vector space. It will be essential to keep in mind the toral example mentioned above. Also, at this stage, we will give some historical background concerning Lie groups and Lie algebras, as well. The theory of Lie groups was introduced by Sophus Lie. The aim of the theory of Lie groups was to apply algebraic concepts to the theory of differential equations and to geometry [Onishchik]. In fact, as we will see throughout this paper, we will deal with many algebraic concepts. Then Michael Shub gave a famous construction of an Anosov diffeomorphism on an infra-nilmanifold , i.e. is covered by a finite number of nilmanifolds in [Dekimpe] (this explains the prefix "infra" lat. beneath). In fact, there are even two examples. One of them is four dimensional and the other is six dimensional, but they are constructed in a very similar way. As I found out, the examples mentioned many times in the subsequent papers (for example [Porteous], [Nitecki]) are not correct and which was affirmed by Michael Shub personally.

Indeed, the only known manifolds admitting Anosov diffeomorphisms are nilmanifolds and infra-nilmanifolds, which lead to the following open conjecture [Gorbatsevitch]:

Conjecture 0.1.1. If a compact manifold M admits Anosov diffeomorphism, then M is homeomorphic to a nilmanifold or an infra-nilmanifold .

0.1.3 Paper Layout

This work is organized in four chapters. Chapter 1 should be regarded as a review of basic algebraic concepts which are needed to understand the material in the subsequent chapters. It starts with the group actions in order to later introduce infra-nilmanifolds and Anosov flows. However, the central part of the first chapter is the introduction of Lie groups and Lie algebras. The chapter thoroughly describes the relation between them concentrating on the notion of the nilpotency and the investigation of the existence of a lattice in a nilpotent Lie group. These concepts lead at the end of the chapter to the definition of a nilmanifold and of an infranil-manifold. The notion of the infranil-manifold is regarded in terms of the completeness and is not discussed and introduced as much as the notion of the nilmanifold. In fact, there are less examples on the algebraic as well topological concepts such as semi-direct product, torsion free, covering space because they can be found in every good book on algebra or topology.

Chapter 2 introduces Anosov flows and Anosov diffeomorphisms in order to give an idea of the hyperbolicity in both cases. More precisely, we will study a geodesic flow on a surface of a constant negative curvature, which turns out to be an Anosov flow, and then draw the appropriate picture. This will not only simplify the problem of imagination, but also help us to motivate the definition of Anosov diffeomorphisms. After having introduced nilmanifolds and infra-nilmanifolds in chapter 1 and Anosov diffeomorphisms in chapter 2, we will discuss the non-toral examples of Anosov diffeomorphisms in chapter 3. More precisely, we will study the example given by Steve Smale on a six-dimensional nilmanifold [Smale] and an example on a six dimensional infranil-manifold by Michael Shub [Shub], which does not work since the lattice Γ_q turns out to be not torsion free. In the last chapter, we will give a short review on known results concerning the classification problem of the manifolds admitting Anosov diffeomorphisms. Most of the results stated in the last chapter are of topological character and are not explained in great detail. It should be seen as a background information on known results.

0.2 Aknowledgements

The conversation with V.V. Gorbatsevich was not only very interesting but more importantly essential for my understanding of the subject. He encouraged me to look at the algebra from a different perspective and study the algebraic geometry in order to gain the complete understanding of the proof of the theorem 4.0.11. Indeed, as he pointed out to me, one needs to have a certain picture in mind to understand what is going on in the abstract level. I would also like to thank Mark Pollicott who helped me to understand the notation in Michael Shub's paper.

Chapter 1

Basic algebraic concepts

First of all, we will introduce the basic algebraic concepts such as group actions, Lie groups, and Lie algebras. Most importantly, we will investigate what the nilpotency of a Lie group and Lie algebra means and when a nilpotent Lie group allows a lattice. This will lead us at the end of this chapter to the definition of a nilmanifold, which will become an essential concept later in the paper.

1.1 Group actions

First, we will give a formal definition of a group action. It turns out that one of the fundamental objects analyzed by the theory of dynamical systems are the flows, which are examples of differentiable group actions and are considered throughout this work. More precisely, we will introduce Anosov flow with certain interesting properties and which performs hyperbolic behavior on the whole compact manifold.

Definition 1.1.1. Group action
Let $(G, *)$ be a group with the neutral element e and M a set. A group action is a map

$$G \times M \to M$$

$$(g, m) \mapsto m^g = m_g = g(m)$$

such that the following holds:

(i) $m^e = m$, $\forall m \in M$

(ii) $m^{g*h} = (m^g)^h$

Remark. More precisely, for each $g \in G$ we get a map

$$\rho_g : M \to M, m \mapsto m^g = \rho_g(m)$$

with

(i) $\rho_e = id$

(ii) $\rho_{g*h} = \rho_g \circ \rho_h$

this implies that $\rho_g : M \to M$ is an isomorphism. This means that a group action of G on M is a group homomorphism $\rho : G \to \mathrm{Iso}(M)$. If M is a linear space and ρ_g is linear for each $g \in G$, we call ρ a representation of G.

First examples:

1. Normal forms

   (a) The action of $\mathcal{GL}(N, \mathbb{R})$ leads to the so called Jordan Normal Form

   $$\mathcal{GL}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | \text{A is invertible, i.e.}\det(A) \neq 0\}$$

   acts on $\mathrm{Mat}(N \times N, \mathbb{R})$ by conjugation, i.e.

   $$\mathcal{GL}(N, \mathbb{R}) \times \mathrm{Mat}(N \times N, \mathbb{R}) \to \mathrm{Mat}(N \times N, \mathbb{R}), (S, A) \mapsto SAS^{-1}$$

2. The action of isometry group of a geometry

   (a) The $PSL(2, \mathbb{R}) = \mathcal{SL}(2, \mathbb{R})/\{\pm \mathrm{id}\}$ of Lobachevsky's plane

   (b) The $SE(2, \mathbb{R})$ of Euclidean plane

3. The *flows* (also known as dynamical systems with continuous time) are actions of the group $(\mathbb{R}, +)$ on a differentiable manifold M. This is the most important term in the theory of dynamical systems.

   As an example for a flow we will introduce the geodesic flow of the Lobachevsky's plane, which turns out to be an Anosov flow. We will return to this example because of its beauty when we introduce Anosov flows in chapter 2.

1.2 Lie groups

Initially, the Lie groups were invented for constructing the analogue of Galois theory for differential equations by Sophus Lie, see for example [Onishchik]. They will become very important for our study concerning the manifolds admitting Anosov diffeomorphisms.

Definition 1.2.1. Lie group[Onishchik]
A Lie group over a field $K = (\mathbb{R}, \mathbb{C})$ is a group $(\mathcal{G}, \mu)$ equipped with the structure of a differentiable manifold over K in such a way that the map

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$

$$(x, y) \mapsto xy$$

is differentiable.

Remarks 1.2.2.

1. We call Lie groups over the field $\mathbb{C}$ complex Lie groups and over the field $\mathbb{R}$ real Lie groups.
2. Instead of differentiability of μ, it is also possible to require its analyticity in a neighbourhood $U \subset \mathcal{G}$ of the neutral element $e = id \in \mathcal{G}$.

Examples 1.2.3.

1. The general linear group

   $$\mathcal{GL}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | \text{A is invertible, i.e.} \det(A) \neq 0\}$$

   is a Lie group.
2. The following subgroups of $GL(N, \mathbb{R})$ are Lie groups. More precisely that are:
   (a) $\mathcal{SO}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | AA^T = Id, \det(A) = 1\}$, the special orthogonal.
   (b) $\mathcal{O}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | AA^T = A^T A = Id\}$, the orthogonal.
   (c) $\mathcal{SE}(N, \mathbb{R}) = SO(N, \mathbb{R}) + \text{translations}$
   (d) $\mathcal{SU}(N, \mathbb{C}) = \{A \in Mat(N \times N; \mathbb{C}) | A^* A = Id, A^* := \overline{A}^T, \det(A) = 1\}$, the special unitary.
   (e) $\mathcal{U}(N, \mathbb{C}) = \{A \in Mat(N \times N; \mathbb{C}) | A^* A = AA^* = Id, A^* := \overline{A}^T\}$, the unitary.
   (f) $\mathcal{SL}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | det(A) = 1\}$, the special linear.
   (g) $\mathcal{HEIS}(3, \mathbb{R}) = \{A \in Mat(3 \times 3; \mathbb{R}) | A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c \in \mathbb{R}\}$, the Heisenberg group. For the definition of the Heisenberg Lie group one can slso take the 3×3 lower triangular matrices [see chapter 4 in Shub's example]. We will also learn its Lie algebra, which also appears in the Bianchi classification (named after Luigi Bianchi) applied to cosmology.

1.3 Lie algebras

Now it is important to know the relation between a Lie group and its Lie algebra. We will need the Lie algebra not only in order to understand the nilpotency much more easily, but also in chapter 3 where we introduce Smale's idea. To each Lie group $\mathcal{G}$, we can consider its tangent algebra $\mathfrak{g}$, which determines the group $\mathcal{G}$. This makes the theory of Lie groups attractive in that the problems concerning Lie groups can be translated in the language of linear algebra by regarding the tangent algebra $\mathfrak{g}$ corresponding to the Lie group $\mathcal{G}$ [Onishchik].

Definition 1.3.1. Let $\mathcal{G}$ be a Lie group, $g(t)$ and $h(s)$ (s,t $\in \mathbb{R}$) be differentiable paths on $\mathcal{G}$ such that $g(0) = h(0) = e$, $g'(0) = \xi$ $h'(0) = \eta$.
Define the Lie bracket $[\xi, \eta] := \frac{\partial^2}{\partial t \partial t}(g(t), h(s))|_{t=s=0} := \frac{\partial^2}{\partial t \partial t} g(t)h(s)g^{-1}(t)h^{-1}(s)|_{t=s=0}$ such that the Lie bracket defined as above has the following properties:
For $\xi, \xi_1, \xi_2, \eta, \eta_1, \eta_2, \mu \in T_e\mathcal{G}, a, b \in \mathbb{R}$

(i) $[,]$ is a bilinear form, i.e. linear in both components:

$$[a(\xi_1 + \xi_2), \eta] = a([\xi_1, \eta] + [\xi_2, \eta])$$

$$[\xi, b(\eta_1 + \eta_2)] = b([\xi, \eta_1] + [\xi, \eta_2])$$

(ii) $[,]$ is antisymmetric, i.e $[\xi, \eta] = -[\eta, \xi]$

(iii) The Lie bracket satisfies the Jacobi-identity:
$[\xi, [\mu, \eta]] + [\eta, [\xi, \mu]] + [\mu, [\eta, \xi]] = 0.$

The space $T_e\mathcal{G}$ with the operation $[,]$ defined in this way, is the *tangent algebra of the group* $\mathcal{G}$ and will be denoted by the gothic letter $\mathfrak{g}$. (see [Onishchik])

Remark. 1 If the Lie group $\mathcal{G}$ consists of matrices, then so does the corresponding Lie algebra $\mathfrak{g}$ and the following holds for the Lie bracket:

$$[X, Y] = XY - YX, \qquad X, Y \in \mathfrak{g}$$

Example 1.3.2. Now we give the corresponding Lie algebras to the Lie groups given in example 1.2.2. via differentiation. In the next paragraph, we will try to figure out which of them are nilpotent.

1. $\mathcal{O}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | AA^T = A^T A = Id\}$
   $\mathfrak{o}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | A^T = -A\}$, see appendix.

2. $\mathcal{SO}(N, \mathbb{R}) = \{A \in O(N, \mathbb{R}) | \det(A) = 1\}$
   $\mathfrak{so}(N, \mathbb{R}) = \{A \in \mathfrak{o}(N, \mathbb{R}) | \mathrm{trace}(A) = 0\}$

3. $\mathcal{U}(N, \mathbb{C}) = \{A \in Mat(N \times N; \mathbb{C}) | A^*A = AA^* = Id, A^* = \overline{A}^T\}$
   $\mathfrak{u}(N, \mathbb{C}) = \{A \in Mat(N \times N; \mathbb{C}) | A^* = -A, A^* = \overline{A}^T\}$

4. $\mathcal{SU}(N, \mathbb{C}) = \{A \in U(N, \mathbb{C}) | \det(A) = 1\}$
   $\mathfrak{su}(N, \mathbb{C}) = \{A \in \mathfrak{u}(N, \mathbb{C}) | \text{trace}(A) = 0\}$

5. $\mathcal{SL}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | \det(A) = 1\}$
   $\mathfrak{sl}(N, \mathbb{R}) = \{A \in Mat(N \times N; \mathbb{R}) | \text{trace}(A) = 0\}$

6. $\mathcal{SE}(N, \mathbb{R}) = \mathcal{SO}(N, \mathbb{R}) + \text{translations}$
   $\mathfrak{se}(N, \mathbb{R}) = \mathfrak{so}(N, \mathbb{R}) + \text{translations}$

7. $\mathcal{HEIS}(3, \mathbb{R}) = \{A \in Mat(3 \times 3; \mathbb{R}) | A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},\ a, b, c \in \mathbb{R}\}$

   $\mathfrak{heis}(3, \mathbb{R}) = \{A \in Mat(3 \times 3; \mathbb{R}) | A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix},\ a, b, c \in \mathbb{R}\}$

If we wish to go the other direction, i.e. from a Lie algebra to a Lie group, we can do this by the exponential map, which is defined in the following way [Onishchik]:

Definition 1.3.3. Let $\xi \in \mathfrak{g}$, $g_\xi(t)$ the one-parameter subgroup with velocity $\xi(t) \equiv \xi$. We will refer to the vector ξ as its dirctin vector. For any Lie group $\mathcal{G}$, we set by definition $\mathfrak{exp}\xi = g_\xi(1)$. The map $\mathfrak{exp} : \mathfrak{g} \to \mathcal{G}$ defined in this way is known as the *exponential map.*

Remark. The exponential map $\mathfrak{exp} : \text{Mat}(n, \mathbb{C}) \to \text{GL}(n, \mathbb{C})$ is defined by $\mathfrak{exp}(X) := \Sigma_{n=0}^{\infty} \frac{X^n}{n!} = \text{Id} + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \dots$, see [Bump].

1.4 Nilpotency

Generally speaking, the term nilpotency (lat. "nil" from "nihil" nothing, "potentia" power, potency) in mathematics stands generally speaking for vanishing of an object from a certain step in an iteration. As an example, one can consider nilpotency of a matrix $A \in Mat(3 \times 3, \mathbb{R})$,

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

by raising A to a higher power. The matrix A becomes the zero matrix when we raise it to the power of three, $A^3 = 0$. We then say that A is a 2-step nilpotent matrix, because we need exactly two steps to obtain the zero matrix. As we will see, one of the most important results concerning the manifolds admitting the Anosov diffeomorphisms will be the following theorem [Jacobson], for the proof of the theorem see chapter 4:

Theorem 1.4.1. *If a Lie algebra $\mathfrak{g}$ admits the existence of a hyperbolic automorphism, then the Lie group $\mathfrak{g}$ is nilpotent.*

From the above theorem follows:

Theorem 1.4.2. *If a Lie group $\mathcal{G}$ admits the existence of a hyperbolic automorphism, then the Lie group $\mathcal{G}$ is nilpotent.*

Even if we don't know what a hyperbolic automorphism is [see chapter 2] and how it is related to an Anosov diffeomorphism, we have to first understand what it means for the Lie group to be nilpotent. Thus, let us now turn to the formal definitions concerning the nilpotency of Lie groups and Lie algebras. In fact, we now look at the already known Lie groups and find out which of them are nilpotent.

1.4.1 Nilpotency of a Lie group

Definition 1.4.3. nilpotent Lie group
Let G be a group, e the neutral element of G and $G = G_0 \supset G_1 \supset G_2 \supset ...$ the decreasing (or lower) central series, i.e

(i) Every G_i is normal in G.

(ii) $\forall i : G_i/G_{i+1}$ is contained in the center of G/G_{i+1}.

A group G is called nilpotent if there exists an m such that $G_m = \{e\}$.

1.4.2 Nilpotency of a Lie algebra

Definition 1.4.4. nilpotent Lie algebra
Let $\mathfrak{g}$ be a Lie algebra and

$$G = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset ... ,$$

$\mathfrak{g}_{k+1} := [\mathfrak{g}, \mathfrak{g}_k]$. We call $\mathfrak{g}$ nilpotent if there exists $m \in \mathbb{N}$ such that $\mathfrak{g}_m = 0$.[Onishchik]

Remarks 1.4.5. 1. There exists a theorem, which says [Onishchik] :

> **Theorem 1.4.6.** *A connected Lie group G is nilpotent if and only if its tangent algebra $\mathfrak{g}$ is nilpotent. Moreover, $G_m = \{e\}$ if and only if $\mathfrak{g}_m = 0$.*
>
> This theorem simplifies the question whether a Lie group is nilpotent because it is sufficient to prove the nilpotency of its corresponding linear space, namely its Lie algebra.

2. Every subalgebra and every quotient algebra of a nilpotent Lie algebra is nilpotent.

We first give an example of a Lie group for which the lower central ceries stays constant.

Example 1.4.7. The special euclidean group
Consider once again the $\mathcal{SE}(N,\mathbb{R}) = \mathcal{SO}(N,\mathbb{R}) + \text{translations}$ with its tangent algebra $\mathfrak{se}(N,\mathbb{R}) = \mathfrak{so}(N,\mathbb{R}) + \text{translations}$. Now, we want to understand why this group is not nilpotent. For that reason, we consider the 2-dimensional case, namely the $\mathcal{SE}(2,\mathbb{R})$.

Step 1 Identify $\mathcal{SE}(2,\mathbb{R}) = \{ \begin{pmatrix} \cos a & -\sin a & b \\ \sin a & \cos a & c \\ 0 & 0 & 1 \end{pmatrix} | a,b,c \in \mathbb{R}\}\}$. This we can do by considering motions of the plane as projective transformations taking the line at infinity to itself [Onishchik]. Then by differentiating with respect to t in zero, we get the corresponding Lie algebra
$\mathfrak{se}(2,\mathbb{R}) = \{ \begin{pmatrix} 0 & -a & b \\ a & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a,b,c \in \mathbb{R}\}$

Step 2 Write the generators of the Lie algebra $\mathfrak{se}(2,\mathbb{R})$:

$$\text{Rot} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Step 3 Calculate the Lie brackets with each generator, i.e. consider the following computed table:

[•, •]	Rot	X	Y
Rot	0	Y	-X
X	-Y	0	0
Y	X	0	0

For example we have

$$[\text{Rot}, X] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 4 Understand what the lower central series do, i.e. consider $\mathfrak{g}_0 := \mathfrak{se}(2,\mathbb{R}), ..., \mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{se}(2,\mathbb{R})]$. We have:
$\mathfrak{g}_1 = [\mathfrak{se}(2,\mathbb{R}), \mathfrak{se}(2,\mathbb{R})] = \text{span}_{\mathbb{R}}\{X, Y\}$.

$\mathfrak{g}_2 = [\mathfrak{se}(2,\mathbb{R}), \mathfrak{g}_1] = \text{span}_{\mathbb{R}}\{X, Y\} = \mathfrak{g}_1$.

$\mathfrak{g}_3 = [\mathfrak{se}(2,\mathbb{R}), \mathfrak{g}_2] = [\mathfrak{se}(2,\mathbb{R}), \mathfrak{g}_1] = \text{span}_{\mathbb{R}}\{X, Y\} = \mathfrak{g}_1$.
Thus, the lower central series of $\mathfrak{se}(2,\mathbb{R})$ stays constant, i.e. can't vanish. From this fact we derive that the special euclidean group is not nilpotent (see the theorem 1.4.6 in remark 1.4.5).

Example 1.4.8. The three dimensional Heisenberg group
As we had already defined, the three dimensional Heisenberg group is:

$$\mathcal{HEIS}(3,\mathbb{R}) = \{A \in Mat(3\times 3;\mathbb{R}) | A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},\ a,b,c \in \mathbb{R}\}$$

with its tangent algebra:

$$\mathfrak{heis}(3,\mathbb{R}) = \{A \in Mat(3\times 3;\mathbb{R}) | A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix},\ a,b,c \in \mathbb{R}\}$$

Now, we shall proof that the $\mathfrak{heis}(3,\mathbb{R})$ is nilpotent.

Step 1 In the case of the Heisenberg Lie algebra we can immediately begin with naming its generators.

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2 By computing the Lie brackets of the above generators, we get the following table:

$[\bullet,\bullet]$	X	Y	Z
X	0	Z	0
Y	-Z	0	0
Z	0	0	0

Step 3 Consider now the lower central series:
$\mathfrak{h}_1 = \text{span}_{\mathbb{R}}\{Z\}$ $\mathfrak{h}_2 = [\mathfrak{h}, \mathfrak{h}_1] = [\text{span}_{\mathbb{R}}\{X, Y, Z\}, \text{span}_{\mathbb{R}}\{Z\}] = 0$
Thus, the Heisenberg Lie group is 2-step nilpotent.

1.5 Uniform discrete subgroups of a Lie group

A subgroup Γ of a topological group G is said to be a discrete subgroup if Γ is a discrete subset of G as a topological space. This is equivalent that each point $x \in \Gamma$ has a neighbourhood U(x) such that $\Gamma \cap U(x) = \{x\}$ [Onishchik/Vinberg]. A Lie group is always a topological group whereas the converse does not hold. For example, $\mathbb{Z}$ with the cofinite topology is a topological group, but not a Lie group. Let us first give some definitions concerning the subgroups of a Lie group, see [Onishchik/Vinberg] .

Definition 1.5.1. Discrete subgroups of a Lie group
Let $\mathcal{G}$ be a Lie group. $\mathcal{L} \subset \mathcal{G}$ a discrete subgroup in $\mathcal{G}$. We call $\mathcal{L}$ a lattice, if there exists a finite, invariant measure on $\mathcal{G}/\mathcal{L}$. $\mathcal{L}$ is called a uniform subgroup of $\mathcal{G}$, if $\mathcal{G}/\mathcal{L}$ is compact.

Note that if $\mathcal{G}/\mathcal{L}$ is compact i.e. $\mathcal{L}$ is uniform, then $\mathcal{G}/\mathcal{L}$ the measure is finite i.e $\mathcal{L}$ is a lattice. The converse does not hold see example 2.

Example 1.5.2. 1 Let $e_1, \dots e_n$ the canonical basis of $\mathbb{R}^n$. Then the discrete subgroup $\Gamma = \Sigma_{i=1}^{m} \mathbb{Z} e_i \subset \mathbb{R}^n$ is a uniform lattice if and only if $n = m$ [Onishchik/Vinberg].

2 $\mathcal{SL}(N, \mathbb{Z})$ is a lattice in $\mathcal{SL}(N, \mathbb{R})$, but for $N \geq 2$ it is not uniform [Onishchik/Vinberg].

Remark. For nilpotent Lie groups, it turns out that a discrete subgroup $\mathcal{D}$ of $\mathcal{G}$ is a lattice if and only if $\mathcal{D}$ is uniform which means that the lattices of the nilpotent Lie groups are exactly the uniform discrete subgroups [Onishchik/Vinberg].

Now, we have the answer to the question how the compactness of a Lie group can be achieved. However, we do not know in which Lie groups the lattices occur. Thus, we need certain restrictions on a Lie group, which are stated in the theorem of Malcev, see [Smale].

Theorem 1.5.3. [Malcev]

(a) A necessary and sufficient condition for a discrete group Γ to occur as a uniform subgroup of a simply connected nilpotent Lie group is that Γ be finitely generated nilpotent group containing no elements of finite order.

(b) A necessary and sufficient condition on a nilpotent simply connected Lie group $\mathcal{G}$ that there exist a uniform discrete subgroup Γ is that the Lie algebra of $\mathcal{G}$ has rational constants of structure in some basis.

(c) If Γ_i is a uniform discrete subgroup of a simply connected nilpotent group $\mathcal{G}_i$, i=1,2, then any isomorphism $\Gamma_1 \to \Gamma_2$ can be uniquely extended to an isomorphism.

1.6 Nilmanifolds

We will mention that all details are in [Onishchik]. As the simplest example of a nilmanifold, one can consider a connected abelian Lie group acting on itself by left translations, as it is stated in [Onishchik]. Here is the formal definition.

Definition 1.6.1. Let $\mathcal{G}$ be a nilpotent Lie group and $\Gamma \subset \mathcal{G}$ a uniform discrete subgroup. We call the qoutient space $\mathcal{G}/\Gamma$ a nilmanifold.

Example 1.6.2. i The n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ is a nilmanifold, since $\mathbb{R}^n$ is abelian with respect to $+$ and $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$.

ii Let $\mathcal{N} := \mathcal{HEIS}(3, \mathbb{R})$, and $\mathcal{L} := \mathcal{HEIS}(3, \mathbb{Z})$. Then $\mathcal{M} := \mathcal{N}/\mathcal{L}$ is a 3-dimensional nilmanifold.

1.7 Infranil-manifolds

Definition 1.7.1. semi-direct product of abstract groups [Onishchik]
Let G_1, G_2 be arbitrary groups and $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$, $b : G_2 \to \mathcal{AUT}(G_1)$ a homomorphism. Let us recall that by the semi-direct product of the sets G_1 and G_2, we mean the direct product of the sets G_1 and G_2 equipped with a group operation by means of the formula $(g_1, g_2)(h_1, h_2) = (g_1 b(g_2) h_1, g_2 h_2)$. We will denote the semi-direct product by $G_1 \rtimes G_2$ or, more correctly, by $G_1 \rtimes_b G_2$.

Remarks 1.7.2. 1. The semi-direct product for Lie groups is defined as the semi-direct product of the abstract groups with the differentiable structure of the direct product of differentiable manifolds. Moreover, the homomorphism b is required to define a differentiable action of the group G_2 on G_1.(In particular, the automorphism $b(g_2)$ of the group G_1 should be differentiable for any $g_2 \in G_2$.) [Onishchik]

2. Let $\mathcal{N}$ be a connected and simply connected nilpotent Lie group and let $Aut(\mathcal{N})$ be the group of continuous automorphisms of $\mathcal{N}$. Then $\mathrm{Aff}(\mathcal{N}) = \mathcal{N} \rtimes Aut(\mathcal{N})$ acts on $\mathcal{N}$ in the following way:
$\forall (n, \alpha) \in \mathrm{Aff}(\mathcal{N}), \forall x \in N : (n, \alpha) \cdot x = n\alpha(x)$. We call $\mathrm{Aff}(\mathcal{N})$ the affine group of $\mathcal{N}$. So an element of $\mathrm{Aff}(\mathcal{N})$ consists of a translational part $n \in \mathcal{N}$ and a linear part $\alpha \in \mathcal{AUT}(\mathcal{N})$ (as a set $\mathrm{Aff}(\mathcal{N})$ is just $\mathcal{N} \rtimes \mathcal{AUT}(\mathcal{N})$ and $\mathrm{Aff}(\mathcal{N})$ acts on $\mathcal{N}$ by first applying the linear part and then multiplying on the left by the translational part). In this way, $\mathrm{Aff}(\mathcal{N})$ can also be seen as a subgroup of $\mathrm{Diff}(\mathcal{N})$ [Dekimpe].

Definition 1.7.3. [Hirsch] Let $\mathcal{N}$ be a simply-connected Lie group, $\mathcal{C} \subset \mathcal{AUT}(\mathcal{N})$ a finite group of automorphisms of $\mathcal{N}$ and $\mathcal{H} := \mathcal{N} \rtimes \mathcal{C}$ the semidirect product. Let $\Gamma \subset \mathcal{H}$ be a uniform discrete subgroup ($\mathcal{H}/\Gamma$ compact).
Γ acts on the space $\mathcal{H}/\mathcal{C}$ of left cosets of $\mathcal{C}$ in $\mathcal{H}$ by left translations. The space $\mathcal{H}/\mathcal{C}$ is naturally diffeomorphic to $\mathcal{N}$. If this action of Γ on $\mathcal{N}$ is totally discontinuous (i.e. if the map $\mathcal{N} \to \mathcal{N}/\Gamma$ is a covering space) then the double coset space $\Gamma \setminus \mathcal{H}/\mathcal{C} \simeq \mathcal{N}/\Gamma$ is a smooth manifold M_0. Such an M_0 is called an infrahomogeneous space.

Remark. M_0 as defined above has as a covering space the homogeneous space $\mathcal{N}/\mathcal{N} \cap \Gamma$ [Hirsch].

Remarks 1.7.4.
1. The semidirect product of $\mathcal{N} \rtimes \mathcal{C}$ is denoted by $\mathrm{Aff}(\mathcal{N})$ which stands for the affine group. This was not mentioned in [Hirsch], instead he denotes the semi-direct product by $\mathcal{H}$. So the aim is actually to construct $\mathrm{Aff}(\mathcal{N})$ of a nilpotent Lie group $\mathcal{N}$.
2. We could also have assumed that Γ is torsion-free, i.e. if the only element in Γ of finite order is the identity. It can be shown that it follows that Γ acts freely on $\mathcal{N}$ [Franks], i.e $\forall\, x \in \mathcal{N}$, the subgroup $\{g \in \Gamma | g(x) = x\}$ is trivial. If the action is free we sometimes say that Γ acts without fixed points or has no fixed points [Wolf].

   Proof. [Franks] Let $g \in \Gamma$, $x \in \mathcal{N}$ such that $g(x) = x$, which implies $g^n(x) = x$ for all n. For some n, g^n is left translation by an element of $\mathcal{N}$. Thus, $g^n(x) = x$ would imply g^n is the identity of Γ, which contradicts to the assumption that Γ is torsion free. □

   Then Franks claims that $\mathcal{N}/\Gamma$ (the quotient space of $\mathcal{N}$ under the action of Γ) is a compact manifold [Franks].
3. We shall give some warnings concerning the infra-nilmanifold case due to [Dekimpe]. This article is very important, in that sense that it points out the mistakes in the definition of infra-nilmanifold endomorphisms given by Franks. The definitions were given first by M.Hirsch in [Hirsch] and then by Franks in [Franks], whose definition was taken instead of the one by Hirsch. But exactly the definition by Franks caused problems, which was also taken by M.Shub.

Remark.

Definition 1.7.5. Let M_0 be an infrahomogeneous space. If $\mathcal{N}$ is nilpotent, we call M_0 an *infranil-manifold*. In this case $\mathcal{N} \cap \Gamma$ is uniform and of finite index in Γ (see Auslander) and M_0 is covered by the nilmanifold $\mathcal{N}/\mathcal{N} \cap \Gamma$ [Hirsch].

Example 1.7.6. Flat Riemannian manifolds are infranil-manifolds with $\mathcal{N} = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$ [Hirsch].

Chapter 2

Hyperbolicity in the sense of dynamical systems

Historically, hyperbolic dynamical systems have its origin in the theory of Steve Smale [Smale] in 1967 and in the theory of U-systems of Dmitrii Anosov. (In literature one can find the following explanations for the letter U which may stand for structurally stable, moustache, condition. All of the three words start with an "U" in Russian.) The U-systems introduced by Dmitrii Anosov were called by Steve Smale Anosov flows in honour to its inventor.

2.1 Anosov flows

Although this work is dedicated to the Anosov diffeomorphisms and especially the manifolds admitting them, it is essential to first understand the Anosov flows on an example. More precisely, we will study the geodesic flow on a surface of constant negative curvature. This will lead us to a detail understanding of hyperbolic behaviour of an Anosov flow on a unit tangent bundel. The following definition is from [Gorbatsevitch].

Definition 2.1.1. Anosov flow
Let M be a compact manifold without boundary,

$$\Phi : \mathbb{R} \times M \to M, (t, x) \mapsto \Phi(t, x) =: \Phi_t(x)$$

be a flow, i.e. smooth one-parameter group action, of class $C^k, k \geq 1$. We call Φ_t an **Anosov flow**, if the following three conditions hold:

(i) [splitting of the tangentbundle] There exists a continuous decomposition of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u,$$

where E^s stands for the stable eigenspace, E^c for the central eigenspace, E^u for the unstable eigenspace.

(ii) [invariance] The splitting defined in (i) is invariant under the flow Φ_t.

(iii) [contraction/expansion] $\exists$ C, D constants, $\alpha > 0$, such that:

$$|| \, (d\Phi) \, (t) \, (v) \, || \leq \mathrm{C}\exp\,(-\alpha t) \, ||v||, \; \forall v \in E^s, t > 0,$$

$$|| \, (d\Phi) \, (t) \, (v) \, || \geq \mathrm{D}\exp\,(\alpha t) \, ||v||, \; \forall v \in E^u, t > 0$$

in order to understand, what is meant by this definition of an Anosov flow, we give a very famous example.

2.1.1 Example: Geodesic flow on the Lobachevsky's plane

Before we start with the example, let us recall the following definition [Burns/Gidea]

Definition 2.1.2. A Riemannian metric on a smooth manifold M is a mapping g that assigns to every point $p \in M$ an inner product $g_p = \langle \cdot, \cdot \rangle_p$ on the tangent space T_pM, which depends smoothly on $p \in M$, in the sense that, for any two smooth vector fields X and Y on M, the function $p \in M \to \langle X_p, Y_p \rangle_p \in \mathbb{R}$ is smooth. A manifold M endowed with a Riemannian metric is called a Riemannian manifold.

Remarks 2.1.3. [Burns/Gidea]

1. We can write the Riemannian metric in terms of the functions $g_{ij}(p) = g_{ij}(x_1, \ldots, x_m) = \langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p \rangle$. $(g_{ij})_{i,j=1,\ldots,m}$ is a $m \times m$ matrix.
2. For Riemannian metric the following notation is convenient $ds^2 = \Sigma_{i,j=1}^m g_{ij} dx_i dx_j$, where s stands for "arc-length" element.

Let us consider the following two dimensional Riemannian manifold

$$\mathbf{H}_2 := \{z = x + iy \in \mathbb{C} | y > 0\}, \text{ endowed with the metric}$$

$$\mathrm{ds}^2 = \frac{1}{y^2} dx^2 + \frac{1}{y^2} dy^2 = \frac{\mathrm{dx}^2 + \mathrm{dy}^2}{y^2}.$$

$$\text{Thus, } (g_{ij}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

The two dimensional manifold as defined above is called the LOBACHEVSKY[1]'s plane being a two dimensional model of hyperbolic geometry[2], i.e the non-euclidean

[1] LOBACHEVSKY 1792-1856

[2] hyperbolic geometry was the answer to the 2000 years old problem concerning the fifth postulate of Euclid's elements. Until the independence of the fifth postulate from the other postulates was recognized by Janos Bolyai, Nikolai Lobachevsky, Carl Friedrich Gauß many mathematicians were involved in the vicious circle of deducing the fifth postulate from the other axioms.

space in which the sum of three angles is smaller than 180° and to each line there exist infinitely many parallels. The mean curvature of $\mathbf{H}_2$ is -1, i.e. $\mathbf{H}_2$ is a surface of constant negative curvature.

Let us now understand the notion of the geodesic flow on $\mathbf{H}_2$, which turns out to be an Anosov flow. For that reason we need the following definitions:

Definition 2.1.4. geodesic
Let M be a manifold, $I \subset M$. A smooth curve $\gamma : I \to M$ is called a geodesic if locally it is the shortest path between two points on γ, i.e. γ is a length minimizing curve.

Definition 2.1.5. geodesic
Let M be a manifold, $I \subset M$. A smooth curve $\gamma : I \to M$ is called a geodesic if its velocity vector field $\frac{d\gamma}{dt}$ is parallel.

Remarks 2.1.6.

(a) Let $k = 1, \ldots, m$ and define the Christoffel symbols as follows $\Gamma_{ij}^k := \frac{1}{2}\Sigma(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l})g^{lk}$, where g^{lk} are the entries of the inverse of the matrix g_{lk}.
The geodesic equation $\frac{d^2x_k}{dt^2} + \Sigma_{i,j=1}^m \frac{dx_i}{dt}\frac{dx_j}{dt}\Gamma_{ij}^k = 0$, where $k = 1, \ldots, m$ determines $(x_1(t), \ldots, x_m(t))$ which reprsents a geodesic γ.

(b) The geodesics in the Eucledean plane are straight lines.

(c) The geodesics in the LOBACHEVSKY$'s$ plane are straight lines and Euclidean semicircles orthogonal to the x-axis.

Proof. [Arnold],[Burns/Gidea]. Arnold's proof is based on the first definition and uses the trick with the isometry group $\mathrm{PSL}(2,\mathbb{R}) = \mathcal{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$ of the Lobachevsky's plane, whose elements are the Möbius-transformations, i.e. rational mappings of the form:

$$g : \mathbb{C} \to \mathbb{C}, z \mapsto \frac{az+b}{cz+d}.$$

In comparison, Burns and Gidea work with the tools of differential geometry. Precisely, they calculate the so called geodesic equations via (g_{ij}) and the Christoffel symbols, defined as above and from the two geodesic equations they deduce the form of the geodesics in $\mathbf{H}_2$. We will give the proof in appendix. □

Definition 2.1.7. geodesic flow [Katok]
Let γ be a geodesic of a manifold M, T^1M the unit tangent bundle of the manifold M and $(p, \vartheta) \in T^1M$, p lies on γ. We call

$$\Upsilon : \mathbb{R} \times \mathrm{T}^1M \to \mathrm{T}^1M, (t,(p,\vartheta)) \mapsto (p_t, \vartheta_t),$$

the geodesic flow.

Remark. Imagination: The point p moves with the constant unit velocity ϑ on the geodesic γ.

Next, we will see, why the geodesic flow

$$\Phi : \mathbb{R} \times \mathrm{T}^1\mathbf{H}_2 \to \mathrm{T}^1\mathbf{H}_2,\ (t,(p,\vartheta)) \mapsto (p_t,\vartheta_t), \forall t \in \mathbb{R}$$

is an Anosov flow. As we remember, we shall see the splitting of the tangent bundle TP, P $:= \mathrm{T}^1\mathbf{H}_2$ at each tuple (p,ϑ).

Definition 2.1.8. horocycle [Hasselblatt/Katok]
Horizontal lines $\mathbb{R} + ir := \{t + ir | t \in \mathbb{R}\}$ are called horocycles centered at ∞. Circles tangent to $\mathbb{R}$ at $x \in \mathbb{R}$ are called horocycles centered at x. If $\gamma : \mathbb{R} \to \mathbf{H}_2$ is a geodesic then $\gamma(-\infty)$ and $\gamma(\infty) \in \mathbb{R} \cup \{\infty\}$ are the limit points of γ as $t \to -\infty$ and $t \to +\infty$, respectively. If $\vartheta \in \mathrm{T}_z\mathbf{H}_2$ then the footpoint of ϑ is z.

The following notations are due to [Bedford/Keane/Series]
Note, that through each point p on the geodesic γ in $\mathbf{H}_2$ there are two horocycles, $\omega^-(p,\vartheta)$ and $\omega^+(p,\vartheta)$ being orthogonal to the geodesic γ and sharing the tangent vector ϑ. We define $\omega^-(p,\vartheta)$ to pass through the end of the geodesic $\gamma(t)$ corresponding to $t = -\infty$ and so having the unit tangent vector ϑ as the outward normal vector. Analogously, $\omega^+(p,\vartheta)$ passes through the end of the geodesic $\gamma(t)$ corresponding to $t = \infty$ and thus having the unit tangent vector ϑ as the inward normal vector.

With the notion of the horocycles we now define the horocycle flow

$$\Psi^+ : \mathbb{R} \times \mathrm{T}^1\mathbf{H}_2 \to \mathrm{T}^1\mathbf{H}_2$$

as the positive horocycle flow which slides the inward normal vectors to each $\omega^+(p,\vartheta)$ to the right along $\omega^+(p,\vartheta)$ at unit speed. In analogy, we define

$$\Psi^- : \mathbb{R} \times \mathrm{T}^1\mathbf{H}_2 \to \mathrm{T}^1\mathbf{H}_2$$

as the negative horocycle flow which slides the outward normal vectors to each $\omega^-(p,\vartheta)$ to the left along $\omega^-(p,\vartheta)$ at unit speed. We have the following relation between the both above defined flows: $\Psi^+(t,(p,\vartheta)) = -\Psi^-(-t,(p,-\vartheta))$. At each $(p,\vartheta) \in T^1\mathbf{H}_2$ we obtain a three dimensional tangent space $T_{(p,\vartheta)}P$ with the basis given by $\Psi^-(\tau,(p,\vartheta)), \Phi(t,(p,\vartheta)), \Psi^+(\tau,(p,\vartheta))$.
Thus, we obtain:

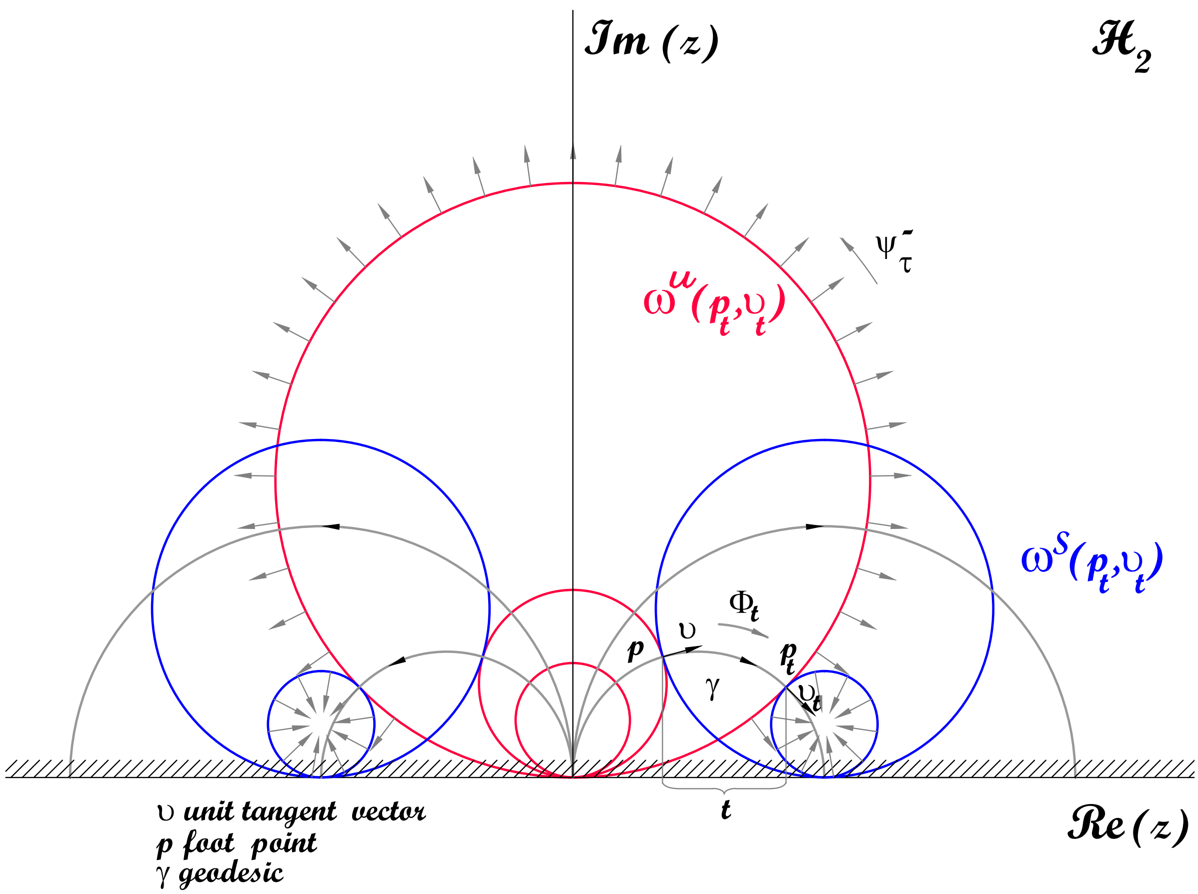
$$\begin{aligned} E^u_{(p,\vartheta)} &:= \operatorname{span}\{\Psi^-(\tau,(p,\vartheta))\}, \\ E^c_{(p,\vartheta)} &:= \operatorname{span}\{\Phi(t,(p,\vartheta))\}, \\ E^s_{(p,\vartheta)} &:= \operatorname{span}\{\Psi^+(\tau,(p,\vartheta))\}. \end{aligned}$$

And finally we obtain the splitting of the tangent bundle of the manifold $\mathrm{T}^1\mathbf{H}_2$:

$$\mathrm{T}_{(p,\vartheta)}\mathrm{P} = E^u_{(p,\vartheta)} \oplus E^c_{(p,\vartheta)} \oplus E^s_{(p,\vartheta)}.$$

In the next picture we set the $\omega^+(p,\vartheta)$ to be $\omega^s(p,\vartheta)$, which stands for its stability and is called a stable leaf and $\omega^-(p,\vartheta)$ to be $\omega^u(p,\vartheta)$, since it is the unstable leaf.

Figure 2.1: Geodesic flow in the upper half plane



Theorem 2.1.9. *Every geodesic flow on a surface of a constant negative curvature is an Anosov flow.*

The proof can be found in [Burns/Gidea]. Initially, the above theorem was first formulated and proved by Anosov.

2.1.2 Structural stability

The most interesting property of Anosov flows is that they are structurally stable, i.e. robust under topological perturbation [Hasselblatt/Katok].

Definition 2.1.10. structural stability of a flow
Let M,N be manifolds $\Phi : \mathbb{R} \times M \to M$, equivalently we write $\Phi_t : M \to M$, be a C^r-flow. We say that Φ_t is C^m structurally stable $(1 \leq m \leq r)$, if any flow sufficiently close to Φ_t in the C^m topology is C^0 orbit equivalent to it, i.e $\exists \epsilon > 0$ such that for any flow $\Psi_t : N \to N$ such that $||\Psi - \Phi||_{C^m} < \epsilon$ there exists a C^m diffeomorphism $h : M \to N$ such that the flow $\chi_t = h^{-1} \circ \Psi_t \circ h$ is a time change of the flow Φ_t.

The next theorem can be found in [Hasselblatt/Katok].

Theorem 2.1.11. *Every Anosov flow is structurally stable.*

This result makes Anosov flows to be so important for the theory of dynamical systems.

2.2 Anosov diffeomorphisms

Definition 2.2.1. hyperbolic set [Hasselblatt/Katok]
Let M be a compact manifold, $U \subset M$ $f : U \to M$ a C^1 diffeomorphism onto its image , and $\Lambda \subset U$ a compact f-invariant set. The set Λ is called a hyperbolic set or hyperbolic structure for the map f if the following holds:
$\exists$ a Riemannian metric called a Lyapunov metric in an open neihgbourhood $U \mathrm{of} \Lambda$ and $\lambda < 1 < \mu$ such that for any point $x \in \Lambda$ the sequence of differentials

$$(Df)_{f^n_x} : T_{f^n_x} M \to T_{f^{n+1}_x} M,\ n \in \mathbb{Z}$$

admits a (λ, μ)-splitting, i.e. $T_\Lambda M = E^s \oplus E^u$.

Remarks 2.2.2. 1. Note that the $(Df)_{f^n_x} : T_{f^n_x} M \to T_{f^{n+1}_x} M$ is linear and E^s denotes the stable eigenspace corresponding to the eigenvalue $\lambda < 1$ and E^u denotes the unstable eigenspace corresponding to the eigenvalue $\mu > 1$.

2. As the simplest example for a hyperbolic set for a diffeomorphism one can consider the hyperbolic fixed points and the hyperbolic periodic points.

Theorem 2.2.3. *[Burns/Gidea] Assume that $f : M \to M$ is a C^k-diffeomorphism of a C^k-smooth manifold M, and Λ is a hyperbolic set for f. There exists $\epsilon > 0$ such that the local stable and unstable manifolds*

$$W_\epsilon^s = \{q \in \Lambda \cap B(p,\epsilon) | d(f^n(q), f^n(p)) \to 0\},$$
$$W_\epsilon^u = \{q \in \Lambda \cap B(p,\epsilon) | d(f^{-n}(q), f^{-n}(p)) \to 0\},$$

are embedded submanifolds of M (local stable and unstable manifolds of p), where d is the distance on M induced by the Riemannian metric, and $B(p,\epsilon) = \{q \in M | d(p,q) < \epsilon\}$. Moreover, $T_pW_\epsilon^s(p) = E_p^s$ and $T_pW_\epsilon^u(p) = E_p^u$. The stable and unstable manifolds are locally unique.

We now establish some basics concerning the Anosov diffeomorphisms, which were first proposed by Dmitrii Anosov. Let us begin with the definition.

Definition 2.2.4. Anosov diffeomorphism [Manning]
Let M be a compact manifold without boundary, $f : M \to M$, be a C^k-diffeomorphism, $k \geq 1$. f is called an Anosov diffeomorphism, if the following holds:

(i) $\exists$ a continuous Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$.

(ii) For any Riemannian metric on M $\exists$ constants $C > 0$,λ, $0 < \lambda < 1$, such that

$$||Df^n(x)|_{E_x^s}|| < C\lambda^n \text{ and } ||Df^{-n}(x)|_{E_x^u}|| < C\lambda^n,$$

for any $n > 0$, $x \in M$.

Remarks 2.2.5.

1. The compactness of M is very important, because of the ergodic property of the Anosov diffeomorphism. As we immidately see from the definition of the Anosov diffeomorphism we do not have the center-part E^c in the splitting, like in the definition of the Anosov flow.

2. We can shorten the above definition by saying, that a diffeomorphism $f : M \to M$, M as above, is an Anosov diffeomorphism if the whole manifold M is a hyperbolic set [Hasselblatt/Katok].

2.2.1 Example: Arnold's cat map

One way to obtain an Anosov diffeomorphism is to start with a hyperbolic automorphism f of a nilpotent Lie group $\mathcal{G}$, which preserves a lattice Γ. Then the quotient space $\mathcal{G}/\Gamma$ is compact and we obtain an induced hyperbolic automorphism, which is Anosov . The same construction is used for non-toral examples and is discussed in chapter 3. The following example is for introductional purpose. Consider the following linear map which was first introduced by Thom.

$$A_0 : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$(x, y) \mapsto (2x + y, x + y),$$

or equivalently $A_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. As it can be calculated , A_0 is a hyperbolic automorphism, which means that all eigenvalues of A_0 are unequal one. More precisely, there are exactly two eigenvalues $0 < \lambda_s < 1$ and $\lambda_u > 1$, see appendix. It can easily be shown that the matrix A_0 preserves the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, i.e. $A_0(\mathbb{Z}^2) = \mathbb{Z}^2$. The invariance of the lattice $\mathbb{Z}^2$ under the map A_0 leads to the following consequence: We obtain an induced map $A : \mathbb{T}^2 \to \mathbb{T}^2$,$(x, y) \mapsto (2x + y, x + y)(\text{mod } 1)$,

or equivalently $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (\text{mod } 1)$, i.e. a map on the compact manifold $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, called a two-dimensional torus.

$\mathbb{R}^2$ is nilpotent Lie group with respect to "+". As it was mentioned in chapter 1, $\mathbb{R}^2/\mathbb{Z}^2$ is a nilmanifold.

This is the easiest example of an Anosov diffeomorphism on a nilmanifold in two dimensions and has been cited many times in the literature. It is named by "Arnold's Cat Map" after the famous russian mathematician Vladimir Igorevitch Arnold who had first described the action of this map on $\mathbb{T}^2$ via an image of a cat head see figure 2.2 in larger size than on the cover, which is shared under this map and after some number of iterations appears again (Poincare Recurrence Theorem).

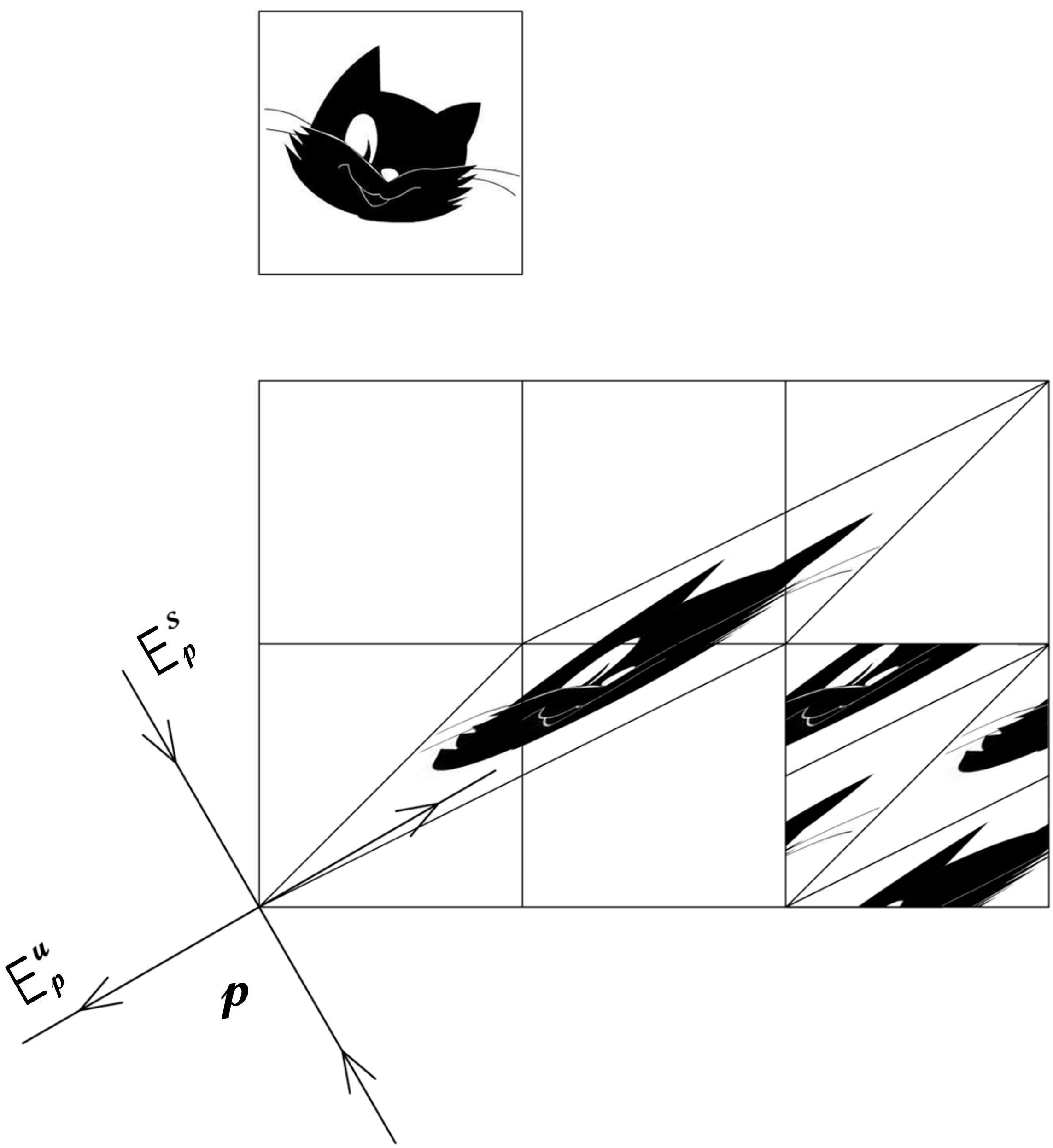
2.2.2 Structural stability of Anosov diffeomorphism

Definition 2.2.6. structural stability of a diffeomorphism[Hasselblatt/Katok] Let M,N be manifolds $f : M \to M$, be a C^r-diffeomorphism. We say that f is C^m structurally stable ($1 \leq m \leq r$), if there exists a neighbourhood U of f in the C^m topology such that every map $g : N \to N$, $g \in U$ is topological conjugate to f, i.e there exists a homeomorphism $H : M \to N$ such that $f = H^{-1} \circ g \circ H$.

Similar to Anosov flows it turns out that the following theorem is true for Anosov diffeomorphisms [Hasselblatt/Katok].

Theorem 2.2.7. *Every Anosov diffeomorphism is structurally stable.*

Figure 2.2: Arnold's cat map, an example of a hyperbolic toral automorphism, note that p is a hyperbolic fixed point



Chapter 3

Examples of non-toral Anosov diffeomorphisms

The question of the existence of compact non-toral Anosov diffeomorphisms was first raised by D.V. Anosov in the Moscower Congress in 1966 [Smale], as it was already mentioned in the introduction. At this time the only known examples were constructed on the torus $\mathbb{T}^n$. But in 1967, as Steven Smale has shown, there are Anosov diffeomorphisms on the nilmanifolds, which can be interpreted as the immediate answer to Anosov's question in 1966. So let us consider the two constructions of Anosov diffeomorphisms on a nilmanifold by Steve Smale [see [Smale]].

3.0.3 Example [Smale]: Anosov diffeomorphism on a nilmanifold of dimension six

General setting [Smale]: Let $\mathcal{G}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and Γ a uniform discrete subgroup (i.e the coset space $\mathcal{G}/\Gamma$ is compact). We know from the theorem 1.5.1 that the Lie group $\mathcal{G}$ must be nilpotent, if $\mathcal{G}$ should admit the existence of a hyperbolic Lie group automorphism $f_0 : \mathcal{G} \to \mathcal{G}$. The nilpotency of a Lie group $\mathcal{G}$ implies the nilpotency of the Lie algebra $\mathfrak{g}$.
Steve Smale took precisely the cartesian product of the three dimensional nilpotent Heisenberg Lie algebra $\mathfrak{heis}(3, \mathbb{R})$, which is 2-step nilpotent, as it has been calculated in chapter 1. If furthermore the discrete subgroup $\Gamma \lhd \mathcal{G}$ is invariant under f_0', i.e. $f_0'(\Gamma_0) = \Gamma_0$, f_0 induces a hyperbolic automorphism

$$f' : \mathfrak{g}/\Gamma_0 \to \mathfrak{g}/\Gamma_0, n\Gamma_0 \mapsto f_0'(n)\Gamma_0.$$

f_0 is called a $\mathfrak{g}$-induced automorphism of $\mathfrak{g}/\Gamma_0$. This is the complete description of the general setting and we will see why it implies the existence of the hyperbolic nilmanifold automorphism $f : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$.
The main idea in this construction is, that it is sufficient to work in the Lie algebra, as it was mentioned in chapter 1. The reason for the permission is roughly

speaking the nilpotency and the connectedness of the Lie group $\mathcal{G}$, because it turns out that for the nilpotent simply connected Lie groups the exponential mapping is surjective.
More presicesly, we can express the main idea of the described construction through the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f_0'} & \mathfrak{g} \\ \downarrow\mathfrak{exp} & & \downarrow\mathfrak{exp} \\ \mathcal{G} & \xrightarrow{f_0} & \mathcal{G} \end{array} \qquad \begin{array}{l} f_0'(\Gamma_0) = \Gamma_0 \Rightarrow f' : \mathfrak{g}/\Gamma_0 \to \mathfrak{g}/\Gamma_0 \\ \\ \\ f_0(\Gamma) = \Gamma \Rightarrow f : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma \end{array}$$

Let us give the proofs of the above claims in the general setting.
Claim 1

i If a discrete subalgebra $\Gamma_0 \subset \mathfrak{g}$ is invariant under a hyperbolic Lie algebra automorphism f_0', i.e. $f_0'(\Gamma_0) = \Gamma_0$, f_0' induces a hyperbolic automorphism

$$f' : \mathfrak{g}/\Gamma_0 \to \mathfrak{g}/\Gamma_0, n + \Gamma_0 \mapsto f_0'(n) + \Gamma_0.$$

ii If a discrete subgroup $\Gamma \subset \mathcal{G}$ is invariant under a hyperbolic Lie group automorphism f_0, i.e. $f_0(\Gamma) = \Gamma$, f_0 induces a hyperbolic nilmanifold automorphism

$$f : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma, x * \Gamma \mapsto f_0(x) * \Gamma.$$

Proof. i Let $n \in \mathfrak{g}$. Then the statement follows immidiately by the property of f_0' to be an automorphism: $f_0'(n+\Gamma_0) = f_0'(n) + f_0'(\Gamma_0)$ and by the invariance of Γ_0 with respect to f_0', it follows $f_0'(n + \Gamma_0) = f_0'(n) + \Gamma_0$, which is the definition of f'. Thus, f' is the induced map, as has been claimed.

ii Let $x \in \mathcal{G}$. In analogy to (i), consider

$$\begin{aligned} f_0(x * \Gamma) &= f_0(x) * f_0(\Gamma) && f_0 \text{ automorphism} \\ &= f_0(x) * \Gamma && f_0(\Gamma) = \Gamma \\ &=: f(x * \Gamma). \end{aligned}$$

□

Claim 2 Let ($\mathcal{G}$,*) be a simply connected nilpotent Lie group then the exponential map $\mathfrak{exp} : \mathfrak{g} \to \mathcal{G}$ is surjective.

Proof.
Step 1 [existence of open neighbouhoods]
$\exists V^{open}(e) \subset \mathfrak{g}, U^{open}(e) \subset \mathcal{G}$ such that

$$\mathfrak{exp} : V^{open}(e) \to U^{open}(e)$$

is a diffeomorphism.
Step 2 Consider the image of the whole Lie algebra $\mathcal{H} := \mathfrak{exp}(\mathfrak{g}) \subset \mathcal{G}$.
Step 3 We use the Baker-Campbell-Hausdorff formula, in order to show that we find to each point $x \in \mathcal{H}$ an open neighbourhood, i.e. $\mathcal{H}$ is open [see step 4]. Furtermore, $\mathcal{H}$ is a subgroup of $\mathcal{G}$, i.e closed under the group operation *.
Step 4 [Campbell-Hausdorff formula] [Onishchik].
Step 5 [connectedness] Consider

$$\mathcal{G} = \bigcup_{g \in \mathcal{G}} g\mathcal{H}.$$

Since $\mathcal{H}$ is open, it follows that either case 1 $g_1\mathcal{H} = g_2\mathcal{H}$ or case 2 $g_1\mathcal{H} \cap g_2\mathcal{H} = \emptyset$ is true. Since $\mathcal{H}$ is connected due to connectedness of $\mathcal{G}$ the case 2 is forbidden. Thus, there exists precisely only one $\mathcal{H}$ such that $\mathcal{H} = \mathcal{G}$. By this result it follows that $\mathfrak{exp}(\mathfrak{g}) = \mathcal{G}$ and especially the surjectivity of $\mathfrak{exp}$. □

1. Simply connected Lie group means that the fundamental group of $\mathcal{G}$ is trivial, i.e. there are no holes in $\mathcal{G}$ or analogously if every closed path can be shrunk to a point [Bump].
2. $f_0 : \mathcal{G} \to \mathcal{G}$ is hyperbolic automorphisms $\iff f_0' : \mathfrak{g} \to \mathfrak{g}$ is hyperbolic, i.e. there exist no eigenvalues of f_0' which are roots of unity.

There exists a very well known theorem by Anthony Manning [Manning], see next chapter :

Theorem 3.0.8. *If $d : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$ is an Anosov diffeomorphism of a nilmanifold then it is topologically conjugate to a hyperbolic nilmanifold automorphism $f : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$, i.e there exists a homeomorphism $h : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$, such that $d = h^{-1} \circ f \circ h$.*

Thus, it will be sufficient to look for a hyperbolic nilpotent Lie group automorphism $f_0 : \mathcal{G} \to \mathcal{G}$ such that $f_0(\Gamma) = \Gamma$.

Let us examine the two non-toral examples of hyperbolic automorphisms. As he points out the second one was given by Borel as well as the explicit number theory approach. First, we shall give the plan of the construction following [Smale].
Plan due to [Smale]
Step 1 Start with a nilpotent Lie algebra $\mathfrak{g} := \mathfrak{g}_1 \times \mathfrak{g}_2$, $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{heis}(3, \mathbb{R})$.
Step 2 Define the hyperbolic automorphisms $f_0' : \mathfrak{g} \to \mathfrak{g}$, $h_0' : \mathfrak{g} \to \mathfrak{g}$.
Step 3 Define the lattice $\Gamma_0 \subset \mathfrak{g}$.
Step 4 Show Γ_0 is a lattice, i.e discrete uniform subgroup in $\mathfrak{g}$.
Step 5 Show $f_0'(\Gamma_0) = \Gamma_0$, which will lead us to the induced hyperbolic automorphism $f_0' : \mathfrak{g}/\Gamma_0 \to \mathfrak{g}/\Gamma_0$ [see Claim 1].
Step 6 Define $\Gamma := \mathfrak{exp}(\Gamma_0) \subset \mathcal{G}$. Show that Γ is a lattice,i.e a discrete uniform

subgroup in the Lie group $\mathcal{HEIS}(3,\mathbb{R}) \times \mathcal{HEIS}(3,\mathbb{R})$.
Step 7 Show $f_0(\mathfrak{exp}(\Gamma_0)) = \mathfrak{exp}(\Gamma_0)$. Due to the results stated in claim 1, we know that f_0 induces a hyperbolic nilmanifold automorphism $f : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$.

To Step 1 Recall that the Heisenberg Lie groups $\mathcal{G}_1 := \mathcal{HEIS}(3,\mathbb{R}) = \mathcal{G}_2$ are simply connected, (non abelian) 2-step nilpotent Lie groups with the corresponding simply connected, 2-step nilpotent Lie algebra $\mathfrak{g}_1 = \mathfrak{heis}(3,\mathbb{R}) = \mathfrak{g}_2$. As in chapter 1: The Heisenberg algebras $\mathfrak{g}_1$,$\mathfrak{g}_1$ are represented by the matrices:

$(\star)$ $\quad A := \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, $a, b, c \in \mathbb{R}$. and $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ is represented by

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

A,B are matrices of the form $(\star)$, see [Smale].

Furthermore, recall from chapter 1 the corresponding basis to the Lie algebra $\mathfrak{g}_1$ X_1, Y_1, Z_1 and to $\mathfrak{g}_2$ X_2, Y_2, Z_2 with the following Lie bracket relations for $i \in \{1, 2\}$:

$[\bullet, \bullet]$	X_i	Y_i	Z_i
X_i	0	Z_i	0
Y_i	$-Z_i$	0	0
Z_i	0	0	0

To Step 2 Define the hyperbolic automorphisms f_0', h_0' on $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Thus, the dimension of $\mathfrak{g}$ is six. Let $\lambda = 2 + \sqrt{3} > 1$. Obviously,
$\lambda^{-1} = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3} < 1$ and $\lambda^{-1} > 0$, $\lambda\lambda^{-1} = 1$.

Example 1	Example 2	Example 2 (modified order)
$f_0' : \mathfrak{g} \to \mathfrak{g}$	$h_0' : \mathfrak{g} \to \mathfrak{g}$	$h_0' : \mathfrak{g} \to \mathfrak{g}$
$X_1 \mapsto \lambda X_1$	$X_1 \mapsto \lambda X_1$	$X_1 \mapsto \lambda X_1$
$Y_1 \mapsto \lambda^2 Y_1$	$Y_1 \mapsto \lambda^{-3} Y_1$	$Z_2 \mapsto \lambda^2 Z_2$
$Z_1 \mapsto \lambda^3 Z_1$	$Z_1 \mapsto \lambda^{-2} Z_1$	$Y_2 \mapsto \lambda^3 Y_2$
$X_2 \mapsto \lambda^{-1} X_2$	$X_2 \mapsto \lambda^{-1} X_2$	$X_2 \mapsto \lambda^{-1} X_2$
$Y_2 \mapsto \lambda^{-2} Y_2$	$Y_2 \mapsto \lambda^3 Y_2$	$Z_1 \mapsto \lambda^{-2} Z_1$
$Z_2 \mapsto \lambda^{-3} Z_2$	$Z_2 \mapsto \lambda^2 Z_2$	$Y_1 \mapsto \lambda^{-3} Y_1$

Note that we are going to discuss only the Example 1. Example 2 follows with the same arguments as the Example 1. In the Example 1, we see that $\mathfrak{g}_1 = \mathfrak{g}^u$, u stands as usual for the unstable part (expanding), because of the factor $\lambda^k > 1, k \in \{1, 2, 3\}$ and $\mathfrak{g}_1 = \mathfrak{g}^s$, s stands for the stable part (contracting), since $0 < \lambda^{-k} < 1$. It follows that f_0' is hyperbolic and $\mathcal{G} = \mathcal{G}^u \times \mathcal{G}^s$. In the Example 2 however, $\mathfrak{g}_1 \neq \mathfrak{g}^u$ and $\mathfrak{g}_1 \neq \mathfrak{g}^s, \mathcal{G} \neq \mathcal{G}^u \times \mathcal{G}^s$.

To Step 3 Define $\sigma : \mathbb{K} \to \mathbb{K}, \sqrt{3} \mapsto -\sqrt{3}$, $\mathbb{K} := \mathbb{Q}[\sqrt{3}]$. Furthermore, define $\Gamma_0 \subset \mathfrak{g}$ as a set of 6×6-matrices of the form

$$\begin{pmatrix} 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(a) & \sigma(b) \\ 0 & 0 & 0 & 0 & 0 & \sigma(c) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$a := k_1 + \sqrt{3}l_1$, $b := k_2 + \sqrt{3}l_2$, $c := k_3 + \sqrt{3}l_3$, $k_j \in \mathbb{Z}, j \in \{1, 2, 3\}$.

To Step 4 We have to show that Γ_0 as defined above is indeed a lattice, i.e. a discrete uniform subgroup in $\mathfrak{g}$.

i Γ_0 is a discrete subgroup.

Proof. Let $A, B \in \Gamma_0$. Define

$$A := \begin{pmatrix} 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(a) & \sigma(b) \\ 0 & 0 & 0 & 0 & 0 & \sigma(c) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B := \begin{pmatrix} 0 & \tilde{\mathrm{a}} & \tilde{\mathrm{b}} & 0 & 0 & 0 \\ 0 & 0 & \tilde{\mathrm{c}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(\tilde{\mathrm{a}}) & \sigma(\tilde{\mathrm{b}}) \\ 0 & 0 & 0 & 0 & 0 & \sigma(\tilde{\mathrm{c}}) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Without loss of generality let

$$\mathrm{a}-\tilde{\mathrm{a}} =: v \neq 0,\ v := k + \sqrt{3}l,$$
$$(k, l) \neq (0, 0), \mathrm{b}-\tilde{\mathrm{b}} = 0, \mathrm{c}-\tilde{\mathrm{c}} = 0.$$

It follows that $\sigma(v) \neq 0$, $\sigma(w) = 0$ and $\sigma(z) = 0$.
Consider: $||A - B||$.
We want to show that $||A - B|| > c$, $c > 0$ fixed, $c \in \mathbb{R}$.
We have

$$\begin{aligned} ||A - B|| &= \Sigma_{i,j=1}^{6} |r_{ij} - s_{ij}| \\ &= |a - \tilde{\mathrm{a}}| + |b - \tilde{\mathrm{b}}| + |c - \tilde{\mathrm{c}}| + |\sigma(a) - \sigma(\tilde{\mathrm{a}})| + |\sigma(b) - \sigma(\tilde{\mathrm{b}})| + |\sigma(c) - \sigma(\tilde{\mathrm{c}})| \\ &= |a - \tilde{\mathrm{a}}| + |\sigma(a - \tilde{\mathrm{a}})| + |b - \tilde{\mathrm{b}}| + |\sigma(b - \tilde{\mathrm{b}})| + |c - \tilde{\mathrm{c}}| + |\sigma(c - \tilde{\mathrm{c}})| \\ &= |v| + |\sigma(v)| \end{aligned}$$

Now consider:
$|v| + |\sigma(v)| \geq 2\sqrt{|v||\sigma(v)|} = 2\sqrt{|k + \sqrt{3}l||k - \sqrt{3}l|} = 2\sqrt{|k^2 - 3l^2|} > 2 * 1.$

Thus, $||A - B|| = |v| + |\sigma(v)| > 2$, i.e. Γ_0 is a discrete subgroup in $\mathfrak{g}$. □

ii Γ_0, as defined above, is a uniform subgroup, i.e. $\mathfrak{g}/\Gamma_0$ is compact.

Proof. To show ii it is sufficient to prove the boundedness of of the funamental domain of Γ_0. Consider the basis of Γ_0.

$$B_{\Gamma_0} = \left\{ \begin{pmatrix} X & [0] \\ [0] & X \end{pmatrix}, \begin{pmatrix} Y & [0] \\ [0] & Y \end{pmatrix} \begin{pmatrix} Z & [0] \\ [0] & Z \end{pmatrix} \begin{pmatrix} \sqrt{3}X & [0] \\ [0] & -\sqrt{3}X \end{pmatrix} \begin{pmatrix} \sqrt{3}Y & [0] \\ [0] & -\sqrt{3}Y \end{pmatrix} \begin{pmatrix} \sqrt{3}Z & [0] \\ [0] & -\sqrt{3}Z \end{pmatrix} \right\},$$

where [0] denotes a 3×3- zero matrix and X, Y, Z are the basis elements of $\mathfrak{heis}(3, \mathbb{R})$ as being defined in chapter 1.
Without loss of generality, let $A = \begin{pmatrix} nX & [0] \\ [0] & nX \end{pmatrix} + \begin{pmatrix} m\sqrt{3}X & [0] \\ [0] & -m\sqrt{3}X \end{pmatrix}$, where $n, m \in \mathbb{Z}$. We define $a := n + m\sqrt{3}$, $b := n - m\sqrt{3}$ and (1) $n := \lfloor \frac{a+b}{2} \rfloor$ and (2) $m := \lfloor \frac{a-b}{2\sqrt{3}} \rfloor$. Consider: $|a - n - m\sqrt{3}| + |b - n + m\sqrt{3}|$
We have to show that there exists a constant c such that

$$|a - n - m\sqrt{3}| + |b - n + m\sqrt{3}| < c,$$

which would imply the required boundedness.
By (1) we obtain: $a - n - m\sqrt{3} \approx a - \frac{a+b}{2} - \sqrt{3}\frac{a-b}{2\sqrt{3}} = 0$

Analogously, by (2) we obtain: $b - n + m\sqrt{3} \approx a - \frac{a+b}{2} + \sqrt{3}\frac{a-b}{2\sqrt{3}} = 0$
$\Rightarrow \exists c$ such that $|a - n - m\sqrt{3}| + |b - n + m\sqrt{3}| < c$.

□

To Step 5 We show that Γ_0 is invariant under the Lie algebra automorphism f_0', i.e.$f_0'(\Gamma_0) = \Gamma_0$.

Proof.
"$\subseteq$": Let $A \in \Gamma_0$, as defined before with the entries defined as follows:

$$\begin{gathered} a := k_1 + \sqrt{3}l_1, \\ b := k_2 + \sqrt{3}l_2, \\ c := k_3 + \sqrt{3}l_3, \\ \sigma(a) := k_1 + \sqrt{3}l_1, \\ \sigma(b) := k_2 - \sqrt{3}l_2, \\ \sigma(c) := k_3 - \sqrt{3}l_3, \\ k_j \in \mathbb{Z},\ j \in \{1, 2, 3\} \end{gathered}$$

Consider

$$f_0'(A) = \begin{pmatrix} 0 & \lambda a & \lambda^3 b & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-1}\sigma(a) & \lambda^{-3}\sigma(b) \\ 0 & 0 & 0 & 0 & 0 & \lambda^{-2}\sigma(c) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We calculate the form of the entry a_{12} of the matrix $f_0'(A)$, the form of the other entries can be calculated analogously.

It is :

$$\begin{aligned}
a_{12} = \lambda a &= (2+\sqrt{3})(k_1+\sqrt{3}l_1)\\
&= 2k_1+3l_1+\sqrt{3}(k_1+2l_1)\\
&= p_1+\sqrt{3}q_1\\
&= \hat{a}\in\mathbb{Z}\oplus\sqrt{3}\mathbb{Z}
\end{aligned}$$

By analogous reasoning it follows that $a_{13}=\lambda^3 b=\hat{b}\in\mathbb{Z}\oplus\sqrt{3}\mathbb{Z}$ and $a_{23}=\lambda^2 c=\hat{c}\in\mathbb{Z}\oplus\sqrt{3}\mathbb{Z}$.

In the right corner of the matrix $f_0'(A)$ we calculate:

$$\begin{aligned}
a_{45} = \lambda^{-1}\sigma(a) &= (2-\sqrt{3})(k_1-\sqrt{3}l_1)\\
&= 2k_1+3l_1-\sqrt{3}(k_1+2l_1)\\
&= p_1-\sqrt{3}q_1\\
&= \sigma(\hat{a})\in\mathbb{Z}\oplus\sqrt{3}\mathbb{Z}
\end{aligned}$$

The form of the remaining entries can be obtained in the similar way, which we do not calculate in the extended way, instead we just mention that: $a_{56}=\lambda^{-2}\sigma(c)=\sigma(\hat{c})\in\mathbb{Z}\oplus\sqrt{3}\mathbb{Z}$ and $a_{46}=\lambda^3\sigma(b)=\sigma(\hat{b})\in\mathbb{Z}\oplus\sqrt{3}\mathbb{Z}$

$$\Rightarrow f_0'(A)=\begin{pmatrix}0&\hat{a}&\hat{b}&0&0&0\\0&0&\hat{c}&0&0&0\\0&0&0&0&0&0\\0&0&0&0&\sigma(\hat{a})&\sigma(\hat{b})\\0&0&0&0&0&\sigma(\hat{c})\\0&0&0&0&0&0\end{pmatrix}\in\Gamma_0.$$

"$\supseteq$": Consider

$$B=\begin{pmatrix}0&x&y&0&0&0\\0&0&z&0&0&0\\0&0&0&0&0&0\\0&0&0&0&\sigma(x)&\sigma(y)\\0&0&0&0&0&\sigma(z)\\0&0&0&0&0&0\end{pmatrix}\in\Gamma_0.$$

Now, consider the entries of $B:=(b_{ij})_{i,j=1,\ldots,6}$.

$$\begin{aligned}
b_{12}=\lambda\lambda^{-1}x &= \lambda((2-\sqrt{3})(v_1+\sqrt{3}w_1))\\
&= \lambda(2v_1-3w_1+\sqrt{3}(2w_1-v_1)\\
&= \lambda(\tilde{v_1}+\sqrt{3}\tilde{w_1})\\
&= \lambda\tilde{x}
\end{aligned}$$

Analogously, $b_{13}=\lambda^3\lambda^{-3}y=\lambda^3\tilde{y}$ and $b_{23}=\lambda^2\lambda^{-2}z=\lambda^2\tilde{z}$.

Now for the other entries, we have:

$$\begin{aligned} b_{45} &= \lambda^{-1}\lambda\sigma(x) = \lambda^{-1}(2+\sqrt{3})(v_1-\sqrt{3}w_1) \\ &= \lambda^{-1}(2v_1 - 3w_1 - \sqrt{3}(2w_1 - v_1) \\ &= \lambda^{-1}(\tilde{v_1} - \sqrt{3}\tilde{w_1}) \\ &= \lambda^{-1}\sigma(\tilde{x}) \end{aligned}$$

Analogously, $b_{56} = \lambda^{-2}\sigma(z) = \lambda^{-2}\sigma(\tilde{z})$ and $b_{46} = \lambda^{-3}\sigma(z) = \lambda^{-3}\sigma(\tilde{y})$.

$$\Rightarrow B = \begin{pmatrix} 0 & \lambda\tilde{x} & \lambda^2\tilde{z} & 0 & 0 & 0 \\ 0 & 0 & \lambda^3\tilde{y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-1}\sigma(\tilde{x}) & \lambda^{-3}\sigma(\tilde{y}) \\ 0 & 0 & 0 & 0 & 0 & \lambda^{-2}\sigma(\tilde{z}) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in f_0'(\Gamma_0).$$

Since "$\subset$" and "$\supset$" hold, it follows that $f_0'(\Gamma_0) = \Gamma_0$, as was claimed.

$\square$

Step 6 Define Γ as the image of Γ_0 under the exponential map $\mathfrak{exp} : \mathfrak{g} \to \mathcal{G}$, i.e.$\Gamma := \mathfrak{exp}(\Gamma_0)$. Show that Γ is a uniform discrete subgroup of $\mathcal{G}$.

Proof. i discretness of Γ. Since the exponential mapping is bilipshitz the inequalities shown in step 4 (discretness of Γ_0) hold.

ii Uniformicity. We can apply theorem 1.3.3.[Malcev] (a) stated in chapter 1. Due to the above theorem we have to show that $\Gamma := \mathfrak{exp}(\Gamma_0)$ is a finitely generated nilpotent group containing no elements of finite order.

$\square$

Step 7 Show $f_0(\Gamma) = \Gamma$.

Proof. Let $A \in \Gamma_0$ as in step 5. Consider $\mathfrak{exp}(A)$.
$\mathfrak{exp}(A) = \mathrm{id} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$. But A^3 is a zero matrix, i.e all entries are zero. Thus, $\mathfrak{exp}(A) = id + A + \frac{A^2}{2!}$ and more precisely

$$\mathfrak{exp}(A) = \begin{pmatrix} 1 & a & \frac{b+ac}{2!} & 0 & 0 & 0 \\ 0 & 1 & c & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sigma(a) & \sigma(\frac{b+ac}{2!}) \\ 0 & 0 & 0 & 0 & 1 & \sigma(c) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider $f_0(\mathfrak{exp}(A)) = \mathfrak{exp}(f_0'(A))$. Since $f_0'(\Gamma_0) = \Gamma_0$ (step 5) it follows $\mathfrak{exp}(f_0'(\Gamma_0)) = \mathfrak{exp}(\Gamma_0)$. By $f_0(\mathfrak{exp}(\Gamma_0)) = \mathfrak{exp}(f_0'(\Gamma_0))$ (see thr diagram at the beginning) we obtain $f_0(\mathfrak{exp}(\Gamma_0)) = \mathfrak{exp}(\Gamma_0)$, as was claimed.

$\square$

Steve Smale not only showed with the above examples that there exist nontoral diffeomorphisms, but also formulated the following problem: What compact M (M denotes a manifold) admits Anosov diffeomorphisms? This problem remains unsolved until now. V.V. Gorbatsevich points out that there is the following conjecture [Gorbatsevitch]:

Conjecture 3.0.9. If a compact manifold M admits Anosov diffeomorphism, then M is homeomorphic to a nilmanifold or an infra-nilmanifold .

Remark. 1. Indeed, all examples of Anosov diffeomorphisms are algebraic, i.e. on nil- or infranil-manifolds. That is mainly the reason why the above conjecture appears. To disprove the above conjecture a counterexample would be sufficient, i.e. one could try to construct a manifold which is not homeomorphic to a nilmanifold. One possible method could be the Dehn chirurgy known from topology, as it was recently communicated to me by V.V. Gorbatsevitch in Moscow.

In the year 1967 [Shub], Michael Shub constructed and calculated, with the help of Jacob Palis, examples of Anosov diffeomorphisms on the infranil-manifolds, as we will see they do not work. In his recent paper [2006], which is not published yet, he formulates the following open problem, which is equivalent to the above conjecture: **"Problem 12: Are all Anosov diffeomorphisms infra-nil?"**

3.0.4 Example [Shub]: Anosov diffeomorphism on a six dimensional infranil-manifold

This example was published in 1967 [Shub], i.e. in the same year as M.Shub's advisor Steve Smale has constructed the above examples of Anosov diffeomorphisms on the six-dimensional nilmanifold. The general idea is in the most parts the same, i.e. the construction is reduced to the definition of the hyperbolic automorphism on the nilpotent Lie group $\mathcal{N}$, which preserves the appropriate lattice Γ in the nilpotent Lie group $\mathcal{N}$. The main difference consists in the definition of the lattice, which in some sense is more complicated and forces the manifold $\mathcal{N}/\Gamma$ to be an infranil-manifold. Notice that this manifold, as M. Shub points out, is not a nilmanifold. Unfortunately, it took me some time until I found a mistake in Michael Shub's example, which will be discussed in the plan of the example. After personally communicating to the author of the example, he admit that it is true that there is indeed a mistake, which has been also found by Jan Willem Nienhuys. The latter told me that he was ignored, so that the mistake has never been corrected in the literature and remained hidden. That is why it is not corrected yet, but maybe in the near future, which seems to be very important.

Example 3.0.10.
We will first give the plan of the example as it is formulated in [Shub] and then turn to discuss why the example is not true, as stated above.

Step 1 We shall give the definitions of the following objects as we can find it in the original paper [Shub]:

1. As in the previous example the nilpotent Lie group will be six dimensional, namely the cartesian product of the three dimensional Heisenberg group: $\mathcal{N} := \mathcal{HEIS}(3,\mathbb{R}) \times \mathcal{HEIS}(3,\mathbb{R})$.
2. $C = \{\mathrm{id}, B\} \subset \mathcal{AUT}(\mathcal{N})$, C is compact, where $B : \mathcal{N} \to \mathcal{N}, (a,b,c,d,e,f) \mapsto (-a,-b,c,-d,-e,f), B^2 = \mathrm{id}$.
Note, (a,b,c,d,e,f) is the abriviate notation for the 6×6-lower triangular matrix with ones on the diagonal and $a,b,c,d,e,f \in \mathbb{R}$ are real entries under the diagonal. More precisely, the notation stands for the following form of the matrix:
$$(a,b,c,d,e,f) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ \mathbf{c} & b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & \mathbf{f} & e & 1 \end{pmatrix}.$$
Thus, the multiplication can be defined in the following way:

$(a,b,c,d,e,f)(a_1,b_1,c_1,d_1,e_1,f_1) :=$
$(a+a_1, b+b_1, c+ba_1+c_1, d+d_1, e+e_1, f+ed_1+f_1)$.
Compare with the usual matrix multiplication:
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ c & b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d & 1 & 0 \\ 0 & 0 & 0 & f & e & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 & 0 \\ c_1 & b_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d_1 & 1 & 0 \\ 0 & 0 & 0 & f_1 & e_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a+a_1 & 1 & 0 & 0 & 0 & 0 \\ c+ba_1+c_1 & b+b_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d+d_1 & 1 & 0 \\ 0 & 0 & 0 & f+ed_1+f_1 & e+e_1 & 1 \end{pmatrix}.$$
3. Let σ be the non-trivial Galois automorphism as in the previous example. Define the uniform discrete subgroup Γ of $\mathcal{N}$, compare with the definition given by Smale:
$$\Gamma := \{(\alpha,\beta,\gamma,\sigma(\alpha),\sigma(\beta),\sigma(\gamma)) | \alpha,\beta,\gamma \in \mathbb{Z} \oplus \sqrt{3}\mathbb{Z}\}.$$
Note, in [Shub] there is a misprint, namely there is the following definition of Γ: $\Gamma := \{(\alpha,\beta,\gamma,\sigma(\alpha),\sigma(\beta),\gamma) | \alpha,\beta,\gamma \in \mathbb{Z} \oplus \sqrt{3}\mathbb{Z}\}$. But this is not a lattice, since the discreteness of Γ is violated.
4. Let $B_q := (0,0,q,0,0,\sigma(q))B$, i.e.
$B_q(a,b,c,d,e,f) = (-a,-b,c+q,-d,-e,\sigma(q)+f)$ due to the matrix multiplication defined above, where $q \in \mathbb{Q}[\sqrt{3}], q \notin \mathbb{Z} \oplus \sqrt{3}\mathbb{Z}, 2q \in \mathbb{Z} \oplus \sqrt{3}\mathbb{Z}, \alpha \in \mathbb{Z} \oplus \sqrt{3}\mathbb{Z}$ and $\lambda q = \alpha + q$. Let Γ_q be the uniform discrete subgroup of $\mathcal{N} \rtimes \mathcal{C}$ generated by Γ and B_q. Then $\Gamma_q = \Gamma \cup B_q\Gamma$.

5. Define a hyperbolic automorphism: Let $\lambda := 2+\sqrt{3}$ as in the previous example. Define $A : \mathcal{N} \to \mathcal{N}, (a,b,c,d,e,f) \mapsto (\sigma(\lambda)a, \lambda^2 b, \lambda c, \lambda d, \sigma(\lambda)^2 e, \sigma(\lambda) f)$, a hyperbolic Lie group automorphism of $\mathcal{N}$.

Step 2 Show $A\Gamma_q A^{-1} = \Gamma_1$, equivalently $A(\Gamma_q) = \Gamma_q$. A induces the map $\overline{A} : \mathcal{N}/\Gamma_q \to \mathcal{N}/\Gamma_q$.
Step 3 Prove $\mathcal{N}/\Gamma_q$ is not a nilmanifold but an infranil-manifold. For this we need to show that Γ_q is torsion free, see chapter 1, infra-nilmanifolds.

To step 1:
1. The nilpotency of $\mathcal{N}$ was shown in chapter 1.
2. Michael Shub introduced this notation for the multiplication, probably in order to save space. Notice, on what places the entries are placed, which is very important for the multiplication as defined above.
3. The definition of Γ is the same like in Smale's example, so we can deduce that Γ is a uniform discrete subgroup in $\mathcal{N}$. Γ is needed for the definition of Γ_q.
4. Note, B_q is not a homomorphism. This we can see by the folllowing analogue definition of B_q: Let $v, w \in \mathbb{R}^6$ with the multiplication definition motivated by the matrix multiplication as above

$$\mu(v,w)_i := \begin{cases} v_i + w_i & \text{for} \quad i = 1,2,4,5 \\ v_3 + w_1 v_2 + w_3 & \text{for} \quad i = 3 \\ v_6 + w_4 v_5 + w_6 & \text{for} \quad i = 3 \end{cases}.$$

Then we can express the above map B_q as follows:

$$B_q : \mathbb{R}^6 \to \mathbb{R}^6, v \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ q \\ 0 \\ 0 \\ \sigma(q) \end{pmatrix}$$

B_q is not a homomorphism, since

$$B_q(\mu(v,w)) = \begin{pmatrix} -(v_1+w_1) \\ -(v_2+w_2) \\ v_3 + w_1 v_2 + w_3 + q \\ -(v_4+w_4) \\ -(v_5+w_5) \\ v_6 + w_4 v_5 + w_6 + \sigma(q) \end{pmatrix} \neq \begin{pmatrix} -(v_1+w_1) \\ -(v_1+w_1) \\ v_3 + q + w_1 v_2 + w_3 + q \\ -(v_4+w_4) \\ -(v_5+w_5) \\ v_6 + \sigma(q) + w_4 v_5 + w_6 + \sigma(q) \end{pmatrix} = \mu(B_q v, B_q w).$$

B_q is not a homomorphism. But $B_q \in Aff(\mathcal{N})$, since $(0,0,q,0,0,\sigma(q)) \in \mathcal{N}$ and $B \in \mathcal{C}$, such that the action of $T := ((0,0,q,0,0,\sigma(q)), B) \in \mathcal{N} \rtimes \mathcal{C}$ on $\mathcal{N}$ is given via the map B_q such that we can conclude that $B_q \in Aff(\mathcal{N})$. $B_q : \mathcal{N} \to \mathcal{N}$ is a map which together with Γ should generate Γ_q. It should be $\Gamma_q \subset \mathcal{N} \rtimes \mathcal{C} := \{(A,B) | A \in \mathcal{N}, B \in \mathcal{C}\}$. Indeed, $\Gamma_q = \Gamma \cup B_q \Gamma$

The above example I found to be cited many times in the subsequent literature, see for examle [Franks] [Nitecki] [Porteous], without correcting it.

Chapter 4

Manifolds admitting Anosov diffeomorphisms: A Survey

Let us give a short survey on known results concerning manifolds admitting Anosov diffeomorphisms.

The earliest result was due to Nathan Jacobson in 1955 in [Jacobson].

We first state the theorem which is needed to prove the theorem 4.0.11.:

Theorem 4.0.11. *generalized Engel theorem If $\mathfrak{W}$ is a weakly closed set of nilpotent linear transformations in a finite-dimensional vector space, then the enveloping associative algebra $\mathfrak{W}^*$ of $\mathfrak{W}$ is nilpotent.*

Theorem 4.0.12. *Let $\mathfrak{l}$ be a Lie algebra which possess an automorphism σ none of whose characteistic roots are roots of unity (i.e. σ is hyperbolic). Then $\mathfrak{l}$ is nilpotent.*

Proof. Assume the base field is algebraically closed. Then $\mathfrak{l} = \Sigma \mathfrak{l}_\zeta$, $\mathfrak{l}_\zeta$ is the eigenspace corresponding to the eigenvalue ζ of σ. Since no ζ is a root of unity the elements $\zeta(\zeta')^k$, $k = 1, 2, \ldots$ are unequal, thus not all of these are roots. Let $Ad(l_\zeta)$ denote the set of adjoint mappings determined by the elements of $\mathfrak{l}_\zeta$. This implies that for $x \in \mathfrak{l}_\zeta$, $a \in \mathfrak{l}_{\zeta'}$ $x(Ada) \equiv [xa]$ is 0 or in $\mathfrak{l}_{\zeta(\zeta')^k}$. Choose k such that $\zeta(\zeta')^k = 1 \rightarrow x(Ada)^k = 0$, which implies the nilpotency of of Ada for every $a \in \mathfrak{l}_{zeta'}$, since ζ' was arbitrary. It follows that every element of $\mathfrak{W} = \bigcup Ad(\mathfrak{l}_\zeta)$ is nilpotent. Conclusion [generalized Engel theorem] $\mathfrak{W}^*$, i.e. the enveloping associative algebra of $\mathfrak{W}$ is nilpotent, and hence $Ad(\mathfrak{l})$ is nilpotent. Thus, the Lie algebra $\mathfrak{l}$ itself should be nilpotent. $\square$

The result in the theorem 4.0.6 implies the nilpotency of a Lie group $\mathcal{L}$ corresponding to a Lie group $\mathfrak{l}$ with a hyperbolic Lie algebra automorphism. Anosov diffeomorphisms are diffeomorphisms which are hyperbolic in each point of a compact manifold, so this fact forces the Lie group to be nilpotent.

As it was mentioned in chapter 3 in 1966 Anosov raised the question whether there exist non-toral examples of Anosov diffeomorphisms. In 1967, S. Smale constructed an example of Anosov diffeomorphism on the six dimensional nilmanifold and M. Shub an infranil-example of Anosov diffeomorphism.

In 1971, Hugh L.Porteous gave a classification of flat manifolds supporting Anosov diffeomorphisms for the dimension less than six, which uses the following theorem:

Theorem 4.0.13. *If M is a flat manifold whose linear holonomy group F is cyclic and $T : F \to GL(n, \mathbb{Z})$ is the natural representation, and if $N = T(g)$ where g is a generator of F, then M supports an Anosov diffeomorphism iff N has none of the following numbers as simple eigenvalues:*$1, -1, i, -i, \omega, \omega^2, -\omega, -\omega^2$.

We can summarize Porteous' results in the following table:

dimension	im/possible flat manifold
1	S^1 does not support an Anosov diffeomorphism since the condition of the theorem 4.0.12 is not satisfied
2	flat torus $\mathbb{T}^2$, in particular the Klein bottle does not support an Anosov diffeomorphism
3	$\mathbb{T}^3$
4	$F = \mathbb{Z}_2$ The qoutients of T^4 by the actions of affine transformations $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \textit{ and } \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$
5	$\mathbb{T}^5$, $\mathbb{T}^5/\mathbb{Z}_2$

The following proposition includes an important restriction on flat manifolds which allow Anosov diffeomorphisms [Porteous].

Proposition 4.0.14. *No flat manifold with first Betti number one supports an Anosov diffeomorphism.*

We recall the definition of the k-th Betti number [Burns/Gidea].

Definition 4.0.15. Let M be a manifold. The vector space

$$H^k(M) := \text{closed k-forms on M/exact k-forms on M}$$

is called the k-th de Rham cohomology group. The dimension β_k of the vector space $H^k(M)$ is called the k-th Betti number (provided it is finite).

Due to the example 6.8.5 in [Burns/Gidea] we obtain the following for the cohomology groups of the n-dimensional sphere ($n \geq 1$):

$$H^k(S^n) = \begin{cases} \mathbb{R} \text{ if } k = 0, \\ 0 \text{ if } 0 < k < n, \\ \mathbb{R} \text{ if } k = n. \end{cases} .$$

Thus the first Betti number of the n-dimensional sphere is one, so there are no Anosov diffeomorphisms on S^n, due to the above proposition.

In 1974, Anthony Manning extended the result known for tori [proved by Franks] to nilmanifolds and infranilmanifolds, namely he proved the following theorem.

Theorem 4.0.16. *If $f : \mathcal{N} \to \mathcal{N}$ is an Anosov diffeomorphism of an infranilmanifold then it is topologically conjugate to a hyperbolic infranilmanifold automorphism $g : \mathcal{N} \to \mathcal{N}$, i.e. there exists a homeomorphism $h : \mathcal{N} \to \mathcal{N}$ such that $f = h^{-1} \circ g \circ h$.*

Remark. 1. Precisely, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{f} & \mathcal{N} \\ \downarrow_h & & \downarrow_h \\ \mathcal{N} & \xrightarrow{g} & \mathcal{N} \end{array}$$

2. A.Manning's result allows to search for hyperbolic automorphism on infranilmanifolds which is equivalent to finding infranil Anosov diffeomorphisms, which is simpler.
3. It turns out that the above theorem remains unproved, see [Dekimpe], because Manning's proof concerning the infra-nilmanifold case is based on a false result by Auslander.

Chapter 5

Appendix

5.1 To chapter 1

We will demonstrate how to obtain a Lie algebra to a corresponding Lie group on the following example:

$$\mathcal{O}(N,\mathbb{R}) = \{A \in Mat(N \times N;\mathbb{R}) | AA^T = A^T A = Id\}.$$

Let A(t) be a one-parameter subgroup of $O(N,\mathbb{R})$ parametrised by t, then differentiating $AA^T = Id$ with respect to t in $t = 0$ we obtain $A'(0) + A'(0)^T = 0$ which leads to $A'(0) = -A'(0)^T$. Thus, the corresponding tangent algebra is:

$$\mathfrak{o}(N,\mathbb{R}) = \{A \in Mat(N \times N;\mathbb{R}) | A^T = -A\}.$$

5.2 To chapter 2

5.2.1 Geodesics of the Lobachevsky's plane

We will give the formal proof for the geodesics of the Lobachevsky's plane with the tools of differential geometry. As it was mentioned before, the reader can find the proof in [Burns/Gidea].

Proof. **Step 1** Calculate the Christoffel symbols

$$\Gamma_{ij}^k := \frac{1}{2}\Sigma_l = 1^2(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l})g^{lk} \tag{5.1}$$

where (g^{lk}) is the inverse of the matrix (g_{lk}).
Step 2 Calculate the gedesic equations using the geodesic equation formula

$$\frac{d^2 x_k}{dt^2} + \Sigma_{i,j=1}^m \frac{dx_i}{dt}\frac{dx_j}{dt}\Gamma_{ij}^k = 0 \tag{5.2}$$

with the results obtained in step 1.
Step 3 Consider two cases and deduce the form of the geodesics.
To step 1 We know that the matrix $(g_{lk}) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$, so $(g^{lk}) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$ is the inverse of (g_{lk}). We use the formula (4.1) in order to compute the Christoffel symbols. We have:

$$\Gamma^1_{11} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{1l}}{\partial x_1} + \frac{\partial g_{1l}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_l})g^{l1}$$
$$= \frac{1}{2}[(\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1})g^{11} + (\frac{\partial g_{12}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2})g^{21}] = 0$$

$$\Gamma^1_{12} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{2l}}{\partial x_1} + \frac{\partial g_{1l}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_l})g^{l1}$$
$$= \frac{1}{2}[(\frac{\partial g_{21}}{\partial x_1} + (\frac{\partial g_{11}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_1})g^{11} + (\frac{\partial g_{22}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_2})g^{21} = -\frac{1}{y}$$

$$\Gamma^1_{21} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{1l}}{\partial x_2} + \frac{\partial g_{2l}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_l})g^{l1}$$
$$= \frac{1}{2}[(\frac{\partial g_{11}}{\partial x_2} + (\frac{\partial g_{21}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_1})g^{11} + \frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_2})g^{21}] = -\frac{1}{y}$$

$$\Gamma^1_{22} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{2l}}{\partial x_2} + \frac{\partial g_{2l}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_l})g^{l1}$$
$$= \frac{1}{2}[(\frac{\partial g_{21}}{\partial x_2} + (\frac{\partial g_{21}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1})g^{11} + \frac{\partial g_{22}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_2})g^{21}] = 0$$

$$\Gamma^2_{11} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{1l}}{\partial x_1} + \frac{\partial g_{1l}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_l})g^{l2}$$
$$= \frac{1}{2}[(\frac{\partial g_{11}}{\partial x_1} + (\frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1})g^{12} + \frac{\partial g_{12}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_2})g^{22}] = \frac{1}{y}$$

$$\Gamma^2_{12} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{2l}}{\partial x_1} + \frac{\partial g_{1l}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_l})g^{l2}$$
$$= \frac{1}{2}[(\frac{\partial g_{21}}{\partial x_1} + (\frac{\partial g_{11}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_1})g^{l2} + \frac{\partial g_{22}}{\partial x_1} + \frac{\partial g_{12}}{\partial x_2} - \frac{\partial g_{12}}{\partial x_2})g^{22}] = 0$$

$$\Gamma^2_{21} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{1l}}{\partial x_2} + \frac{\partial g_{2l}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_l})g^{l2}$$
$$= \frac{1}{2}[(\frac{\partial g_{11}}{\partial x_2} + (\frac{\partial g_{21}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_1})g^{12} + (\frac{\partial g_{12}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_1} - \frac{\partial g_{21}}{\partial x_l})g^{22}] = 0$$

$$\Gamma^2_{22} = \frac{1}{2}\Sigma^2_{l=1}(\frac{\partial g_{2l}}{\partial x_2} + \frac{\partial g_{2l}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_l})g^{l2}$$
$$= \frac{1}{2}(\frac{\partial g_{21}}{\partial x_2} + \frac{\partial g_{21}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_1})g^{12} + (\frac{\partial g_{22}}{\partial x_2} + \frac{\partial g_{22}}{\partial x_2} - \frac{\partial g_{22}}{\partial x_2})g^{22}] = -\frac{1}{y}.$$

To step 2 Using (4.2) we obtain :

$$\frac{d^2x}{dt^2} + \frac{dx}{dt}\frac{dy}{dt}\Gamma^1_{12} + \frac{dy}{dt}\frac{dx}{dt}\Gamma^1_{21} = \frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0 \tag{5.3}$$

$$\frac{d^2y}{dt^2} + \Big(\frac{dy}{dt}\Big)^2\Gamma^2_{22} + \Big(\frac{dx}{dt}\Big)^2\Gamma^2_{11} = \frac{d^2y}{dt^2} - \frac{1}{y}\Big(\frac{dy}{dt}\Big)^2 + \frac{1}{y}\Big(\frac{dx}{dt}\Big)^2 = 0 \tag{5.4}$$

To step 3 Consider the following two cases.
Case 1 $\frac{dx}{dt} = 0\,\forall t$, it follows that (4.3) is satisfied, i.e. vertical lines x=const. are geodesics.
Case 2 There exists a t such that $\frac{dx}{dt} \neq 0$. Locally, we can solve for x as a function of y. Substitute

$$u = \frac{dx}{dy} \tag{5.5}$$

then

$$\frac{dx}{dt} = u\frac{dy}{dt}. \tag{5.6}$$

Then it follows by the chain rule

$$\frac{d^2x}{dt^2} = \frac{du}{dy}\Big(\frac{dy}{dt}\Big)^2 + u\frac{d^2y}{dt^2} \tag{5.7}$$

$$\begin{aligned}
\frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} &= \frac{d^2x}{dt^2} = \frac{du}{dy}\Big(\frac{dy}{dt}\Big)^2 + u\frac{d^2y}{dt^2} && \text{insert} \quad (4.3)\\
&= \frac{du}{dy}\Big(\frac{dy}{dt}\Big)^2 + u\Big(\frac{1}{y}\Big(\frac{dy}{dt}\Big)^2 - \frac{1}{y}\Big(\frac{dx}{dt}\Big)^2\Big) && \text{insert} \quad (4.4)\\
\frac{2u}{y}\Big(\frac{dy}{dt}\Big)^2 &= \frac{du}{dy}\Big(\frac{dy}{dt}\Big)^2 + u\Big(\frac{1}{y}\Big(\frac{dy}{dt}\Big)^2 - \frac{u^2}{y}\Big(\frac{dy}{dt}\Big)^2\Big) && \text{insert} \quad (4.5), (4.6)\\
\frac{2u}{y} &= \frac{du}{dy} + u\Big(\frac{1}{y} - \frac{u^2}{y}\Big)\\
\frac{du}{dy} &= \frac{u^3+u}{y}
\end{aligned}$$

Seperation of variables leads to

$$\frac{\mathrm{d}u}{u^3+u} = \frac{\mathrm{d}y}{y} \tag{5.8}$$

Thus, we have to solve

$$\int \frac{\mathrm{d}u}{u^3+u} = \int \frac{\mathrm{d}y}{y} \tag{5.9}$$

Integrating the left side of (4.9) by partial fraction leads to

$$\begin{aligned}
\int \frac{\mathrm{d}u}{u^3+u} &= \int \Big(\frac{1}{u} - \frac{u}{u^2+1}\Big)\mathrm{d}u \\
&= \log(u) - \frac{1}{2}\log(u^2+1) \\
&= \log\left(\frac{u}{(u^2+1)^{\frac{1}{2}}}\right)
\end{aligned}$$

and the right side of (4.9) leads to

$$\int \frac{\mathrm{d}y}{y} = \log(cy)$$

for some $c \in \mathbb{R}$. Thus,

$$\begin{aligned}
\log\left(\frac{u}{(u^2+1)^{\frac{1}{2}}}\right) &= \log(cy) \\
\frac{u}{(u^2+1)^{\frac{1}{2}}} &= cy \\
u &= cy(u^2+1)^{\frac{1}{2}} \\
u^2 &= (cy)^2(u^2+1) \\
u^2(1-(cy)^2) &= (cy)^2 \\
u &= \pm\frac{cy}{\sqrt{1-(cy)^2}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad \frac{dx}{dy} &= u(y) = \pm\frac{cy}{\sqrt{1-(cy)^2}} \\
\Rightarrow \qquad \int\frac{dx}{dy} &= \mathrm{x} = \pm\int\frac{cy}{\sqrt{1-(cy)^2}}\mathrm{d}y \\
&= \pm\sqrt{\left(\frac{1}{c}\right)^2 - y^2} + d,
\end{aligned}$$

for some $d \in \mathbb{R}$. This geodesic has a form of a semi-circle of equation

$$(x-d)^2 + y^2 = \left(\frac{1}{c}\right)^2,$$

which is centered on the x-axis.

$\square$

5.2.2 Arnold's cat map

Consider once again the linear map, [Hasselblatt/Katok], [Burns/Gidea], [Nitecki]

$$A : \mathbb{T}^2 \to \mathbb{T}^2,$$
$$(x, y) \mapsto (2x + y, x + y)(\text{mod } 1),$$

or equivalently $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (mod 1), i.e. a map on the compact manifold $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, which is a nilmanifold.

cat map, who had first described the action of this map on $\mathbb{T}^2$ via an image of a cat head, which is shared under this map and after some number of iterations appears again. However, the above map itself was first introduced by Thom. group, i.e. nilpotent, and $\mathbb{Z}^2$ is a discrete subgroup of $\mathbb{R}^2$, see chapter 1.

Let us see, why this map is an example for an Anosov diffeomorphism.

Step 1 Calculate the eigenvalues of the matrix A. The characteristic polynomial is: $p(\lambda) = \lambda^2 - \text{trace}(A)\lambda + \det(A) = \lambda^2 - 3\lambda + 1$. Thus, solving $\lambda^2 - 3\lambda + 1 = 0$ we get the following eigenvalues: $\lambda_s = \frac{3-\sqrt{5}}{2} < 1$, $\lambda_u = \frac{3+\sqrt{5}}{2} > 1$, i.e. A is a hyperbolic matrix.

Step 2 Calculate the eigenspaces E^s, corresponding to the eigenvalue λ_s, and E^u, corresponding to the eigenvalue λ_u, i.e. solve the following equations known from the linear algebra course:

$$(A - \lambda_s * Id) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{5.10}$$

$$(A - \lambda_u * Id) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{5.11}$$

Calculation for (2.1):	Conclusion:
$\begin{pmatrix} \frac{4-3+\sqrt{5}}{2} & 1 \\ 1 & \frac{2-3+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	The eigenvectors to the eigenvalue λ_s
$\rightsquigarrow \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	lie on the line $\frac{1+\sqrt{5}}{2} * x + y = 0$
$\rightsquigarrow \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\iff y = \frac{-1-\sqrt{5}}{2} * x$
Calculation for (2.2):	Conclusion:
$\begin{pmatrix} \frac{4-3-\sqrt{5}}{2} & 1 \\ 1 & \frac{2-3-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	The eigenvectors to the eigenvalue λ_u
$\rightsquigarrow \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	lie on the line $\frac{1-\sqrt{5}}{2} * x + y = 0$
$\rightsquigarrow \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\iff y = \frac{-1+\sqrt{5}}{2} * x$

We conclude, the corresponding eigenvectors are :
$s = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$ and $u = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix}$.
Thus $E_p^s = \{ts \mid t \in \mathbb{R}\}$ and $E_p^u = \{tu \mid t \in \mathbb{R}\}$

Step 3 We conclude with the aid of step 2 that in each non-fixed point $p \in \mathbb{T}^2$there exists a splitting of the tangent bundle of the two-dimensional torus $\mathbb{T}^2$, i.e $\mathrm{T}_p\mathbb{T}^2 = E_p^s \oplus E_p^u$, which is invariant under $D(A)_p : \mathrm{T}_p\mathbb{T}^2 \to \mathrm{T}_p\mathbb{T}^2$.

Step 4 $D(A)_p : E_p^s \to E_p^s$ is contracting and $D(A)_p : E_p^u \to E_p^u$ is expanding due to the result in step 1: $\lambda_s = \frac{3-\sqrt{5}}{2} < 1, \lambda_u = \frac{3+\sqrt{5}}{2} > 1$.

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5.3 Selbständigkeitserklärung

Hiermit erkläre ich, dass ich die vorgelegte Bachelorarbeit eigenständig verfasst und keine anderen als die im Literaturverzeichnis angegebenen Quellen, Darstellungen und Hilfsmittel benutzt habe. Ich erkläre hiermit weiterhin, dass die vorgelegte Arbeit zuvor weder von mir noch -soweit mir bekannt ist - von einer anderen Person an dieser oder einer anderen Hochschule eingereicht wurde.

Darüber hinaus ist mir bekannt, dass die Unrichtigkeit dieser Erklärung eine Benotung der Arbeit mit der Note "nicht ausreichend" zur Folge hat und dass Verletzungen des Urheberrechts strafrechtlich verfolgt werden können.

(Datum) (Unterschrift)