Stable, Oscillatory Viscous Profiles of Weak Shocks in Systems of Stiff Balance Laws

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Abstract

This paper is devoted to a phenomenon in hyperbolic balance laws, first described by Fiedler and Liebscher [2], which is similar in spirit to the Turing instability. The combination of two individually stabilising effects can lead to quite rich dynamical behaviour, like instabilities, oscillations, or pattern formation.

Our problem is composed of two ingredients. First, we have a strictly hyperbolic conservation law. The second part is a source term which, alone, would describe a simple, stable kinetic behaviour: all trajectories end by converging monotonically to an equilibrium. The balance law, constructed of these two parts, however, can support viscous shock profiles with oscillatory tails. They emerge from a Hopf-like bifurcation point in the associated travelling-wave system.

The main result establishes convective stability of oscillatory viscous profiles to weak shocks with extreme speed: if the speed of the wave exceeds any characteristic speed, then the profile is linearly stable in a suitable exponentially weighted space. For intermediate speeds, the profiles are absolutely unstable.

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1 Travelling Waves

Searching for viscous shock profiles of the RIEMANN problem, we consider systems of hyperbolic balance laws of the form

$$u_t + f(u)_x \;=\; \varepsilon^{-1} g(u) + \varepsilon\delta u_{xx}, \tag{1.1}$$

with $u = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}^n$, $f \in C^6$, $g \in C^5$, $\delta > 0$ fixed, and with real time t and space x. The balance law is assumed to be hyperbolic, i.e. $f'(u)$ has only real eigenvalues.

The case of conservation laws, $g \equiv 0$, has been studied extensively. Here, we consider stiff source terms $\varepsilon^{-1}g$, $\varepsilon > 0$ small. Stiff sources arise frequently in models of chemical reactions and combustion processes and in relaxation systems.

We are looking for travelling-wave solutions with wave speed s of the form

$$u \;=\; u\left(\frac{x - st}{\varepsilon}\right). \tag{1.2}$$

To solve (1.1) they must satisfy

$$-s\dot{u} + A(u)\dot{u} \;=\; g(u) + \delta\ddot{u}. \tag{1.3}$$

Here $A(u) = f'(u)$ denotes the Jacobian of the flux and the dot indicates the differentiation with respect to the comoving, rescaled coordinate $\xi = (x - st)/\varepsilon$. Note that (1.3) is independent of ε.

Viscous profiles are travelling waves (1.2) which connect two asymptotic states $u^\pm$. They are solutions of (1.3) for which asymptotic states

$$\lim_{\xi \to \pm\infty} u(\xi) = u^\pm \tag{1.4}$$

exist. In the limit $\varepsilon \searrow 0$ they give rise to a solution of the RIEMANN problem of (1.1) with values $u = u^\pm$ connected by a shock front travelling with shock speed s.

The second-order travelling-wave equation (1.3) can be rewritten as a first-order system

$$\begin{aligned} \dot{u} \;&=\; v, \\ \delta\dot{v} \;&=\; -g(u) + (A(u) - s)v. \end{aligned} \tag{1.5}$$

Viscous profiles are now heteroclinic orbits connecting equilibria $(u = u^\pm, v = 0)$ of (1.5). Note that any such equilibrium is also a zero of the source term,

$$g(u^\pm) \;=\; 0. \tag{1.6}$$

In general, the source term $g(u)$ may depend on some but not all components of u. For instance a chemical reaction depends on concentrations and temperature, but not on velocities.

In this work, we assume that the u_0-component does not contribute to the reaction terms and that the origin is an equilibrium state:

$$\begin{aligned} g \;=\; g(u_1,\ldots,u_{n-1}) \;&=\; (g_0,g_1,\ldots,g_{n-1})^T, \\ g(0) \;&=\; 0. \end{aligned} \tag{1.7}$$

This gives rise to a line of equilibria

$$u_0 \in \mathbb{R}, \qquad u_1 = \cdots = u_{n-1} = 0, \qquad v = 0, \tag{1.8}$$

of our viscous-profile system (1.5).

The asymptotic behaviour of viscous profiles $u(\xi)$ for $\xi \to \pm\infty$ depends on the linearisation L of (1.5) at $u = u_\pm$, $v = 0$. In block-matrix notation corresponding to coordinates (u, v) we have

$$L \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ -\delta^{-1}g' & \delta^{-1}(A-s) \end{pmatrix}. \tag{1.9}$$

Here $A = A(u)$ and $g' = g'(u)$ describe the Jacobi matrices of flux and source at $u = u^\pm$. To abbreviate (1.9), we wrote s rather than $s \cdot \mathrm{id}$.

We now investigate the failure of normal hyperbolicity of L along the line of equilibria $u = (u_0, 0, \ldots, 0)$, $v = 0$. Although our method applies in complete generality, we just present the analysis for systems of the smallest possible dimension, $n = 3$, for which purely imaginary eigenvalues of L arise when $\delta > 0$ is fixed small enough.

2 Bifurcation from Lines of Equilibria

We restrict ourselves to the three-dimensional example

$$\begin{aligned} A(u_0,0,0) \;&=\; A_0 + u_0 \cdot A_1 + \mathcal{O}(u_0^2), \\ A_0 \;&=\; \begin{pmatrix} \alpha_0 & & \\ & \alpha_1 & \\ & & \alpha_2 \end{pmatrix}, \qquad A_1 \;=\; (\beta_{ij})_{0\le i,j\le 2} \quad \text{symmetric}, \\ g'(0) \;&=\; \begin{pmatrix} 0 & & \\ & \gamma_{11} & \gamma_{12} \\ & \gamma_{21} & \gamma_{22} \end{pmatrix}, \qquad \gamma_{ij} \neq 0, \end{aligned} \tag{2.1}$$

with omitted entries being zero.

Proposition 1 *Consider the linearisation $L = L(u_0)$, see (1.9), along the line $u_0 \in \mathbb{R}, u_1 = u_2 = 0$ of equilibria of the viscous-profile system (1.5) in $\mathbb{R}^3$. Assume (2.1) holds.*

For $s \notin \{\alpha_0, \alpha_1, \alpha_2\}$, small $|u_0|$, and $\delta \searrow 0$ the spectrum of L then decouples into two parts:

1. *an unboundedly growing part* $\operatorname{spec}_\infty L \;=\; \delta^{-1}\operatorname{spec}(A-s) + \mathcal{O}(1)$
2. *a bounded part* $\operatorname{spec}_{\mathrm{bd}} L \;=\; \operatorname{spec}((A-s)^{-1}g') + \mathcal{O}(\delta)$

Here A and g are evaluated at $u = (u_0, 0, 0)$. For $\alpha_1 \neq \alpha_2$, $\gamma_{11} + \gamma_{22} \neq 0$, $\delta \searrow 0$, the bounded part, $\operatorname{spec}_{\mathrm{bd}} L$, *at $u_0 = 0$ limits for values*

$$\begin{aligned} s \;=\; s^{\mathrm{crit}} \;&=\; \frac{\alpha_1\gamma_{22} + \alpha_2\gamma_{11}}{\gamma_{11} + \gamma_{22}}, \\ 1 \;&<\; \frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}} \end{aligned} \tag{2.2}$$

onto simple eigenvalues

$$\begin{aligned} \mu_0 \;&\in\; \{0, \pm i\omega_0\}, \\ \omega_0 \;&=\; \frac{\gamma_{11} + \gamma_{22}}{\alpha_1 - \alpha_2}\sqrt{\frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}} - 1}. \end{aligned} \tag{2.3}$$

with eigenvectors $\binom{\tilde{u}}{\tilde{v}}$ given by $\tilde{v} = \mu_0 \tilde{u}$ and

$$\begin{aligned} \tilde{u} \;&=\; (1, 0, 0)^T && \text{for } \mu_0 = 0, \\ \tilde{u} \;&=\; \left(0, 1, -\tfrac{\gamma_{11}}{\gamma_{12}} \pm i\tfrac{\gamma_{11}}{\gamma_{12}}\sqrt{\tfrac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}} - 1}\right)^T && \text{for } \mu_0 = \pm i\omega_0. \end{aligned} \tag{2.4}$$

Proposition 1 describes a situation which looks like a Hopf bifurcation. But, of course, no parameter is involved. Unusually, there is no foliation of the vector field near the critical point, transversal to the line of equilibria.

Bifurcations from lines of equilibria in absence of parameters have been investigated in [6, 4] from a theoretical point of view. We shall apply the abstract result of [4] to our problem of viscous profiles of systems of hyperbolic balance laws near Hopf points of the kind observed above. For a related application to binary oscillators in discretised systems of hyperbolic balance laws see [5]. The case of Takens-Bogdanov-type bifurcation from planes of equilibria can be found in [3].

3 Oscillatory Shock Profiles

Theorem 2 (Liebscher [7]) *Consider the problem (1.4, 1.5) of finding viscous profiles with shock speed s to the system of viscous hyperbolic balance laws (1.1). Let assumptions*

(1.7, 2.1) hold. Suppose further that the following conditions are satisfied:

$$\begin{array}{llcl}
(i) & \alpha_1 & \neq & \alpha_2, \\
(ii) & 0 & \neq & \gamma_{11} + \gamma_{22}, \\
(iii) & 1 & < & \frac{\gamma_{12}\gamma_{21}}{\gamma_{11}\gamma_{22}}, \\
(iv) & \alpha_0 & \neq & s^{\text{crit}} = \frac{\alpha_1\gamma_{22}+\alpha_2\gamma_{11}}{\gamma_{11}+\gamma_{22}}, \\
(v) & 0 & \neq & \beta_{11}\gamma_{22} + \beta_{22}\gamma_{11} - \beta_{12}\gamma_{21} - \beta_{21}\gamma_{12}, \\
(vi) & 0 & \neq & g_0''(0)[\tilde{u}, \overline{\tilde{u}}].
\end{array} \tag{3.1}$$

The first four conditions are these of proposition 1. Therefore, a pair of purely imaginary simple eigenvalues occurs at the HOPF *point* $(u_0^\delta, 0, \ldots, 0)$ *near the origin for the linearisation* L*, in the limit* $\delta \to 0$*, for a fixed speed* $s = s^{\text{crit}}$*. Assumption (v) is a transversality condition. It guarantees that the* HOPF *eigenvalues cross the imaginary axis with nonconstant speed,* $\frac{\mathrm{d}}{\mathrm{d}u_0}\mu(0) \neq 0$*. The nondegeneracy condition (vi) excludes the case of usual* HOPF *bifurcation; no invariant foliation transversal to the line of equilibria is possible. Here* $\tilde{u}$ *denotes the first part of the* HOPF *eigenvector (2.4) and* $\overline{\tilde{u}}$ *its complex conjugate.*

Under the above assumptions the result of [4] applies. A bifurcation occurs at the origin. Depending on the sign

$$\begin{aligned} \eta \;=\; & \operatorname{sign}(\beta_{12}\gamma_{21} + \beta_{21}\gamma_{12} - \beta_{11}\gamma_{22} - \beta_{22}\gamma_{11}) \operatorname{sign}(\gamma_{11}\gamma_{22}) \cdot \\ & \cdot \operatorname{sign}(\alpha_0 - s^{\text{crit}}) \operatorname{sign}(g_0''(0)[\tilde{u}, \overline{\tilde{u}}]). \end{aligned} \tag{3.2}$$

we call the bifurcation elliptic, *if* $\eta = -1$*, and* hyperbolic, *for* $\eta = +1$*. Then the following holds true in a neighbourhood* U *of the* HOPF *point within a three-dimensional centre manifold. See figure 3.1.*

In the hyperbolic case, $\eta = +1$*, all non-equilibrium trajectories leave the neighbourhood* U *in positive or negative time direction, or both. The stable and unstable sets of the* HOPF *point, respectively, form cones around the positive/negative* u_0*-axis, with asymptotically elliptic cross section near their tips. These cones separate regions with different convergence behaviour.*

In the elliptic case all non-equilibrium trajectories starting in U *are heteroclinic between equilibria* $u^\pm = (u_0^\pm, 0, \ldots, 0)$*,* $v = 0$*, on opposite sides of the* HOPF *point. If* f*,* g *are real analytic near the origin, then the two-dimensional strong stable and strong unstable manifolds,* $W^s(u^+)$ *and* $W^u(u^-)$*, of the asymptotic states within the centre manifold intersect at an angle that possesses an exponentially small upper bound in terms of* $|u^\pm|$*.*

Since the conditions (3.1) define open regions, the results persist for shock speeds s *in a small open interval around* s^{crit}*, even when* f*,* g *remain fixed. The* HOPF *bifurcation will then occur at a point* $(\breve{u}_0^\delta, 0, \ldots, 0)$*, with* $\breve{u}_0^\delta$ *converging to a point near the origin, for* $\delta \searrow 0$*.*

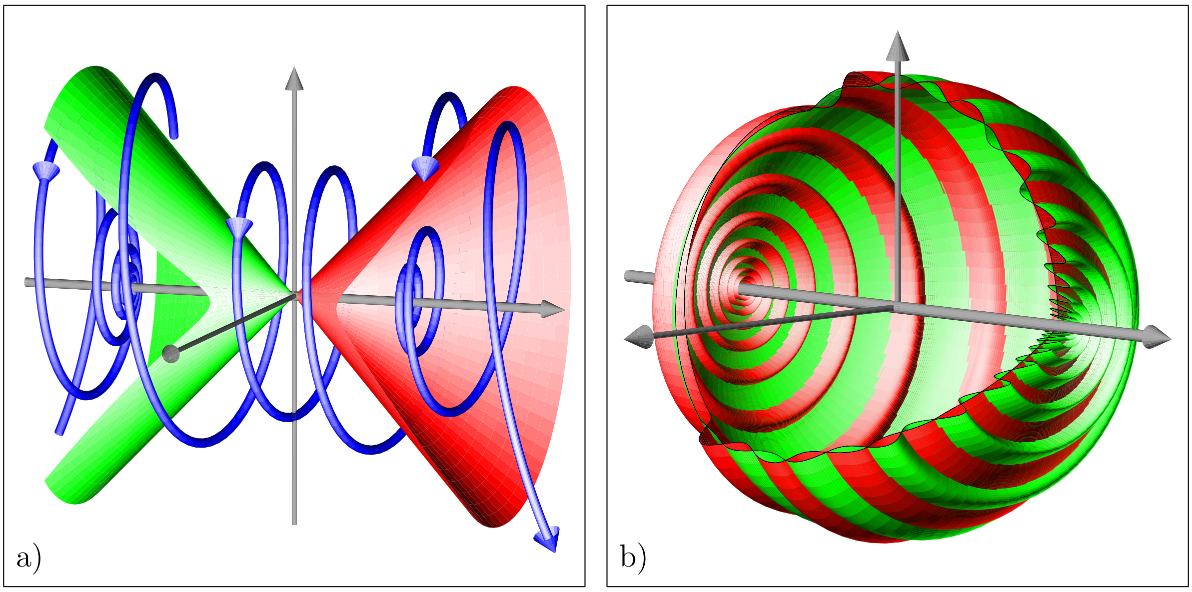


Figure 3.1: Dynamics near HOPF bifurcation from lines of equilibria. Hyperbolic (a), $\eta = +1$, and elliptic (b), $\eta = -1$, case.

In the elliptic case, $\eta = -1$, we observe (at least) pairs of weak shocks with oscillatory tails, connecting u^- and u^+. In the hyperbolic case, $\eta = +1$, viscous profiles leave the neighbourhood U and thus, possibly, represent large shocks.

We shall focus on the elliptic case, $\eta = -1$. The heteroclinic connections, figure 3.2a, which fill an entire neighbourhood of the HOPF point in the centre manifold then lead to travelling waves of the balance law (1.1, 1.2). In figure 3.2b, such a wave is shown, and a generic projection of the n-dimensional space of u-values onto the real line was used. The oscillations imposed by the purely imaginary eigenvalues now look like a GIBBS phenomenon. But here, they are an intrinsic property of the analytically derived solution.

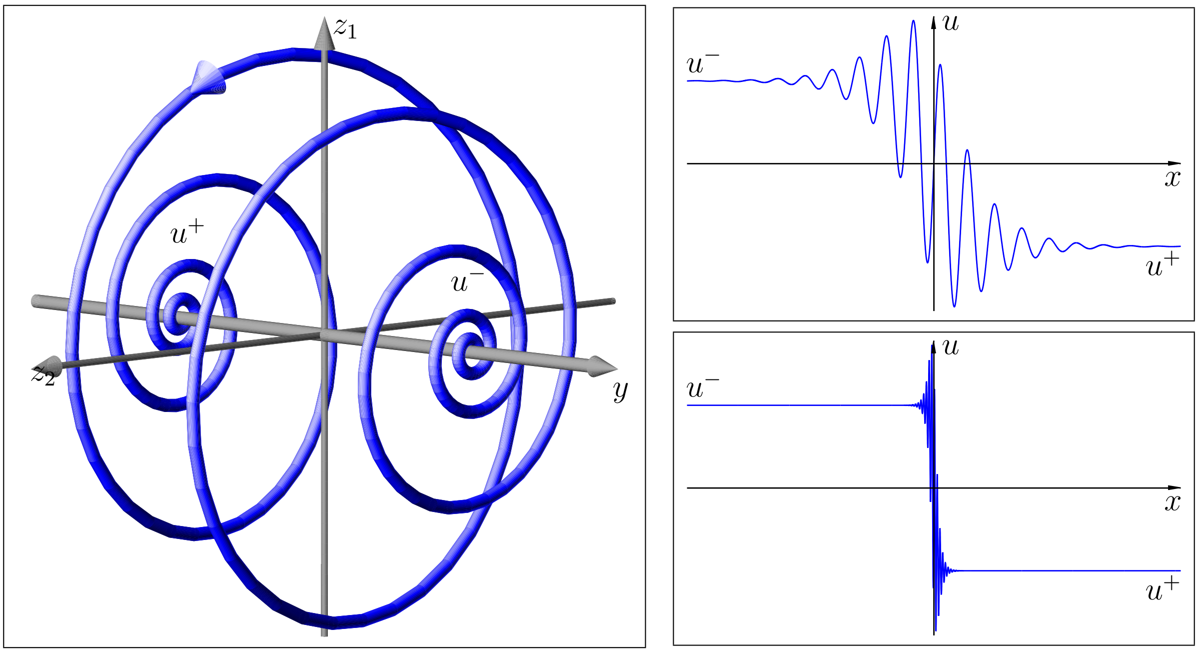
Corollary 3 *Theorem 2 hold true for $\eta = \pm 1$ with the following specific choices of a gradient flux term $f(u) = \nabla\Phi(u)$,*

$$\Phi(u) \;=\; -\frac{1}{2}u_0^2 + u_1^2 + \frac{1}{2}u_2^2 - \frac{1}{2}u_0u_1^2, \tag{3.3}$$

and a reaction term $g(u)$,

$$g(u) \;=\; \begin{pmatrix} -\eta u_1^2 & - & \eta u_2^2 \\ \frac{1}{4}u_1 & + & \frac{3}{2}u_2 \\ -\frac{3}{2}u_1 & - & 4u_2 \end{pmatrix}. \tag{3.4}$$

These choices correspond to values $\alpha_0 = -1$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_{11} = -1$, all other $\beta_{ij} = 0$,



a) heteroclinic orbit near the Hopf point. b) profile for two values of ε.

Figure 3.2: Oscillatory viscous profiles emerging from the Hopf point in the elliptic case of example (3.3, 3.4).

$\gamma_{11} = 1/4$, $\gamma_{12} = 3/2$, $\gamma_{21} = -3/2$, $\gamma_{22} = -4$. *In addition,* g_0'' *is positive* ($\eta = -1$) *or negative* ($\eta = 1$) *definite. The critical shock speed is* $s^{\text{crit}} = \frac{31}{15}$.

In the above example the flux function is a gradient. For the conservation-law part without the source g, the travelling-wave equation (1.3) would be represented by a gradient flow without any possible oscillatory behaviour. The pure kinetics, $\dot{u} = g(u)$, too, does not support oscillations. On the u_1, u_2 subspace, the flow $g'(u)$ is linear with real and negative eigenvalues. All trajectories end by converging monotonically to the equilibrium line. Anyhow, the interaction of the conservation law with the source produces oscillatory viscous profiles. These profiles can be even stable, in some sense, as we shall in the following section.

4 Stability, Main Result

Our next aim is to find stability of the shock profiles (1.2) with respect to perturbations in the parabolic equation (1.1).

To achieve this, we have to study the spectrum of the linearisation of the p.d.e. operator associated to (1.1). The spectrum is a disjoint union of the *point spectrum* and the *essential spectrum*. If the entire spectrum is contained in the left half plain, we speak of spectral stability, which can be extended to linear stability by establishing suitable estimates of the resolvent. Parts of the essential spectrum in the right half plane typically correspond to continua of unstable modes. In some cases the growth of these modes may be dominated by their convection to either $-\infty$ or $+\infty$. Perturbations may grow but travel away. Pointwise, the perturbations decay to zero. We call this situation *convective stability* in contrast to *absolute instability*, where growing modes exist which do not travel away.

Convective stability can be investigated through the introduction of exponentially weighted norms $\| \cdot \|_\nu$,

$$\|u\|_\nu^2 \;=\; \int_{-\infty}^{\infty} |e^{\nu x} u(x)|^2 \, \mathrm{d}x, \tag{4.1}$$

instead of the usual L^2 norm. Choose, for instance, $\nu > 0$. Then, perturbations of fixed L^2 norm near $+\infty$ will be very large in the weighted norm. Near $-\infty$, however, a perturbation of the same L^2 norm will be very small due to the small weight. Modes that travel towards $-\infty$ are multiplied by $e^{\nu x}$ which gets smaller as x goes to $-\infty$. In fact, growing modes might even become decaying. Absolute instabilities are, in contrast, not affected by supplementary weights. Convective stability is defined as spectral stability with respect to a appropriate exponentially weighted norm.

Theorem 4 (Liebscher [7]) *Consider the viscous balance law (1.1) under the conditions (1.7, 2.1, 3.1). Then theorem 2 applies; suppose the elliptic case holds. Then for small* δ, $0 < \delta < \delta_0$, *there exist viscous profiles (1.2, 1.4) to weak shocks, see figure 3.2. These profiles travel with speed* s *near* s^{crit}, *as defined in theorem 2, and connect asymptotic states* $u^\pm$ *of small distance,* $|u^+ - u^-| < \epsilon$.

Suppose further, that the considered profiles have extreme speed s *that exceeds all characteristic speeds, in particular*

$$s^{\mathrm{crit}} \;>\; \alpha_0, \alpha_1, \alpha_2 \qquad \text{or} \qquad s^{\mathrm{crit}} \;<\; \alpha_0, \alpha_1, \alpha_2. \tag{4.2}$$

We call the corresponding shocks hypersonic. Then, possibly after further reduction of δ_0 *and* ϵ, *all those travelling waves are convectively stable.*

For intermediate wave speeds, within the range of characteristic speeds, the constructed profiles cannot be stabilised by any exponential weight. They are absolutely unstable.

Sketch of the proof. A crucial point of the proof is the reformulation of the eigenvalue problem to the elliptic p.d.e. on the real line as a dynamical system with the spatial coordinate as the time-like variable.

We have to calculate the spectrum of the linearisation of the p.d.e. (1.1) at a travelling wave u in co-moving, rescaled coordinates

$$u_\tau \;=\; \delta u_{\xi\xi} + s u_\xi - A(u)u_\xi + g(u). \tag{4.3}$$

with respect to an exponentially weighted space L_ν. Alternatively, we can consider an appropriately modified operator $\mathcal{M}_\nu$ in L_2. The associated eigenvalue problem $0 = (\mathcal{M}_\nu - \lambda)\tilde{u}$ can be recast as a first order system

$$\mathcal{T}_\nu(\lambda)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} - M_\nu(\cdot,\lambda)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} \tag{4.4}$$

the so called "spatial dynamics". Here M is a ξ-dependent matrix.

The essential spectrum consists of all complex values λ, for which $\mathcal{T}_\nu(\lambda)$ is Fredholm with nonzero index or not Fredholm at all. The point spectrum is the set of eigenvalues, i.e. of values λ, for which $\mathcal{T}_\nu(\lambda)$ is Fredholm with index zero and nontrivial kernel.

Now, the Fredholm properties of T depend only on the asymptotic states $M(\pm\infty,\lambda) = L(u^\pm) + \nu$, see (1.9). $\mathcal{T}_\nu(\lambda)$ is Fredholm with index 0 if, and only if, both asymptotic matrices are hyperbolic with coinciding unstable dimensions. [1, 8] Together with the constraint imposed by the pair of conjugate complex eigenvalues near the Hopf point, this proves the absolute instability of waves with intermediate speeds.

For hypersonic waves, on the other hand, one can find appropriate exponential weights ν that push the essential spectrum to the negative half plane. It remains to exclude the possibility of a point spectrum with nonnegative real part. This can be done by estimating the spectrum of the linearisation around the trivial solution and using structural-stability arguments.

5 Discussion

We have constructed viscous profiles with oscillatory tails in a three-dimensional system of hyperbolic conservation laws with stiff source terms. They arise near a Hopf-like bifurcation point along a line of equilibria in the associated travelling-wave equation.

The oscillations arise despite the fact that our problem is composed of two ingredients which, if considered separately, resist oscillations. The conservation-law part is strictly hyperbolic, the flux can even be a gradient. Without the source, the conservation law gives rise to monotone viscous profiles of weak shocks. The source, taken alone, stabilises the dynamics: all trajectories of the pure kinetics end by converging monotonically to equilibria. The unsuspected existence of oscillations due to the combination of non-oscillatory ingredients is similar to the TURING instability observed in reaction-diffusion equations.

The oscillatory waves can be constructed for any wave speed, regardless of the characteristic speeds of the conservation law. In three dimensional systems, they are convectively stable only for hypersonic shocks, i.e. for profiles with extreme speeds larger or smaller than all characteristic speeds. For intermediate speeds the waves are absolutely unstable.

Note how the oscillatory profiles contradict the common expectations that arise from the study of hyperbolic conservation laws. Viscous profiles to weak shocks, especially stable ones, should respect the LAX criterion. The profiles which we have constructed are of non-LAX nature. In particular, they are stable only for hypersonic speed (that exceeds all characteristic speeds).

The exponential weights which are needed to prove convective stability are strong in relation to the rates at which the profiles converge to their asymptotic states. This has the following reason. Because we investigate the stability of a particular wave, the considered norm must separate it from the continuum of profiles that emerge from the HOPF point. We recall that, for fixed speed, an entire neighbourhood of the bifurcation point within its centre manifold is filled with heteroclinic orbits. For nearby wave speeds, a HOPF bifurcation occurs at a slightly shifted critical point. In addition, the travelling-wave equation is equivariant with respect to translations. Therefore, we obtain at least a four-parameter family of oscillatory profiles. The exponential weights centred at an oscillatory travelling wave push any other profile out of the examined space.

For example, consider an oscillatory profile $u(\xi)$. Then the linearisation of the balance law around $u(\cdot)$ has a zero eigenvalue with the eigenvector $u_\xi(\cdot)$, due to the translational symmetry. This eigenvector does not belong to the exponentially weighted space that we used to prove convective stability, and the trivial eigenvalue has been eliminated. Perturbations that would lead to a shift of the wave are likewise eliminated from our stability analysis.

So, what does convective stability mean? Take any small perturbation with bounded support of a convectively stable oscillatory profile, for example. Then, in the linearised equation, this perturbation will pointwise decay to zero. On any bounded subinterval it will decay uniformly.

We have proven linear stability in the exponentially weighted space. Nonlinear stability is a much more difficult problem due to the exponential weights. Perturbations that are convected to $-\infty$ may increase (for weights with positive exponents) exponentially in the L^2 or supremum norm. If they become large enough, the linearisation of the balance law is no longer a sufficient approximation of the whole system. Nonlinear effects become dominant. The nonlinearity could accelerate the growth of the perturbation. The growth need not to be dominated by the convection to $-\infty$, anymore. The perturbation may even spread out in the opposite direction and destroy the very wave structure itself. Completely different patterns may evolve.

A bridge from the linear stability, as we proved here, to nonlinear stability on bounded domains is provided by the results of SANDSTEDE and SCHEEL [9]. Indeed, with suitable separated boundary conditions, the spectrum of the linearisation at an oscillatory profile is bounded to the left of the imaginary axis with positive gap. In fact, the boundary conditions must not introduce additional unstable point eigenvalues. Appropriate conditions are given in [9] This is sufficient to ensure even nonlinear stability on any fixed bounded subinterval. Perturbations can grow only slightly and then disappear through the boundary. Actually, on bounded subintervals, all weighted norms are equivalent to the L^2-norm.

In numerical simulations, the calculations must be carried out on bounded domains. The described effects should be nonlinearly stable phenomena. In the limit $\varepsilon \searrow 0$ the observed oscillations manifest themselves as overshooting at the shock layer. If such a phenomenon occurs in numerical simulations then, typically, an incapable numerical scheme is blamed for it. Numerical viscosities are used to smooth out unwanted oscillations. However, the oscillations in our example are intrinsic properties of the solution and should be represented by a numerical scheme. In fact, our example could become a test-case for numerical schemes that are designed for systems with stiff source terms. Further work has to be done is this direction.

References

[1] W. A. Coppel. *Dichotomies in Stability Theory*, volume 629 of *Lecture Notes in Mathematics*. Springer, New York, 1978.

[2] B. Fiedler and S. Liebscher. Generic Hopf bifurcation from lines of equilibria without parameters: II. Systems of viscous hyperbolic balance laws. *SIAM Journal on Mathematical Analysis*, 31(6):1396–1404, 2000.

[3] B. Fiedler and S. Liebscher. Takens-Bogdanov bifurcations without parameters, and oscillatory shock profiles. In H. Broer, B. Krauskopf, and G. Vegter, editors, *Global*

Analysis of Dynamical Systems, Festschrift dedicated to Floris Takens for his 60th birthday. IOP, to appear 2001.

[4] B. Fiedler, S. Liebscher, and J. C. Alexander. Generic Hopf bifurcation from lines of equilibria without parameters: I. Theory. *Journal of Differential Equations*, 167:16–35, 2000.

[5] B. Fiedler, S. Liebscher, and J. C. Alexander. Generic Hopf bifurcation from lines of equilibria without parameters: III. Binary oscillations. *Internatianal Journal on Bifurcation and Chaos in Applied Sciences and Engineering*, 10(7):1613–1622, 2000.

[6] S. Liebscher. Stabilität von Entkopplungsphänomenen in Systemen gekoppelter symmetrischer Oszillatoren. Diploma thesis, Free University Berlin, 1997.

[7] S. Liebscher. *Stable, Oscillatory Viscous Profiles of Weak, non-Lax Shocks in Systems of Stiff Balance Laws*. PhD thesis, Free University Berlin, 2000.

[8] K. J. Palmer. Exponential dichotomies and Fredholm operators. *Proceedings of the American Mathematical Society*, 104:149–156, 1988.

[9] B. Sandstede and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Physica D*, 145:233–277, 2000.